LECTURE NOTES 7.5

Dispersion: The Frequency-Dependence of the Electric Permittivity \( \varepsilon = \varepsilon(\omega) \)

and the Electric Susceptibility \( \chi_\varepsilon(\omega) = \varepsilon(\omega)/\varepsilon_0 - 1 \)

Over the entire EM frequency interval \( \{0 \leq f \leq \infty \text{ Hz}\} \), the speed of propagation \( v_{\text{prop}} \) of monochromatic (i.e. single-frequency) EM waves in matter is often not constant, not independent of frequency: \( v_{\text{prop}} \neq \text{constant} \); \( = fcn(f) = v_{\text{prop}}(f) \), because matter - at the microscopic scale - is composite - comprised of atoms/molecules – which have resonances in energy/energy levels – which are governed by the laws of quantum mechanics...

The frequency-dependence of the wavelength \( \lambda = \lambda(\omega) \), or wavenumber \( k(\omega) = 2\pi/\lambda(\omega) \), and linear momentum \( p(\omega) \) associated with macroscopic EM waves propagating in a dispersive medium arises from the frequency-dependence of the macroscopic electric permittivity \( \varepsilon(\omega) \) (or equivalently the electric susceptibility \( \chi_\varepsilon(\omega) \)) since:

\[
\varepsilon(\omega) = \varepsilon_0(1 + \chi_\varepsilon(\omega)).
\]

The frequency-dependence of the macroscopic electric permittivity \( \varepsilon(\omega) \) is known as dispersion; a medium that has \( \varepsilon = fcn(\omega) \) is known as a dispersive medium.

For non-magnetic/non-conducting linear/homogeneous/isotropic media, the index of refraction \( n = \sqrt{\varepsilon/\varepsilon_0} \). Thus, if: \( \varepsilon = \varepsilon(\omega) = \varepsilon_0(1 + \chi_\varepsilon(\omega)) \) then: \( n(\omega) = \sqrt{\varepsilon(\omega)/\varepsilon_0} \).

For a wave packet (= a group \{= superposition/linear combination\} of waves of many frequencies – as explained by Mssr. Fourier), the envelope of the wave packet travels with (in general, frequency-dependent) group speed = speed at which energy in the wave flows:

\[
v_g(\omega) \equiv \frac{d\omega}{dk(\omega)} = \frac{1}{dk(\omega)/d\omega} = \left[ \frac{dk(\omega)}{d\omega} \right]^{-1}
\]

A propagating wave packet: \( \hat{f}(kz - \omega t) \)

\[
v_p(\omega) \equiv \frac{\omega}{k(\omega)}
\]

\[
v_g(\omega) \equiv \left[ \frac{dk(\omega)}{d\omega} \right]^{-1}
\]

A space-point \( z(t) \) on the waveform moves - with constant phase \( \Phi \equiv (kz - \omega t) \) with (in general, frequency-dependent) phase speed \( v_\phi(\omega) \equiv \omega/k(\omega) \).

Hence: \( z(t, \omega) = \Phi/k(\omega) + v_\phi(\omega)t \). Note that \{in general\}: \( v_g(\omega) \neq v_\phi(\omega) \).
If \( v_\phi (\omega) = \omega/k(\omega) \) is frequency-dependent, and \( v_g (\omega) = \left[ \frac{dk(\omega)}{d\omega} \right]^{-1} \neq v_\phi (\omega) \) \{e.g. as in the case for surface waves on water, where \( v_\phi = 2v_g \} \) the relationship between \( v_\phi (\omega) \) and \( v_g (\omega) \) depends on the detailed physics of the medium (as we shall soon see . . . ). Note that in certain circumstances, \( v_\phi \) can exceed \( c \) \{= speed of light in the vacuum\} but in these situations, no energy (and/or information) is transmitted at super-luminal speeds – energy/information is transmitted at \( v_g < c \) always, by causality . . .

**A physical/mechanical example:** calculate the phase speed of the intersection point of the two halves of a scissors as the blades of the scissors are closed. \{Answer: \( v_s^{scissors} \rightarrow \infty !! \} \}

**Dispersion Phenomena in Linear Dielectrics**

In a non-conducting, linear, homogeneous, isotropic medium there are no free electrons \( (i.e. \rho_{free}(\vec{r}) = 0) \). Atomic electrons are permanently bound to nuclei of atoms comprising the medium. \( \Rightarrow \exists \) no preferential direction / no preferential directions in such an \{isotropic\} medium.

Suppose each atomic electron (charge \(-e\)) in a dielectric is displaced by a small distance \( \vec{r} \) from its equilibrium position, e.g. by application of a static electric field \( \vec{E}(\vec{r}) \parallel \vec{r} \) direction.

The resulting macroscopic electric polarization (aka electric dipole moment per unit volume) is:

\[
\vec{P}(\vec{r}) = n_b^e \vec{p}(\vec{r})
\]

where: \( n_e^b = \text{bound atomic electron number density} \\left( \# e/m^3 \right) \)

and the \{induced\} atomic/molecular electric dipole moment is: \( \vec{p}(\vec{r}) = -e\vec{r} \) \{here\}, where \( \vec{r} \) is the \{vector\} displacement of the atomic electron from its equilibrium \{\( \vec{r} = 0 \)\} position.

Thus: \( \vec{P}(\vec{r}) = n_b^e \vec{p}(\vec{r}) = -n_e^b e\vec{r} \)

The atomic electrons are each elastically bound to their equilibrium positions with a force constant \( k_e (N/m) \). The force equation for each atomic electron is thus: \( \vec{F}_e(\vec{r}) = -e\vec{E}(\vec{r}) = k_e\vec{r} \).

Hence: \( \vec{r} = -e\vec{E}(\vec{r})/k_e \).

The static polarization is therefore given by:

\[
\vec{P}(\vec{r}) = n_b^e \vec{p}(\vec{r}) = -n_e^b e\vec{r} = -n_b^e e \frac{-e\vec{E}(\vec{r})}{k_e} = +n_b^e e^2 \vec{E}(\vec{r})
\]

However, if the \( \vec{E} \)-field e.g. varies harmonically with time, \{i.e. \( \vec{E} = \vec{E}(\vec{r},t;\omega) = \vec{E}_o e^{(kz - \omega t)} \) \}
due to a monochromatic EM plane wave incident on an atom, the above relation is incorrect!

A more correct \{but “semi-classical”\} approach to treat this situation is to consider the bound atomic electrons as classical, damped, forced harmonic oscillators (driven by the incident electric field), as mathematically described by the following differential equation:

\[
\boxed{m_\gamma \ddot{\vec{r}}(t;\omega) + m_\gamma \gamma \dot{\vec{r}}(t;\omega) + k_\gamma \vec{r}(t;\omega) = -e\vec{E}(\vec{r},t;\omega)} \leftarrow \text{inhomogeneous 2\textsuperscript{nd}-order differential eqn.}
\]

The damping constant \( \gamma (\text{radians/sec}) \) represents the effect of EM re-radiation by the atom \{here\}.  

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\[
m_e \frac{\partial^2 \tilde{r}(t, \omega)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t, \omega)}{\partial t} + k_e \tilde{r}(t, \omega) = -e \tilde{E}(\tilde{r}, t, \omega)
\]

\[
\text{Velocity-dependent damping term} \quad \gamma \equiv \text{damping constant}
\]

\[
\text{Potential Force} \quad \text{(binding of atomic electrons to atom)}
\]

\[
\text{Driving Force} \quad m_e = \text{electron mass} = 9.1 \times 10^{-31} \text{kg}
\]

Suppose the driving/forcing term varies sinusoidally/is harmonic/periodic with angular frequency \(\omega\), i.e. \(\tilde{F}_e(\tilde{r}, t; \omega) = -e \tilde{E}(\tilde{r}, t; \omega) = -e \tilde{E}_o e^{-i \omega t} \tilde{r}\) because \(\tilde{E}(\tilde{r}, t; \omega) = \tilde{E}_o e^{-i \omega t}\)

\(n.b.\) The electric field \(\tilde{E}\) is complex \(\tilde{E}\) and plane-polarized in the \(\hat{r}\)-direction.

The inhomogeneous force equation becomes: \(m_e \frac{\partial^2 \tilde{r}(t, \omega)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t, \omega)}{\partial t} + k_e \tilde{r}(t, \omega) = -e \tilde{E}_o e^{-i \omega t} \tilde{r}\)

with complex time-domain vector displacement amplitude: \(\tilde{r}(t; \omega) = \tilde{r}(t; \omega) \hat{r}\). In the steady state, we have:

\[
m_e \frac{\partial^2 \tilde{r}(t; \omega)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t; \omega)}{\partial t} + k_e \tilde{r}(t; \omega) = -e \tilde{E}_o e^{-i \omega t} \tilde{r}
\]

Since \(\tilde{r}(t; \omega)\) physically represents the complex vector spatial displacement of an atomic electron from its equilibrium \(\{ \tilde{r} = 0 \}\) position, then:

\[
\tilde{r}(t; \omega) = \tilde{r}(t; \omega) \hat{r} = \tilde{r}_o(\omega) e^{-i \omega t} \hat{r}
\]

Thus:

\[
m_e \frac{\partial^2 \tilde{r}(t; \omega)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t; \omega)}{\partial t} + k_e \tilde{r}(t; \omega) = -e \tilde{E}_o e^{-i \omega t} \tilde{r}
\]

\[
\left( m_e \omega^2 - k_e + i m_e \gamma \omega \right) \tilde{r}_o(\omega) = -e \tilde{E}_o e^{-i \omega t} \hat{r}
\]

\(\omega^2 \equiv \frac{k_e}{m_e} \) or \(\omega_0 = \sqrt{\frac{k_e}{m_e}}\) = characteristic/natural resonance \{angular\} frequency. Then:

\[
\tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) \tilde{E}_o \left[ \frac{1}{(\omega^2 - \omega_0^2) + i \gamma \omega} \right]
\]

Note that the complexity of \(\tilde{r}_o(\omega)\) in the denominator.
We can move it to the numerator using the following standard “trick”/procedure:

If: \(\tilde{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}\)

where: \(\text{Re}\{\tilde{z}\} = \frac{x}{x^2 + y^2}\) and: \(\text{Im}\{\tilde{z}\} = \frac{y}{x^2 + y^2}\)
Thus:

$$\tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) \tilde{E}_o \frac{(\omega^2 - \omega_0^2) - i\gamma \omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \tilde{r}_o^{\text{real}}(\omega) + i\tilde{r}_o^{\text{imag}}(\omega) = \tilde{r}_o(\omega)e^{i\phi(\omega)}$$

Note that: $\tilde{E}_o = \tilde{E}_o e^{i\delta} \equiv E_o e^{i\delta}$. However, we can always “rotate away”/absorb the phase $\delta \to \delta = 0 \text{ e.g. by a } \text{global} \text{ redefinition of the zero of time, or a } \text{global} \text{ redefinition/translation of our coordinate system. Hence, we can equivalently write, without any loss of generality:}$

$$\tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) E_o \frac{(\omega^2 - \omega_0^2) - i\gamma \omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \tilde{r}_o^{\text{real}}(\omega) + i\tilde{r}_o^{\text{imag}}(\omega) = \tilde{r}_o(\omega)e^{i\phi(\omega)}$$

The {frequency-dependent} real part of the frequency-domain displacement amplitude $\tilde{r}_o(\omega)$ – i.e. the component of $\tilde{r}_o(\omega)$ that is in-phase with the driving force $-e\tilde{E}(\tilde{r},t;\omega) = -e\tilde{E}_0 e^{-i\omega \tilde{r}}$ is:

$$r_o^{\text{real}}(\omega) = \text{Re} \{\tilde{r}_o(\omega)\} = \left( \frac{e}{m_e} \right) \tilde{E}_o \frac{(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}$$

The {frequency-dependent} imaginary part of the frequency-domain displacement amplitude $\tilde{r}_o(\omega)$ – i.e. the component of $\tilde{r}_o(\omega)$ that is 90°-out-of-phase with the driving force $-e\tilde{E}(\tilde{r},t;\omega) = -e\tilde{E}_0 e^{-i\omega \tilde{r}}$ is:

$$r_o^{\text{imag}}(\omega) = \text{Im} \{\tilde{r}_o(\omega)\} = -\left( \frac{e}{m_e} \right) \tilde{E}_o \frac{\gamma \omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}$$

The {frequency-dependent} phase of the frequency-domain displacement amplitude $\tilde{r}_o(\omega)$ is:

$$\varphi_r(\omega) = \tan^{-1} \left( \frac{\text{Im} \{\tilde{r}_o(\omega)\}}{\text{Re} \{\tilde{r}_o(\omega)\}} \right) = \tan^{-1} \left( \frac{r_o^{\text{imag}}(\omega)}{r_o^{\text{real}}(\omega)} \right) = \tan^{-1} \left( \frac{-\gamma \omega}{(\omega^2 - \omega_0^2)} \right)$$

When: $\omega < \omega_0 = \sqrt{\frac{k}{m_e}}, \varphi_r > 0 \Rightarrow \tilde{r} \text{ lags } \tilde{E}$. When: $\omega > \omega_0 = \sqrt{\frac{k}{m_e}}, \varphi_r < 0 \Rightarrow \tilde{r} \text{ leads } \tilde{E}$.
The {frequency-dependent} magnitude of the frequency-domain displacement amplitude $\tilde{r}_o(\omega)$ is:

$$
|\tilde{r}_o(\omega)| = \sqrt{\tilde{r}_o(\omega) \cdot \tilde{r}_o^*(\omega)} = \sqrt{(r_o^{real}(\omega))^2 + (r_o^{imag}(\omega))^2} = \left(\frac{e}{m_c}\right)E_o \sqrt{\frac{\left[(\omega^2 - \omega_o^2) - i\gamma\omega\right]\left[(\omega^2 - \omega_o^2) + i\gamma\omega\right]}{\left[(\omega^2 - \omega_o^2)^2 + \gamma^2\omega^2\right]}^2}
$$

The 8-figure plot below shows the behavior of a mechanical resonance at $f_o = 30 \, \text{KHz}$, and FWHM $\Gamma \equiv \gamma/2\pi = 30 \, \text{Hz}$ in the vicinity of the resonance for unit amplitude $\left(\frac{e}{m_c}\right)E_o = 1$. The resonance frequency $f_o$ and lo/hi frequencies $f_1 < f_o < f_2$ that define the FWHM, $\Gamma \equiv f_2 - f_1$ \{i.e. $\gamma = 2\pi \Gamma \equiv 2\pi(f_2 - f_1) = (\omega_2 - \omega_1)$\} are also indicated in each plot:

Note that the so-called $Q$ of {this} resonance {here} is: $Q \equiv \omega_o/\gamma = f_o/\Gamma = 1000$. For small damping $\gamma \ll \omega_o \,(Q \gg 1)$, it can be shown that $f_1 = f_o - \frac{1}{2}(\gamma/2\pi)$, $f_2 = f_o + \frac{1}{2}(\gamma/2\pi)$ and that $\phi_r(f_o) = \tan^{-1}(\infty) = 90^\circ$, $\phi_r(f_1) = \tan^{-1}(-1) = 135^\circ$ and $\phi_r(f_2) = \tan^{-1}(+1) = 45^\circ$. 

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The 3-D plot below shows the CW path taken in the complex plane in passing through the resonance:

Now:  \( \tilde{P}(\vec{r},t;\omega) = -n_e^b e \tilde{r}(t;\omega) = -n_e^b e \tilde{r}_o(\omega)e^{-i\omega t} \hat{r} \)

where:

\[
\tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) E_o \frac{\left( \omega^2 - \omega_0^2 \right) - i\gamma \omega}{\left( \omega^2 - \omega_0^2 \right)^2 + \gamma^2 \omega^2} \]

Thus:

\[
\tilde{P}(\vec{r},t;\omega) = -n_e^b e \left( \frac{e}{m_e} \right) E_o \frac{\left( \omega^2 - \omega_0^2 \right) - i\gamma \omega}{\left( \omega^2 - \omega_0^2 \right)^2 + \gamma^2 \omega^2} e^{-i\omega t} \hat{r} = n_e^b \left( \frac{e^2}{m_e} \right) E_o \frac{\left( \omega_0^2 - \omega^2 \right) + i\gamma \omega}{\left( \omega_0^2 - \omega^2 \right)^2 + \gamma^2 \omega^2} e^{-i\omega t} \hat{r}
\]

Since the complex time-domain \( \tilde{P}(\vec{r},t;\omega) = \tilde{P}(\vec{r};\omega) \cdot e^{-i\omega t} \hat{r} \), we see that the complex frequency-domain vector amplitude is:

\[
\tilde{P}(\vec{r};\omega) = n_e^b \left( \frac{e^2}{m_e} \right) E_o \frac{\left( \omega_0^2 - \omega^2 \right) + i\gamma \omega}{\left( \omega_0^2 - \omega^2 \right)^2 + \gamma^2 \omega^2} \hat{r}.
\]

In the static limit (i.e. \( \omega = 0 \)):  

\[
\tilde{P}(\vec{r};\omega = 0) = \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{\omega_0^2} E_o \hat{r} = \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \frac{E_o}{m_e} \hat{r} \Rightarrow \text{Static polarization } \tilde{P}(\omega = 0) \text{ is in-phase with } \hat{E}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]

\[
\text{Still valid if static limit (i.e. } \omega = 0) \quad \Rightarrow \left( \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{k_e} \right) \left( \frac{E_o}{m_e} \right) \hat{r}
\]
Note that the \textbf{phase} of the \textit{frequency-domain} vector amplitude \( \tilde{P}(\vec{r};\omega) \) is frequency dependent, \( \tilde{P}(\omega) \) \textit{lags} behind (or \textit{leads}) \( \tilde{E}(\omega) \) by a frequency-dependent phase angle of:

\[
\phi_p(\omega) = \tan^{-1}\left(\frac{\Im\{\tilde{P}(\vec{r},\omega)\}}{\Re\{\tilde{P}(\vec{r},\omega)\}}\right) = \tan^{-1}\left(\frac{\gamma \omega}{(\omega_0^2 - \omega^2)}\right)
\]

\textit{n.b. The damping constant} \( \gamma \) has the same units as \( \omega^2 \): \textit{radians/sec}.

When: \( \omega < \omega_0 = \sqrt{\frac{k_m}{\varepsilon_0}} \), \( \phi_p > 0 \) \( \Rightarrow \tilde{P} \text{ lags} \tilde{E} \). When: \( \omega > \omega_0 = \sqrt{\frac{k_m}{\varepsilon_0}} \), \( \phi_p < 0 \) \( \Rightarrow \tilde{P} \text{ leads} \tilde{E} \).

From the above formula, note that if the damping constant \( \gamma = 0 \), then \( \phi_p = 0 \), the polarization \( \tilde{P}(\omega) \) is \textit{always} in-phase with \( \tilde{E}(\omega) \), because if \( \gamma = 0 \), then \( \Im\{\tilde{P}(\vec{r},\omega)\} = 0 \), \textit{i.e.} the polarization \( \tilde{P}(\omega) \) is purely \textit{real}! A damping constant of \( \gamma = 0 \) also means that the \textit{width} (FWHM) \( \Gamma = \gamma / 2 \pi \text{ (Hz)} \) of the atomic/molecular \textit{resonance} is \textit{infinitely narrow}, and thus there are \textit{no dissipative} processes (\textit{i.e.} energy \textit{loss} mechanisms) present at the microscopic atomic/molecular level in this macroscopic medium! Note also that \( \gamma \) has physical/SI units of radians/second.

Note further that \( \tilde{E} \) in the above expression is actually \( \tilde{E}_{\text{int}} \) – the \textit{internal} macroscopic electric field of the dielectric: \( \tilde{E} = \tilde{E}_{\text{int}} = \tilde{E}_{\text{ext}} + \tilde{E}_p \), the \textit{sum} of the macroscopic \textit{external applied} electric field and the macroscopic electric field \textit{due to the polarization of the dielectric medium}.

The electric field due to polarization of the medium is:

\[
\tilde{E}_p = -\frac{1}{3\varepsilon_0} \tilde{P}
\]

Thus: \( \tilde{E} = \tilde{E}_{\text{int}} = \tilde{E}_{\text{ext}} - \frac{1}{3\varepsilon_0} \tilde{P} \) Therefore:

\[
\tilde{P} = n_c^b \left(\frac{e^2}{m_c}\right) \frac{(\omega_0^2 - \omega^2) + i\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \tilde{E}_{\text{ext}} - \frac{1}{3\varepsilon_0} \tilde{P}
\]

\[
\omega_0^2 = \frac{k_c}{m_c}
\]

Now solve for \( \tilde{P} \): Skipping writing out some \{tedious\} complex algebra, we obtain:

\[
\tilde{P} = n_c^b \left(\frac{e^2}{m_c}\right) \frac{(\omega_1^2 - \omega^2) + i\gamma \omega}{(\omega_1^2 - \omega^2)^2 + \gamma^2 \omega^2} \tilde{E}_{\text{ext}}
\]

where:

\[
\omega_1 = \sqrt{\omega_0^2 - \left(\frac{n_c^b e^2}{3\varepsilon_0 m_c}\right)^2} < \omega_0
\]

\textit{effective angular resonance frequency of bound atomic electrons}

Note that this formula is essentially identical \textit{e.g.} to the \{complex\} displacement amplitude formula for a driven harmonic oscillator, and/or that for the \{complex\} AC voltage amplitude in an LCR circuit, and for many other physical systems exhibiting a \{damped\} resonance-type behavior.

Now if \( \tilde{E}_{\text{ext}} = \tilde{E} \)-field associated with a monochromatic plane \textit{EM} wave propagating in a dielectric medium: \( \tilde{E}_{\text{ext}}(z,t) = \tilde{E}_e e^{i(kz - \omega t)} \), then because of the \textit{linear} relationship between the polarization \( \tilde{P} \) and \textit{e.g.} \( \tilde{E}_{\text{ext}}(z,t) = \tilde{E}_e e^{i(kz - \omega t)} \hat{x} \), Gauss’ Law becomes (since \( \rho_{\text{free}}(\vec{r}) = 0 \)):

\[
\tilde{\nabla} \cdot \tilde{E}_{\text{ext}} = -\frac{1}{\varepsilon_0} \tilde{\nabla} \cdot \tilde{P} = \tilde{\rho}_{\text{bound}} = 0
\]

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The wave equation for a dielectric medium with \( \rho_{\text{free}}(\vec{r}) = 0 \) and \( J_{\text{free}} = 0 \) becomes:

\[
\nabla^2 \tilde{E}_{\text{ext}} - \frac{1}{c^2} \frac{\partial^2 \tilde{E}_{\text{ext}}}{\partial t^2} = \frac{\mu_o}{c^2} \frac{\partial^2 \tilde{P}}{\partial t^2} = \mu_o n_e^b \left( \frac{e^2}{m_e} \right) \left[ \frac{(\omega_k^2 - \omega_e^2) + i\gamma \omega_e}{(\omega_k^2 - \omega_e^2)^2 + \gamma^2 \omega_e^2} \right] \frac{\partial^2 \tilde{E}_{\text{ext}}}{\partial t^2}
\]

with:

\[
\frac{1}{c^2} = \varepsilon_o \mu_o
\]

Or:

\[
\nabla^2 \tilde{E}_{\text{ext}} = \frac{1}{c^2} \left[ 1 + \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \left( \frac{(\omega_k^2 - \omega_e^2) + i\gamma \omega_e}{(\omega_k^2 - \omega_e^2)^2 + \gamma^2 \omega_e^2} \right) \right] \frac{\partial^2 \tilde{E}_{\text{ext}}}{\partial t^2}
\]

with:

\[
\omega_1 = \sqrt{\omega_0^2 - \left( \frac{n_b e^2}{3\varepsilon_o m_e} \right)}
\]

The general solution to this \textit{dispersive} wave equation is of the form:

\[
\tilde{E}_{\text{ext}}(\vec{r}, t; \omega) = \tilde{E}_e e^{i(k-\omega t)}
\]

with \textbf{complex} \( k = k + ik \) and:

\[
k^2 = \frac{\omega^2}{c^2} \left[ 1 + \frac{n_b e^2}{\varepsilon_o m_e} \left( \frac{(\omega_k^2 - \omega_e^2) + i\gamma \omega_e}{(\omega_k^2 - \omega_e^2)^2 + \gamma^2 \omega_e^2} \right) \right].
\]

Thus, we also see that here \{again\} the complex wavenumber \( k = k + ik \) is \textbf{explicitly} dependent on the angular frequency \( \omega \), \textit{i.e.} \( \tilde{k}(\omega) = k(\omega) + ik(\omega) \).

We further see that monochromatic plane \textit{EM} waves propagating in a \textit{dispersive} dielectric medium are \textbf{exponentially} attenuated, because:

\[
\tilde{E}_{\text{ext}}(\vec{r}, t; \omega) = \tilde{E}_e e^{i(k-\omega t)} = \tilde{E}_e e^{-\kappa t} e^{i(k-\omega t)},
\]

\textit{i.e.} the \( \kappa(\omega) = \Im \tilde{k}(\omega) \) term corresponds to \textbf{absorption/dissipation} in the macroscopic dielectric, and is physically related to/is proportional to the \textbf{damping constant} \( \gamma \).

Note that we also have:

\[
\tilde{P}(\vec{r}, t; \omega) = \varepsilon_o \tilde{\chi}_e(\omega) \tilde{E}_{\text{ext}}(\vec{r}, t; \omega),
\]

thus the susceptibility \( \tilde{\chi}_e(\omega) \) \{here\} is also complex, and frequency-dependent:

\[
\tilde{\chi}_e(\omega) = \chi_e(\omega) + \frac{\kappa_e(\omega)}{i \omega}
\]

The \( \chi_e(\omega) = \Im \tilde{\chi}_e(\omega) \) term corresponds to \textbf{absorption/dissipation} in the dielectric, and is physically related to/is proportional to the \textbf{damping constant} \( \gamma \). The corresponding dissipative energy \textbf{losses} at the microscopic, atomic/molecular level in the dielectric ultimately wind up as \textbf{heat}!

Since:

\[
\tilde{P}(\vec{r}, t; \omega) = \varepsilon_o \tilde{\chi}_e(\omega) \tilde{E}_{\text{ext}}(\vec{r}, t; \omega) = n_b \left( \frac{e^2}{m_e} \right) \left[ \frac{(\omega_k^2 - \omega_e^2) + i\gamma \omega_e}{(\omega_k^2 - \omega_e^2)^2 + \gamma^2 \omega_e^2} \right] \tilde{E}_{\text{ext}}(\vec{r}, t; \omega)
\]

\[
= \varepsilon_o \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \left[ \frac{(\omega_k^2 - \omega_e^2) + i\gamma \omega_e}{(\omega_k^2 - \omega_e^2)^2 + \gamma^2 \omega_e^2} \right] \tilde{E}_{\text{ext}}(\vec{r}, t; \omega)
\]

where:

\[
\omega_1 = \sqrt{\omega_0^2 - \left( \frac{n_b e^2}{3\varepsilon_o m_e} \right)}
\]
We see that the complex susceptibility associated with a single resonance is:

\[
\tilde{\chi}_e(\omega) = \left( \frac{n_e^b e^2}{\varepsilon_0 m_e} \right) \frac{\omega^2 - \omega^2 + i\gamma\omega}{\left(\omega^2 - \omega^2\right)^2 + \gamma^2\omega^2} = \chi_e(\omega) + i\zeta_e(\omega)
\]

Hence:

\[
\chi_e(\omega) = \Re \{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\varepsilon_0 m_e} \right) \frac{\omega^2 - \omega^2}{\left(\omega^2 - \omega^2\right)^2 + \gamma^2\omega^2}
\]

And:

\[
\zeta_e(\omega) = \Im \{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\varepsilon_0 m_e} \right) \frac{\gamma\omega}{\left(\omega^2 - \omega^2\right)^2 + \gamma^2\omega^2}
\]

Now before we go much further with this, we need to discuss another aspect of our model – namely that in most linear dielectric materials, the atoms comprising the material are multi-electron atoms, and consequently there are many different binding energies – the outer shell atomic electrons are weakly bound, hence have small \(k_e\), and thus small \(\omega_0 = \sqrt{k_e/m_e}\), whereas the inner-shell electrons are much more tightly bound, hence have larger \(k_e\), larger \(\omega_0 = \sqrt{k_e/m_e}\).

Furthermore, in complex media, i.e. dielectrics with more than one kind of atom, electrons can be shared between atoms – i.e. they are bound to molecules e.g. the π-electrons in benzene ring / aromatic hydrocarbon-type compounds, which can be weakly bound in some molecules.

Thus, there can be also be {molecular} resonances e.g. in the microwave and infra-red regions of the EM spectrum – atomic resonances are typically in the optical and UV regions {for the outer-most shell electrons}, as well as in the far UV and x-ray regions {for the inner-shell electrons}!

Allowing for all such resonances, we can write the {complex} electric polarization \(\vec{P}\) as a summation over all of the resonances present in the linear dielectric as follows:

\[
\vec{P}(\vec{r}, t; \omega) = \varepsilon_0 \left( \frac{n_e^b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} f_j^{osc} \left[ \frac{\left(\omega_j^2 - \omega^2\right) + i\gamma_j \omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2} \right] \vec{E}_{ext}(\vec{r}, t; \omega)
\]

where: \(\omega_j \equiv \sqrt{\frac{\omega^2 - \frac{n_e^b e^2}{3\varepsilon_0 m_e}}{m_e}}\) and: \(\omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}\)

and where: \(f_j^{osc} \equiv \) oscillator strength of \(j\)th resonance, defined such that: \(\sum_{j=1}^{n} f_j^{osc} = 1\)

Physically: \(f_j^{osc} \equiv \) fractional strength of the \(j\)th resonance and \(\gamma_j = 2\pi \times \)width \(\Gamma_j\) of the \(j\)th resonance.
Thus, we see that the multi-resonance complex electric susceptibility \( \tilde{\chi}_e(\omega) = \chi_e(\omega) + i\zeta_e(\omega) \) is:

\[
\tilde{\chi}_e(\omega) = \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\left( \omega_{j}^2 - \omega^2 \right) + i\gamma_j \omega}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right] \equiv \chi_e(\omega) + i\zeta_e(\omega)
\]

Hence:

\[
\chi_e(\omega) = \Re \{ \tilde{\chi}_e(\omega) \} = \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\omega_{j}^2 - \omega^2}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right]
\]

And:

\[
\zeta_e(\omega) = \Im \{ \tilde{\chi}_e(\omega) \} = \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\gamma_j \omega}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right]
\]

The complex electric permittivity \( \tilde{\varepsilon}(\omega) = \varepsilon_o \left( 1 + \tilde{\chi}_e(\omega) \right) \equiv \varepsilon(\omega) + i\zeta(\omega) \) of a dispersive, linear dielectric medium is:

\[
\tilde{\varepsilon}(\omega) = \varepsilon_o \left( 1 + \tilde{\chi}_e(\omega) \right) = \varepsilon_o \left[ 1 + \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\left( \omega_{j}^2 - \omega^2 \right) + i\gamma_j \omega}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right] \right] \equiv \varepsilon(\omega) + i\zeta(\omega)
\]

with the relations:

\[
\varepsilon(\omega) = \Re \{ \tilde{\varepsilon}(\omega) \} = \Re \{ \varepsilon_o \left( 1 + \tilde{\chi}_e(\omega) \right) \} = \varepsilon_o \left( 1 + \chi_e(\omega) \right)
\]

\[
= \varepsilon_o \left[ 1 + \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\left( \omega_{j}^2 - \omega^2 \right) + i\gamma_j \omega}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right] \right]
\]

and:

\[
\zeta(\omega) = \Im \{ \tilde{\varepsilon}(\omega) \} = \Im \{ \varepsilon_o \left( 1 + \tilde{\chi}_e(\omega) \right) \} = \varepsilon_o \zeta_e(\omega)
\]

\[
= \varepsilon_o \left( \frac{n_e^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} f_{j}^{\text{osc}} \left[ \frac{\gamma_j \omega}{\left( \omega_{j}^2 - \omega^2 \right)^2 + \gamma_j^2 \omega^2} \right]
\]
Monochromatic plane EM wave solutions to the **dispersive** wave equation are of the form:

\[
\tilde{E}(\vec{r}, t, \omega) = \tilde{E}_o e^{i k (z - \omega t)}
\]

with complex wavenumber

\[
\vec{k}(\omega) = k(\omega) + i \kappa(\omega) \equiv \sqrt{\varepsilon(\omega) \mu_o \omega}.
\]

for a \(\approx\) **non-magnetic** medium, i.e. \(\mu = \mu_o\).

Thus:

\[
\tilde{E}_{ext}(\vec{r}, t, \omega) = \tilde{E}_o e^{i k (z - \omega t)} = \tilde{E}_o e^{-kz} e^{i k z}
\]

Introducing a \{frequency-dependent\} complex wavenumber \(\vec{k}(\omega) = k(\omega) + i \kappa(\omega)\) is **equivalent** to introducing a \{frequency-dependent\} complex index of refraction \(\vec{n}(\omega) = n(\omega) + i \eta(\omega)\).

For a dispersive dielectric, the complex index of refraction and complex wavenumber are \{simply\} related to each other by:

\[
\vec{k}(\omega) = \left(\frac{\omega}{c}\right) \vec{n}(\omega)
\]

\[
\therefore \quad (k(\omega) + i \kappa(\omega)) = \left(\frac{\omega}{c}\right) (n(\omega) + i \eta(\omega)) = \left(\frac{\omega}{c}\right) n(\omega) + i \left(\frac{\omega}{c}\right) \eta(\omega)
\]

\[
\Rightarrow k(\omega) = \left(\frac{\omega}{c}\right) n(\omega) \quad \text{and} \quad \kappa(\omega) = \left(\frac{\omega}{c}\right) \eta(\omega)
\]

For a \(\approx\) **non-magnetic** medium (\(\mu = \mu_o\)), the complex index of refraction \(\vec{n}(\omega)\) is related to the complex electric permittivity \(\varepsilon(\omega) = \varepsilon_o (1 + \tilde{\chi}(\omega))\) and thus the complex electric susceptibility \(\tilde{\chi}(\omega)\) via the relation:

\[
\vec{n}(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_o}} = \sqrt{1 + \tilde{\chi}(\omega)}
\]

Squaring both sides:

\[
\vec{n}^2(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_o} = 1 + \tilde{\chi}(\omega) = 1 + \left(\frac{n^2 e^2}{\varepsilon_o m_e} \sum_{j=1}^{n} f_{j} \left[ \frac{(\omega^2_j - \omega^2) + i \gamma_j \omega}{(\omega^2_j - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \right)
\]

But:

\[
\vec{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 \left[ 1 + \left(\frac{n^2 e^2}{\varepsilon_o m_e} \sum_{j=1}^{n} f_{j} \left[ \frac{(\omega^2_j - \omega^2) + i \gamma_j \omega}{(\omega^2_j - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \right) \right]
\]

\[
= (k(\omega) + i \kappa(\omega))^2 = k^2(\omega) + 2i k(\omega) \kappa(\omega) - k^2(\omega)
\]
Since: \( \hat{n}(\omega) = \left( \frac{c}{\omega} \right) \tilde{k}(\omega) \) then:

\[
\hat{n}^2(\omega) = \left( \frac{c}{\omega} \right)^2 \tilde{k}^2(\omega) = 1 + \left( \frac{n_b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \]

\[
= (n(\omega) + i \eta(\omega))^2 = n^2(\omega) + 2i n(\omega) \eta(\omega) - \eta^2(\omega)
\]

Equating the real and imaginary parts of the LHS & RHS of the above equation, we obtain:

\[
n^2(\omega) - \eta^2(\omega) = 1 + \left( \frac{n_b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \]

\[
2n(\omega) \eta(\omega) = \left( \frac{n_b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \]

First define: \( \alpha_s(\omega) \equiv \left( \frac{n_b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \)

\( \beta_s(\omega) \equiv \left( \frac{n_b e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \) \( \text{(n.b. } \beta_s(\omega) > 0, \text{ is always positive)} \)

Then: \( n^2(\omega) - \eta^2(\omega) = 1 + \alpha_s(\omega) \) and \( 2n(\omega) \eta(\omega) = \beta_s(\omega) \) \( \implies \eta(\omega) = \beta_s(\omega)/2n(\omega) \)

Thus: \( n^2(\omega) - \left( \frac{\beta_s(\omega)}{2n(\omega)} \right)^2 = (1 + \alpha_s(\omega)) \) \( \iff \) multiply equation through by \( n^2(\omega) \)

\( n^4(\omega) - \left( \frac{\beta_s(\omega)}{2} \right)^2 = (1 + \alpha_s(\omega)) n^2(\omega) \)

Or: \( n^4(\omega) - (1 + \alpha_s) n^2(\omega) - \left( \frac{\beta_s(\omega)}{2} \right)^2 = 0 \) \( \iff \) \text{n.b. This may look like a \textbf{quartic} equation, but it is actually a \textbf{quadratic} equation !!!}

Define: \( x \equiv n^2(\omega) \). We temporarily suppress the \( (\omega) \)-dependence in the following:

Then: \( x^2 - (1 + \alpha_s) x - \left( \frac{\beta_s(\omega)}{2} \right)^2 = 0 \) \( \implies ax^2 + bx + c = 0 \) with: \( a = 1, \ b = -(1 + \alpha_s), \ c = -\left( \frac{\beta_s}{2} \right)^2 \)

The solutions / roots of this quadratic equation are of the general form:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[ x = \frac{+(1+\alpha_x)\pm \sqrt{(1+\alpha_x)^2 + 4\left(\frac{\beta_x}{2}\right)^2}}{2} = \frac{1}{2} \left[ (1+\alpha_x)\pm \sqrt{(1+\alpha_x)^2 + \beta_x^2} \right] \]

i.e. \[ x = \frac{1}{2} (1+\alpha_x) \left[ 1 \pm \sqrt{1 + \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2} \right] \]  

n.b. the term: \[ \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2 > 0 \]

\[ \Rightarrow \textbf{Must select +ve root on physical grounds, since } x \equiv n^2 > 0. \]

\[ x = n^2 = \frac{1}{2} (1+\alpha_x) \left[ 1 + \sqrt{1 + \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2} \right] \]

Finally, we obtain:

\[ n(\omega) \equiv \Re \{\tilde{n}(\omega)\} = \sqrt{\frac{1+\alpha_x(\omega)}{2}} \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2} \right] \]  

Complex index of refraction: \[ \tilde{n}(\omega) = n(\omega) + i\eta(\omega) \]

\[ \eta(\omega) \equiv \Im \{\tilde{n}(\omega)\} = \frac{\beta_x(\omega)/2}{2n(\omega)} \sqrt{\frac{1+\alpha_x(\omega)}{2}} \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2} \right] \]

Where:

\[ \alpha_x(\omega) \equiv \left( \frac{n_x^4 e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} (\omega_j^3 - \omega^3)}{\left(\omega_j^3 - \omega^3\right)^2 + \gamma_j^3 \omega^2} \]

\[ \beta_x(\omega) \equiv \left( \frac{n_x^4 e^2}{\varepsilon_0 m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc} (\gamma_j \omega)}{\left(\omega_j^3 - \omega^3\right)^2 + \gamma_j^3 \omega^2} \]

Obviously, \textbf{explicitly} writing out the full mathematical formulae for \( n(\omega) \) and \( \eta(\omega) \) is quite tedious – but these can be reasonably-easily coded up \{i.e. a computer program\} and plots of \( n(\omega) \) vs. \( \omega \) and \( \eta(\omega) \) vs. \( \omega \) can be obtained. We can also then obtain the following:

The complex relations: \[ \tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega) \text{ and } \tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \frac{\omega}{c} \tilde{n}(\omega) \]

and thus: \[ k(\omega) = \left( \frac{\omega}{c} \right) n(\omega) \text{ and } \kappa(\omega) = \left( \frac{\omega}{c} \right) \eta(\omega). \]
The \{frequency-dependent\} intensity/irradiance \( I(\vec{r}, \omega) = \left\langle \tilde{S}(\vec{r}, t; \omega) \right\rangle \) associated with a monochromatic plane EM wave propagating in a linear, dispersive dielectric is also exponentially decreased by a factor of \( 1/e = e^{-1} \) of its original value in going a characteristic distance of: 
\[ z = 1/\alpha(\omega) = 1/2\kappa(\omega) \equiv \ell_{\text{attenu}}(\omega) \]  i.e. defining: \( \ell_{\text{attenu}}(\omega) \equiv 1/\alpha(\omega) = 1/2\kappa(\omega) \) = intensity attenuation length – which is ~ analogous to the skin depth, \( \delta_{sc} \equiv 1/\kappa \) for metals / conductors. However, note that \( \delta_{sc} \equiv 1/\kappa \) is associated with the attenuation of the \( \vec{E} \) and \( \vec{B} \) -fields, whereas attenuation effects in intensity/irradiance, \( I \) varies as the square of the \( \vec{E} \)-field:
\[ I(\vec{r}, \omega) = \left\langle \tilde{S}(\vec{r}, t; \omega) \right\rangle \propto \left\langle E^2_{\text{ext}}(\vec{r}, t; \omega) \right\rangle \]  hence: 
\[ I(\vec{r}, \omega) \propto E^2_{\text{ext}} e^{-2\kappa(\omega)z} = E^2 e^{-\alpha(\omega)z} \]

In the exponential \( z \)-dependent term \( e^{-2\kappa(\omega)z} \), since the energy densit(ies) \( \left\langle u_{E,M}(\vec{r}, t; \omega) \right\rangle \) and intensity \( I(\vec{r}, \omega) = \left\langle \tilde{S}(\vec{r}, t; \omega) \right\rangle \) are both proportional to \( E^2 \) i.e. both proportional to \( e^{-2\kappa(\omega)z} \), we define the \{frequency-dependent\} absorption coefficient:
\[ \alpha(\omega) \equiv 2\kappa(\omega) = 1/\ell_{\text{attenu}}(\omega). \]

Similarly, for the \{frequency-dependent\} complex index of refraction \( \tilde{n}(\omega) = n(\omega) + \imath \eta(\omega) \) we can also define the \{frequency-dependent\} extinction coefficient:
\[ \xi(\omega) \equiv 2\eta(\omega) \]

Since: 
\[ \kappa(\omega) = \left( \frac{\omega}{c} \right) \eta(\omega) \Rightarrow 2\kappa(\omega) = \left( \frac{\omega}{c} \right) 2\eta(\omega) \]  thus: 
\[ \alpha(\omega) = \left( \frac{\omega}{c} \right) \xi(\omega) = 2\left( \frac{\omega}{c} \right) \eta(\omega) \]

The absorption coefficient: 
\[ \alpha(\omega) \equiv 2\kappa(\omega) = \left( \frac{\omega}{c} \right) 2\eta(\omega) = \left( \frac{\omega}{c} \right) \xi(\omega) = 1/\ell_{\text{attenu}}(\omega) \]

The extinction coefficient: 
\[ \xi(\omega) \equiv 2\eta(\omega). \]

Typical values of the (real) index of refraction \( n(\omega) \) for solids and liquids are \( n(\omega) \approx 1.3 - 1.7 \) in the visible light region of EM spectrum, e.g. \( n_{\text{glass}}(\omega) = 1.5 \), \( n_{\text{H}_2\text{O}}(\omega) = 1.3 \), \( n_{\text{plastic}}(\omega) = 1.7 \).

Then if: 
\[ n(\omega) = \sqrt{\frac{1 + \alpha_x(\omega)}{2} \left[ 1 + \sqrt{1 + \left( \frac{\beta_x(\omega)}{1 + \alpha_x(\omega)} \right)^2} \right]} = 1.5 \]  \( \Leftarrow \) index of refraction of glass in the visible light region

Then:
\[ n^2(\omega) = \left( \frac{1 + \alpha_x(\omega)}{2} \right) 1 + \sqrt{1 + \left( \frac{\beta_x(\omega)}{1 + \alpha_x(\omega)} \right)^2} = (1.5)^2 = 2.25 \]

Thus: 
\[ (1 + \alpha_x(\omega)) \left[ 1 + \sqrt{1 + \left( \frac{\beta_x(\omega)}{1 + \alpha_x(\omega)} \right)^2} \right] = 4.50 \]  \( \Leftarrow \) One equation & two unknowns: \( \alpha_x(\omega) \) and \( \beta_x(\omega) \)

\( \Rightarrow \) Need another relation / independent constraint!!

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Note that glass doesn’t have significant absorption in the **visible light** region, typical such solid/liquid materials have (measured) absorption coefficients for visible light in the range of:

\[
\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \approx 10^{-2} - 10^{-1}\text{m}^{-1}
\]

Intensity \(I(z)\) decreases to \(1/e \approx 0.368\) of initial \(I(z=0)\) value after light travels a distance \(z = 1/\alpha(\omega)\) ~ 10 – 100 m in glass.

So suppose:

\[
\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) = 10^{-1}\text{m}^{-1}
\]

in glass for visible light, \(\omega_{vis} = 10^{16}\text{radians/sec}\)

\[
\Rightarrow \eta(\omega) = \left(\frac{c}{\omega}\right)\alpha(\omega) = \left(\frac{3 \times 10^8}{10^{16}}\right)10^{-1} = 3 \times 10^{-9} \ll 1
\]

Now:

\[
\eta(\omega) = \left(\frac{\beta_s(\omega)}{2n(\omega)}\right)
\]

and \(n(\omega) = 1.5\) for glass in visible light range of EM spectrum.

\[
\Rightarrow \eta(\omega) = \frac{1}{3}\beta_s(\omega) \quad \text{or:} \quad \beta_s(\omega) = 3\eta(\omega) = 9 \times 10^{-7} \ll 1 \text{ in the visible light range for glass}
\]

Then:

\[
(1 + \alpha_s(\omega))\left[1 + \sqrt{1 + \left(\frac{\beta_s(\omega)}{1 + \alpha_s(\omega)}\right)^2}\right] = 4.50 \quad \text{and:} \quad \beta_s(\omega) = 9 \times 10^{-9}.
\]

Can now solve for \(\alpha_s(\omega)\):

\[
1 + \sqrt{1 + \left(\frac{\beta_s(\omega)}{1 + \alpha_s(\omega)}\right)^2} = \frac{4.50}{1 + \alpha_s(\omega)} \quad \Rightarrow \quad 1 + \left(\frac{\beta_s(\omega)}{1 + \alpha_s(\omega)}\right)^2 = \left[\frac{4.50}{1 + \alpha_s(\omega)} - 1\right]^2
\]

\[
\Rightarrow \frac{\beta_s(\omega)}{1 + \alpha_s(\omega)} = \sqrt{\left[\frac{4.50}{1 + \alpha_s(\omega)} - 1\right]^2 - 1} \quad \Rightarrow \quad \beta_s(\omega) = (1 + \alpha_s(\omega))\sqrt{\left[\frac{4.50}{1 + \alpha_s(\omega)} - 1\right]^2 - 1}
\]

This has a solution when: \(\alpha_s(\omega) = 1.25\) for: \(\beta_s(\omega) = 9 \times 10^{-9} \ll \alpha_s(\omega)\) \(\Rightarrow\) Obtained via numerical methods using a computer

Thus, for \(n(\omega) = 1.5\) for glass in the visible light region of the EM spectrum, with \(\alpha_s(\omega) = 1.25\) and \(\beta_s(\omega) = 9 \times 10^{-9}\) (i.e. \(\beta_s(\omega)/(1 + \alpha_s(\omega)) \ll 1\)), as an explicit check, we see that:

\[
n(\omega) = \sqrt{\frac{1 + \alpha_s(\omega)}{2}}\left[1 + \sqrt{1 + \left(\frac{\beta_s(\omega)}{1 + \alpha_s(\omega)}\right)^2}\right] = \sqrt{\left(\frac{1 + 1.25}{2}\right)}\left[1 + \sqrt{1 + \left(9 \times 10^{-9}\right)^2}\right] \approx \sqrt{\left(\frac{1 + 1.25}{2}\right)}\left[1 + \sqrt{1}\right]
\]

\[= \sqrt{\left(\frac{1 + 1.25}{2}\right)} = 1.5 \quad \checkmark
\]
Thus we also see that:
\[
\alpha_x(\omega) \approx n^2(\omega) - 1 \equiv \left( \frac{n_e e^2}{\varepsilon_0 m_e} \right) \left[ \sum_{j=1}^{n} \frac{f_{j,\text{osc}}^2 (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \\
i.e. \ n^2(\omega) = 1 + \alpha_x(\omega)
\]
for typical materials – glass, water, plastic – in the visible light region of the EM spectrum, \(\omega \approx 10^{16}\) radians / sec.

Whereas:
\[
\beta_x(\omega) \equiv \left( \frac{n_e e^2}{\varepsilon_0 m_e} \right) \left[ \sum_{j=1}^{n} \frac{f_{j,\text{osc}} \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \ll 1
\]
for these same materials – glass, water, plastic – in the visible light region of the EM spectrum, \(\omega \approx 10^{16}\) radians / sec.

Our original equations were: \(n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)\) and \(2\eta(\omega)\eta(\omega) = \beta_x(\omega)\) or:
\[
\eta(\omega) = \beta_x(\omega)/2\eta(\omega)
\]
with \(\alpha_x(\omega) = 1.25\) and \(\beta_x(\omega) = 9 \times 10^{-9}\) for \(n(\omega) = 1.5\) (for glass) with visible light and: \(\eta(\omega) = \beta_x(\omega)/2n(\omega) = 3 \times 10^{-9}\).

We now see more clearly that: \(\eta(\omega) \ll n(\omega)\) in the visible light region of the EM spectrum for glass, i.e. the complex index of refraction \(\tilde{n}(\omega) = n(\omega) + i\eta(\omega) = 1.25 + 9 \times 10^{-9}i\) for glass is predominantly real in the visible light region of the EM spectrum.

Thus, for glass in the visible light region of the EM spectrum:
\[
n^2(\omega) - \eta^2(\omega) = n^2(\omega) = 1 + \alpha_x(\omega) = 1 + \left( \frac{n_e e^2}{\varepsilon_0 m} \right) \left[ \sum_{j=1}^{n} \frac{f_{j,\text{osc}}^2 (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] = (1.5)^2 = 2.25
\]
and:
\[
2\eta(\omega)\eta(\omega) = \beta(\omega) = \left( \frac{n_e e^2}{\varepsilon_0 m} \right) \left[ \sum_{j=1}^{n} \frac{f_{j,\text{osc}} \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx 9 \times 10^{-9}
\]
\[
\alpha_x(\omega) = \left( \frac{n_e e^2}{\varepsilon_0 m} \right) \left[ \sum_{j=1}^{n} \frac{f_{j,\text{osc}} (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] = 1.25
\]

Note that these results that we just obtained for glass in the visible light region of the EM spectrum do not hold true for all frequencies of EM waves {visible light region is in fact only a narrow portion of the EM spectrum}!!! In particular, these results do not hold at {or near} an atomic (or molecular) resonance!
Let us consider a “simplified” atomic/molecular system, that of having only a **single** resonance frequency (*i.e.* a single bound-state quantum energy level), then:

\[
\alpha_i = \sqrt{\frac{\omega_i^2}{3\varepsilon_0 m_e}} = 2\pi f_i
\]

with:

\[
\omega_b = \sqrt{\frac{k}{m_e}}
\]

or:

\[
f_i = \frac{1}{2\pi} \omega_i = \frac{1}{2\pi} \sqrt{\frac{n_i^2 e^2}{3\varepsilon_0 m_e}}
\]

Then:

\[
n(\omega) = \sqrt{\frac{1 + \alpha_i^2(\omega)}{2}} \left[ 1 + \sqrt{1 + \left( \frac{\beta_i(\omega)}{1 + \alpha_i(\omega)} \right)^2} \right]
\]

\[
h(\omega) = \frac{\beta_i(\omega)}{2n(\omega)} = \frac{\beta_i^4(\omega)/2}{\sqrt{\left( \frac{1 + \alpha_i^2(\omega)}{2} \right) \left[ 1 + \sqrt{1 + \left( \frac{\beta_i(\omega)}{1 + \alpha_i(\omega)} \right)^2} \right]}}
\]

\[
\alpha_i^2(\omega) \equiv \left( \frac{n_i^2 e^2}{\varepsilon_0 m_e} \right) \left( \frac{f^{\text{osc}}_i (\omega_i^2 - \omega^2)}{\left( \omega^2_i - \omega^2 \right)^2 + \gamma^2 \omega_i^2} \right)
\]

\[
\beta_i^4(\omega) \equiv \left( \frac{n_i^2 e^2}{\varepsilon_0 m_e} \right) \left( \frac{f^{\text{osc}}_i \gamma_i \omega}{\left( \omega^2_i - \omega^2 \right)^2 + \gamma^2 \omega_i^2} \right)
\]

The figure on the left (immediately below) shows the behavior of the real and imaginary parts of the complex index of refraction of a dispersive, linear medium, *i.e.* \{\(n(\omega)\) vs. \(\omega\)\} and \{\(h(\omega)\) vs. \(\omega\)\} for a single atomic resonance. The figure on the right (immediately below) shows the behavior of \{\(n(\omega)-1\) vs. \(\omega\)\} and the **absorption coefficient** \{\(\alpha(\omega) \equiv 2\kappa(\omega)\) vs. \(\omega\)\} for a single atomic resonance.

---

**Figure 5.1.** Graphs of the index of refraction and extinction coefficient versus frequency near a single resonance line.
n.b. The above curves are “classic” features of a complex resonance – with center / resonance frequency $\omega_i$ and damping constant $\gamma_j = \omega_2 - \omega_i = 2\pi \Gamma_j = 2\pi \times \text{width } \Gamma_j$ (= FWHM) of the $j^{\text{th}}$ resonance. Note that the width (FWHM) $\Gamma \equiv f_2 - f_1 \,(Hz)$. In the $\eta$ - $n$ complex plane:

In the visible light region of the EM spectrum, the graph below shows both the frequency and wavelength behavior of the \{real\} index of refraction of glass, i.e. $n(f)$ vs. $f$ \{dotted line\} and $n(\lambda)$ vs. $\lambda$ \{solid line\}. Note that since $v_{\text{prop}} = f \lambda \Rightarrow f = v_{\text{prop}} / \lambda$ or: $\lambda = v_{\text{prop}} / f$.

$n.b.$ Media which are very transparent e.g. in the visible light region are often almost (or are) opaque in the so-called anomalous dispersion region of a resonance, $\omega_i < \omega_n < \omega_2$ - i.e. in the FWHM region of the atomic resonance, where the extinction coefficient $\eta(\omega)$ becomes very large – EM waves near the resonance frequency $\omega_n$ are very rapidly exponentially attenuated!

**The General Behavior of “Classic” Complex Resonance:**

\[ z(\omega) = x(\omega) + iy(\omega) \]

$n.b.$ in some complex systems e.g. the resonance of a LCR circuit, $\Re \{z\}$ & $\Im \{z\}$ are interchanged from what is drawn here!

\[ i.e. \Re \{z\} \iff \Im \{z\} \]
Note that the shape of the curve for the magnitude of $\tilde{z}(\omega) = |\tilde{z}(\omega)| = \sqrt{\tilde{z}(\omega) \cdot \tilde{z}^*(\omega)} = \sqrt{x^2(\omega) + y^2(\omega)}$, is very similar to shape of the $\Im m\{z(\omega)\}$ curve (as shown here).

The trajectory of $\tilde{z}(\omega)$ in the complex plane:

A more realistic microscopic picture of an atomic system – with many electrons with many quantum bound states $\Rightarrow \exists$ many resonances in a dispersive, linear macroscopic dielectric!!!

Exercise(s): Draw out the corresponding trajectories of complex $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ and $\tilde{\varepsilon}(\omega) = \varepsilon(\omega) + i\xi(\omega)$ for the above triple-resonance cases in the complex plane!
In the high-frequency region, above the highest resonant frequency (typically in UV or x-ray region), the index of refraction is predicted to be \( n(\omega) < 1.0 \) (i.e. actually less than that of the vacuum).

Indeed, this phenomenon has explicitly been observed/measured e.g. in quartz (SiO\(_2\)) using x-rays:

Note

Suppressed Zero!

![Figure 5.3. Measured index of refraction of quartz in the x-ray region.](image)

Note that physically the damping constant \( \gamma_j = width of the \ j^{th} \ resonance \) is inversely related to the lifetime \( \tau_j \) associated with the corresponding excited state of the constituent atoms/molecules of the dispersive, linear dielectric, since at the microscopic level, the {real} photons associated with the monochromatic plane EM wave have energy \( E_r = hf \) and {assuming the atoms/molecules of the dispersive, linear dielectric to all be in their ground state, with ground state energy \( E_1 \)}, then if the monochromatic plane EM wave has {angular} frequency \( \omega = \omega_R = 2\pi f_R \) = the resonance frequency of the bound atomic electrons, then we see that \( \Delta E_j = E_j - E_1 = E_r = hf_R = h\omega_R \) at resonance!

At a resonance, e.g. when \( \omega = \omega_j \), the {real} photons in the monochromatic plane EM wave easily stimulate the atomic electrons, causing them to resonate – the {real} photons are absorbed, enabling the atomic electron to make a transition from the ground state {with energy \( E_1 \)} to the \( j^{th} \) excited state {with energy \( E_j \)} via an electric dipole transition, if so allowed by quantum-mechanical selection rules. The \( j^{th} \) excited atomic state has {mean} lifetime \( \tau_j \) associated with it, thus the atomic electron de-excites back to the ground state by emitting a {real} photon of this same frequency. The miracle of all of this is that {real} photons associated with the EM field are actually interacting simultaneously with all of the atoms in the dispersive linear dielectric (within the coherence length of the photon \( c/\gamma \)) at any given instant in time, thus the resultant “scattered” photon that is {ultimately} emitted, actually must be summed over the response of the ensemble of many atoms – the miraculous result of which is forward scattering of the photons associated with the macroscopic EM wave, but with a {frequency-dependent} phase shift, which is related to the resonance lineshape {and the finite lifetime \( \tau_j \)} of the excited state of the atom!
At a resonance, e.g. when \( \omega = \omega_k \), a large, transitory/transient \{complex and frequency-dependent\} electric dipole moment \( \vec{p}(\vec{r}, \omega) = -e \vec{r}(\omega) = -e \vec{r}(\omega) \hat{r} \) is induced in the atom, where:

\[
\vec{r}(\omega) = r(\omega) + \rho(\omega)
\]

Note here we can also make a direct connection with quantum mechanics – the electric dipole moment operator \( \hat{p}(\vec{r}, \omega) = -e \hat{r}(\omega) \) and position operator \( \hat{r}(\omega) \) operating e.g. on the ground state wave function of the atom/molecule \( |\psi_1(\vec{r})\rangle \), i.e. \( \hat{p}(\vec{r}, \omega)|\psi_1(\vec{r})\rangle \) and \( \hat{r}(\omega)|\psi_1(\vec{r})\rangle \).

We can e.g. compute the expectation value of the modulus squared of the electric dipole moment \( \langle \psi_j(\vec{r})|\psi_1(\vec{r})\rangle \) of the atom/molecule. Inserting a complete set of states \( \sum_{j=1}^{\infty} \langle \psi_j(\vec{r})|\psi_1(\vec{r})\rangle = 1 \) into this expression, we can then obtain the quantum mechanical predictions for the {squares} the oscillator strengths \( f_{j,\text{osc}} \):

\[
\sum_{j=1}^{\infty} \langle \psi_j(\vec{r})|\vec{p}^*(\vec{r}, \omega)|\psi_1(\vec{r})\rangle \langle \psi_1(\vec{r})|\vec{p}(\vec{r}, \omega)|\psi_j(\vec{r})\rangle = \sum_{j=1}^{\infty} \langle \psi_1(\vec{r})|\vec{p}(\vec{r}, \omega)|\psi_j(\vec{r})\rangle^2
\]

The transition rate \( \Gamma_{ij} = \gamma_j / 2\pi \) for the excited states of atoms are \{identical\} related to the widths \( \gamma_j \) of the \( j \)th resonances/widesths of the \( j \)th excited states by the Heisenberg uncertainty principle: \( \Delta E \Delta t \geq \hbar \) where \( \hbar \equiv h/2\pi \) and \( h = \text{Planck’s constant} \).

Note further that the lifetimes \( \tau_j \) of the excited states of atoms are \{inversely\} related to the widths \( \gamma_j \) of the \( j \)th resonances/widesths of the \( j \)th excited states by the Heisenberg uncertainty principle: 

\[
(h \gamma_j) \ast \tau_j = \hbar \Rightarrow \hbar \gamma_j = \hbar / \tau_j \quad \text{or:} \quad 1/\tau_j = \Gamma_{ij} \equiv \Gamma_{ji} = \gamma_j / 2\pi
\]
If one stays **well away/far** from \{all\} of the resonance frequencies of bound-state atomic electrons, the resonance factor becomes:

\[
\left[ \frac{(\omega_j^2 - \omega^2) + i \gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx \frac{1}{(\omega_j^2 - \omega^2)}
\]

**i.e. far** from a resonance: \((\omega_j^2 - \omega^2) \gg \gamma_j \omega\)

Thus, **far** from a resonance / all resonances, relatively little absorption/dissipation occurs **i.e.** \(n^2(\omega) \ll n^2(\omega)\), such that \(\tilde{n}(\omega) = n(\omega) + \tilde{\gamma}(\omega) = n(\omega)\) is predominantly **real** and hence:

\[
n^2(\omega) - \tilde{n}^2(\omega) = n^2(\omega) = 1 + \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^n f_{j}^{osc} \left( \frac{\omega_j^2}{\omega_j^2 - \omega^2} \right)
\]

Now:

\[
\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{1 - (\omega^2 / \omega_j^2)} \approx \frac{1}{\omega_j^2} \left( 1 + \frac{\omega^2}{\omega_j^2} \right) = \frac{1}{\omega_j^2} + \frac{\omega^2}{\omega_j^2}
\]

Then:

\[
n^2(\omega) - \tilde{n}^2(\omega) = n^2(\omega) = 1 + \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^n \left( f_{j}^{osc} \right) + \omega^2 \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^2} \right)
\]

If \(n^2 = 1 + \epsilon\) and \(\epsilon \ll 1\) \(\Rightarrow n = \sqrt{1 + \epsilon} = 1 + \frac{1}{2} \epsilon\)

Thus, **far** from a resonance/resonances:

\[
n(\omega) \approx 1 + \frac{1}{2} \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^2} \right) + \omega^2 \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^4} \right)
\]

But:

\[
\lambda_o = \frac{2\pi}{k_o} = \frac{2\pi c}{\omega} = \text{vacuum} \text{ wavelength, hence:} \quad \omega = \frac{2\pi c}{\lambda_o}
\]

\[
n(\lambda_o) \approx 1 + \frac{1}{2} \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^2} \right) + \left( \frac{2\pi c}{\lambda_o} \right)^2 \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^4} \right)
\]

\(\Rightarrow \text{we obtain Cauchy's Formula:}\)

\[
n(\lambda_o) \approx 1 + A \left( 1 + \frac{B}{\lambda_o^2} \right)
\]

Where: \(A = \text{Coefficient of Refraction}\) and: \(B = \text{Coefficient of Dispersion}\).

Comparing the 2 equations, we see that:

\[
A = \frac{1}{2} \left( \frac{n_b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^2} \right) \quad \text{and:} \quad B = \left( \frac{2\pi c}{\lambda_o} \right)^2 \sum_{j=1}^n \left( \frac{f_{j}^{osc}}{\omega_j^4} \right)
\]
Since:
\[
\tilde{n}(\omega) = n(\omega) + i\eta(\omega)
\]
and/or:
\[
\tilde{k}(\omega) = k(\omega) + ik(\omega)
\]
and:
\[
\tilde{k}(\omega) = \left(\frac{\omega}{c}\right)\tilde{n}(\omega)
\]

Then:
\[
k(\omega) = \left(\frac{\omega}{c}\right)n(\omega)
\]
and:
\[
\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega),
\]
thus:

The phase speed:
\[
v_p(\omega) = \frac{\omega}{k} = \frac{c}{n(\omega)} > c \quad \text{if} \quad n(\omega) < 1
\]

The group speed:
\[
v_g(\omega) = \left(\frac{d\omega}{dk}\right)^{-1} = \left[\frac{dk(\omega)}{d\omega}\right]^{-1}
\]

Note that at the “turning points” of either the \( n(\omega) \) vs. \( \omega \) or: \( k(\omega) \) vs. \( \omega \) graphs, i.e. at \{angular\} frequencies \( \omega = \omega_1 \) and/or \( \omega = \omega_2 \) where the slope \( dk(\omega)/d\omega = 0 \) \( \Rightarrow \) \( v_g = \infty \) !!!

Note further that in the \{angular\} frequency region \( \omega_1 < \omega < \omega_2 \) \{the “anomalous dispersion” region\}, since the slope \( dk(\omega)/d\omega < 0 \) then the group speed \( v_g(\omega) = 1/(dk(\omega)/d\omega) < 0 \) !!! \{Hence the name anomalous dispersion…\}

This phenomenon has been experimentally verified (see e.g. C.G.B. Garrett & D.E. McCumber, Phys. Rev. A, 1, p. 305 (1970). If the dispersive medium is not too thick, a Gaussian pulse with a central frequency \textit{near an absorption line} (i.e. \textit{near a resonance}, \( \omega_n \)) and with pulse width \( \Delta t \gg \tau_n = 1/\gamma_n \) propagates with appreciable absorption, but (more or less) retains its \textit{shape}.

The peak of the Gaussian pulse propagates at \( v_g \) even when the group speed is negative!!!

\( \Rightarrow \) Useful for \textit{pulse re-shaping} applications - leading edge is less attenuated than trailing edge.
⇒ Can actually have the peak of a greatly attenuated pulse emerge from the absorber before the peak of the incident pulse enters the absorber (≡ definition of negative group speed)!!!

{i.e. microscopically, if the absorber is not too thick, then some photons can make it all the way through the absorber w/o interacting at all – this probability is exponentially suppressed.}

⇒ Has applications/uses e.g. in optical mammography/breast cancer screening for women...

⇒ See e.g. J.D. Jackson’s Electrodynamics, 3rd Edition, pages 325-26 for more details!

Finally, if we set \( \omega = 0 \), then we obtain the static (i.e. zero-frequency) limit of \{all of\} these quantities. Note that they also \{all\} become purely real in this limit:

\[
\textbf{Static Polarization:} \quad \mathbf{P}(0) = \left( \frac{n^b e^2}{m} \right) \sum_{j=1}^{n} \frac{f_j^{osc}}{\alpha_{ij}^2} \quad \text{where} \quad \alpha_{ij} \equiv \sqrt{\omega_0^2 - \left( \frac{n^b e^2}{3 \varepsilon_a m_e} \right)}
\]

and since \( \mathbf{P} = \varepsilon_o \chi_e \mathbf{E}_{\text{ext}} \).

\[
\textbf{Static Electricity Susceptibility:} \quad \chi_e(0) = \left( \frac{n^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc}}{\alpha_{ij}^2} \quad \text{and} \quad \omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}
\]

\[
\textbf{Static Index of Refraction:} \quad n(0) = \sqrt{1 + \alpha_s(0)} = \sqrt{1 + \chi_e(0)} = \sqrt{K_e(0)}
\]

But: \( K_e(\omega) = \varepsilon(\omega)/\varepsilon_o = (1 + \chi_e(\omega)) \quad \Rightarrow \quad \varepsilon(0) = \varepsilon_o (1 + \chi_e(0)) \) and thus:

\[
\textbf{Static Dielectric Constant:} \quad K_e(0) = \varepsilon(0)/\varepsilon_o = (1 + \chi_e(0)) = 1 + \alpha_s(0) = n^2(0)
\]

But: \( \alpha_s(0) = \left( \frac{n^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc}}{\alpha_{ij}^2} = \chi_e(0) \)

\[
\therefore \quad K_e(0) = 1 + \left( \frac{n^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc}}{\alpha_{ij}^2} = n^2(0) \quad \text{and:} \quad \chi_e(0) = K_e(0) - 1 = \left( \frac{n^b e^2}{\varepsilon_o m_e} \right) \sum_{j=1}^{n} \frac{f_j^{osc}}{\alpha_{ij}^2}
\]

⇒ Note that the static dielectric constant \{as measured at \( f = 0 \) Hz/DC\} is \( K_e(0) > 1.0 \) because it contains information about all of the \{quantum mechanical\} resonances/excited states \( \omega_{ij} > 0 \) present in the dispersive, linear medium, even into the x-ray region at \( \omega_{ij} \approx 10^{18-19} \) Hz and beyond !!!

⇒ Equivalently, armed now with this knowledge of the microscopic behavior of a dispersive, linear medium, an electric susceptibility \( \chi_e(0) > 0 \) \{or equivalently, a dielectric constant \( K_e(0) > 1\} \textbf{instantly} tells us that there are indeed \{quantum mechanical\} resonances/excited states present in the \{composite\} atoms/molecules that make up the macroscopic material of the dispersive, linear medium!!!
⇒ A wonderful macroscopic example of dispersion in nature is the rainbow. At the microscopic level, the frequency-dependence of the index of refraction of light \( n(\omega) \) arises as a consequence of the resonant behavior of quantum mechanical bound states of electrons in the atoms of the water molecule (H\(_2\)O) responding to EM light waves (= visible light photons) coming from our sun.

⇒ If no such composite behavior existed at the microscopic level, there would be no rainbows to enjoy in the macroscopic everyday world!

![Rainbow Image]

**Experimental measurements of the absorption coefficient of pure water:**

![Absorption Coefficient Graph]

Absorption coefficient of pure water - \( \alpha_{H_2O}(\omega) \) is large in UV and IR regions due to **resonances**!