LECTURE NOTES 19

LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS
(SLIGHT RETURN)

Before continuing on with our onslaught of the development of relativistic electrodynamics via tensor analysis, I want to briefly discuss an equivalent, simpler method of Lorentz transforming the EM fields $\vec{E}$ and $\vec{B}$ from one IRF($S$) to another IRF($S'$), which also sheds some light (by contrast) on how the EM fields Lorentz transform vs. “normal” 4-vectors.

In P436 Lecture Notes 18.5 {p. 18-22} we discussed the tensor algebra method for Lorentz transformation of the electromagnetic field e.g. in the lab frame IRF($S$), represented by the $EM$ field strength tensor $F^{\mu\nu}$ to another frame IRF($S'$), represented by the $EM$ field strength tensor $F'^{\mu\nu}$ via the relation:

$$F'^{\mu\nu} = \Lambda^{\mu}_{\lambda} F^{\lambda\beta} \Lambda'^{\beta}_{\sigma}$$

n.b. in matrix form: $F' = \Lambda F \Lambda^T = \Lambda F \Lambda$ since $\Lambda$ is symmetric.

Analytically carrying out this tensor calculation by hand can be tedious and time-consuming. If such calculations are to be carried out repeatedly/frequently, we encourage people to code this up and simply let the computer do the repetitive work, which it excels at.

For 1-dimensional Lorentz transformations (only) there is a simpler, less complicated, perhaps somewhat more intuitive method. Starting with the algebraic rules for Lorentz-transforming $\{ \vec{E} \text{ and } \vec{B} \}$ in one IRF($S$) to $\{ E' \text{ and } B' \}$ in another IRF($S'$) e.g. moving with relative velocity $\vec{v} = +v\hat{\nu}$ with respect to IRF($S$):

|| component(s): $E'_x = E_x$  \hspace{1cm} $B'_x = B_x$  \hspace{1cm} $\gamma \equiv 1/\sqrt{1-\beta^2}$  \hspace{1cm} $\beta \equiv v/c$

\hspace{1cm} $E'_y = \gamma (E_y - \beta B_x)$  \hspace{1cm} $B'_y = \gamma (B_y + \beta E_x/c)$  \hspace{1cm} $\gamma \equiv 1/\sqrt{1-\beta^2}$  \hspace{1cm} $\beta \equiv v/c$

\hspace{1cm} $E'_z = \gamma (E_z + \beta B_y)$  \hspace{1cm} $B'_z = \gamma (B_z + \beta E_y/c)$

We can write these relations more compactly and elegantly by resolving them into their $||$ and $\perp$ components relative to the boost direction: here, $||$ is along $\vec{v} = +v\hat{\nu}$ and $\perp$ is perpendicular to $\vec{v}$, defined as follows {n.b in general, $\vec{v}$ could be $||$ e.g. to $\hat{x}, \hat{y}, \hat{z}$ or $\hat{r}$}:

$E'^{||} = E'^{\perp}$

$E'^{\perp} = \gamma \left(E^{\perp} + \vec{v} \times B^{\perp}\right) = \gamma \left(E^{\perp} + \vec{\beta} \times B^{\perp}\right)$

$B'^{||} = B'^{\perp}$

$B'^{\perp} = \gamma \left(B^{\perp} - \frac{1}{c^2} \vec{v} \times E^{\perp}\right) = \gamma \left(B^{\perp} - \frac{1}{c} \vec{\beta} \times E^{\perp}\right)$

Now since $\vec{v} = +v\hat{\nu}$ {here} then: $E^{||} \equiv E_x$  \hspace{1cm} $B^{||} \equiv B_x$  \hspace{1cm} and: $B^{\perp} \equiv B_y \hat{\nu} + B_z \hat{\nu}$  \hspace{1cm} $E^{\perp} \equiv E_y \hat{\nu} + E_z \hat{\nu}$

{and similarly for corresponding quantities in IRF($S'$)}.
Since \( v = +v \hat{x} \) and \( E \parallel E_x \), then: \( \vec{v} \times E = \vec{v} \times E_x \hat{x} = 0 \).

And likewise, since \( B \parallel B_x \), then: \( \vec{v} \times B = \vec{v} \times B_x \hat{x} = 0 \).

Thus, we can \( \text{safely} \) write: \( \vec{v} \times E = \vec{v} \times E_{\perp} \) and \( \vec{v} \times B = \vec{v} \times B_{\perp} \), as long as \( v \) is \( \text{always} \parallel \) to one of the components of \( E \) and \( B \) - e.g. \( \hat{x} \) or \( \hat{y} \) or \( \hat{z} \).

Then we \( \text{can} \) write the Lorentz transformation of \( EM \) fields as:

\[
\begin{align*}
E_{\parallel}' &= \gamma E \\
E_{\perp}' &= \gamma \left( E_{\perp} + \frac{1}{c} \vec{v} \times \vec{B} \right) \\
B_{\parallel}' &= B \\
B_{\perp}' &= \gamma \left( B_{\perp} - \frac{1}{c} \vec{v} \times \vec{E} \right)
\end{align*}
\]

This can be written more compactly in 2-D matrix form as:

\[
\begin{pmatrix}
E_{\parallel}' \\
E_{\perp}' \\
B_{\parallel}' \\
B_{\perp}'
\end{pmatrix} =
\begin{pmatrix}
\gamma & 0 \\
0 & \gamma \left( 1 - \frac{\vec{v} \cdot \vec{B}}{c^2} \right)
\end{pmatrix}
\begin{pmatrix}
E_{\parallel} \\
E_{\perp} \\
B_{\parallel} \\
B_{\perp}
\end{pmatrix}
\]

Thus, we see that for the \( EM \) fields vs. the 3-D space-part of a “normal” 4-vector, the \( \parallel \) vs. \( \perp \) components are \( \text{switched} \). \( \vec{B} \) transforms “sort of” like time \( t \), but \( 2 \times 2 \) Lorentz boost matrices for \( ( \vec{E} \) and \( \vec{B} \)) vs. 4-vectors are \( \text{not} \) the same (they are \( \text{similar} \), but they are \( \text{not} \) identical).

We can also write compact \( \text{inverse} \) Lorentz transformations (e.g. from IRF\((S')\) rest frame → IRF\((S)\) lab frame):

\[
\begin{align*}
E_{\parallel} &= E_{\parallel}' \\
E_{\perp} &= \gamma E_{\perp}' + \gamma \vec{v} \times \vec{B}' \\
B_{\parallel} &= B_{\parallel}' \\
B_{\perp} &= \gamma \left( B_{\perp}' - \frac{1}{c} \vec{v} \times \vec{E}' \right)
\end{align*}
\]

This can be written more compactly in 2-D matrix form as:

\[
\begin{pmatrix}
E_{\parallel} \\
E_{\perp} \\
B_{\parallel} \\
B_{\perp}
\end{pmatrix} =
\begin{pmatrix}
\gamma & 0 \\
0 & \gamma \left( 1 + \frac{\vec{v} \cdot \vec{B}}{c^2} \right)
\end{pmatrix}
\begin{pmatrix}
E_{\parallel}' \\
E_{\perp}' \\
B_{\parallel}' \\
B_{\perp}'
\end{pmatrix}
\]

Unit Matrix
Operator Matrix
Scalar Matrix

Thus, we see that for the \( EM \) fields vs. the 3-D space-part of a “normal” 4-vector, the \( \parallel \) vs. \( \perp \) components are \( \text{switched} \). \( \vec{B} \) transforms “sort of” like time \( t \), but \( 2 \times 2 \) Lorentz boost matrices for \( ( \vec{E} \) and \( \vec{B} \)) vs. 4-vectors are \( \text{not} \) the same (they are \( \text{similar} \), but they are \( \text{not} \) identical).
For a general Lorentz transformation (i.e. no restriction on the orientation of \( \vec{v} \) {arbitrary}):

A.) Lorentz transformation from IRF(\( S \)) \( \rightarrow \) IRF(\( S' \)):

\[
\begin{aligned}
\vec{E}' &= \gamma (\vec{E} + \vec{\beta} c \times \vec{B}) - \frac{\gamma^2}{\gamma + 1} \vec{\beta} (\vec{\beta} \cdot \vec{E}) \\
\vec{B}' &= \gamma (\vec{B} - \frac{1}{c} \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma + 1} \vec{\beta} (\vec{\beta} \cdot \vec{B})
\end{aligned}
\]

\[ \vec{\beta} = \vec{v}/c \quad \vec{\beta} = \vec{v}/c \]

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]

Or:

\[
\begin{pmatrix}
\vec{E}' \\
c\vec{B}'
\end{pmatrix} = \begin{pmatrix}
\gamma \left( 1 - \frac{\gamma}{\gamma + 1} \vec{\beta} \vec{\beta} \right) & + \gamma \vec{\beta} \times \\
-\gamma \vec{\beta} \times & \gamma \left( 1 - \frac{\gamma}{\gamma + 1} \vec{\beta} \vec{\beta} \right)
\end{pmatrix}
\begin{pmatrix}
\vec{E} \\
c\vec{B}
\end{pmatrix}
\]

operator matrix

B.) Inverse Lorentz transformation from IRF(\( S' \)) \( \rightarrow \) IRF(\( S \)):

\[
\begin{aligned}
\vec{E} &= \gamma (\vec{E}' - \vec{\beta} c \times \vec{B}') - \frac{\gamma^2}{\gamma + 1} \vec{\beta} (\vec{\beta} \cdot \vec{E}') \\
\vec{B} &= \gamma (\vec{B}' + \frac{1}{c} \vec{\beta} \times \vec{E}') - \frac{\gamma^2}{\gamma + 1} \vec{\beta} (\vec{\beta} \cdot \vec{B}')
\end{aligned}
\]

\[ \vec{\beta} \rightarrow -\vec{\beta} \quad \vec{E} \rightarrow \vec{E}' \quad \vec{B} \rightarrow -\vec{B}' \]

in above relations

Or:

\[
\begin{pmatrix}
\vec{E} \\
c\vec{B}
\end{pmatrix} = \begin{pmatrix}
\gamma \left( 1 - \frac{\gamma}{\gamma + 1} \vec{\beta} \vec{\beta} \right) & - \gamma \vec{\beta} \times \\
+ \gamma \vec{\beta} \times & \gamma \left( 1 - \frac{\gamma}{\gamma + 1} \vec{\beta} \vec{\beta} \right)
\end{pmatrix}
\begin{pmatrix}
\vec{E}' \\
c\vec{B}'
\end{pmatrix}
\]

operator matrix

**Electrodynamics in Tensor Notation**

So now that we know how to represent the EM field in relativistic tensor notation (as \( F^{\mu\nu} \) or \( G^{\mu\nu} \)), we can also reformulate all laws of electrodynamics (e.g. Maxwell’s equations, the Lorentz force law, the continuity equation {expressing electric charge conservation}, etc. . . ) in the mathematical language of tensors.

In order to begin this task, we must first determine how the sources of the EM fields – the electric charge density \( \rho \) (a scalar quantity) and the electric current density \( \vec{J} \) (a 3-D vector quantity) Lorentz transform.

The electric charge density \( \rho = Q/V \) = charge per unit volume (Coulombs/m\(^3\))
Imagine a cloud of electric charge drifting by. Concentrate on an infinitesimal volume $V$ containing charge $Q$ moving at (ordinary) velocity $\vec{u}$:

Then: $\rho = \frac{Q}{V} = \text{charge density (Coulombs/m}^3\text{)}$

And: $\vec{J} = \rho \vec{u} = \text{current density (Amps/m}^2\text{)}$.

A subtle, but important detail:

If there is only one species (i.e. kind / type) of charge carrier contained within the infinitesimal volume $V$, then all charge carriers travel at the same (average / mean) speed $\vec{u}$.

However, if there are multiple species (kinds / types) of charge carriers (e.g. with different masses) and/or different signs of charge carriers contained within the infinitesimal volume $V$ (e.g. electrons $e^-$ with rest masses $m_e c^2$ and protons $p$ with rest masses $m_pc^2$) then the different constituents / species must be treated separately in the following:

If $\exists$ $N$ species:

Current density $\vec{J}_i = \rho_i \vec{u}$ for the $i^{th}$ species ($i = 1, \ldots, N$), the electric charge density $\rho_i = \frac{Q_i}{V}$

And: $\vec{J} = \sum_{i=1}^{N} \vec{J}_i = \sum_{i=1}^{N} \rho_i \vec{u}_i$

We also need to express $\rho$ and $\vec{J}$ in terms of the proper charge density $\rho_0 = \text{volume charge density defined in the rest frame of the charge } Q$, IRF($S_0$).

The infinitesimal rest volume / proper volume = $V_0$ {defined in the rest/proper frame IRF($S_0$)}

The proper charge density: $\rho_0 = \frac{Q}{V_0}$ ← Recall that electric charge $Q$ (like $c$) is a Lorentz invariant scalar quantity

Because the longitudinal direction of motion undergoes Lorentz contraction from the rest frame IRF($S_0$) in the Lorentz transformation $\rightarrow$ another reference frame, e.g. lab frame IRF($S$)

Then: $V = \frac{1}{\gamma_u} V_0$ where: $V_0 = \ell_0 w_0 d_0$ and: $V = \ell wd$, where: $\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}}$ and: $\beta_u = \frac{u}{c}$

If the Lorentz transformation is along (i.e. || to) the length $\ell$, $\ell_0$ of the infinitesimal volumes

Then: $\ell = \frac{1}{\gamma_u} \ell_0$ and the ⊥ components of the volumes are unchanged: $w_0 = w$, $d_0 = d$

Then if: $V = \frac{1}{\gamma_u} V_0$ → $\rho = \frac{Q}{V} = \gamma_u \left( \frac{Q}{V_0} \right) = \gamma_u \rho_0$:

$\vec{J} = \rho \vec{u} = \gamma_u \rho_0 \vec{u} = \rho_0 (\gamma_u \vec{u})$

Recall that the 3-D vector associated with the proper velocity is: $\vec{n} = \gamma_u \vec{u}$ (≡ $\frac{d\ell}{d\tau}$) ⇒ $\vec{J} = \rho_0 \vec{n}$
The zeroth (i.e. temporal/scalar) component of the proper 4-velocity is:

\[ \eta^0 \equiv \frac{dx^0}{d\tau} = \frac{c}{c^\gamma} = \gamma_c \]

The corresponding zeroth (i.e. temporal/scalar) component of the current density 4-vector \( J^\mu \) is:

\[ J^0 = \rho \eta^0 = \rho \gamma_c = \gamma c = \rho_in \]

The current density 4-vector is:

\[ \left( J^0, J^1, J^2, J^3 \right) = \left( \rho c, J_x, J_y, J_z \right) \]

(SI units: Amps/m^2)

Then:

\[ J^\mu = \rho \eta^\mu = \left( \rho \eta^0, \rho \eta^1, \rho \eta^2, \rho \eta^3 \right) \]  

where:

\[ \eta^\mu = \left( \eta^0, \eta^1, \eta^2, \eta^3 \right) = \left( \gamma_c, \gamma_u \bar{u} \right) = \gamma_u \left( c, \bar{u} \right) = \gamma_u \left( c, u_x, u_y, u_z \right) \]

constant scalar quantity

.: \( J^\mu \) is a proper four vector, i.e. \( J^\mu = \text{proper} \) current density 4-vector.

Thus:

\[ J^\mu J_\mu = J^\mu J^\mu \]

is a Lorentz invariant quantity. Is it ???

\[ J^\mu J_\mu = J^\mu J^\mu = \rho^2 \eta^\mu \eta_\mu = \rho^2 \gamma^2 \left( -c^2 + u_x^2 + u_y^2 + u_z^2 \right) = \rho^2 \left( \frac{-c^2 + u^2}{c^2} \right) = -\left( \rho_c \right)^2 \left( \frac{1-u^2/c^2}{c^2} \right) = -\left( \rho_c \right)^2 \]

\[ J^\mu J_\mu = J^\mu J^\mu = -\left( \rho_c \right)^2 = \rho^2 \eta^\mu \eta_\mu = \rho^2 \eta^\mu \eta_\mu = -\rho^2 c^2 \]

Since:

\[ \eta^\mu \eta_\mu = \eta^0 \eta^0 = c^2 \]

Yes, \( J^\mu J_\mu = J^\mu J^\mu \) is a Lorentz invariant quantity!

The 3-D continuity equation mathematically expresses local conservation of electric charge (using differential vector calculus):

\[ \nabla \cdot \vec{J} \left( \vec{r}, t \right) = -\frac{\partial \rho \left( \vec{r}, t \right)}{\partial t} \]  

\( \rho \left( \vec{r}, t \right) = \text{scalar point function, } \vec{J} = \vec{J} \left( \vec{r}, t \right) = 3-D \text{ vector point function} \)

\[ \vec{\nabla} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \]  

(in Cartesian coordinates)

We can also express the continuity equation in 4-vector tensor notation:

\[ \vec{\nabla} \cdot J = \sum_{i=1}^{3} \frac{\partial \rho}{\partial x^i} + \frac{\partial J^i}{\partial x^i} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} \]

and:

\[ \frac{\partial \rho}{\partial t} = \frac{1}{c} \frac{\partial (c \rho)}{\partial t} = \frac{1}{c} \frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial \vec{\nabla}^0} \]

\[ \left( J^0 = c \rho \right) \quad \text{n.b., repeated indices implies summation!} \]

Then:

\[ \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} + \frac{\partial J^0}{\partial x^0} = \frac{\partial J^0}{\partial \vec{\nabla}^0} + \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} = 0 \]

Thus:

\[ \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \text{ or } \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial J^\mu}{\partial x^\mu} = 0 \]

Continuity equation (local charge conservation)
Physically, note that \( \frac{\partial J^\mu}{\partial x^\mu} \) is the 4-dimensional space-time divergence of the current density 4-vector \( J^\mu = (c \rho, \vec{J}) \). The 4-current density \( J^\mu = (c \rho, \vec{J}) \) is divergenceless because \( \frac{\partial J^\mu}{\partial x^\mu} = 0 \).

The 4-vector operator \( \frac{\partial}{\partial x^\mu} \) is called the 4-D gradient operator, (a.k.a the quad operator \( \Box_\mu \) or “quad” for short). However, because the 4-D gradient operator \( \frac{\partial}{\partial x^\mu} \) functions like a covariant 4-vector, e.g. when it operates on contravariant \( J^\mu \) (or any other contravariant 4-vectors), it is often alternatively given the shorthand notation \( \frac{\partial}{\partial x^\mu} \equiv \Box_\mu = \partial_\mu \). Similarly, because the 4-D gradient operator \( \frac{\partial}{\partial x^\mu} \) functions like a contravariant 4-vector, e.g. when it operates on covariant \( J^\mu \) (or any other covariant 4-vectors), it is given the shorthand notation \( \frac{\partial}{\partial x_\mu} \equiv \Box^\mu = \partial^\mu \).

{See/work thru Griffiths Problem 12.55 (p. 543) for more details.}

The contravariant quad/gradient 4-vector operator:
\[
\frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) \equiv \Box_\mu = \partial_\mu
\]

The covariant quad/gradient 4-vector operator:
\[
\frac{\partial}{\partial x_\mu} = \left( -\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) \equiv \Box^\mu = \partial^\mu
\]

Then:
\[
\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} = \frac{\partial^2}{c^2 \partial t^2} + \nabla^2 = \Box_\mu \Box_\mu = \Box^\mu \Box^\mu = \partial_\mu \partial_\mu = \partial^\mu \partial^\mu = \Box^2
\]

= D’Alembertian 4-vector operator = 4-D Laplacian operator = Lorentz-invariant quantity!

The 4-vector product of any two (bona-fide) relativistic 4-vectors is a Lorentz invariant quantity (i.e. the same value in any/all IRF’s)!

We can equivalently write the relativistic 4-D continuity equation as:
\[
\frac{\partial J^\mu}{\partial x^\mu} \equiv \Box_\mu J^\mu = \partial_\mu J^\mu = 0 \quad \Leftarrow \text{n.b. A Lorentz-invariant quantity}!!
\]

⇒ Electric charge is (locally) conserved in any/all IRF’s (as it must be!!!)
Maxwell’s Equations in Tensor Notation

Maxwell’s Equations:

1) Gauss’ law:
\[ \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \]

2) No Magnetic Monopoles:
\[ \nabla \cdot B = 0 \]

3) Faraday’s law:
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

4) Ampere’s law:
\[ \nabla \times B = \mu_o J + \frac{1}{c^2} \frac{\partial E}{\partial t} \]  
[with Maxwell’s Displacement Current]

Can be written as 4-derivatives of the relativistic EM field strength tensors \( F^{\mu\nu} \) and \( G^{\mu\nu} \):

\[ \partial_v F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_o J^\mu \]  
[with Maxwell’s Displacement Current]

\[ \partial_v G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^v} = 0 \]

n.b. summation over \( \nu = 0:3 \) is implied

1) Gauss’ law \[ \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \]. If \( \mu = 0 \) in:
\[ \partial_v F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_o J^\mu \]  
[i.e. \( \partial_v F^{0\nu} = \frac{\partial F^{0\nu}}{\partial x^v} = \mu_o J^0 \), \( \nu = 0:3 \)]

Physically, \( \mu = 0 \) is the temporal/scalar component of any space-time 4-vector.

Then:

\[ \frac{\partial F^{0\nu}}{\partial x^v} = \frac{\partial F^{00}}{\partial x^v} + \frac{\partial F^{01}}{\partial x^v} + \frac{\partial F^{02}}{\partial x^v} + \frac{\partial F^{03}}{\partial x^v} \]

\[ \mu = 0 \] (first row of \( F^{\mu\nu} \))

Row #

\[ F^{\mu\nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & -B_z & B_y \\
-E_y/c & B_z & 0 & B_x \\
-E_z/c & -B_y & -B_x & 0
\end{pmatrix} \]

Column #

\[ \frac{\partial F^{0\nu}}{\partial x^v} = 0 + \frac{1}{c} \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \frac{1}{c} \nabla \cdot E \]  
[\( \mu_o J^0 = \mu_o (c \rho) \)]

\[ \nabla \cdot E = \mu_o c \rho \]  
[\( \nabla \cdot E = \mu_o c^2 \rho \) but: \( \mu_o c^2 = \frac{1}{\varepsilon_0} \) \( \therefore \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \)]

Gauss’ law arises from the \( \mu = 0 \) (scalar / temporal) component of the 4-vector relation:

\[ \partial_v F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^v} = \mu_o J^\mu \]
4) Ampere’s law: 
\[ \nabla \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]

If \( \mu = 1 \) in 
\[ \frac{\partial F_{\mu \nu}}{\partial x^\nu} = \partial_x F_{\mu \nu} = \mu_o J^\mu \]

Then:
\[ \frac{\partial F_{\mu \nu}}{\partial x^\nu} = \frac{\partial F_{10}}{\partial x^0} + \frac{\partial F_{11}}{\partial x^1} + \frac{\partial F_{12}}{\partial x^2} + \frac{\partial F_{13}}{\partial x^3} \]

\( \mu = 1 \) (second row of \( F_{\mu \nu} \))

\[ \frac{\partial F_{11}}{\partial x^0} = -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + 0 + 0 = -\frac{1}{c^2} \frac{\partial E_y}{\partial z} = \left( -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + \nabla \times \vec{B} \right) \]

and: 
\[ \mu_o J^1 = \mu_o J_x \]

Then for \( \mu = 2 \) and \( \mu = 3 \) (third and fourth rows of \( F_{\mu \nu} \)), likewise we find that:

\[ \frac{\partial F_{21}}{\partial x^0} = -\frac{1}{c^2} \frac{\partial E_y}{\partial t} + \nabla \times \vec{B} \]

\[ \frac{\partial F_{31}}{\partial x^0} = -\frac{1}{c^2} \frac{\partial E_z}{\partial t} + \nabla \times \vec{B} \]

\[ \mu_o J^2 = \mu_o J_y \]

\[ \mu_o J^3 = \mu_o J_z \]

3-D spatial components of 4-vector \( J^\mu \)

Or:

\[ \nabla \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]

Ampere’s law with Maxwell’s displacement current term !!!

\[ \Rightarrow \text{Ampere’s law arises from the } \mu = 1:3 \text{ (3-D spatial / vector component) of 4-vector relation:} \]

\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_o} \rho \] \( (\mu = 0 \text{ temporal / scalar component}) \)

\[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_o \vec{J} \] \( (\mu = 1:3 \text{ 3-D spatial / vector component}) \)

Thus, Gauss’ law and Ampere’s law form a 4-vector:

\[ \mu_o J^\mu = \left( \begin{array}{c}
\frac{1}{c} \nabla \cdot \vec{E} = \frac{1}{\varepsilon_o} \rho, \\
\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_o \vec{J}
\end{array} \right) = \left( \begin{array}{c}
\frac{\partial F_{\mu \nu}}{\partial x^\nu} = \partial_x F_{\mu \nu} = \mu_o J^\mu
\end{array} \right) \]

Gauss’ law temporal / scalar
Ampere’s law 3-D spatial / vector
component \( \mu_o J^0 \) of 4-vector \( \mu_o J^\mu \)
component \( \mu_o \vec{J} \) of 4-vector \( \mu_o J^\mu \)

And:

\[ J^\mu J_\mu = J^\mu J^\mu = -\rho^2 c^2 \]

Lorentz invariant quantity \{from above, page 5\}. 

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2) \( \nabla \cdot \mathbf{B} = 0 \) no magnetic monopoles / no magnetic charges. If \( \mu = 0 \) in \( \partial_v G^{0v} = \frac{\partial G^{0v}}{\partial x^v} = 0 \)

\( \mu = 0 \) is the temporal (scalar) component of space-time “null” 4-vector

Then: \( \frac{\partial G^{0v}}{\partial x^v} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = 0 \) \( \mu = 0 \) (First row of \( G^{\mu v} \))

\[
G^{0v} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & -E_z/c & E_y/c \\
-B_y & E_z/c & 0 & -E_x/c \\
-B_z & -E_y/c & E_x/c & 0 \\
\end{pmatrix}
\]

\( \nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial G^{0v}}{\partial x^v} = 0 \)

3) Faraday’s law:

\( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) If \( \mu = 1 \) in \( \partial_v G^{0v} = \frac{\partial G^{0v}}{\partial x^v} = 0 \)

Then: \( \frac{\partial G^{1v}}{\partial x^v} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = 0 \) \( \mu = 1 \) (Second row of \( G^{\mu v} \))

\[
\frac{\partial G^{1v}}{\partial x^v} = -\frac{1}{c} \frac{\partial B_x}{\partial t} + 0 - \frac{1}{c} \frac{\partial E_z}{\partial y} + \frac{1}{c} \frac{\partial E_y}{\partial z} = -\frac{1}{c} \left( \frac{\partial B_x}{\partial t} + \nabla \times \mathbf{E} \right) = 0
\]

\( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)

Likewise, for \( \mu = 2 \) and \( \mu = 3 \) (third and fourth rows of \( G^{\mu v} \))

\[
\frac{\partial G^{2v}}{\partial x^v} = 0 \quad \text{gives} \quad \left( \frac{\partial B_y}{\partial t} + \nabla \times \mathbf{E} \right)_y = 0
\]

\( \frac{\partial G^{3v}}{\partial x^v} = 0 \quad \text{gives} \quad \left( \frac{\partial B_z}{\partial t} + \nabla \times \mathbf{E} \right)_z = 0
\]

Thus: \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) (\( \mu = 1:3 \) 3-D spatial / vector component)

Arise from temporal (\( \mu = 0 \)) and spatial (\( \mu = 1:3 \)) component of the “null” 4-vector \( \frac{\partial \mathbf{G}^{\mu v}}{\partial x^v} = 0 \)
Thus, in relativistic 4-vector / tensor notation, Maxwell’s 4 equations (written in language of 3-D differential vector calculus):

**Maxwell’s Equations:**

1) Gauss’ law:
\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho
\]

2) No Magnetic Monopoles:
\[
\nabla \cdot \mathbf{B} = 0
\]

3) Faraday’s law:
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

4) Ampere’s law:
\[
\nabla \times \mathbf{B} = \mathbf{\mu}_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]

are elegantly represented by two simple 4-vector equations:

\[
\begin{align*}
\mu = 0 & \quad \text{temporal/scalar component:} \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \\
\mu = 1:3 & \quad \text{spatial/vector component:} \quad \nabla \times \mathbf{B} = 0
\end{align*}
\]

\[
\begin{align*}
\mu = 0 & \quad \text{temporal/scalar component:} \quad \nabla \cdot \mathbf{B} = 0 \\
\mu = 1:3 & \quad \text{spatial/vector component:} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0
\end{align*}
\]

**Griffiths Problem 12.53:**

We can show that Maxwell’s two equations \(\nabla \cdot \mathbf{B} = 0\) and \(\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0\) that are contained in

\[
\frac{\partial G_{\mu\nu}}{\partial x^\nu} = 0
\]

can also be obtained from (the more cumbersome/inelegant relation):

\[
\frac{\partial G_{\mu\nu}}{\partial x^\nu} = 0 \quad \Leftrightarrow \quad \nabla \cdot \mathbf{F}_\mu + \nabla \cdot \mathbf{F}_\nu + \nabla \cdot \mathbf{F}_\lambda = \frac{\partial F_{\mu\nu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\nu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0
\]

Since there are 3 indices in the latter equation \(\mu = 0:3, \nu = 0:3, \lambda = 0:3\), there are actually 64 (= 4³) separate equations!!! However, many of these 64 equations are either trivial or redundant.
Suppose two indices are the same (e.g. $\mu = v$)

Then:  
\[
\frac{\partial F_{\mu\mu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\mu} = 0.
\]
But the EM field strength tensor $F_{\mu\nu}$ (like $F^{\mu\nu}$) is anti-symmetric.

$\therefore F_{\mu\nu} = 0$ and $F_{\mu\lambda} = -F_{\lambda\mu}$. Thus, $\geq 2$ indices the same gives the trivial relation: $0 = 0$.

Thus, in order to obtain a / any non-trivial result, $\mu$, $v$, and $\lambda$ must all be different from each other.

1) The indices $\mu$, $v$, and $\lambda$ could all be spatial indices, such as: $\mu = 1$ ($x$), $v = 2$ ($y$), $\lambda = 3$ ($z$) (or permutations thereof).

Or:

2) One index could be temporal, and two indices could be spatial, such as:

   $\mu = 0$,
   $v = 1$,
   $\lambda = 2$ (or permutations thereof), or:

   $\mu = 0$,
   $v = 1$,
   $\lambda = 3$

1) For the case(s) of all spatial indices, e.g. $\mu = 1$, $v = 2$, $\lambda = 3$:

\[
\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0
\]

\[
\frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0
\]

All other permutations involving the all-spatial indices $\{1, 2, 3\}$ yield the same relation $\nabla \cdot \vec{B} = 0$ or minus it: i.e. $-\nabla \cdot \vec{B} = 0$.

2) For the case of one temporal and two spatial indices, e.g. $\mu = 0$, $v = 1$, $\lambda = 2$:

\[
\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0
\]

\[
\frac{\partial B_z}{\partial c} + \frac{\partial B_x}{\partial c} + \frac{\partial B_y}{\partial c} = 0
\]

\[
\nabla \times \vec{E} = 0
\]

Other Permutations:

For $\nu = 0$, $\mu$ & $\lambda = 1:3$ and $\lambda = 0$, $\mu$ & $\nu = 1:3$ get redundant results (same as above).

If $\mu = 0$, $\nu = 1$, $\lambda = 3$ get $y$ – component of above relation!
If $\mu = 0$, $\nu = 2$, $\lambda = 3$ get $x$ – component of above relation!

$\therefore \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$ contains: $\nabla \cdot \vec{B} = 0$ and: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Duality Transformation of the Relativistic EM Field Strength Tensors $F^{\mu\nu}$ and $G^{\mu\nu}$:

The duality transformation for the specific case of space-time “rotating” $\vec{E} \rightarrow c\vec{B}$ and $c\vec{B} \rightarrow -\vec{E}$ (duality = 90°) takes $F^{\mu\nu} \rightarrow G^{\mu\nu}$, and can be mathematically represented in tensor notation as:

$$G^{\mu\nu} = \frac{1}{2} \varepsilon^{\nu\lambda\sigma} F_{\lambda\sigma}$$

where: $F_{\lambda\sigma}$ is the {doubly} covariant form of the contravariant tensor $F^{\lambda\sigma}$.

and: $\varepsilon^{\nu\lambda\sigma}$ is the totally anti-symmetric rank-four tensor.

$$\varepsilon^{\nu\lambda\sigma} = \begin{cases} +1 & \text{for all even permutations of } \nu = 0, \lambda = 1, \sigma = 2, \text{ and } \sigma = 3 \\ 0 & \text{if any two indices are equal/identical/the same.} \\ -1 & \text{for all odd permutations of } \nu = 0, \lambda = 1, \sigma = 2, \text{ and } \sigma = 3 \end{cases}$$

Since $\varepsilon^{\nu\lambda\sigma}$ is a rank-four tensor (= 4-dimensional “matrix”) we can’t write it down on 2-D paper all at once! $\varepsilon^{\nu\lambda\sigma}$ has $(\mu, \nu, \lambda, \sigma = 0:3) \rightarrow 4^4$ elements = 256 elements!!

We could write out 16 {4×4} matrices – e.g. one $\mu$-$\nu$ matrix for each unique combination of $\lambda$ and $\sigma$:

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{bmatrix} \lambda=0 \nu \rightarrow \mu ; \lambda=0 \nu \rightarrow \mu ; \lambda=0 \nu \rightarrow \mu ; \lambda=0 \nu \rightarrow \mu \\ \sigma=0 (4\times4) \Psi ; \sigma=1 (4\times4) \Psi ; \sigma=2 (4\times4) \Psi ; \sigma=3 (4\times4) \Psi \\ \lambda=1 \nu \rightarrow \mu ; \lambda=1 \nu \rightarrow \mu ; \lambda=1 \nu \rightarrow \mu ; \lambda=1 \nu \rightarrow \mu \\ \sigma=0 (4\times4) \Psi ; \sigma=1 (4\times4) \Psi ; \sigma=2 (4\times4) \Psi ; \sigma=3 (4\times4) \Psi \\ \lambda=2 \nu \rightarrow \mu ; \lambda=2 \nu \rightarrow \mu ; \lambda=2 \nu \rightarrow \mu ; \lambda=2 \nu \rightarrow \mu \\ \sigma=0 (4\times4) \Psi ; \sigma=1 (4\times4) \Psi ; \sigma=2 (4\times4) \Psi ; \sigma=3 (4\times4) \Psi \\ \lambda=3 \nu \rightarrow \mu ; \lambda=3 \nu \rightarrow \mu ; \lambda=3 \nu \rightarrow \mu ; \lambda=3 \nu \rightarrow \mu \\ \sigma=0 (4\times4) \Psi ; \sigma=1 (4\times4) \Psi ; \sigma=2 (4\times4) \Psi ; \sigma=3 (4\times4) \Psi \end{bmatrix}$$

Define $\varepsilon^{\lambda\sigma} = 4\times4$ totally anti-symmetric rank-two tensor (aka the 4×4 Levi-Civitá symbol):

$$\varepsilon^{\lambda\sigma} = \begin{bmatrix} 0 & +1 & +1 & +1 \\ -1 & 0 & +1 & +1 \\ -1 & -1 & 0 & +1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Then, we can define $\varepsilon^{\mu\nu\lambda\sigma}$ in terms of the product of two $\varepsilon^{\lambda\sigma}$'s: $\varepsilon^{\mu\nu\lambda\sigma} = \varepsilon^{\mu\nu} \varepsilon^{\lambda\sigma}$
The Minkowski / Proper Force on a Point Electric Charge

The Minkowski force (a.k.a. proper force) \( K^\mu \) acting on a point electric charge \( q \) can be written in 4-vector / tensor notation in terms of the EM field strength tensor \( F^{\mu\nu} \) and the proper 4-velocity \( \eta^\mu \). Recall that:

\[
\left[ \frac{dp^\mu}{d\tau} = \gamma_u \frac{dp^\mu}{dt} = \gamma_u F^{\mu} \right]
\]

where the ordinary force:

\[
F^{\mu} = \frac{dp^\mu}{dt}
\]

and:

\[
\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}}
\]

However, we can equivalently write the Minkowski/proper force as:

\[
K^\mu = q\eta_v F^{\mu\nu}
\]

where \( \eta_v \) is the covariant form of the contravariant proper 4-velocity \( \eta^\nu \). i.e. we contract the EM field strength tensor \( F^{\mu\nu} \) with the covariant proper 4-velocity \( \eta_v \).

Since:

\[
K^\mu = \gamma_u F^{\mu}
\]

and:

\[
\eta_v = \gamma_u u^\mu_v \]

where:

\[
\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}} \quad \text{and} \quad \beta_u = u/c
\]

\[
\therefore \quad K^\mu = q\gamma_u F^{\mu\nu} \Rightarrow \gamma_u F^{\mu} = q\gamma_u u^\mu_v
\]

or:

\[
F^{\mu} = qu_{\nu} F^{\mu\nu} \leftarrow \text{ordinary force on point charge}
\]

If \( \mu = 1 \) (i.e. row #1):

\[
K^1 = q\gamma_u F^{1\nu} = q\left(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}\right)
\]

\[
u_v \equiv (c,\bar{u}) \leftrightarrow \eta_v \equiv (\gamma_u c, \gamma_u \bar{u}) = \gamma_u u^\nu_v
\]

where:

\[
\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}} \quad \text{and} \quad \beta = u/c
\]

Row #

\[
F^{\mu\nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix}
\]

Column #

\[
K^0 = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
K^1 = q\gamma_u \left[-c\left(-E_x/c\right) + u_x \cdot 0 + u_y B_z + u_z \left(-B_y\right)\right] = q\gamma_u \left[E_x + u_y B_z - u_z B_y\right] = q\gamma_u \left(\vec{E} + \vec{u} \times \vec{B}\right)
\]

\[
K^1 = q\gamma_u \left(\vec{E} + \vec{u} \times \vec{B}\right) \leftarrow \text{Minkowski 3-D Force law}
\]

Similarly, for \( \mu = 2, \mu = 3 \):

\[
K^2 = q\gamma_u \left(\vec{E} + \vec{u} \times \vec{B}\right)
\]

\[
K^3 = q\gamma_u \left(\vec{E} + \vec{u} \times \vec{B}\right)
\]

But:

\[
\vec{K} = \gamma_u \vec{F}
\]

\[
\therefore \quad \vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right) \leftarrow \text{Lorentz 3-D Force law}
\]
For \( \mu = 0 \) (the temporal / scalar component) {see/work Griffiths Problem 12.54, page 541}:

\[
K^0 = q\gamma_u \left( -c \cdot 0 + u_x (E_x / c) + u_y (E_y / c) + u_z (E_z / c) \right) = q\gamma_u \bar{u} \cdot \bar{E} / c
\]

\[
= -\eta^0 F^{00} + \eta^1 F^{01} + \eta^2 F^{02} + \eta^3 F^{03} = \eta \cdot F^{0\nu}
\]

\( n.b. \) this relation explicitly shows that \( E_x, E_y, E_z \) are temporal-spatial (or spatial-temporal) components of \( F^{\mu\nu} \), whereas \( B_x, B_y, B_z \) are pure spatial-spatial components of \( F^{\mu\nu} \)!!!

We also know that:

\[
K^0 \equiv \frac{dp^0}{d\tau} = \frac{1}{c} dE = \frac{1}{c} \frac{dt}{d\tau} dE = \frac{1}{c} \gamma_u \frac{dE}{dt} \Rightarrow \quad K^0 = q\gamma_u \bar{u} \cdot \bar{E} / c = \frac{1}{c} \gamma_u \frac{dE}{dt}
\]

or:

\[
P_q = \frac{dE}{dt} = q \left( \bar{u} \cdot \bar{E} \right) = \{ \text{ordinary} \} \text{ relativistic power delivered to point charged particle (> 0)}
\]

\[
P_q = \frac{dE}{dt} = q \left( \bar{u} \cdot \bar{E} \right) = \left( q\bar{E} \right) \cdot \bar{u} = \bar{F} \cdot \bar{u} = \frac{dW}{dt} = \text{Time rate of change of work done on changed particle by EM field}
\]

**Note:** The \( \{ \text{ordinary} \} \) Lorentz force:

\[
\bar{F} = q\bar{E} + q \left( \bar{u} \times \bar{B} \right)
\]

\[
\Rightarrow \quad \bar{F} \cdot \bar{u} = q \left( \bar{u} \cdot \bar{E} \right) + q \left[ \bar{u} \cdot (\bar{u} \times \bar{B}) \right] = q \left( \bar{u} \cdot \bar{E} \right)
\]

But: \( (\bar{u} \times \bar{B}) \perp \bar{u} \) \( \Rightarrow \quad \bar{u} \cdot (\bar{u} \times \bar{B}) \equiv 0 \) \( \Leftarrow \) magnetic forces do no work !!!

Thus we have the relations: \( K^\mu = \gamma_u F^\mu = K^\mu = q\eta_v F^{\mu\nu} \) and also: \( F^\mu = qu_v F^{\mu\nu} \) with \( \eta_v = \gamma_u \).

---

**The Relativistic 4-Vector Potential \( A^\mu \)**

We know that the electric and magnetic fields \( \bar{E} \) and \( \bar{B} \) can be expressed in terms of a scalar potential \( V \) and a vector potential \( \bar{A} \) as:

\[
\bar{E}(\bar{r},t) = -\nabla V(\bar{r},t) - \frac{\partial \bar{A}(\bar{r},t)}{\partial t} \quad \text{and:} \quad \bar{B}(\bar{r},t) = \nabla \times \bar{A}(\bar{r},t)
\]

Thus, it should not be surprising to realize that the scalar potential \( V \) and the vector potential \( \bar{A} \) form the temporal and spatial components (respectively) of the relativistic 4-vector potential \( A^\mu \):

The 4-Vector Potential: \( A^\mu \equiv \left( V/c, \bar{A} \right) = \left( V/c, A_x, A_y, A_z \right) \)

\( \text{SI Units: Newtons/Ampere = "p/q"} \)

\( \text{=} \) momentum per Coulomb!}

\( n.b. \) SI units of \( V \): \( \text{Volts} = \frac{\text{Newton-meters}}{\text{Coulomb}} \) then:

\[
V = \frac{N \cdot \text{m}}{c \cdot \text{Coul}} / \text{m} = \frac{N \cdot \text{sec}}{\text{Coul}} = \frac{\text{Newton}}{\text{Amp}}
\]
The EM field strength tensor $F^{\mu\nu}$ can be written in terms of covariant space-time derivatives of the 4-vector potential field $A^\mu$ as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} = n.b. \text{ covariant differentiation here!!}$$

n.b. We must change the sign of the temporal/scalar component of the covariant derivatives $\partial^\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial^\nu \equiv \frac{\partial}{\partial x^\nu}$ relative to that of the contravariant derivatives $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial_\nu \equiv \frac{\partial}{\partial x^\nu}$.

Explicitly evaluate a few terms: $A^\mu = \left( V/c, \vec{A} \right)$

For $\mu = 0$ and $\nu = 1$:

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_t}{\partial x^0} - \frac{\partial (V/c)}{\partial x^1} = -\frac{1}{c^2} \left( \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} V \right) = \frac{1}{c} \left( -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right) = E_x$$

Likewise, for ($\mu = 0$, $\nu = 2$) and ($\mu = 0$, $\nu = 3$) we obtain:

$$F^{02} = \frac{1}{c} \left( -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{E_y}{c} \quad \text{and:} \quad F^{03} = \frac{1}{c} \left( -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{E_z}{c}$$

For $\mu = 1$ and $\nu = 2$:

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = \left( \vec{\nabla} \times \vec{A} \right)_z = B_z$$

Likewise, for ($\mu = 1$, $\nu = 3$) and ($\mu = 2$, $\nu = 3$) we obtain:

$$F^{13} = -\left( \vec{\nabla} \times \vec{A} \right)_y = -B_y \quad \text{and:} \quad F^{23} = \left( \vec{\nabla} \times \vec{A} \right)_x = B_x$$

Note that the relativistic 4-potential formulation automatically takes care of the homogeneous Maxwell equation $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \{ \text{it gives} \ \vec{\nabla} \cdot \vec{B} = 0 \}$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

because $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ is equivalent to $\frac{\partial F_{\mu\nu}}{\partial x^x} + \frac{\partial F_{\nu\mu}}{\partial x^x} + \frac{\partial F_{\nu\mu}}{\partial x^\nu} = 0$.

{See/read pages 10-11 of these lecture notes – also see/work Griffiths Problem 12.53, page 541}.

And since: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \left( \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right) \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)$
Thus:

\[
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\nu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0
\]

\[
= \frac{\partial}{\partial x^\lambda} \left( \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) + \frac{\partial}{\partial x^\mu} \left( \frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\lambda} \right) + \frac{\partial}{\partial x^\nu} \left( \frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\mu} \right) = 0
\]

\[
= \frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu} + \frac{\partial}{\partial x^\mu} \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\nu} \frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial}{\partial x^\nu} \frac{\partial A_\lambda}{\partial x^\mu} = 0
\]

\[
= \frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu} + \frac{\partial}{\partial x^\mu} \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\nu} \frac{\partial A_\mu}{\partial x^\lambda} = 0
\]

\[
\frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu} + 0 = 0
\]

But:

\[
\frac{\partial}{\partial x^\lambda} \frac{\partial A_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu}
\]

\[
\therefore \text{i.e. can change the order of differentiation} \quad \text{– has no effect!}
\]

\[
= \frac{\partial^2}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2}{\partial x^\mu \partial x^\lambda}
\]

\[
\therefore 0 = 0
\]

\[
. \quad \text{The relativistic 4-potential formulation} \quad F_{\mu\nu} = \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)
\]

\[
\text{does indeed automatically satisfy}
\]

\[
\partial_\nu G_{\mu\nu} = \frac{\partial G_{\mu\nu}}{\partial x^\nu} = 0
\]

\[
\text{because} \quad \frac{\partial F_{\mu\nu}}{\partial x^\nu} + \frac{\partial F_{\nu\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\nu}}{\partial x^\nu} = 0
\]

\[
\text{[shown to be equivalent to} \quad \frac{\partial G_{\mu\nu}}{\partial x^\nu} = 0
\]

\[
\text{is satisfied / obeyed for}
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)
\]

\[
\text{Does the relativistic 4-potential formulation} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)
\]

\[
\text{satisfy the inhomogeneous Maxwell relation} \quad \partial_\nu F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^\nu} = \mu_0 J_\mu
\]

\[
\partial_\nu F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) = \frac{\partial^2 A_\nu}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 A_\mu}{\partial x^\mu \partial x^\nu} = \mu_0 J_\mu
\]

\[
\text{Switching the order of derivatives:}
\]

\[
\partial_\nu F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) = \mu_0 J_\mu
\]

\[
\text{This is an intractable equation, as it stands now…}
\]
However, from our formulation of $F^\mu_\nu$ in terms of (differences) in space-time derivatives of the 4-vector potential $A^\mu$: 
\[ F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu = \left( \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right) \]

it is clear that we can add to the 4-vector potential $A^\mu$ the space-time gradient of any scalar function $\lambda$: 
\[ A^\mu \rightarrow A^\mu = A^\mu + \frac{\partial \lambda}{\partial x^\mu} \]
!!!

The scalar and vector potentials $V$ and $A$ are not uniquely determined by the EM fields $\vec{E}$ and $\vec{B}$. Thus:
\[ \Box = \nabla^2 = \frac{\partial^2}{\partial x^\mu \partial x^\mu} \]

We can exploit the gauge invariant properties of $F^\mu_\nu$ to simplify the seemingly intractable relation:
\[ \partial_\mu F^\mu_\nu = \frac{\partial F^\mu_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \left( \int A^\nu \right) - \frac{\partial}{\partial x^\nu} \left( \int A^\mu \right) = \mu_0 J^\mu \]

Using the **Lorenz gauge condition**: 
\[ \nabla \cdot A = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Rightarrow \nabla \cdot A + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \Rightarrow \partial_\mu A^\mu = \partial_\mu \frac{\partial A^\mu}{\partial x^\mu} = 0 \]

Thus: 
\[ \partial_\mu F^\mu_\nu = \frac{\partial F^\mu_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \left( \int A^\nu \right) - \frac{\partial}{\partial x^\nu} \left( \int A^\mu \right) = \mu_0 J^\mu \]

or:
\[ \partial^\nu \partial_\sigma A^\sigma = \frac{\partial^2 A^\nu}{\partial x^\sigma \partial x^\sigma} = -\mu_0 J^\nu \]

But: 
\[ \Box = \partial_\nu \equiv \frac{\partial}{\partial x^\nu}, \quad \Box^\nu = \partial_\nu \equiv \frac{\partial}{\partial x^\nu} \quad \text{and:} \quad \Box^\nu \equiv \partial_\nu \partial^\nu = \partial_\nu \partial_\nu = \frac{\partial^2}{\partial x^\nu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\nu} = \int^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]

D’Alembertian operator (4-dimensional Laplacian operator)

\[ \Box^\nu A^\mu = \partial^\nu \partial_\sigma A^\sigma = -\mu_0 J^\mu \]

\[ \frac{\partial^2 A^\nu}{\partial x^\sigma \partial x^\sigma} = -\mu_0 J^\mu \]

\[ \Box^\nu = \partial_\nu J^\mu = 0 \]

Taken together with the continuity equation (charge conservation): 
\[ \Box^\nu J^\mu = \partial_\nu J^\mu = 0 \]
these two relations compactly describe virtually all of {non-matter/free space} EM phenomena!!!
Note that the choice of the \textit{(instantaneous)} Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ is a \textbf{bad} one for use in \textit{relativistic} electrodynamics, because $\vec{\nabla} \cdot \vec{A}$ \{alone\} is \textbf{not} a Lorentz invariant quantity!

However: \begin{equation*} \vec{\nabla} \cdot \vec{A} (\vec{r}, t) + \frac{1}{c^2} \frac{\partial V (\vec{r}, t)}{\partial t} = \partial_\mu A^\mu (\vec{r}, t) = \frac{\partial A^\mu (\vec{r}, t)}{\partial x^\mu} = 0 \end{equation*} \textbf{is} a Lorentz invariant quantity because it is the product of two relativistic 4-vectors: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $A^\mu$.

\textit{n.b.} $\vec{\nabla} \cdot \vec{A} = 0$ is “destroyed” by \textit{any} Lorentz transformation from one IRF(S) to another IRF(S’) !!!

⇒ In order to \textit{restore} $\vec{\nabla} \cdot \vec{A} = 0$, one must perform an appropriate gauge transformation for \textit{each} new inertial system entered, \textit{in addition} to carrying out the Lorentz transformation itself !!!

In the Coulomb gauge $A^\mu$ is \textbf{not} a “true” relativistic 4-vector, because $A^\mu A_\mu = A_\mu A^\mu$ is actually \textbf{not} a Lorentz invariant quantity in the Coulomb gauge !!!

\textit{n.b.} The Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ \textbf{is} useful when $v \ll c$, \textit{i.e.} for \textit{non-relativistic} problems.