LECTURE NOTES 11

POTENTIALS & FIELDS

The Potential Formulation: Scalar and Vector Potentials \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \)

How do the instantaneous sources \( \rho_{\text{tot}}(\vec{r},t) \) and \( \vec{J}_{\text{tot}}(\vec{r},t) \) (total electric charge density and total electric current density) generate the electric and magnetic fields \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \)?

We seek general solutions to the full Maxwell equations:

1. Gauss’ Law:
   \[
   \nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r},t)
   \]
   where:
   \[
   \rho_{\text{tot}}(\vec{r},t) = \rho_{\text{free}}(\vec{r},t) + \rho^p_{\text{bnd}}(\vec{r},t)
   \]

2. No Magnetic Charges:
   \[
   \nabla \times \vec{B}(\vec{r},t) = 0
   \]

3. Faraday’s Law:
   \[
   \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}
   \]
   where:
   \[
   \vec{J}_{\text{tot}}(\vec{r},t) = \vec{J}_{\text{free}}(\vec{r},t) + \vec{J}_{\text{bnd}}^p(\vec{r},t) + \vec{J}_{\text{bnd}}^M(\vec{r},t)
   \]

4. Ampere’s Law:
   \[
   \nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}_{\text{tot}}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t}
   \]

If \textit{time-dependent} \( \rho_{\text{tot}}(\vec{r}',t) \) and \( \vec{J}_{\text{tot}}(\vec{r}',t) \) are given/specified at \textit{source} position(s) \( \vec{r}' \), what are the corresponding \textit{EM} fields \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \) at the \textit{observer} position \( \vec{r} \)?

\[ \Rightarrow \text{For the } \textit{static} \text{ case } \{ \text{i.e. } \rho, \vec{J}, \vec{E}, \vec{B} \neq \text{fcns}(t) \}: \] Coulomb’s law
   \[
   \vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int_{\vec{r}'} \frac{\rho_{\text{tot}}(\vec{r}')}{\hat{k}''} \frac{1}{\varepsilon_0} d\tau'
   \]
   and the Biot-Savart law
   \[
   \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\vec{r}'} \frac{\vec{J}_{\text{tot}}(\vec{r}') \times \hat{k}'}{\varepsilon_0} d\tau'
   \]
   provide the answers. \{n.b. \( \varepsilon_0 = |\vec{r} - \vec{r}'| \), where \( \vec{r} = \textit{observer’s} \text{ position at field point } P(\vec{r}) \) and \( \vec{r}' = \textit{source} \text{ position at point } S(\vec{r}') \}.

\[ \Rightarrow \text{We seek } \textit{time-dependent generalizations} \text{ of Coulomb’s law and the Biot-Savart law.} \]

This is \textbf{not} an easy problem!! The approach we will take is to represent the \textit{time-dependent} \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \) fields in terms of their corresponding \textit{time-dependent potentials} \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \), which are in turn related to the \textit{time-dependent sources} \( \rho_{\text{tot}}(\vec{r}') \), \( \vec{J}_{\text{tot}}(\vec{r}') \).

In electro\textit{statics}, \( \nabla \times \vec{E}(\vec{r},t) = 0 \) enabled us to write:
   \[
   \vec{E}(\vec{r},t) = -\nabla V(\vec{r},t).
   \]

In electro\textit{dynamics}, we \textbf{cannot} do this because:
   \[
   \nabla \times \vec{E}(\vec{r},t) \neq 0. \quad \{ \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \}
   \]
   However, in electro\textit{dynamics} \( \nabla \times \vec{B}(\vec{r},t) = 0 \) \{still\}. \[ \Rightarrow \vec{B}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \]
   where \( \vec{A}(\vec{r},t) = \text{magnetic vector potential} \), as we saw in the case of magnetostatics.
Insert $\tilde{B}(\tilde{r},t) = \nabla \times \tilde{A}(\tilde{r},t)$ into Faraday’s Law:  
$$\nabla \times \tilde{E}(\tilde{r},t) = -\frac{\partial \tilde{B}(\tilde{r},t)}{\partial t} = -\nabla \left( \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t} \right)$$

Then we see that: $\nabla \times \left[ \tilde{E}(\tilde{r},t) + \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t} \right] = 0$ i.e. the curl of $\left[ \tilde{E}(\tilde{r},t) + \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t} \right]$ does vanish for any/all $(\tilde{r},t)$, ∴ we can define: 
$$\tilde{E}(\tilde{r},t) = -\nabla V(\tilde{r},t)$$

Then we see that:  
$$\nabla \times \tilde{F}(\tilde{r}) = 0 \ \forall (\tilde{r})$$

and $\tilde{F}(\tilde{r}) = -\nabla f(\tilde{r})$ since: $\nabla \times \nabla f(\tilde{r}) = 0$ always.

Thus: 
$$\tilde{E}(\tilde{r},t) = -\nabla V(\tilde{r},t) - \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t} \quad \leftarrow \text{n.b. reduces to } \tilde{E}(\tilde{r}) = -\nabla V(\tilde{r}) \text{ for the electrostatics case.}$$

Since we used $\nabla \cdot \tilde{B} = 0$ and $\nabla \times \tilde{E} = -\partial \tilde{B}/\partial t$ in deriving the above relation, Maxwell’s equations (2) $\nabla \cdot \tilde{B} = 0$ and (3) Faraday’s Law: $\nabla \times \tilde{E} = -\partial \tilde{B}/\partial t$ are automatically satisfied. What about equation (1) Gauss’ Law: $\nabla \cdot \tilde{E} = -\rho_{\text{tot}}/\varepsilon_0$ and (4) Ampère’s Law: $\nabla \times \tilde{B} = \mu_0 \tilde{J}_{\text{tot}} + \varepsilon_0 \mu_o \partial \tilde{E}/\partial t$?

(1) Gauss’ Law:
$$\nabla \cdot \tilde{E}(\tilde{r},t) = \frac{1}{\varepsilon_0} \rho_{\text{tot}}(\tilde{r},t) \quad \text{with:} \quad \tilde{E}(\tilde{r},t) = -\nabla V(\tilde{r},t) - \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t}$$

Electrodynamics version of the Poisson equation:
$$\nabla^2 V(\tilde{r},t) + \frac{\partial}{\partial t} \left( \nabla \cdot \tilde{A}(\tilde{r},t) \right) = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\tilde{r},t)$$

(4) Ampère’s Law:
$$\nabla \times \tilde{B}(\tilde{r},t) = \mu_0 \tilde{J}_{\text{tot}}(\tilde{r},t) + \varepsilon_0 \mu_o \frac{\partial \tilde{E}(\tilde{r},t)}{\partial t} \quad \text{with:} \quad \tilde{E}(\tilde{r},t) = -\nabla V(\tilde{r},t) - \frac{\partial \tilde{A}(\tilde{r},t)}{\partial t}$$

and: $\tilde{B}(\tilde{r},t) = \nabla \times \tilde{A}(\tilde{r},t)$

$$\nabla \times \left( \nabla \times \tilde{A}(\tilde{r},t) \right) = \mu_0 \tilde{J}_{\text{tot}}(\tilde{r},t) - \mu_0 \varepsilon_0 \nabla \left( \frac{\partial V(\tilde{r},t)}{\partial t} \right) - \frac{\partial^2 \tilde{A}(\tilde{r},t)}{\partial t^2}$$

but: $\nabla \times \left( \nabla \times \tilde{A} \right) \equiv \nabla \left( \nabla \cdot \tilde{A} \right) - \nabla^2 \tilde{A}$

$$\left( \nabla^2 \tilde{A}(\tilde{r},t) - \frac{\partial^2 \tilde{A}(\tilde{r},t)}{\partial t^2} \right) - \nabla \left( \nabla \cdot \tilde{A}(\tilde{r},t) + \mu_0 \varepsilon_0 \frac{\partial V(\tilde{r},t)}{\partial t} \right) = -\mu_0 \tilde{J}_{\text{tot}}(\tilde{r},t) \quad \leftarrow \text{n.b. this relation is actually 3 separate eqns. for } x, y \text{ and } z!
These two equations may seem “ugly” at this point in time, but watch what we can do with them:

a.) Add: \( 0 = -\varepsilon_\alpha \mu_\alpha \frac{\partial^2 V(\vec{r},t)}{\partial t^2} + \varepsilon_\alpha \mu_\alpha \frac{\partial^2 V(\vec{r},t)}{\partial t^2} \) to the first equation, and group terms as follows:

\[
\begin{align*}
1\text{st eqn}: & \quad \nabla^2 V(\vec{r},t) - \varepsilon_\alpha \mu_\alpha \frac{\partial^2 V(\vec{r},t)}{\partial t^2} + \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A}(\vec{r},t) + \varepsilon_\alpha \mu_\alpha \frac{\partial V(\vec{r},t)}{\partial t} \right) = -\frac{1}{\varepsilon_\alpha} \rho_{\text{tot}}(\vec{r},t) \\
2\text{nd eqn}: & \quad \nabla^2 \vec{A}(\vec{r},t) - \varepsilon_\alpha \mu_\alpha \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} - \nabla \left( \nabla \cdot \vec{A}(\vec{r},t) + \varepsilon_\alpha \mu_\alpha \frac{\partial V(\vec{r},t)}{\partial t} \right) = -\mu_\alpha \vec{J}_{\text{tot}}(\vec{r},t)
\end{align*}
\]

b.) Define: 
\[
L(\vec{r},t) \equiv \left( \nabla \cdot \vec{A}(\vec{r},t) + \varepsilon_\alpha \mu_\alpha \frac{\partial V(\vec{r},t)}{\partial t} \right)
\]

c.) Define the D’Alembertian (aka “Box^2”) operator:
\[
\Box^2 \equiv \nabla^2 - \varepsilon_\alpha \mu_\alpha \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}
\]

Then:
\[
\begin{align*}
1\text{st eqn}: & \quad \Box^2 V(\vec{r},t) + \frac{\partial L(\vec{r},t)}{\partial t} = -\frac{1}{\varepsilon_\alpha} \rho_{\text{tot}}(\vec{r},t) \\
2\text{nd eqn}: & \quad \Box^2 \vec{A}(\vec{r},t) - \nabla L(\vec{r},t) = -\mu_\alpha \vec{J}_{\text{tot}}(\vec{r},t)
\end{align*}
\]

d.) Divide 1st eqn. by \( c \), then use \( \varepsilon_\alpha \mu_\alpha = 1/c^2 \) relation:

\[
\begin{align*}
1\text{st eqn}: & \quad \Box^2 \frac{1}{c} V(\vec{r},t) + \frac{1}{c} \frac{\partial L(\vec{r},t)}{\partial t} = -\frac{\mu_\alpha}{c \varepsilon_\alpha \mu_\alpha} \rho_{\text{tot}}(\vec{r},t) = -\mu_\alpha c \rho_{\text{tot}}(\vec{r},t) \\
2\text{nd eqn}: & \quad \Box^2 \frac{1}{c} \vec{A}(\vec{r},t) - \nabla \frac{1}{c} L(\vec{r},t) = -\mu_\alpha \frac{1}{c} \vec{J}_{\text{tot}}(\vec{r},t)
\end{align*}
\]

Thus:
\[
\begin{align*}
1\text{st eqn}: & \quad \Box^2 \frac{1}{c} V(\vec{r},t) + \frac{1}{c} \frac{\partial L(\vec{r},t)}{\partial t} = -\mu_\alpha c \rho_{\text{tot}}(\vec{r},t) \quad \text{“time-like”} \\
2\text{nd eqn}: & \quad \Box^2 \frac{1}{c} \vec{A}(\vec{r},t) - \nabla \frac{1}{c} L(\vec{r},t) = -\mu_\alpha \frac{1}{c} \vec{J}_{\text{tot}}(\vec{r},t) \quad \text{“space-like”}
\end{align*}
\]

Do you see/can you see any interesting parallels between these two equations???
If you can, it’s in fact not coincidental!!

We shall see later on in the semester (when we get to relativistic electrodynamics), that relativistic four vectors exist {valid in any inertial reference frame}. The relativistic 4-vector potential: \( A^\mu(\vec{r},t) \equiv \left( V(\vec{r},t)/c, \vec{A}(\vec{r},t) \right) \) \{where \( \mu = 0,1,2,3 ; \mu = 0 \) is the temporal component of the relativistic 4-vector, and \( \mu = 1,2,3 \) are e.g. the \( x,y,z \) spatial components of the 4-vector\}.

The relativistic 4-current density: \( J_{\text{tot}}^\mu(\vec{r},t) \equiv (c \rho_{\text{tot}}(\vec{r},t), \vec{J}_{\text{tot}}(\vec{r},t)) \).

The covariant relativistic 4-gradient operator: \[
\partial^\mu = \frac{\partial}{\partial x^\mu} \equiv \left( -\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right),
\]
whereas

the contravariant relativistic 4-gradient operator: \[
\partial^\mu = \frac{\partial}{\partial x_\mu} \equiv \left( \frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right).
\]
The D’Alembertian \{aka “Box\} operator can be written in relativistic 4-vector notation as:

\[ \Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = \partial_v \partial^v \]

We also can write:

\[ L(\vec{r},t) = \frac{1}{c^2} \frac{\partial V(\vec{r},t)}{\partial t} + \nabla \cdot \vec{A}(\vec{r},t) = \frac{\partial A^\mu(\vec{r},t)}{\partial x^\mu} = \partial_v A^v(\vec{r},t) \]

We can also write:

\[ \frac{1}{c} \frac{\partial L(\vec{r},t)}{\partial t} - \nabla L(\vec{r},t) = \left( - \frac{1}{c} \frac{\partial}{\partial t} + \vec{v} \right) \frac{\partial L(\vec{r},t)}{\partial x_\mu} = -\partial_v L(\vec{r},t) = -\partial_v A^v(\vec{r},t) \]

Thus, the two equations:

1st eqn: \[ \Box^2 V(\vec{r},t) = -\mu_o \epsilon_{tot}(\vec{r},t) \]

2nd eqn: \[ \Box^2 A(\vec{r},t) - \nabla L(\vec{r},t) = -\mu_o J_{tot}(\vec{r},t) \]

can be written as a single equation, very compactly (& elegantly!) in relativistic electrodynamics as:

\[ \partial_v A^v(\vec{r},t) - \partial_v \partial_v A^v(\vec{r},t) = -\mu_o J_{tot}(\vec{r},t) \]

Very shortly, we will learn that as a consequence of the gauge-invariant nature of the electromagnetic interaction, we can choose to work in the so-called Lorenz gauge, namely that:

\[ \partial_v A^v(\vec{r},t) = -\mu_o J_{tot}(\vec{r},t) \]

If \[ L(\vec{r},t) = \partial_v A^v(\vec{r},t) = 0 \quad \forall \ (\vec{r},t), \] then separately both \[ \frac{\partial L(\vec{r},t)}{\partial t} = 0 \] and \[ \nabla L(\vec{r},t) = 0 \quad \forall \ (\vec{r},t). \]

Hence:

\[ \frac{1}{c} \frac{\partial L(\vec{r},t)}{\partial t} - \nabla L(\vec{r},t) = \left( - \frac{1}{c} \frac{\partial}{\partial t} + \vec{v} \right) \frac{\partial L(\vec{r},t)}{\partial x_\mu} = -\partial_v L(\vec{r},t) = -\partial_v A^v(\vec{r},t) = 0 \quad \forall \ (\vec{r},t). \]

Thus, in the Lorenz gauge \[ \partial_v A^v(\vec{r},t) = 0, \] and hence our single equation (above) becomes:

\[ \partial_v \partial_v A^v(\vec{r},t) = -\mu_o J_{tot}(\vec{r},t) \]

or equivalently: \[ \Box^2 A^v(\vec{r},t) = -\mu_o J_{tot}(\vec{r},t) \]

This single relativistic 4-potential equation (actually 4 separate equations, since \( \mu = 0,1,2,3 \)) contain(s) all of the information that is contained in the four Maxwell field equations !!!

(1) Gauss’ Law: \[ \nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\epsilon_o} \rho_{tot}(\vec{r},t) \]

(2) No Magnetic Charges: \[ \nabla \times \vec{B}(\vec{r},t) = 0 \]

(3) Faraday’s Law: \[ \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \]

(4) Ampere’s Law: \[ \nabla \times \vec{B}(\vec{r},t) = \mu_o \vec{J}_{tot}(\vec{r},t) + \mu_o \epsilon_o \frac{\partial \vec{E}(\vec{r},t)}{\partial t} \]
Gauge Transformations

Using \( \partial_v \partial^\nu A^\nu (\vec{r},t) - \partial^\nu \partial_v A^\nu (\vec{r},t) = -\mu_o J^\nu_{\text{tot}} (\vec{r},t) \) enables us to reduce the problem of finding the six components associated with the two vectors \( \vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \) and \( \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \) down to four components – the scalar potential \( V \) and the vector potential \( \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \).

As we saw last semester in P435, \( \vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t) \) and \( \vec{E}(\vec{r},t) = -\vec{\nabla} V(\vec{r},t) - \partial \vec{A}(\vec{r},t)/\partial t \) do not enable us to uniquely define / specify / determine the scalar and vector potentials \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \); only potential differences \( V_2 - V_1 \) and \( A_2 - A_1 \) are physically meaningful…

We are free to impose extra conditions on \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \) as long as \( \vec{B}(\vec{r},t) \) and \( \vec{E}(\vec{r},t) \) remain unchanged by the imposition of these extra conditions…

This freedom to impose extra/additional conditions on \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \) without changing \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \) is known as gauge freedom or (more formally) as gauge invariance.

Suppose we have e.g. two sets of potentials \( \{V(\vec{r},t), \vec{A}(\vec{r},t)\} \) and \( \{V'(\vec{r},t), \vec{A}'(\vec{r},t)\} \) that correspond to the same physical fields \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \). However, these two sets of potentials must be related to each other:

\[
\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \alpha(\vec{r},t)
\]

and:

\[
V'(\vec{r},t) = V(\vec{r},t) + \beta(\vec{r},t)
\]

Because:

\[
\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t) = \vec{\nabla} \times \vec{A}'(\vec{r},t) = \vec{\nabla} \times (\vec{A}(\vec{r},t) + \alpha(\vec{r},t)) \Rightarrow \vec{\nabla} \times \alpha(\vec{r},t) \equiv 0
\]

But if:

\[
\left( \vec{\nabla} \times \alpha(\vec{r},t) \right) = 0 \quad \forall (\vec{r},t),
\]

then since:

\[
\left( \vec{\nabla} \times \vec{f}(\vec{r},t) \right) \equiv 0 \quad \{\text{always}\}
\]

\[
\Rightarrow \text{We can always write } \alpha(\vec{r},t) \text{ as the gradient of a scalar function } \lambda(\vec{r},t): \alpha(\vec{r},t) \equiv \vec{\nabla} \lambda(\vec{r},t).
\]

Then:

\[
\vec{E}(\vec{r},t) = -\vec{\nabla} V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}
\]

\[
= -\vec{\nabla} V'(\vec{r},t) - \frac{\partial \vec{A}'(\vec{r},t)}{\partial t} = -\vec{\nabla} V(\vec{r},t) - \vec{\nabla} \beta(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \frac{\partial \alpha(\vec{r},t)}{\partial t}
\]

The scalar function \( \beta(\vec{r},t) \) must be related to \( \alpha(\vec{r},t) \) by:

\[
\vec{\nabla} \beta(\vec{r},t) + \frac{\partial \alpha(\vec{r},t)}{\partial t} = 0
\]

But:

\[
\alpha(\vec{r},t) \equiv \vec{\nabla} \lambda(\vec{r},t) \Rightarrow \vec{\nabla} \beta(\vec{r},t) + \frac{\partial \left( \vec{\nabla} \lambda(\vec{r},t) \right)}{\partial t} = 0 \Rightarrow \vec{\nabla} \left( \beta(\vec{r},t) + \frac{\partial \lambda(\vec{r},t)}{\partial t} \right) = 0
\]

which must hold for arbitrary/any/all space-time points \((\vec{r},t)\) ⇒

\[
\beta(\vec{r},t) + \frac{\partial \lambda(\vec{r},t)}{\partial t} = 0.
\]
Note that \( \nabla \left( \beta(\vec{r},t) + \frac{\partial \lambda(\vec{r},t)}{\partial t} \right) = 0 \) can also be satisfied if:

\[
\beta(\vec{r},t) + \frac{\partial \lambda(\vec{r},t)}{\partial t} = \kappa(t)
\]

i.e. the scalar fcn \( \kappa(t) \) depends only on time, \( t \). Then we see that:

\[
\beta(\vec{r},t) = -\frac{\partial \lambda(\vec{r},t)}{\partial t} + \kappa(t)
\]

However, we can always “absorb” \( \kappa(t) \) into \( \lambda(\vec{r},t) \) by adding \( \int_{t=0}^{t} \kappa(t') dt' \) to \( \lambda(\vec{r},t) \), i.e. \( \lambda'(\vec{r},t) = \lambda(\vec{r},t) + \int_{t=0}^{t} \kappa(t') dt' \). We can then redefine the “new” \( \lambda'(\vec{r},t) \). \( \Rightarrow \) “old” \( \lambda(\vec{r},t) \).

Note also that since the scalar function \( \kappa(t) \) depends only on time \( t \), this will not affect the gradient of \( \nabla \lambda(\vec{r},t) \) in any way, and hence \( \vec{\alpha}(\vec{r},t) = \nabla \lambda(\vec{r},t) \) is completely unaffected by this!

Thus:

\[
\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \alpha(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla \lambda(\vec{r},t)
\]

or:

\[
\nabla \lambda(\vec{r},t) = \vec{A}'(\vec{r},t) - \vec{A}(\vec{r},t) \equiv \Delta \vec{A}(\vec{r},t)
\]

And:

\[
V'(\vec{r},t) = V(\vec{r},t) - \frac{\partial \lambda(\vec{r},t)}{\partial t}
\]

or:

\[
-\frac{\partial \lambda(\vec{r},t)}{\partial t} = V'(\vec{r},t) - V(\vec{r},t) \equiv \Delta V(\vec{r},t)
\]

Hence, for any scalar function \( \lambda(\vec{r},t) \), we can always add \( \nabla \lambda(\vec{r},t) \) to \( \vec{A}(\vec{r},t) \) provided that we simultaneously subtract \( \partial \lambda(\vec{r},t)/\partial t \) from \( V(\vec{r},t) \).

\( \Rightarrow \) This “prescription” will leave the \( \vec{E}(\vec{r},t) \) and \( \vec{B}(\vec{r},t) \)-fields unchanged / invariant under this so-called gauge transformation!

- Gauge transformations can be exploited to adjust \( \nabla \ast \vec{A}(\vec{r},t) \).
- In magneto\textit{statics}, we chose \( \nabla \ast \vec{A}(\vec{r},t) = 0 \) (= the Coulomb gauge).
- In electro\textit{dynamics}, the situation is not always so clear cut!! The most convenient “gauge” choice depends on the detailed nature of the problem! \( \exists \) Many gauges to work with….

The two most popular gauges for use in \( E&M \)

\begin{align*}
\text{Coulomb Gauge:} & \quad \nabla \ast \vec{A}(\vec{r},t) = 0 & \text{(Most useful in electro/magneto\textit{statics})} \\
\text{Lorenz Gauge:} & \quad \nabla \ast \vec{A}(\vec{r},t) = -\mu_0 \varepsilon_0 \frac{\partial V(\vec{r},t)}{\partial t} & \text{(Most useful in electro\textit{dynamics})}
\end{align*}

Ludwig Valentin Lorenz, Danish physicist – a contemporary of J.C. Maxwell, ca. 1867 – not to be confused with Hendrick A. Lorentz, Dutch physicist & contemporary of Albert Einstein…

{See/read J.D. Jackson & L.B. Okun’s “Historical Roots of Gauge Invariance” Rev. Mod. Phys. 73, 663 (2001).}

On the web at: \url{http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.73.663}


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Gauge Transformation(s) are intimately connected to the choice of an {inertial} reference frame. “Hints” of this connection, in the context of special relativity – length contraction and time-dilation FX between two {inertial} reference frames – are the spatial gradient $\vec{\nabla} \lambda(\vec{r},t)$ and temporal “gradient” $\partial \lambda'(\vec{r},t)/\partial t$ terms \{where $\lambda'(\vec{r},t) = \lambda(\vec{r},t) + \int_{t_0}^{t} \kappa(t')dt'$\} that can be added to {subtracted from} the vector potential $\vec{A}(\vec{r},t)$ \{scalar potential $V(\vec{r},t)$\}, respectively.

**The Coulomb Gauge:** \[\vec{\nabla} \cdot \vec{A}(\vec{r},t) = 0\]

Here, the scalar potential $V(\vec{r},t)$ is “easy” to calculate, but in comparison, the vector potential $\vec{A}(\vec{r},t)$ is “difficult” to calculate. The inhomogeneous differential equation for $V(\vec{r},t)$ is:

$$\nabla^2 V(\vec{r},t) + \frac{\partial}{\partial t} \left( \vec{\nabla} \cdot \vec{A}(\vec{r},t) \right) = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r},t)$$

Then for:

$$\nabla^2 V(\vec{r},t) = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r},t)$$

if we can set: $V(\vec{r} = \infty, t) = 0$ \{i.e. $\rho_{\text{tot}}(\vec{r},t)$ is a local chg. dist’n\}

the solution to 3-D Poisson’s equation is:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_{\text{tot}}(\vec{r}',t)}{\kappa} d\tau'$$

with: $\kappa = |\vec{r} - \vec{r}'| = \sqrt{r^2 - r'^2}$

But $V(\vec{r},t)$ alone does not determine $\vec{E}(\vec{r},t)$ in electrodynamics – we must also know $\vec{A}(\vec{r},t)$ as well! In the Coulomb gauge, the differential equation for the vector potential $\vec{A}(\vec{r},t)$ is:

$$\nabla^2 \vec{A}(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} - \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A}(\vec{r},t) + \mu_0\varepsilon_0 \frac{\partial V(\vec{r},t)}{\partial t} \right) = -\mu_0\vec{J}_{\text{tot}}(\vec{r},t)$$

becomes:

$$\nabla^2 \vec{A}(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{J}_{\text{tot}}(\vec{r},t) + \mu_0\varepsilon_0 \vec{\nabla} \left( \frac{\partial V(\vec{r},t)}{\partial t} \right)$$

In the Coulomb Gauge, the scalar potential at time $t$, $V(\vec{r},t)$ is determined by the distribution of electric charge at the “right now” time $t$ – which is acausal, because EM signals/information cannot propagate faster than the speed of light $c$! However, changes in the electric charge density distribution $\rho_{\text{tot}}(\vec{r}')$ at the source point $\vec{r}'$ take a finite time to be observed at the observation point $\vec{r}$ !!!

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_{\text{tot}}(\vec{r}',t)}{\kappa} d\tau'$$

Thus, in the Coulomb Gauge, the scalar potential $V(\vec{r},t)$ instantaneously reflects all changes in $\rho_{\text{tot}}(\vec{r}',t)$ . However, recall that $V(\vec{r},t)$ by itself is not a physically measurable quantity – only potential differences such as $\Delta V_{\text{fl}}(t) \equiv V(\vec{r}_2,t) - V(\vec{r}_1,t)$ or $\Delta V(\vec{r},\Delta t) \equiv V(\vec{r}_2,t) - V(\vec{r}_1,t_1)$ are physically measurable quantities!
An astronaut standing on the surface of the moon can only e.g. directly measure \( \vec{E}(\vec{r},t) \), which manifestly involves both \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \): 
\[
\vec{E}(\vec{r},t) = -\nabla V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t},
\]
thus in the Coulomb gauge, while the scalar potential \( V(\vec{r},t) \) instantaneously reflects all changes in \( \rho_{\text{tot}}(\vec{r},t) \), the combination 
\[
-\nabla V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}
\]
does not have this instantaneous, “right-now” behavior in the Coulomb gauge, i.e. the combination 
\[
-\nabla V(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}
\]
is causal in the Coulomb Gauge, and thus \( \vec{E}(\vec{r},t) \) changes in a causally-connected manner only after EM “news”/information arrives at the appropriate/ causally-related time interval \( \Delta t \), as a direct consequence of the propagation speed (= \( c \) in free space/vacuum) of this EM “news”/information. Thus, in the Coulomb gauge, the causal behavior is carried by/encrypted into the vector potential \( \vec{A}(\vec{r},t) \), in satisfying the above differential equation for \( \vec{A}(\vec{r},t) \).

**The Lorenz Gauge:**
\[
\nabla \cdot \vec{A}(\vec{r},t) = -\mu_o \epsilon_o \frac{\partial V(\vec{r},t)}{\partial t} \implies L(\vec{r},t) \equiv \nabla \cdot \vec{A}(\vec{r},t) + \mu_o \epsilon_o \frac{\partial V(\vec{r},t)}{\partial t} = 0
\]
Here we obtain:

(a) \[
\Box^2 V(\vec{r},t) + \frac{\partial L(\vec{r},t)}{\partial t} = -\frac{1}{\epsilon_o} \rho_{\text{tot}}(\vec{r},t)
\]
\[
\implies \Box^2 V(\vec{r},t) = -\frac{1}{\epsilon_o} \rho_{\text{tot}}(\vec{r},t)
\]
and:

(b) \[
\Box^2 \vec{A}(\vec{r},t) - \nabla \times \nabla \times \vec{A}(\vec{r},t) = -\mu_o \vec{J}_{\text{tot}}(\vec{r},t)
\]
\[
\implies \Box^2 \vec{A}(\vec{r},t) = -\mu_o \vec{J}_{\text{tot}}(\vec{r},t)
\]
where: \( \Box^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \).

Thus, we see from \( \Box^2 V(\vec{r},t) = -\rho_{\text{tot}}(\vec{r},t)/\epsilon_o \) and \( \Box^2 \vec{A}(\vec{r},t) = -\mu_o \vec{J}_{\text{tot}}(\vec{r},t) \) that the Lorenz gauge puts the \{time-like\} scalar potential \( V(\vec{r},t) \) and the \{space-like\} vector potential \( \vec{A}(\vec{r},t) \) on an equal footing! (i.e. a “democratic” treatment of \( V(\vec{r},t) \) and \( \vec{A}(\vec{r},t) \)).

In special relativity, \( V(\vec{r},t)/c \) and \( \vec{A}(\vec{r},t) \) respectively are the temporal and spatial components of the relativistic 4-vector potential \( A^\mu(\vec{r},t) = (V(\vec{r},t)/c, \vec{A}(\vec{r},t)) \), and \( c \rho_{\text{tot}}(\vec{r},t) \) and \( \vec{J}_{\text{tot}}(\vec{r},t) \) respectively are the temporal and spatial components the relativistic 4-vector current density \( J^\mu_{\text{tot}}(\vec{r},t) = (c \rho_{\text{tot}}(\vec{r},t), \vec{J}_{\text{tot}}(\vec{r},t)) \), where the index \( \mu = 0:3 = \{0,1,2,3\} \).

In tensor notation: \( \Box^2 A^\mu(\vec{r},t) = -\mu_o J^\mu_{\text{tot}}(\vec{r},t) \) or equivalently: \( \partial_\nu \partial^\nu A^\mu(\vec{r},t) = -\mu_o J^\mu_{\text{tot}}(\vec{r},t) \).

where: \( A^\mu \equiv (V/c, A_x, A_y, A_z) \) and \( J^\mu_{\text{tot}} \equiv (c \rho_{\text{tot}}, J_{\text{tot}}, J_{\text{tot}}, J_{\text{tot}}) \) and \( \Box^2 \equiv \frac{\partial^2}{\partial x^\nu \partial x^\nu} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \).

This is the 4-D \{space-time\} Poisson eqn (aka inhomogeneous wave eqn) in the Lorenz gauge! n.b. **Convention:** repeated indices (here = \( \nu \)) are summed over (i.e. summed over \( \nu = 0:3 \)).

n.b. Griffiths claims that he will use the Lorenz gauge for the remainder of his book {we will also!}.

The whole of electrodynamics thus reduces to solving the \{inhomogeneous\} 4-D Poisson’s equation: \( \Box^2 A^\mu(\vec{r},t) = -\mu_o J^\mu_{\text{tot}}(\vec{r},t) \) or: \( \partial_\nu \partial^\nu A^\mu(\vec{r},t) = -\mu_o J^\mu_{\text{tot}}(\vec{r},t) \) for specified sources!
Griffiths Example 10.1

Find the electric charge and current density distributions \( \rho_{\text{tot}}(\vec{r}, t) \) and \( \vec{J}_{\text{tot}}(\vec{r}, t) \) that produce:

\[
\vec{V}(\vec{r}, t) = 0
\]

\[
\vec{A}(\vec{r}, t) = \begin{cases} \frac{\mu_o k}{4c} (ct - |x|)^2 \hat{z} & \text{for } |x| \leq ct \\ 0 & \text{for } |x| > ct \end{cases}
\]

where \( k = \text{constant} \) and \( c = 1/\sqrt{\epsilon_o \mu_o} \)

Note that the parabola: \( (ct - |x|)^2 = (ct)^2 - 2ct|x| + |x|^2 = 0 \) at \( ct = |x| \).

At \( t = 0 \): \( A_z(x, t = 0) = \frac{\mu_o k}{4c} |x|^2 \) but this is also subject to \( |x| \leq ct \), hence: \( A_z(x, t = 0) = 0 \ \forall \ x \).

For \( t > 0 \): \( A_z(x, t > 0) = \begin{cases} \frac{\mu_o k}{4c} (ct - |x|)^2, & \text{for } |x| \leq ct \\ 0, & \text{for } |x| > ct \end{cases} \)

And:

\[
\vec{E}(\vec{r}, t) = \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} = -\frac{\mu_o k}{2} (ct - |x|) \hat{z} \quad \text{for } |x| < ct, \ \text{and: } \vec{E}(\vec{r}, t) = 0 \quad \text{for } |x| \geq ct.
\]

Hence:

\[
\vec{B}(\vec{r}, t) = \vec{V} \times \vec{A}(\vec{r}, t) = -\frac{\mu_o k}{4c} \frac{\partial}{\partial x} (ct - |x|)^2 \hat{y} = \pm \frac{\mu_o k}{2c} (ct - |x|) \hat{y} \quad \text{for } |x| < ct, \ \text{and } \vec{B} = 0 \quad \text{for } |x| \geq ct.
\]

n.b. For \( |x| \geq ct \), \( \vec{E}(\vec{r}, t) \) and \( \vec{B}(\vec{r}, t) = 0 \) n.b. \( \vec{B}_y(x, t) \) has a discontinuity for \( t > 0 \) at \( x = 0 \) !!

A discontinuity in \( \vec{B} \Rightarrow \exists \ \text{a free surface current } \vec{K}_{\text{free}} \!!

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It can be shown that indeed $\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 0$, $\vec{\nabla} \times \vec{B}(\vec{r}, t) = 0$ \{and $\vec{\nabla} \cdot \vec{A}(\vec{r}, t) = 0$\} \{n.b. please explicitly check/work these out yourselves!\}, and that indeed:

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = \pm \frac{\mu_o \kappa}{2} \hat{\gamma}$$

$$\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} = \pm \frac{\mu_o \kappa}{2} \hat{\gamma}$$

Maxwell’s eqn’s all satisfied, with $\rho_{tot}(\vec{r}, t) = 0$ and $\vec{J}_{tot}(\vec{r}, t) = 0$.

- However, we have seen before \{déjà vu!\} that the discontinuity in $\vec{B}(\vec{r}, t)$ @ $x = 0$ for $t > 0$ heralds/signals the presence of a free surface current $\vec{K}_{free}(\vec{r}, t)$.

- In our case \{here\}, the free surface current $\vec{K}_{free}(\vec{r}, t)$ lies in the $y$-$z$ plane \{n.b. $\vec{A} \parallel \vec{K}_{free} \parallel \hat{z}$ \}

The boundary condition associated with a free surface current (see P435 Lect. Notes 24, p. 13-17 and/or Griffiths equation 7.63 (iv), p. 333) is:

$$H^1 - H^2 = 1 \frac{B^1}{\mu_1} - 1 \frac{B^2}{\mu_2} = \vec{K}_{free} \times \hat{n}$$

with: $\mu_1 = \mu_2 = \mu_o$ \{here\}

Region 1: $x < 0$
Region 2: $x > 0$

⇒ $B^1 - B^2 = \mu_o \vec{K}_{free} \times \hat{n}$ at $x = 0$

$i.e.$: $B_y(x < 0, t)\big|_{x=0} - B_y(x > 0, t)\big|_{x=0} = \mu_o \vec{K}_{free} \times \hat{n}\big|_{x=0}$

n.b. $\hat{n}$ points from medium 2 → 1 ($\hat{n} = -\hat{x}$)

See Griffiths p. 331-2, figs. 7.46 and 7.47

But: $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$ ⇒ $kt\hat{y} = \vec{K}_{free} \times \hat{x}$

$$\vec{K}_{free}(t) = kt\hat{z}$$ at $x = 0$ in $y$-$z$ plane.

Physically, this corresponds to a uniform, but time-dependent free surface current $\vec{K}_{free}(t) = kt\hat{z}$ which flows in the $+\hat{z}$ direction at $x = 0$ in the $y$-$z$ plane. It starts up from zero at time $t = 0$, and its strength \{magnitude\} $|\vec{K}_{free}(t)| = kt$ increases linearly with time $t$. The EM “news” travels outward at speed $c$, note that $\vec{E}$ and $\vec{B}$ are still zero for points $|x| \geq ct$ !!!

Note also that $\vec{A}(\vec{r}, t)$, $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$ → $\infty$ for $|x| < ct$ as $t \to \infty$ because we have an \{∞\} extended \{not finite\} free surface current source $\vec{K}_{free}(t) = kt\hat{z}$ in this problem…
Continuous Electric Charge and Current Density Distributions
Retarded and Advanced Potentials

We derived (see above) the following relations for the potentials in the Lorenz gauge:

\[ L(\vec{r},t) \equiv \nabla \cdot \vec{A}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial V(\vec{r},t)}{\partial t} = 0 \]

\[ \Box^2 A(\vec{r},t) = -\mu_0 J_{\text{tot}}(\vec{r},t) \]

\[ \Box^2 \vec{A}(\vec{r},t) = -\mu_0 \vec{J}_{\text{tot}}(\vec{r},t) \]

Or:

\[ \Box^2 V(\vec{r},t) = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r},t) \]

\[ \Box^2 \vec{A}(\vec{r},t) = -\mu_0 \vec{J}_{\text{tot}}(\vec{r},t) \]

Or:

\[ \nabla^2 V(\vec{r},t) - \mu_0 \varepsilon_0 \frac{\partial^2 V(\vec{r},t)}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r},t) \]

\[ \nabla^2 \vec{A}(\vec{r},t) - \mu_0 \varepsilon_0 \frac{\partial^2 V(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{J}_{\text{tot}}(\vec{r},t) \]

For static situations (no time dependence) these reduce to the familiar “3-D” Poisson equation:

\[ \nabla^2 V(\vec{r}) = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r}) \]

\[ \Rightarrow \text{solution is:} \quad V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int_{V'} \frac{\rho_{\text{tot}}(\vec{r}')}{\vec{r} - \vec{r}'} d\tau' \]

\[ \nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}_{\text{tot}}(\vec{r}) \]

\[ \Rightarrow \text{solution is:} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{j}_{\text{tot}}(\vec{r}')}{\vec{r} - \vec{r}'} d\tau' \]

where:

\[ \vec{h} = \vec{r} - \vec{r}' \]

\[ \vec{h}' = \vec{r}' - \vec{r} \]

\[ \vec{h} = \vec{h}'/\vec{h} = \vec{h}'/\vert \vec{h}' \vert = (\vec{r} - \vec{r}')/\vert \vec{r} - \vec{r}' \vert \]
Now consider what happens if the sources $\rho_{tot}(\vec{r}^\prime, t)$ and $\vec{J}_{tot}(\vec{r}^\prime, t)$ in the volume $v'$ are \textit{time-dependent}:

An observer at the field point $P(\vec{r}, t)$ detects \textit{changes} in the potentials and/or the EM fields at the time $t$. However, those changes observed at the field point $P(\vec{r}, t)$ result from changes in $\rho_{tot}(\vec{r}^\prime, t_r)$ and/or $\vec{J}_{tot}(\vec{r}^\prime, t_r)$ that occurred at \textit{earlier} time(s) $t_r \equiv t - \frac{\lambda}{c}$, because it takes a \textit{finite} amount of time for EM “news” (i.e. changes) occurring at a source point $S(\vec{r}^\prime, t_r)$ to \textit{propagate} to the \textit{observation/field} point $P(\vec{r}, t)$.

In terms of a space-time light-cone diagram (for propagation of EM news in free space = vacuum):

$$c\Delta t = c(t - t_r) = \lambda$$

or:
$$\Delta t = (t - t_r) = \frac{\lambda}{c}$$

or:
$$t = t_r + \frac{\lambda}{c}$$

or:
$$t_r = t - \frac{\lambda}{c} = \text{retarded time}$$

with:
$$\lambda = |\vec{r}(t) - \vec{r}^\prime(t_r)|$$

and:
$$\vec{h} = \vec{r}(t) - \vec{r}^\prime(t_r)$$

and:
$$\hat{h} = \frac{\vec{h}}{|\vec{h}|} = \frac{|\vec{r}(t) - \vec{r}^\prime(t_r)|}{|\vec{r}(t) - \vec{r}^\prime(t_r)|}$$

The Light-Cone in Space-Time:

$$c\Delta t = \sqrt{\lambda^2 + c^2(t - t_r)^2}$$

$$t = t - \frac{\lambda}{c}$$

$$t_r = \sqrt{(\vec{r} - \vec{r}^\prime)^2 + c^2(t - t_r)^2}$$

$$P(\vec{r}, t)$$

Field/Observation Point

$$(0, 0)$$

Source Point $S(\vec{r}^\prime, t_r)$

space

c-time

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Due to causality, it takes a finite time $\Delta t = t - t_r = \lambda/c = |\vec{r}(t) - \vec{r}(t_r)|/c$ for a change e.g. in the electric charge density $\rho_{tot}(\vec{r}, t_r)$ at the source point $S(\vec{r}, t_r)$ at the earlier, retarded time $t_r$ to propagate to the observation/field point $P(\vec{r}, t)$ at the later time, $t > t_r$: $t = t_r + \lambda/c$. In free space (the vacuum) this EM “news” / information propagates with speed $c = 3 \times 10^8$ m/sec.

We point out {here, for completeness’ sake} that both the source point $S(\vec{r}, t_r)$ and observation/field point $P(\vec{r}, t)$ are at rest (e.g. in the lab frame – an inertial {i.e. a non-accelerating} reference frame). The electrodynamics of this situation will be different e.g. if the observer is moving relative to the source, or if both the source and the observer are moving with respect to a chosen reference frame (e.g. the lab frame). Special relativity deals with these situations...

Thus, for non-static source volume charge density and/or current density distributions $\rho_{tot}(\vec{r}, t)$ and $J_{tot}(\vec{r}, t)$, the scalar and vector potentials $V(\vec{r}, t)$ and $A(\vec{r}, t)$ at the observation/field point $P(\vec{r}, t)$ at the later, causal time $t = t_r + \lambda/c$ ($t > t_r$) are causally related to the sources $\rho_{tot}(\vec{r}, t_r)$ and $J_{tot}(\vec{r}, t_r)$ at the source point(s) $S(\vec{r}, t_r)$ at the earlier, so-called retarded time, $t_r = t - \lambda/c$ by the following relations:

\[
\begin{align*}
V(\vec{r}, t) &= \frac{1}{4\pi \varepsilon_0} \int_0^{t_r} \frac{\rho_{tot}(\vec{r}, t)}{\lambda} d\tau' \\
A(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int_0^{t_r} \frac{J_{tot}(\vec{r}, t)}{\lambda} d\tau'
\end{align*}
\]

with $\lambda = \frac{|\vec{r}(t) - \vec{r}(t_r)|}{c}$

These expressions for the potentials are known as retarded potentials because changes in the source volume charge density and/or current density distributions $\rho_{tot}(\vec{r}, t)$ and $J_{tot}(\vec{r}, t)$ at source point(s) $S(\vec{r})$ occurring at the (earlier) “retarded time”, $t_r < t$, take a time interval $\Delta t = t - t_r = \lambda/c = |\vec{r} - \vec{r}_{\text{source}}|/c$ to propagate from the source point $S(\vec{r}, t_r)$ to the observation/field point $P(\vec{r}, t)$ arriving there at the later, causal time $t = t_r + \lambda/c$, where $\lambda = |\vec{r}(t) - \vec{r}(t_r)|$.

- This is exactly the situation where an observer is looking out into the night sky. Light (= EM radiation) from a star a distance $\lambda_{\text{star}} = |\vec{r}_{\text{obs}}(t) - \vec{r}_{\text{star}}(t_r)|$ away, arriving on Earth at time $t$ had to have left the surface of that star at an earlier time: $t_r = t - \lambda_{\text{star}}/c$.

The transit/propagation time of the light from the star to the Earth is: $\Delta t = t - t_r = \lambda_{\text{star}}/c$.

- From our own star (the sun), this time interval is:

\[
\Delta t = \frac{\lambda_{\odot - \odot}}{c} = \frac{\lambda_{\text{Earth-Sun}}}{c} = \frac{1.496 \times 10^{11} \text{m}}{3 \times 10^8 \text{m/s}} = 500 \text{ seconds} = 8.3 \text{ minutes}
\]

Thus, we see that causality over astronomical distances is significant, but it is also important even for laboratory/everyday distance scales.
In the previous semester’s E&M course (P435) we saw that, for static sources:

\[
\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = -\nabla \left( \frac{\rho_{\text{tot}}(\vec{r})}{\varepsilon_0} \right) \quad \text{where: } \rho_{\text{tot}} = fcn(\vec{r}) \quad \text{only}
\]

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \nabla \left( \frac{\rho_{\text{tot}}(\vec{r}')}{\varepsilon_0} \right) \, d\tau' \quad \text{where: } \rho_{\text{tot}}(\vec{r}') = fcn(\vec{r}') \quad \text{only}
\]

\[
\vec{E}(\vec{r}) = + \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \nabla \left( \frac{\rho_{\text{tot}}(\vec{r}')}{\varepsilon_0} \right) \hat{\tau} \, d\tau'
\]

\[
\vec{B}(\vec{r}) = -\nabla \times \vec{A}(\vec{r}) = -\nabla \times \left( \frac{\vec{J}_{\text{tot}}(\vec{r})}{\mu_0} \right) \quad \text{where: } \vec{J}_{\text{tot}}(\vec{r}) = fcn(\vec{r}) \quad \text{only}
\]

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\tau'} \nabla \times \left( \frac{\vec{J}_{\text{tot}}(\vec{r}')}{\mu_0} \right) \, d\tau' \quad \text{where: } \vec{J}_{\text{tot}}(\vec{r}') = fcn(\vec{r}') \quad \text{only}
\]

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\tau'} \vec{J}_{\text{tot}}(\vec{r}') \times \hat{\tau} \, d\tau'
\]

However, we **cannot** simply “generalize” these to the time-dependent case **merely** by adding \(t\) and \(t'\) arguments to the \{\(\vec{E}\) and \(\vec{B}\)\} and \{\(\rho_{\text{tot}}\) and \(\vec{J}_{\text{tot}}\)\} (field and source) variables respectively!!

\[
\text{i.e.} \quad \vec{E}_t(\vec{r},t) \neq \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \rho_{\text{tot}}(\vec{r}',t) \hat{\tau} \, d\tau' \quad \text{**Nyet** !!!}
\]

\[
\text{i.e.} \quad \vec{B}_t(\vec{r},t) \neq \frac{\mu_0}{4\pi} \int_{\tau'} \vec{J}_{\text{tot}}(\vec{r}',t) \times \hat{\tau} \, d\tau' \quad \text{**Nyet** 2 !!!}
\]

The reason **why** these expressions are **not** correct is simple: The **causal** connection between \(t\) and \(t'\) has **not** been properly taken into account in the above two formulae: \(t_c = t - \lambda/c\) with \(\lambda = |\vec{r}(t) - \vec{r}'(t_c)|\).

Properly taking into account this **causal** connection we need to realize that:

\[
\rho_{\text{tot}}(\vec{r}',t) = \rho_{\text{tot}}(\vec{r}',t - \lambda/c) \quad \text{via the causal relation } t_c = t - \lambda/c \quad \text{and hence are **implicit** functions of } \lambda = |\vec{r}(t) - \vec{r}'(t_c)|
\]

\[
\vec{J}_{\text{tot}}(\vec{r}',t) = \vec{J}_{\text{tot}}(\vec{r}',t - \lambda/c) \quad \text{functions of } \lambda \text{ because } \lambda = |\vec{r}(t) - \vec{r}'(t_c)| = c\Delta t = c(t - t_c) !!!
\]

Thus, in order to correctly / properly calculate \(\vec{E}(\vec{r},t)\) and \(\vec{B}(\vec{r},t)\) we need to back up and calculate these relations **much** more carefully!!!
In calculating \( e.g. \) the Laplacian of the retarded scalar potential \( V (\vec{r}, t) \), it is **critical** to realize that the integrand in \( V (\vec{r}, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \rho_{tot}(\vec{r}', t_{\tau}) \frac{d\tau'}{h} \) depends on \( \vec{r} \) in **two** places:

**Explicitly**, in the **denominator** of the integrand, because: \( h = |\vec{r} - \vec{r}'| \), and **also**

**Implicitly**, in the **numerator** of the integrand, because: \( t_{\tau} = t - h/c = t - |\vec{r} - \vec{r}'|/c \).

Since:

\[
V (\vec{r}, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \rho_{tot}(\vec{r}', t_{\tau}) \frac{d\tau'}{h} = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \rho_{tot}(\vec{r}', t \frac{|\vec{r} - \vec{r}'|}{c}) \frac{d\tau'}{|\vec{r} - \vec{r}'|}
\]

Then since \( \vec{V} \) = \( fcn(\vec{r}) \) only:

\[
\vec{V} V (\vec{r}, t) = \vec{V} \left( \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \rho_{tot}(\vec{r}', t_{\tau}) \frac{d\tau'}{h} \right)
= \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \vec{V} \left( \frac{\rho_{tot}(\vec{r}', t_{\tau})}{h} \right) d\tau' = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \vec{V} \left( \frac{\rho_{tot}(\vec{r}', t_{\tau})}{|\vec{r} - \vec{r}'|} \right) d\tau'
\]

But \( \rho_{tot}(\vec{r}', t_{\tau}) \) is an **explicit** \( fcn(\vec{r}') \) and **also** an **implicit** \( fcn(\vec{r}) \) because \( t_{\tau} = t - h/c = t - |\vec{r} - \vec{r}'|/c \).

By the chain rule:

\[
\frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \vec{V} \left( \frac{\rho_{tot}(\vec{r}', t_{\tau})}{h} \right) d\tau' = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \left[ \frac{1}{h} \vec{V} \rho_{tot}(\vec{r}', t_{\tau}) + \rho_{tot}(\vec{r}', t_{\tau}) \vec{V} \left( \frac{1}{h} \right) \right] d\tau'
\]

And:

\[
\vec{V} \rho_{tot}(\vec{r}', t_{\tau}) = \frac{\partial \rho_{tot}(\vec{r}', t_{\tau})}{\partial t_{\tau}} \vec{V} t_{\tau} = \frac{\partial \rho_{tot}(\vec{r}', t_{\tau})}{\partial \tau} \vec{V} t_{\tau}
\]

**n.b.** In the last step we used the fact that \( t_{\tau} = t - h/c \) with \( h = |\vec{r} - \vec{r}'| \) \( \neq fcn(t, t_{\tau}) \) (because \{here\} the **source** and the **observer** are not moving relative to each other – \( i.e. \) they are at fixed/stationary points, \( e.g. \) in the lab reference frame), therefore: \( \frac{\partial}{\partial t_{\tau}} = \frac{\partial}{\partial \tau} \) \{here\}.

What is \( \vec{V} t_{\tau} \)? Since \( \vec{V} = fcn(\vec{r}) \) only:

\[
\vec{V} t_{\tau} = \vec{V} \left( t_{\tau} - \frac{h}{c} \right) = \vec{V} \left( \frac{1}{c} \right) = -\frac{1}{c} \vec{V} h = -\frac{1}{c} \vec{V} |\vec{r} - \vec{r}'| = -\frac{1}{c} \hat{h}
\]

where: \( \vec{V} h = \vec{V} |\vec{r} - \vec{r}'| = \hat{h} \) and: \( \vec{V} \left( \frac{1}{h} \right) = \vec{V} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\hat{h}}{h^2} \)

Thus:

\[
\vec{V} \rho_{tot}(\vec{r}', t_{\tau}) = \frac{\partial \rho_{tot}(\vec{r}', t_{\tau})}{\partial t_{\tau}} \vec{V} t_{\tau} = \frac{\partial \rho_{tot}(\vec{r}', t_{\tau})}{\partial \tau} \left( -\frac{\vec{V} h}{c} \right) = \rho_{tot}(\vec{r}', t_{\tau}) \left( -\frac{\hat{h}}{c} \right) = -\frac{h}{c} \rho_{tot}(\vec{r}', t_{\tau}) \hat{h}
\]

where: \( \rho_{tot}(\vec{r}', t_{\tau}) \equiv \frac{\partial \rho_{tot}(\vec{r}', t_{\tau})}{\partial t} \)

**See Appendices A & B**
Thus:
\[ \vec{\nabla} V_t(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\vec{\nabla} \rho_{tot}(\vec{r}', t) + \rho_{tot}(\vec{r}', t) \vec{\nabla} \left( \frac{1}{\varepsilon} \right)}{r'} d\tau' \]

but: \[ \vec{\nabla} \rho_{tot}(\vec{r}', t) = -\frac{1}{c} \dot{\rho}_{tot}(\vec{r}', t) \hat{c} \]
and: \[ \vec{\nabla} \left( \frac{1}{\varepsilon} \right) = -\frac{\hat{c}}{\varepsilon^2} \]
{from above}

Therefore:
\[ \vec{\nabla} V_t(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \left[ -\frac{\dot{\rho}_{tot}(\vec{r}', t)}{\hbar c} \hat{c} - \rho_{tot}(\vec{r}', t) \left( \frac{\hat{c}}{\hbar^2} \right) \right] d\tau' \]

If we now take the divergence of \( \vec{\nabla} V_t(\vec{r}, t) \), i.e. the Laplacian of \( V_t(\vec{r}, t) \):

\[ \nabla^2 V_t(\vec{r}, t) = \vec{\nabla} \cdot \left( \vec{\nabla} V_t(\vec{r}, t) \right) = \frac{1}{4\pi\varepsilon_o} \int \vec{\nabla} \cdot \left[ -\frac{\dot{\rho}_{tot}(\vec{r}', t)}{\hbar c} \vec{\nabla} \left( \frac{\hat{c}}{\hbar} \right) - \rho_{tot}(\vec{r}', t) \vec{\nabla} \left( \frac{\hat{c}}{\hbar^2} \right) \right] d\tau' \]

Using the chain rule:
\[ = \frac{1}{4\pi\varepsilon_o} \int \left[ -\frac{1}{c} \left( \frac{\hat{c}}{\hbar} \right) \vec{\nabla} \dot{\rho}_{tot}(\vec{r}', t) + \dot{\rho}_{tot}(\vec{r}', t) \vec{\nabla} \left( \frac{\hat{c}}{\hbar^2} \right) \right] d\tau' \]

Since: \[ \vec{\nabla} \rho_{tot}(\vec{r}', t) = -\frac{1}{c} \dot{\rho}_{tot}(\vec{r}', t) \hat{c} \]
{from above}

Then:
\[ \vec{\nabla} \dot{\rho}_{tot}(\vec{r}', t) = \vec{\nabla} \dot{\rho}_{tot}(\vec{r}', t - \frac{\vec{r} - \vec{r}'}{c}) = \vec{\nabla} \dot{\rho}_{tot}(\vec{r}', t) \frac{\vec{r} - \vec{r}'}{c} = -\frac{1}{c} \dot{\rho}_{tot}(\vec{r}', t) \vec{\nabla} \hat{c} = -\frac{1}{c} \dot{\rho}_{tot}(\vec{r}', t) \hat{c} \]

where: \( \dot{\rho}_{tot}(\vec{r}', t) = \frac{\partial \rho_{tot}(\vec{r}', t)}{\partial t} = \frac{\partial^2 \rho_{tot}(\vec{r}', t)}{\partial t^2} \)
and: \( \vec{\nabla} \hat{c} = \hat{c} \)

See Appendix A

\[ \vec{\nabla} \left( \frac{\hat{c}}{\hbar^2} \right) = 4\pi \delta^3 \left( \frac{\vec{r} - \vec{r}'}{\hbar} \right) \]
See Appendix C

\( \delta^3 \left( \frac{\vec{r} - \vec{r}'}{\hbar} \right) = 3\text{-D delta fcn for } \vec{r} = \vec{r} - \vec{r}' \)

\[ \nabla^2 V_t(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \left[ \frac{1}{c^2} \left( \frac{\dot{\rho}_{tot}(\vec{r}', t)}{\hbar} \right) - 4\pi \rho_{tot}(\vec{r}', t) \delta^3 \left( \frac{\vec{r} - \vec{r}'}{\hbar} \right) \right] d\tau' \]

Thus:
\[ = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[ \frac{1}{4\pi\varepsilon_o} \int_{\tau' = \tau} \frac{\rho_{tot}(\vec{r}', t)}{\hbar} d\tau' \right] - \frac{1}{\varepsilon_o} \int_{\rho_{tot}(\vec{r}, t)} \frac{\rho_{tot}(\vec{r}', t)}{\hbar} \delta^3 \left( \frac{\vec{r} - \vec{r}'}{\hbar} \right) d\tau' \]

Or:
\[ \square V_t(\vec{r}, t) = \nabla^2 V_t(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 V_t(\vec{r}, t)}{\partial t^2} = -\frac{1}{\varepsilon_o} \rho_{tot}(\vec{r}, t) \]
Thus, we see that the retarded scalar potential $V_r(\vec{r}, t)$ does indeed satisfy the \textit{inhomogeneous wave equation} / the 4-D Poisson’s equation!

The same methodology can be carried out for the retarded vector potential $A_r(\vec{r}, t)$ with the same results \{\textit{please work through this yourselves} !!!\}, such that:

\[
\Box^2 V_r(\vec{r}, t) = \nabla^2 V_r(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 V_r(\vec{r}, t)}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r}, t)
\]

\[
\Box^2 A_r(\vec{r}, t) = \nabla^2 A_r(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 A_r(\vec{r}, t)}{\partial t^2} = -\mu_0 J_{\text{tot}}(\vec{r}, t)
\]

where:

\[
V_r(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \rho_{\text{tot}}(\vec{r}', t) \frac{1}{|\vec{r} - \vec{r}'|} d\mathcal{V}'
\]

\[
A_r(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \mathcal{J}_{\text{tot}}(\vec{r}', t) \frac{1}{|\vec{r} - \vec{r}'|} d\mathcal{V}'
\]

Note that because the D’Alembertian operator $\Box^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ explicitly involves the second derivative with respect to time, $\partial^2/\partial t^2$ (i.e. note that it is \textit{quadratic} \{not \textit{linearly}\} dependent in the time variable $t$), therefore the D’Alembertian operator a.)\textit{manifestly} obeys time-reversal invariance ($t \to -t$) and b.) nor does it distinguish \textit{past} from \textit{future}!

Thus, there exist equally mathematically-acceptable, but physically \textit{unacceptable}, \textit{acausal} solutions (\textit{i.e.} ones which \textit{violate causality}), known as the so-called \textit{advanced potentials} (n.b. the above proof(s) are also valid for the \textit{advanced potential} solutions) where:

\textbf{Advanced Time}: $t_a \equiv t + \hbar/c$ with $t_a > t$ and thus: $t = t_a - \hbar/c$.

and:

\[
V_a(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \rho_{\text{tot}}(\vec{r}', t_a) \frac{1}{|\vec{r} - \vec{r}'|} d\mathcal{V}'
\]

\[
A_a(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \mathcal{J}_{\text{tot}}(\vec{r}', t_a) \frac{1}{|\vec{r} - \vec{r}'|} d\mathcal{V}'
\]

with:

\[
\Box^2 V_a(\vec{r}, t) = \nabla^2 V_a(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 V_a(\vec{r}, t)}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho_{\text{tot}}(\vec{r}, t)
\]

\[
\Box^2 A_a(\vec{r}, t) = \nabla^2 A_a(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 A_a(\vec{r}, t)}{\partial t^2} = -\mu_0 J_{\text{tot}}(\vec{r}, t)
\]

The \textit{advanced} potentials are entirely consistent with Maxwell’s equations, but \textit{violate causality} – because they predict potentials \textit{now} (at time $t$) that depend on the charge and current distributions at a \textit{future} time $t_a \equiv t + \hbar/c \Leftrightarrow$ We do \textit{not} observe such things in our universe!\{\textit{n.b.} this has \textit{not} stopped physicists from seriously looking for such things as tachyons, \textit{etc.}\}
• In our universe, direct/empirical observation tells us that electromagnetic influences / changes / disturbances propagate with time going **forward**, not going **backward** in time – i.e. the universe that we live in behaves **causally**.

• The **macroscopic** theory of electrodynamics **must** be **manifestly time-reversal invariant** (i.e. under the operation of time reversal, \( t \to -t \)) because at the **microscopic/elementary particle physics** level, the electromagnetic interaction **manifestly** obeys time-reversal invariance. This is **not** a trivial point, because e.g. the **microscopic** weak interaction **violates** time-reversal invariance in **certain** situations, e.g. the weak decays of neutral strange, charmed and \( b \)-mesons \((K^0/\bar{K}^0, D^0/\bar{D}^0, B^0/\bar{B}^0, B_s^0/\bar{B}_s^0)!! \)

**Griffiths Example 10.2:**

An infinitely long straight wire carries a time-dependent current \( I(t) = 0 \) for \( t < 0 \), \( I(t) = I_0 \) for \( t \geq 0 \).

Find/determine the resulting \( \vec{E} \) and \( \vec{B} \) fields at an observation point \( P(\rho, t) \) at a radial distance \( \rho \) from the axis of the wire. We choose the current-carrying long wire to lie along the \( \hat{z} \)-axis as shown in the figure below:

\[ \vec{\kappa} = \vec{r}(t) - \vec{r}(t_r) \]
\[ \kappa = |\vec{\kappa}| = |\vec{r} - \vec{r}'| = \sqrt{\rho^2 + z^2} \]

n.b. We assume that the \( \infty \)-long line current is \{always\} electrically **neutral**, hence the **retarded** scalar potential \( V_r(\rho, t) = 0 \) everywhere \( \forall (\rho, t) \).

The **retarded** vector potential is:
\[
\vec{A}_r(\rho, t) = \frac{\mu_0}{4\pi} \int_{z=-\infty}^{z=\infty} \frac{I(t)}{\kappa} \, dz \, \hat{z} \quad \text{where: } \kappa = \sqrt{\rho^2 + z^2}
\]
and \( t = t_r + \kappa/c \), where \( t_r = 0 \) is the **retarded** time that the current is switched on.
If \( t = t_x + \frac{\lambda}{c} \) and \( \lambda = \sqrt{\rho^2 + z^2} \), then for times \( t < \frac{\lambda}{c} \), from the observer’s perspective, at the position \((\vec{r} = \rho \hat{\rho})\) the current \( I(t) \) has not yet been switched on. For \( t \geq \frac{\lambda}{c} \), only the segment \( |z| \leq \sqrt{(ct)^2 - \rho^2} \) contributes (because outside of this range the retarded time \( t_r < 0 \), so \( I(t_r < 0) = 0 \)).

Thus, for observer times \( t \geq \frac{\lambda}{c} \) (i.e. \( t_x \geq 0 \)) the retarded vector potential is:

\[
\vec{A}_r(\rho, t) = \frac{\mu_0 I_a}{2\pi} \ln \left( \frac{ct + \sqrt{(ct)^2 - \rho^2}}{\rho} \right) \hat{z} \quad \text{n.b.} \quad \vec{A}_r(\rho, t) \text{ has no explicit } z \text{ or } \phi \text{-dependence.}
\]

Noting that:
\[
\frac{\partial}{\partial x} \ln u(x) = \frac{1}{u(x)} \frac{\partial u(x)}{\partial x},
\]
after carrying out the needed differentiation(s) and ensuing algebra, the retarded electric and magnetic fields at the observer’s position \( P(\vec{r} = \rho \hat{\rho}) \) for times \( t \geq \frac{\lambda}{c} \) (i.e. \( t_x \geq 0 \)) are:

\[
\vec{E}_r(\rho, t) = -\frac{\partial \vec{A}_r(\rho, t)}{\partial t} = -\frac{\mu_0 I_a c}{2\pi \sqrt{(ct)^2 - \rho^2}} \hat{z} \quad \text{(n.b.} \quad \vec{E}_r(\rho, t) \text{ is anti-} \parallel I(t) )
\]

and:

\[
\vec{B}_r(\rho, t) = \vec{V} \times \vec{A}_r(\rho, t) = \left[ \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{\rho} + \left[ \frac{\partial A_\phi}{\partial z} - \frac{\partial A_z}{\partial \phi} \right] \hat{\phi} + \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} \left( \rho A_\phi \right) - \frac{\partial A_\phi}{\partial \rho} \right] \hat{z}
\]

Only surviving term:

\[
\vec{B}_r(\rho, t) = -\frac{\partial A_z(\rho, t)}{\partial \rho} \hat{\phi} = \frac{\mu_0 I_a}{2\pi \rho} \frac{ct}{\sqrt{(ct)^2 - \rho^2}} \hat{\phi}
\]

Thus:

\[
\vec{E}_r(\rho, t \rightarrow \infty) \rightarrow 0 \quad \text{These are } \vec{E} \text{ and } \vec{B} \text{ fields associated with a steady current } I_a \text{ flowing down wire – i.e. the static fields!}
\]

Note that \( \vec{A}_r(\rho, t) \rightarrow \infty \) (logarithmically) as \( t \rightarrow \infty \) because we have an \( \{ \infty \} \) extended (not finite) source in this problem... However, note also that as \( t \rightarrow \infty \):

\[
\vec{E}_r(\rho, t \rightarrow \infty) \rightarrow 0 \quad \text{and} \quad \vec{B}_r(\rho, t \rightarrow \infty) \rightarrow \frac{\mu_0 I_a}{2\pi \rho} \hat{\phi}
\]
Jefimenko’s Equations:

Time-Dependent Generalization of Coulomb’s Law and the Biot-Savart Law

Given the retarded potentials:

\[
V_r(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int_{\mathbb{V}} \frac{\rho_{\text{tot}}(\vec{r'}, t')}{|\vec{r} - \vec{r'}|} \, d\tau' \\
A_r(\vec{r}, t) = \frac{\mu_o}{4\pi} \int_{\mathbb{V}} \frac{\vec{J}_{\text{tot}}(\vec{r'}, t')}{|\vec{r} - \vec{r'}|} \, d\tau'
\]

where:

\[
t = t_r + \frac{\lambda}{c}
\]

and:

\[
\lambda = |\vec{r} - \vec{r'}(t_r)|
\]

We can determine the corresponding \textit{retarded} electric and magnetic fields from:

\[
\vec{E}_r(\vec{r}, t) = -\vec{\nabla} V_r(\vec{r}, t) - \frac{\partial A_r(\vec{r}, t)}{\partial t} \\
\vec{B}_r(\vec{r}, t) = \vec{\nabla} \times \vec{A}_r(\vec{r}, t)
\]

However, the (gory) micro-details of obtaining \( \vec{E}_r \) and \( \vec{B}_r \) from \( V_r \) and \( A_r \) are \textbf{not} completely trivial, and must be done/carried out with great care/attention to detail…

We have previously/already calculated \(-\vec{\nabla} V_r(\vec{r}, t)\) \{on pages 14-16 of these lecture notes\}:

\[
-\vec{\nabla} V_r(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int_{\mathbb{V}} \left[ \hat{\rho}_{\text{tot}}(\vec{r'}, t') \frac{\lambda}{c} + \rho_{\text{tot}}(\vec{r'}, t') \left( \frac{\dot{\lambda}}{\lambda^2} \right) \right] d\tau'
\]

\text{(p. 16 at top) with } t_r = t - \frac{\lambda}{c}

Calculating \( \frac{\partial A_r(\vec{r}, t)}{\partial t} \) is easy \{assuming \( \exists \) \textit{no relative} motion of source vs. observer\):

\[
\frac{\partial A_r(\vec{r}, t)}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\mu_o}{4\pi} \int_{\mathbb{V}} \frac{\vec{J}_{\text{tot}}(\vec{r'}, t')}{|\vec{r} - \vec{r'}|} \, d\tau' \right) = \mu_o \int_{\mathbb{V}} \frac{1}{4\pi} \frac{\partial \vec{J}_{\text{tot}}(\vec{r'}, t')}{\partial t} \, d\tau'
\]

where:

\[
\vec{J}(\vec{r'}, t_r) = \frac{\partial}{\partial t} \left( \vec{J}(\vec{r'}, t_r) \right) \text{ with } t_r = t - \frac{\lambda}{c}
\]

Thus, the Time-Dependent Generalization of Coulomb’s Law is:

\[
\vec{E}_r(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int_{\mathbb{V}} \left[ \frac{\rho_{\text{tot}}(\vec{r'}, t')}{\lambda^2} \hat{\rho}_{\text{tot}}(\vec{r'}, t') \frac{\dot{\lambda}}{c} + \frac{\hat{\rho}_{\text{tot}}(\vec{r'}, t')}{\lambda} - \frac{\dot{\vec{J}}_{\text{tot}}(\vec{r'}, t')}{\lambda^2} \right] d\tau'
\]

\text{with: } c^2 = \frac{1}{\varepsilon_o\mu_o} \text{ in free space}

Note that in the \textbf{static limit} \( \{ \hat{\rho} = \dot{\vec{J}} = 0 \} \), this expression reduces to the familiar form:

\[
\vec{E}_r = \frac{1}{4\pi\varepsilon_o} \int_{\mathbb{V}} \frac{\rho_{\text{tot}}(\vec{r'})}{\lambda^2} \, d\tau' \text{ with } \lambda = |\vec{r} - \vec{r'}|.
\]
Let’s work on obtaining $\vec{B}_t(\vec{r}, t)$:

$$\vec{B}_t(\vec{r}, t) = \nabla \times \vec{A}_t(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \hat{n} \times \left( \frac{\vec{J}_{tot}(\vec{r}', t_t)}{\lambda} \right) \, d\tau'$$

From Griffiths Product Rule # 7: $\nabla \times \vec{A} = f \left( \nabla \times \vec{A} \right) - \vec{A}(\nabla f)$

Then:

$$\vec{B}_t(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \left[ \frac{1}{\lambda} \left( \nabla \times \vec{J}_{tot}(\vec{r}', t_t) \right) - \vec{J}_{tot}(\vec{r}', t_t) \times \nabla \left( \frac{1}{\lambda} \right) \right] \, d\tau'$$

Let’s look at just a single component of the curl of $\vec{J}_{tot}(\vec{r}', t_t)$:

$$\left( \nabla \times \vec{J}_{tot}(\vec{r}', t_t) \right)_s = \frac{\partial J_{tot}(\vec{r}', t_t)}{\partial y} - \frac{\partial J_{tot}(\vec{r}', t_t)}{\partial z}$$

And:

$$\frac{\partial J_{tot}(\vec{r}', t_t)}{\partial y} = J_{tot}(\vec{r}', t_t) \frac{\partial t_t}{\partial y} = -\frac{1}{c} \vec{J}_{tot}(\vec{r}', t_t) \frac{\partial t}{\partial y}$$

And:

$$\frac{\partial J_{tot}(\vec{r}', t_t)}{\partial z} = J_{tot}(\vec{r}', t_t) \frac{\partial t_t}{\partial z} = -\frac{1}{c} \vec{J}_{tot}(\vec{r}', t_t) \frac{\partial t}{\partial z}$$

Therefore:

$$\left( \nabla \times \vec{J}_{tot}(\vec{r}', t_t) \right)_s = \frac{1}{c} \left( \vec{J}_{tot}(\vec{r}', t_t) \frac{\partial \vec{h}}{\partial y} - \vec{J}_{tot}(\vec{r}', t_t) \frac{\partial \vec{h}}{\partial z} \right) = \frac{1}{c} \left[ \frac{\dot{\vec{J}}(\vec{r}', t_t) \times (\nabla \vec{h})}{\lambda} \right]$$

But: $\nabla \vec{h} = \vec{\lambda}$

Thus:

$$\nabla \left( \frac{1}{\lambda} \right) = \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|^2} \right) = -\frac{\hat{r}}{|\vec{r} - \vec{r}'|^2} = -\frac{\vec{\lambda}}{\lambda^2}$$

Now:

$$\vec{\lambda} \frac{1}{\lambda} = -\frac{\hat{r}}{|\vec{r} - \vec{r}'|^2} = -\frac{\hat{r}}{\lambda^2}$$

Thus, the Time-Dependent Generalization of the Biot-Savart Law is:

$$\vec{B}_t(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \left( \frac{\vec{J}_{tot}(\vec{r}', t_t)}{\lambda^2} + \frac{\dot{\vec{J}}_{tot}(\vec{r}', t_t)}{\lambda C} \times \hat{\lambda} \right) \, d\tau'$$

Note again that in the static limit $\{ \dot{\vec{J}} = 0 \}$, this expression also reduces to the familiar form:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\vec{J}_{tot}(\vec{r}') \times \hat{\lambda}}{\lambda^2} \, d\tau'$$

with $\hat{\lambda} = |\vec{r} - \vec{r}'|$ and $t_t = t - \frac{\lambda}{C}$. Only if there is no relative motion between source & observer!
The **retarded** electric and magnetic field relations:

\[
\tilde{E}_r(\vec{r}, t) = \frac{1}{4\pi \varepsilon_0} \int \left( \frac{\rho_{\text{tot}}(\vec{r}', t_t)}{\lambda^2} \hat{\lambda} + \frac{\rho_{\text{tot}}(\vec{r}', t_t)}{\lambda c} \hat{\lambda} - \frac{\tilde{J}_{\text{tot}}(\vec{r}', t_t)}{\lambda^2} \right) d\tau'
\]

\[
\tilde{B}_r(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \left( \frac{\tilde{J}_{\text{tot}}(\vec{r}', t_t)}{\lambda^2} + \frac{\tilde{J}_{\text{tot}}(\vec{r}', t_t)}{\lambda c} \right) \times \hat{\lambda} d\tau'
\]

with: \( \lambda = |\vec{v}(t) - \vec{v}'(t_t)| \)

and:

\[ t_t = t - \frac{\lambda}{c} \]

\[ \tilde{\kappa} = \frac{\tilde{\lambda}}{\lambda} = \frac{\tilde{\lambda}}{\lambda} \]

are known as **Jefimenko’s equations** (in honor of Oleg Jefimenko, who first worked these out in 1966 – n.b. he also has recently written some new E&M books – Google these, if interested!)

We can use Jefimenko’s equations for **retarded** \( \tilde{E}_r(\vec{r}, t) \) and \( \tilde{B}_r(\vec{r}, t) \) to obtain specializations of these formulae for a **point electric charge** \( q \) moving with **retarded** velocity \( \vec{v}(\vec{r}'(t_t)) \).

Let:

\[ \rho(\vec{r}'(t_t)) = q \delta^3(\vec{r}'(t_t)) \]

where \( \vec{r}'(t_t) \) = instantaneous position of the electric charge \( q \)

\[ \tilde{J}(\vec{r}'(t_t)) = \rho(\vec{r}'(t_t)) \vec{v}(\vec{r}'(t_t)) \]

at the **source** point \( \vec{r}'(t_t) \) at the **retarded** time \( t_t \).

It can be shown {n.b. after much work!} for a **moving** point electric charge \( q \):

\[
\tilde{E}_r(\vec{r}, t) = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{\kappa}{\lambda^2} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\kappa}{\lambda^2} \right] - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[ \frac{\kappa}{\lambda^2} \right] \right\}
\]

\[
\tilde{B}_r(\vec{r}, t) = \frac{\mu_0 q}{4\pi} \left\{ \frac{\kappa}{\lambda^2} \frac{\vec{v}(\vec{r}, t) \times \hat{\lambda}}{\kappa} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\kappa}{\lambda^2} \right] \frac{\vec{v}(\vec{r}, t) \times \hat{\lambda}}{\kappa} \right\}
\]

with:

\[ t_t = t - \frac{\lambda}{c} = t - \frac{\vec{v}(t) - \vec{v}'(t_t)}{c} \]

where: \( \kappa \equiv 1 - (\vec{v}(t_t) \cdot \hat{\lambda})/c = \text{retardation factor} \)

Due to the explicit \( \vec{r}'(t_t) \) time-dependence associated with the **moving** charge \( q \), e.g. \( \vec{r}'(t_t) = \vec{v}(t_t) t_t \) we **must** be very careful in evaluating the **time derivatives**! The results (after much additional careful work) are Richard Feynman’s expression for the retarded electric field \( \tilde{E}_r(\vec{r}, t) \) and Oliver Heaviside’s expression for the retarded magnetic field \( \tilde{B}_r(\vec{r}, t) \) associated with a **moving** point charge \( q \):

\[
\tilde{E}_r(\vec{r}, t) = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{\kappa}{\lambda^2} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\kappa}{\lambda^2} \right] - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[ \frac{\kappa}{\lambda^2} \right] \right\}
\]

\[
\tilde{B}_r(\vec{r}, t) = \frac{\mu_0 q}{4\pi} \left\{ \frac{\kappa}{\lambda^2} \frac{\vec{v}(\vec{r}, t) \times \hat{\lambda}}{\kappa} + \frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\kappa}{\lambda^2} \right] \frac{\vec{v}(\vec{r}, t) \times \hat{\lambda}}{\kappa} \right\}
\]

with:

\[ t_t = t - \frac{\lambda}{c} = t - \frac{\vec{v}(t) - \vec{v}'(t_t)}{c} \]

where: \( \kappa \equiv 1 - (\vec{v}(t_t) \cdot \hat{\lambda})/c = \text{retardation factor} \).
In the static limit, we (again) see that Feynman’s expression for the retarded electric field associated with a moving point charge \( q \):

\[
\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{\hat{\mathbf{k}}}{\lambda^2} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{\hat{\mathbf{k}}}{\lambda^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{k}}{\partial t^2} \right) \right\}
\]

with \( \lambda = |\vec{r} - \vec{r}'(t)| \) and \( \hat{\mathbf{k}} = \frac{\hat{\mathbf{k}}}{\lambda} = \frac{\hat{\mathbf{k}}}{|\hat{\mathbf{k}}|} \)

reduces to the familiar form of Coulomb’s Law:

\[
\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 \lambda^2}
\]

In the quasi-static/non-relativistic limit (i.e. \( v \ll c \)), we also see that Heaviside’s expression for the retarded magnetic field associated with a moving point charge \( q \):

\[
\vec{B}_r(\vec{r},t) = \frac{\mu_0 q}{4\pi} \left\{ \frac{\vec{v}(\vec{r},t) \times \hat{\mathbf{k}}}{\kappa^2 \lambda^2} + \frac{1}{\lambda c} \frac{\partial}{\partial t} \left( \frac{\vec{v}(\vec{r},t) \times \hat{\mathbf{k}}}{\kappa} \right) \right\}
\]

with \( \lambda = |\vec{r} - \vec{r}'(t)| \)

and \( \hat{\mathbf{k}} = \frac{\hat{\mathbf{k}}}{\lambda} = \frac{\hat{\mathbf{k}}}{|\hat{\mathbf{k}}|} \) and retardation factor \( \kappa \equiv 1 - \left( \vec{v}(\vec{r},t) \cdot \hat{\mathbf{k}} \right) / c \)

also reduces to the familiar Lorentz formula:

\[
\vec{B}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \frac{q\vec{v}(\vec{r}) \times \hat{\mathbf{k}}}{\lambda^2} \quad \text{n.b. for } v \ll c \text{, the retardation factor } \kappa = 1
Appendices:

Appendix A:

Show: \[ \mathbf{\nabla} \mathbf{h} = \hat{h} \] where: \[ \mathbf{h} = |\hat{r}(t) - \hat{r}'(t)|, \quad \hat{h} = \hat{r}(t) - \hat{r}'(t) \] and: \[ \hat{h} = \frac{\mathbf{h}}{|\mathbf{h}|} \]

In Cartesian coordinates: \[ \mathbf{h} = |\hat{r}(t) - \hat{r}'(t)| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \]

Thus:
\[
\mathbf{\nabla} \mathbf{h} = \left( \frac{\partial}{\partial x} \mathbf{\hat{x}} + \frac{\partial}{\partial y} \mathbf{\hat{y}} + \frac{\partial}{\partial z} \mathbf{\hat{z}} \right) \mathbf{h} = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \frac{\Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}}{\sqrt{x-x'}^2 + (y-y')^2 + (z-z')^2}
\]

But: \[ \hat{h} = \hat{r}(t) - \hat{r}'(t) = (x-x') \hat{x} + (y-y') \hat{y} + (z-z') \hat{z} = \Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z} \]

And:
\[ \mathbf{h} = |\hat{r}(t) - \hat{r}'(t)| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} = \sqrt{x-x'}^2 + \Delta y^2 + \Delta z^2 \]

Thus:
\[ \mathbf{\nabla} \mathbf{h} = \frac{\mathbf{h}}{|\mathbf{h}|} = \hat{h} / |\hat{h}| = \hat{h} \]

Appendix B:

Show: \[ \mathbf{\nabla} \left( \frac{1}{\mathbf{h}} \right) = -\hat{h} \frac{1}{h^2} \]

In Cartesian coordinates:
\[
\mathbf{\nabla} \left( \frac{1}{h} \right) = \left( \frac{\partial}{\partial x} \mathbf{\hat{x}} + \frac{\partial}{\partial y} \mathbf{\hat{y}} + \frac{\partial}{\partial z} \mathbf{\hat{z}} \right) \frac{1}{h} = -\frac{1}{h^2} \cdot \frac{\Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \frac{\Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}
\]

\[ = -\frac{\hat{h}}{h^2} \]

Thus:
\[ \mathbf{\nabla} \left( \frac{1}{h} \right) = -\hat{h} \frac{1}{h^2} \]
Appendix C:

Following on from the result obtained in Appendix B above, note that in fact:

\[ \nabla^2 \left( \frac{1}{r} \right) = \nabla \cdot \nabla \left( \frac{1}{r} \right) = \nabla \cdot \left( -\frac{\mathbf{r}}{r^2} \right) = -\nabla \cdot \left( \frac{\mathbf{r}}{r^2} \right) = -4\pi \delta^3 \left( \mathbf{r} \right) \]

From the **divergence theorem**, we know that:

\[ \int_{\mathcal{V}} \nabla^2 \left( \frac{1}{r} \right) d\tau = \int_{\mathcal{S}} \nabla \cdot \nabla \left( \frac{1}{r} \right) \left( \mathbf{r} \right) d\mathbf{a} = \oint_{\partial \mathcal{V}} \left( -\frac{\mathbf{r}}{r^2} \right) \cdot d\mathbf{a} = \frac{4\pi}{r^2} \cdot r^2 d\Omega = -4\pi \]

Which implies that:

\[ \int_{\mathcal{V}} \nabla^2 \left( \frac{1}{r} \right) d\tau = \int_{\mathcal{V}} \left(-4\pi \delta^3 \left( \mathbf{r} \right) \right) d\tau = -4\pi \]

Thus, we also have:

\[ \int_{\mathcal{V}} \nabla^2 \left( \frac{1}{r} \right) d\tau = \int_{\mathcal{V}} \left(-4\pi \delta^3 \left( \mathbf{r} \right) \right) d\tau = -4\pi \]

Indeed, if \( \mathbf{r} \neq \mathbf{r}' \), then:

\[ \nabla^2 \left( \frac{1}{r} \right) = \nabla \cdot \nabla \left( \frac{1}{r} \right) = \nabla \cdot \left( -\frac{\mathbf{r}}{r^2} \right) = \nabla \cdot \left( \frac{\mathbf{r}}{r^2} \right) = -\nabla \cdot \left( \frac{\mathbf{r}}{r^2} \right) = -4\pi \delta^3 \left( \mathbf{r} \right) \]

Work on just the x-component:

\[ -\mathbf{\nabla} \cdot \left( \frac{\mathbf{r}}{r^3} \right) = -\frac{\partial}{\partial x} \left( \frac{(x-x')}{\left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}} \right) = -\frac{1}{\left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}} \cdot \frac{\partial}{\partial x} \left( x-x' \right) \]

\[ = -\frac{\partial}{\partial x} \left[ \left( x-x' \right)^2 + (y-y')^2 + (z-z')^2 \right]^{3/2} \cdot \left( x-x' \right) \]

Thus, we see that:

\[ -\mathbf{\nabla} \cdot \left( \frac{\mathbf{r}}{r^3} \right) = -\frac{\partial}{\partial x} \left( \left( x-x' \right)^2 + \left( y-y' \right)^2 + \left( z-z' \right)^2 \right)^{3/2} \cdot \left( x-x' \right) + \frac{\partial}{\partial y} \left( \left( x-x' \right)^2 + \left( y-y' \right)^2 + \left( z-z' \right)^2 \right)^{3/2} \cdot \left( y-y' \right) + \frac{\partial}{\partial z} \left( \left( x-x' \right)^2 + \left( y-y' \right)^2 + \left( z-z' \right)^2 \right)^{3/2} \cdot \left( z-z' \right) \]

\[ = 0 !!! \]
However, if \( \vec{r} = \vec{r}' \) then we see that the denominator in the above expression is also simultaneously equal to zero: 
\[
\kappa^{5/2} = \left[ \Delta x^2 + \Delta y^2 + \Delta z^2 \right]^{5/2} = 0
\]
and thus when \( \vec{r} = \vec{r}' \) we actually have:
\[
\nabla^2 \left( \frac{1}{\kappa} \right) = \nabla \cdot \nabla \left( \frac{1}{\kappa} \right) = \frac{\vec{\nabla} \cdot \left( \frac{-\hat{\kappa}}{\kappa^2} \right)}{0} = \frac{0}{0} = -4\pi \delta^3(\vec{\kappa})
\]

**Appendix D:**

Show that: 
\[
\nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right) = \frac{1}{\kappa^2}
\]
In Cartesian coordinates:
\[
\nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right) = \nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right) = \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \left[ \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{(x-x')^2 + (y-y')^2 + (z-z')^2} \right] = \frac{1}{(x-x')^2 + (y-y')^2 + (z-z')^2}
\]

Work on just the \( x \)-component:
\[
\nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right)_x = \frac{\partial}{\partial x} \left[ \frac{(x-x')^2}{(x-x')^2 + (y-y')^2 + (z-z')^2} \right] = \frac{1}{(x-x')^2 + (y-y')^2 + (z-z')^2} \cdot \frac{2(x-x')^2}{(x-x')^2 + (y-y')^2 + (z-z')^2} = \frac{2\Delta x^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} - \frac{\Delta x^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} = \frac{\Delta y^2 + \Delta z^2 - \Delta x^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2}
\]

Thus, we see that:
\[
\nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right) = \frac{\Delta y^2 + \Delta z^2 - \Delta x^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} + \frac{\Delta x^2 + \Delta y^2 - \Delta z^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} + \frac{\Delta x^2 + \Delta z^2 - \Delta y^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} = \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^2} = \frac{1}{\kappa^2} = \frac{1}{h^2}
\]

Thus:
\[
\nabla \cdot \left( \frac{\vec{\kappa}}{\kappa^2} \right) = \frac{1}{\kappa^2}
\]