Supplemental Handout # 6

Symmetry Properties of Electromagnetism

The various field and source quantities, such as $\vec{E}, \vec{D}, \vec{P}, \vec{H}, \vec{B}, \vec{M}$ e.g. $\vec{j}_m, \vec{v}, \epsilon, \mu, \vec{L}, \vec{S}, \ldots$ etc. have various symmetry properties under symmetry operations such as:

$P \equiv$ Parity (Space-Inversion, $\vec{r} \rightarrow -\vec{r}$) $\rightarrow$ Reflection in a mirror

$T \equiv$ Time Reversal (e.g. particle motion, but run backwards in time)

$C \equiv$ Electric Charge Conjugation (Charge of Particle $\rightarrow$ Charge of Antiparticle, e.g. $e^- \rightarrow e^+$)

$M \equiv$ Magnetic Charge Conjugation (Magnetically Charged Particle $\rightarrow$ Magnetically charged antiparticle, e.g. $g_N \rightarrow g_\bar{N}$)

The field and source quantities mentioned above fall into various generic mathematical classes of objects, or quantities:

1) Scalar quantities under a given symmetry transformation, designated $\phi$.
2) Pseudoscalar quantities under a given symmetry transformation, designated $p$.
3) Polar Vector quantities under a given symmetry transformation, designated $V$.
4) Axial, or Pseudo-Vector quantities under a given symmetry transformation, designated $A$.
5) $N^{th}$ rank covariant / contravariant tensors under a given symmetry transformation, $T_{\mu\nu}, T^{\mu\nu}, T_{\mu}, T_{\nu}, T^{\mu\nu}, T_{\alpha\beta\gamma}$, ...

Note that, e.g.:  
Electric charge $q$ is ODD under electric charge conjugation: $Ce^- = e^+$ (i.e. $q$ behaves as a pseudoscalar quantity $p$ under $C$)

Magnetic charge $g_m$ is ODD under magnetic charge conjugation: $Mg_m^- = g_m^+$ (i.e. $g_m$ behaves as a pseudoscalar quantity $p$ under $M$)

However note also that, e.g.:  
Electric charge is EVEN under magnetic charge conjugation: $Me^- = e^-$ (i.e. $q$ behaves as a scalar quantity $\phi$ under $M$.)

Magnetic charge is EVEN under electric charge conjugation: $Cg_m^- = g_m^-$ (i.e. $g_m$ behaves as a scalar quantity $\phi$ under $C$.)

Note also that (if $\exists$ no magnetic charges) the combined operations $CPT = 1$ (in any order) (i.e. $CPT = \text{identity operator}$). If have magnetic charges, then $CPTM = 1$ (in any order).
In order to understand the distinction between Polar Vectors and Axial (or Pseudo)-Vectors under a specific symmetry transformation, consider parity $P$ (i.e. mirror-reflection) operation:

**Polar Vector $\vec{V}$**

- e.g. direction vector $\vec{r}$
  - Parity is space-inversion
  - i.e. $\vec{r} \rightarrow -\vec{r}$
  - $x \rightarrow -x$
  - $y \rightarrow -y$
  - $z \rightarrow -z$

  $\parallel$-component of $\vec{V}$ unchanged under Parity
  i.e. $PV_\parallel = V_\parallel$

  $\perp$-component of $\vec{V}$ changes sign under Parity
  i.e. $PV_\perp = -V_\perp$

**Axial (or Pseudo) Vector $\vec{A}$**

- e.g. spinning top – angular momentum $\vec{L}$

  $\parallel$-components of $\vec{A}$ reversed under Parity
  i.e. $PA_\parallel = -A_\parallel$

  $\perp$-components of $\vec{A}$ unchanged under Parity
  i.e. $PA_\perp = A_\perp$

**Time Reversal:**

Particle Moving with Velocity $\vec{v}$

\[ \vec{v} = T\vec{v} = -\vec{v} \] (velocity is odd under Time reversal)
Electric Current Flowing in a Long Wire:

\[
\begin{align*}
\mathbf{B} & \quad \mathbf{E} \\
I, \mathbf{v}, \mathbf{j}^f & \quad \text{Time Reversed:} \quad I', \mathbf{v}', \mathbf{j}^f' \\
\end{align*}
\]

\[
\begin{align*}
\therefore \quad T \mathbf{v}_d &= -\mathbf{v}_d & \mathbf{j}_e^{\text{free}} &= n_e q \mathbf{v}_d \\
T \mathbf{j}_e^{\text{free}} &= -\mathbf{j}_e^{\text{free}} & Te^- &= e^- (q \text{ is even under } T) \\
TI = -I & \quad \text{all} & \mathbf{B} &= \frac{\mu_0}{4\pi} \left( \mathbf{I} \times d\mathbf{l} \right) \\
T\mathbf{B} &= -\mathbf{B} & \text{odd} & T\mathbf{B} &= \mu \mathbf{H} \\
T\mathbf{H} &= -\mathbf{H} & \text{under} & T\mathbf{H} &= \mu_0 \mathbf{H} + \mathbf{M} \\
T\mathbf{M} &= -\mathbf{M} & T & \mathbf{M} &= -\mathbf{M} \\
\end{align*}
\]

Current Flowing in a Loop:

\[
\begin{align*}
\therefore \quad T\mathbf{m} &= -\mathbf{m} & (\text{mag. dipole moment}) \\
\text{And} \quad T\mathbf{M} &= -\mathbf{M} & (\text{magnetization}) \\
\end{align*}
\]

By considering a parallel-plate capacitor, it can be seen that \( \mathbf{E}, \mathbf{D} \) and \( \mathbf{P} \) fields, \( \mathbf{p} \) = electric dipole moment, \( \varepsilon \) = permittivity, etc. are all even under time reversal.

\[
\begin{align*}
T\mathbf{E} &= \mathbf{E} & (\text{e.g. } \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{r^2} \right) \hat{r} ) \\
Tq &= +q & (\text{even under } T) \\
T\mathbf{P} &= \mathbf{P} \\
T\mathbf{D} &= \mathbf{D} = \varepsilon \mathbf{E} \\
T \varepsilon &= \varepsilon \\
\end{align*}
\]
Parity and Magnetic Fields

Electric Charge Conjugation & Magnetic Fields

Time Reversal & Magnetic Fields

Magnetic Charge Conjugation & Magnetic Fields
### Summary of Symmetry Properties of Kinematic & Electromagnetic Quantities

$\phi \equiv$ Scalar Quantity  \quad $p \equiv$ Pseudoscalar Quantity  \quad $\vec{V} =$ Polar Vector  \quad $\vec{A} =$ Axial Vector (Pseudo-Vector)

<table>
<thead>
<tr>
<th>Kinematic and/or Electromagnetic Quantity</th>
<th>Parity ($\mathcal{P}$)</th>
<th>Charge Reversal ($\mathcal{C}$)</th>
<th>Time Reversal ($\mathcal{T}$)</th>
<th>Magnetic Charge Reversal ($\mathcal{M}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$\vec{r} = e\vec{r}$</td>
<td>$-$</td>
<td>$+$</td>
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<td>$+$</td>
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<tr>
<td>$\vec{p} = m\vec{r}$</td>
<td>$+$</td>
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<tr>
<td>$\vec{a} = \frac{d\vec{v}}{dt}$</td>
<td>$-$</td>
<td>$+$</td>
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</tr>
<tr>
<td>$\vec{E} = \nabla \phi$</td>
<td>$+$</td>
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</tr>
<tr>
<td>$\vec{B} = \nabla \times \vec{A}$</td>
<td>$+$</td>
<td>$-$</td>
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<tr>
<td>$\vec{E} = \rho / \varepsilon$</td>
<td>$+$</td>
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</tr>
<tr>
<td>$\vec{B} = \mu / \varepsilon$</td>
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<tr>
<td>$\vec{E} \cdot \vec{A} = \vec{E} \times \vec{A}$</td>
<td>$-$</td>
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</tr>
</tbody>
</table>

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