LECTURE NOTES 7

LAPLACE’S EQUATION

As we have seen in previous lectures, very often the primary task in an electrostatics problem is to determine the electric field \( \vec{E}(\vec{r}) \) of a given stationary/static charge distribution — e.g. via Coulomb’s Law:

Charge density \( \rho(\vec{r'}) \)

![Diagram of electrostatics problem](image)

\[ \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r'})}{|\vec{r}|^2} d\tau' \]

\[ \vec{r} = \vec{r} - \vec{r'} \quad \vec{f} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|} \]

\[ |\vec{r}| = |\vec{r} - \vec{r'}| = \sqrt{(x_p - x_s)^2 + (y_p - y_s)^2 + (z_p - z_s)^2} \]

Oftentimes \( \rho(\vec{r'}) \) is complicated, and analytic calculation of \( \vec{E}(\vec{r}) \) is painful / tedious (or just plain hard). (Numerical integration on a computer is likely faster/easier. . . )

Oftentimes it is easier to first calculate the potential \( V(\vec{r}) \), and then use \( \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \)

Here: \( V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r'})}{|\vec{r}|} d\tau' \)

But even doing this integral analytically often can be very challenging. . .

Furthermore, often in problems involving conductors, \( \rho(\vec{r'}) \) may not apriori (i.e. beforehand) be known! Charge is free to move around, and often only the total free charge \( Q_{\text{free}} \) is controlled / known in the problem.

In such cases, it is usually better to recast the problem in DIFFERENTIAL form, using Poisson’s equation:

\[ \vec{\nabla} \cdot \vec{E}(\vec{r}) = -\vec{\nabla} \cdot \vec{\nabla} V(\vec{r}) = -\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o} \]

Or:

\[ \nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o} \]

\( \Leftarrow \) Poisson’s Equation
Poisson’s equation, together with the *boundary conditions* associated with the value(s) allowed for $V(\vec{r})$ e.g. on various conducting surfaces, or at $r = \infty$, etc. enables one to **uniquely** determine $V(\vec{r})$ (we’ll see how / why shortly...).

The Poisson equation is an **inhomogeneous** second-order differential equation – its solution consists of a particular solution for the inhomogeneous term (RHS of Poisson’s Equation) plus the general solution for the **homogeneous** second-order differential equation:

$$\nabla^2 V(\vec{r}) = 0 \iff \text{Laplace’s Equation}$$

commensurate with the boundary conditions for the specific problem at hand.

Very often, in fact, we are interested in finding the potential $V(\vec{r})$ in a *charge-free* region, containing no electric charge, *i.e.* where $\rho(\vec{r}) = 0$.

If $\rho(\vec{r}) = 0$, then $\nabla^2 V(\vec{r}) = 0$ and the **TRIVIAL** solution is $V(\vec{r}) = 0 \forall \vec{r}$, which is boring / useless!

We seek *physically meaningful / non-trivial* solutions $V(\vec{r}) \neq 0$ that satisfy $\nabla^2 V(\vec{r}) = 0$ and the boundary conditions on $V(\vec{r})$ for a given physical problem.

Now, before we go any further on this discussion, let’s back up a bit and take a (very) broad generalized **MATHEMATICAL** view (or approach) to find $V(\vec{r})$.

First, let’s simplify the discussion, by talking about *one-dimensional* problems:

If $\rho(x) = 0$, Laplace’s Equation in one-dimension becomes (in rectangular/Cartesian coordinates):

$$\nabla^2 V(\vec{r}) = 0 \quad \Rightarrow \quad \frac{d^2V(x)}{dx^2} = 0 \iff \text{Note the total (not partial) derivative with regards to } x.$$  

Integrating this equation (both sides) once, we have:

$$\int \frac{d^2V(x)}{dx^2} dx = \int d\left( \frac{dV(x)}{dx} \right) dx = \int d\left( \frac{dV(x)}{dx} \right) = \int \frac{dV(x)}{dx} = \int dx = m = 1^{st} \text{ constant of integration}$$

Then:  

$$\int \frac{dV(x)}{dx} \text{d}x = \int m \text{d}x = m \int \text{d}x$$

Or:  

$$\int dV(x) = V(x) = mx + b \iff 2^{nd} \text{ constant of integration}$$

So:  

$$V(x) = b + mx \quad (\text{equation for a straight line}) \text{ is the general solution for } \frac{d^2V(x)}{dx^2} = 0.$$

* y-intercept  
  slope
Depending on the boundary conditions for the problem, e.g. suppose $V(x=5)=0$ Volts and $V(x=1)=4$ Volts, then together, these two boundary conditions uniquely specify what $b$ and $m$ must be – we have two equations, and two unknowns ($m$ & $b$) – solve simultaneously:

$$V(x) = b + mx \quad \text{← equation for a straight line}$$

y-intercept \hspace{1cm} \text{slope}

1. $V(x=5) = 0 = b + 5m \quad \rightarrow \quad b = -5m$
2. $V(x=1) = 4 = b + 1m \quad \rightarrow \quad 4 = -5m + 1m = -4m$

\therefore $V(x) = 5 - 1x$ or: $m = -1$ and $b = 5$.

$V(x) = 5 - 1x$ is the equation of a straight line for this problem.
General features of 1-D Laplace’s Equation $\nabla^2 V(x) = 0$ and potential $V(x)$:

1. From above one-dimensional case $V(x) = b + mx$ (general solution = straight line eqn.) we can see that:

   $V(x)$ is the **average** of $V(x+a)$ and $V(x-a)$ *i.e.* $V(x) = \frac{1}{2} [V(x+a) + V(x-a)]$

   $\Rightarrow$ Laplace’s Equation is a kind of **averaging instruction**

   The solutions of $V(x)$ are as “boring” as possible, but fit the endpoints (boundary conditions) properly.

   This may be “obvious” in one-dimension, but it is also true / also holds in 2-D and 3-D cases of $\nabla^2 V(\vec{r}) = 0$.

2. $\nabla^2 V(\vec{r})$ tolerates / allows NO local maxima or minima – extrema **must** occur at endpoints *i.e.* $\nabla^2 V(\vec{r}) = 0$ requires the **second spatial derivative(s)** of $V(\vec{r})$ to be zero.

   - Not a proof, because e.g. $\exists$ fcns $(x)$ where the second derivative **vanishes** other than at endpoints - e.g. $f(x) = x^4$ (has a minimum at $x = 0$).

   **Laplace’s Equation in Two Dimensions (in Rectangular/Cartesian Coordinates)**

   If $V = V(x,y)$ then $\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

   *n.b. now have partial derivatives of $V(\vec{r})$.

   Because $\nabla^2 V = 0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x,y)$ now contains partial derivatives, the general solution does **not** contain just two arbitrary constants or any finite number - $\exists$ an **infinite** number of possible solutions (in general)

   – the most general solution is a linear combination of harmonic functions (sine and cosine functions of $x$ and $y$ in rectangular coordinates and other functions (Bessel Functions) in cylindrical coordinates).

   Nevertheless, $V(x,y)$ will still wind up being the **average value of $V$** around a point $(x,y)$ within a **circle** of radius $R$ **centered** on the point $(x,y)$.
The Method of Relaxation - Iterative Computer Algorithm for Finding $V(x, y)$:

- Start with $V(x, y)$ as specified on boundary (fixed)
- Choose reasonable “interpolated” values of $V(x, y)$ (from boundary) on interior $(x, y)$ points away from the boundaries.
- 1st pass reassigns $V(x, y) = \text{average value at interior point } (x, y)$ of its nearest neighbors.
- 2nd pass repeats this process . . .
- 3rd pass repeats this process . . .
- etc. . . .

After few iterations, $V(x, y)$ of $n^{th}$ iteration settles down, e.g. when:

$$\Delta V(x, y) = \left| V_{n}(x, y) - V_{n-1}(x, y) \right| \leq \text{tolerance}$$

then QUIT iterating, $V(x, y)$ is determined after $n^{th}$ iteration is “good enough”.

$V(x, y)$ again will have no local maxima or minima – all extrema will occur on boundaries. $\nabla^2 V(x, y) = 0$ has solution $V(x, y)$ which is the most featureless function – as smooth as possible.
Laplace’s Equation in Three Dimensions

Can’t draw this on 2-D sheet of paper (because now this is a 4 dimensional problem!), but:

\[ V(x, y, z) = V(\vec{r}) = \text{average value of } V \text{ over a spherical surface of radius } R \text{ centered on } \vec{r}. \]

i.e. \[ V(\vec{r}) = \frac{1}{4\pi R^2} \int_{\text{sphere at } \vec{r} \text{ of radius } R} V \, da \]

Again \( V(\vec{r}) \) will have no local maxima or minima
- all extrema must occur at boundaries of problem (see work-through proof in Griffiths, p. 114)
- The average potential produced by a collection of charges, averaged over a sphere of radius \( R \) is equal to the value of the potential at the center of that sphere!

Boundary Conditions on the Potential \( V(\vec{r}) \)

**Dirichlet** Boundary Conditions on \( V(\vec{r}) \):

\( V(\vec{r}) \) itself is specified (somewhere) on the boundary - *i.e.* the value of \( V(\vec{r}) \) is specified (somewhere) on the boundary.

**Neumann** Boundary Conditions on \( V(\vec{r}) \):

The normal derivative of \( V(\vec{r}) \) is specified somewhere on the boundary - *i.e.*

\[ \nabla V(\vec{r}) \cdot \hat{n} = -E_{\perp}(\vec{r}) \] is specified somewhere on the boundary.
**Uniqueness Theorem(s):**

Suppose we have two solutions of Laplace’s equation, \( V_1(\vec{r}) \) and \( V_2(\vec{r}) \), each satisfying the same boundary condition(s), \emph{i.e.} the potentials \( V_1(\vec{r}) \) and \( V_2(\vec{r}) \) are specified on the boundaries. We assert that the two solutions can at most differ by a constant. \emph{(n.b. Only differences in the scalar potential \( V(\vec{r}) \) are important / physically meaningful!)}

**Proof:** Consider a closed region of space with volume \( v \) which is exterior to \( n \) charged conducting surfaces \( S_1, S_2, S_3, \ldots S_n \) that are responsible for generating the potential \( V \).

The volume \( v \) is bounded (outside) by the surface \( S \).

Suppose we have \emph{two} solutions \( V_1(r) \) and \( V_2(r) \) both satisfying \( \nabla^2 V(\vec{r}) = 0 \) \emph{i.e.} \( \nabla^2 V_1(\vec{r}) = 0 \) and \( \nabla^2 V_2(\vec{r}) = 0 \) in the \emph{charge-free} region(s) of the volume \( v \).

\( V_1(\vec{r}) \) and \( V_2(\vec{r}) \) satisfy either Dirichlet boundary conditions or satisfy Neumann boundary conditions \( \nabla V(\vec{r}) \cdot \hat{n} \) on the surfaces \( S_1, S_2, S_3, \ldots S_n \). We also demand that \( V(r) \) be \emph{finite} at \( r = \infty \).

Let us define: \( V_\Delta(\vec{r}) \equiv V_1(\vec{r}) - V_2(\vec{r}) = \) difference in the two potential solutions at the point \( \vec{r} \).

Since both \( \nabla^2 V_1(\vec{r}) = 0 \) and \( \nabla^2 V_2(\vec{r}) = 0 \) then:

\[
\nabla^2 V_\Delta(\vec{r}) = \nabla^2 (V_1(\vec{r}) - V_2(\vec{r})) = \nabla^2 V_1(\vec{r}) - \nabla^2 V_2(\vec{r}) = 0
\]

\emph{Note that:} \( \nabla^2 V(\vec{r}) = \nabla \cdot (\nabla V(\vec{r})) \)

The potentials \( V_{i=1,2} \) are uniquely specified on charged (equipotential) surfaces \( S_1, S_2, S_3, \ldots S_n \) in the volume \( v \).

Now apply the \emph{divergence theorem} to the quantity \( (V_\Delta \nabla V_\Delta) \); we also define:\( \vec{E}_\Delta(\vec{r}) \equiv -\nabla V_\Delta(\vec{r}) \)

\[
\int_V \nabla \cdot (V_\Delta \nabla V_\Delta) d\tau = \int_{S_+} (V_\Delta \nabla V_\Delta) \cdot d\vec{A} = \int_{S_1+S_2+S_3+\ldots S_n} V_\Delta (\vec{E}_\Delta) \cdot d\vec{A}
\]

Volume integral over enclosing volume \( v \)  
Surface integral over ALL surfaces in \( v \)
Then: 
\[- \int_{S_1 + S_2 + S_3 + \ldots + S_n} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}) = -\int_{S} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_S) - \int_{S_1} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_1}) - \int_{S_2} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_2}) - \ldots - \int_{S_n} V_{\Delta} (\vec{E}_{\Delta} \cdot d\vec{A}_{S_n})\]

Recognizing that:

1. The conducting surfaces $S_1, S_2, S_3, \ldots S_n$, are equipotentials.
   Thus: $V_{\Delta}(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) (= \text{a constant on surfaces } S_1, S_2, S_3, \ldots S_n)$ must = 0 at/on those surfaces!!!

2. The volume $v$ is arbitrary, so let’s choose volume $v \to \infty$, and thus surface area $S \to \infty$ as well.

3. $\int_{S_i} \vec{E}_{\Delta} \cdot d\vec{A}_i = \Phi_{E_i}$ = electric flux through $i^{th}$ surface.

4. $V_{\Delta}(r \to \infty) = V_1(r \to \infty) - V_2(r \to \infty)$ (= constant on surface $S \to \infty$) must = 0
   because $V_1(r \to \infty) = V_2(r \to \infty)$.

\[
\therefore \int_{v} \nabla \cdot (V_{\Delta} \nabla V_{\Delta}) d\tau = \left[ -V_{\Delta}^{\infty} \right]_{=0}^{\infty} \int_{all \ space} \left[ \vec{E}_{\Delta} \cdot d\vec{A}_S - V_{\Delta}^{S_1} \right]_{=0}^{S_1} \int_{S_1} \left[ \vec{E}_{\Delta} \cdot d\vec{A}_{S_1} - V_{\Delta}^{S_2} \right]_{=0}^{S_2} \int_{S_2} \ldots \int_{S_n} \left[ \vec{E}_{\Delta} \cdot d\vec{A}_{S_n} \right]_{=0}^{S_n} - \ldots - \int_{S_2} \left[ \vec{E}_{\Delta} \cdot d\vec{A}_{S_2} \right]_{=0}^{S_2} - \int_{S_1} \left[ \vec{E}_{\Delta} \cdot d\vec{A}_{S_1} \right]_{=0}^{S_1} - \int_{all \ space} V_{\Delta}^{\infty} = 0
\]

Thus: 
\[
\int_{v} \nabla \cdot (V_{\Delta} \nabla V_{\Delta}) d\tau = 0
\]

However, using the identity $\nabla \cdot (V_{\Delta} \nabla V_{\Delta}) = V_{\Delta} \left( \nabla^2 V_{\Delta} \right) + (\nabla V_{\Delta})^2$

Then: 
\[
\int_{v} \nabla \cdot (\nabla^2 V_{\Delta}) d\tau = \int_{v} V_{\Delta} \left( \nabla^2 V_{\Delta} \right) d\tau + \int_{v} (\nabla V_{\Delta})^2 d\tau = 0
\]

\[
= \int_{all \ space \ mathematically} (\nabla V_{\Delta})^2 d\tau = \int_{all \ space} (\nabla V_{\Delta} \cdot \nabla V_{\Delta}) d\tau = 0
\]

The only way $\int_{v} (\nabla V_{\Delta})^2 d\tau = 0$ is iff (i.e. if and only if) the integrand $\left( \nabla V_{\Delta}(\vec{r}) \right)^2 = (\nabla V_{\Delta}(\vec{r}) \cdot \nabla V_{\Delta}(\vec{r})) = 0$.

If $\left( \nabla V_{\Delta}(\vec{r}) \right)^2 = (\nabla V_{\Delta}(\vec{r}) \cdot \nabla V_{\Delta}(\vec{r})) = 0$, then: $\nabla V_{\Delta}(\vec{r})$ itself must be = 0
(i.e. $\vec{A}(\vec{r}) \cdot \vec{A}(\vec{r}) = 0 \Rightarrow \vec{A}(\vec{r}) \equiv \vec{0}$) for all points $(\vec{r})$ in volume $v$.

If $\nabla V_{\Delta}(\vec{r}) = 0$ for all points $\vec{r}$ in volume $v$, then $V_{\Delta}(\vec{r})$ = (same) constant at all points in volume $v$. ∴ $V_{\Delta}(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) =$ constant at all points in volume $v$. 

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Dirichlet Boundary Conditions \((V\) specified on surfaces \(S_1, S_2, S_3, \ldots, S_n\))

If \(V_1(\vec{r})\) and \(V_2(\vec{r})\) are specified on the surfaces \(S_1, S_2, S_3, \ldots, S_n\) in the volume \(v\) enclosed by surface \(S\) (Dirichlet boundary conditions), then: \(V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = 0\) 
(i.e. the problem is over-determined).

\[
\therefore \quad V_\Delta(\vec{r}) = 0 \quad \text{throughout the volume } \quad v \quad \text{and } \quad V_1(\vec{r}) = V_2(\vec{r}) \quad \text{throughout the volume } \quad v.
\]

\(i.e.\) the two solutions \(V_1(\vec{r})\) and \(V_2(\vec{r})\) for \(\nabla^2 V(\vec{r}) = 0\) are *identical* – there is only *one* unique solution.

Neumann Boundary Conditions \((E^\perp\) specified on surfaces \(S_1, S_2, S_3, \ldots, S_n\))

If \(\vec{\nabla}V_1 \cdot \hat{n} = -\vec{E}^1\) and \(\vec{\nabla}V_2 \cdot \hat{n} = -\vec{E}^2\) are specified on the surfaces \(S_1, S_2, S_3, \ldots, S_n\) in the volume \(v\) enclosed by surface \(S\) (Neumann boundary conditions), then \(\vec{\nabla}V_\Delta(\vec{r}) = \vec{\nabla}V_1(\vec{r}) - \vec{\nabla}V_2(\vec{r}) = 0\) at all points in volume \(v\) and \(\vec{\nabla}V_\Delta \cdot \hat{n} = 0\).

Then \(V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = \text{constant, but is not necessarily } = 0!!!\)

Here, solutions \(V_1(\vec{r})\) and \(V_2(\vec{r})\) *can* differ, but *only* by a constant \(V_o\).

\(e.g.\) \(V_1(\vec{r}) = V_2(\vec{r}) + V_o \quad \Rightarrow \quad \text{problem is NOT over-determined for } \quad V(\vec{r}).\)

\((\vec{E}(\vec{r}) \quad \text{is over-determined / unique, but not } \quad V(\vec{r}).\)

Physical Example:

<table>
<thead>
<tr>
<th>The Parallel Plate Capacitor: (E = \Delta V/d = 100 \ V/m)</th>
<th>(+ 100 \ V)</th>
<th>(\downarrow)</th>
<th>(\hat{E})</th>
<th>0 (V)</th>
<th>(d = 1 \ m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Or: (E = \Delta V/d = 100 \ V/ m)</td>
<td>(+500 \ V)</td>
<td>(\downarrow)</td>
<td>(\hat{E})</td>
<td>(+400 \ V)</td>
<td>(d = 1 \ m)</td>
</tr>
</tbody>
</table>

\(\Delta V = 100 \ V\) in both cases – thus \(E\)-field is same/identical in both cases!
If we instead specify the charge density \( \rho(r) \) within the volume \( V \) (see figure below), then we also have a uniqueness theorem for the electric field associated with Poisson’s equation \( \nabla \times \vec{E}(\vec{r}) = - \nabla^2 V(\vec{r}) = \rho(r)/\varepsilon_o \).

![Charged Conducting Surfaces](image)

Suppose there are two electric fields \( \vec{E}_1(\vec{r}) \) and \( \vec{E}_2(\vec{r}) \), both satisfying all of the boundary conditions of this problem. Both obey Gauss’ law in differential and integral form everywhere within the volume \( V \):

\[
\nabla \times \vec{E}_1(\vec{r}) = \rho(r)/\varepsilon_o \quad \text{and} \quad \nabla \times \vec{E}_2(\vec{r}) = \rho(r)/\varepsilon_o
\]

At the outer boundary (enclosing surface \( S \)) we also have:

\[
\int_S \vec{E}_1 \cdot da = \frac{1}{\varepsilon_o} Q_{\text{net}}^{\text{encl}} \quad \text{and} \quad \int_S \vec{E}_2 \cdot da = \frac{1}{\varepsilon_o} Q_{\text{net}}^{\text{encl}}
\]

We define the difference in electric fields: \( \vec{E}_\Lambda(\vec{r}) \equiv \vec{E}_1(\vec{r}) - \vec{E}_2(\vec{r}) \) which, in the region between the conductors, obeys \( \nabla \times \vec{E}_\Lambda(\vec{r}) = \nabla \times \vec{E}_1(\vec{r}) - \nabla \times \vec{E}_2(\vec{r}) = \rho(r)/\varepsilon_o - \rho(r)/\varepsilon_o = 0 \), and obeys

\[
\int_{S_i} \vec{E}_\Lambda \cdot da = \int_{S_i} \vec{E}_1 \cdot da - \int_{S_i} \vec{E}_2 \cdot da = \frac{1}{\varepsilon_o} Q^{\text{encl}}_i - \frac{1}{\varepsilon_o} Q^{\text{encl}}_i = 0 \quad \text{over each boundary surface} \ S_i.
\]

Even though we do not know how the charge \( Q_i \) on the \( i \)th conducting surface \( S_i \) is distributed, we do know that each surface \( S_i \) is an equipotential, hence the scalar potential \( V_\Lambda \equiv V_i - V_2 \) on each surface is at least a constant on each surface \( S_i \) (n.b. \( V_\Lambda \) may not necessarily be = 0, since in general \( V_2 \) may not in general be equal to \( V_1 \) on each/every surface \( S_i \)).
Using Griffith’s product rule #5: \( \nabla \cdot (f \vec{A}) = f \left( \nabla \cdot \vec{A} \right) + \vec{A} \cdot \left( \nabla f \right) \), then:

\[
\nabla \cdot (V_\Delta \vec{E}_\Delta) = V_\Delta \left( \nabla \cdot \vec{E}_\Delta \right) + \vec{E}_\Delta \cdot \left( \nabla V_\Delta \right)
\]

However, in the region between conductors, we have shown (above) that \( \nabla \cdot \vec{E}_\Delta (\vec{r}) = 0 \), and \( \vec{E}_\Delta \equiv -\nabla V_\Delta \), hence: \( \nabla \cdot (V_\Delta \vec{E}_\Delta) = \vec{E}_\Delta \cdot \left( \nabla V_\Delta \right) = -\vec{E}_\Delta \cdot \vec{E}_\Delta = -E_\Delta^2 \).

If we integrate this relation over the entire volume \( v \) (with associated enclosing surface \( S \)):

\[
\int_v \nabla \cdot (V_\Delta \vec{E}_\Delta) \, d\tau = \oint_{all \, S} V_\Delta \vec{E}_\Delta \cdot d\vec{a} = -\int_v E_\Delta^2 d\tau
\]

Note that the surface integral covers all boundaries of the region in question – the enclosing outer surface \( S \) and all of the \( S_i \) inner surfaces associated with the \( i \) conductors. Since \( V_\Delta \) is a constant on each surface, it can be pulled outside of the surface integral (\( n.b. \) if the outer surface \( S \) is at infinity, then for localized sources of charge, \( V_\Delta (r = \infty) = 0 \)). Thus:

\[
V_\Delta \oint_{all \, S} \vec{E}_\Delta \cdot d\vec{a} = -\int_v E_\Delta^2 d\tau
\]

But since we have shown above that \( \int_{S_i} \vec{E}_\Delta \cdot d\vec{a} = 0 \) for each surface \( S_i \), then \( \oint_{all \, S} \vec{E}_\Delta \cdot d\vec{a} = 0 \).

Therefore: \( \int_v E_\Delta^2 d\tau = 0 \). Note that the integrand \( E_\Delta^2 (\vec{r}) = \vec{E}_\Delta (\vec{r}) \cdot \vec{E}_\Delta (\vec{r}) \) is always non-negative.

Hence, in general, the only way that this integral can vanish is if \( \vec{E}_\Delta (\vec{r}) \equiv \vec{E}_1 (\vec{r}) - \vec{E}_2 (\vec{r}) = 0 \) everywhere, thus, we must have \( \vec{E}_1 (\vec{r}) = \vec{E}_2 (\vec{r}) \).
Solving Laplace’s Equation \( \nabla^2 V(\bar{r}) = 0 \) in 3-D, 2-D and 1-D Situations

In general, when solving the potential \( V(\bar{r}) \) problems in 3 (or less) dimensions, first note the symmetries associated with the problem. Then, if you have:

\[
\begin{align*}
\text{Rectangular} & \quad \Rightarrow \quad \text{Solve} \quad \text{Problem} \quad \text{Using} \quad \text{Rectangular} \\
\text{Cylindrical} & \quad \Rightarrow \quad \text{Coordinates} \\
\text{Spherical} & \quad \Rightarrow \quad \text{Coordinates}
\end{align*}
\]

In 2-D and 3-D problems, the general solutions to \( \nabla^2 V(\bar{r}) = 0 \) are the harmonic functions (an \( \infty \)-series solution, in principle) e.g. of sines and cosines, Bessel functions, or Legendre Polynomials and/or Spherical Harmonics.

The boundary conditions / symmetries will select a subset of the \( \infty \)-solutions.

We will now work through derivations of finding solutions to Laplace’s Equations in 3-dimensions in rectangular (i.e. Cartesian) coordinates, cylindrical coordinates, and spherical coordinates. We will also use / show the method of separation of variables.

Laplace’s Equation \( \nabla^2 V(x,y,z) = 0 \) and Potential Problems with Rectangular Symmetry

(Rectangular / Cartesian coordinates)

In Three Dimensions: Solve Laplace’s equation in rectangular / Cartesian coordinates:

\[
\nabla^2 V(x,y,z) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0
\]

The solutions of \( \nabla^2 V = 0 \) in rectangular coordinates are known as harmonic functions (i.e. sines and cosines) \( \rightarrow \) Fourier Series Solutions.

It is usually (but not always) possible to find a solution to the Laplace Equation, \( \nabla^2 V = 0 \) which also satisfies the boundary conditions, via separation of variables technique, i.e. try a product solution of the form:

\[
V(x,y,z) = X(x)Y(y)Z(z)
\]

where:

\[
\begin{align*}
\{ X(x) \} & \quad \text{are functions only of} \quad \{ x \} \\
\{ Y(y) \} & \quad \text{respectively.} \quad \{ y \} \\
\{ Z(z) \} & \quad \text{respectively.} \quad \{ z \}
\end{align*}
\]

Then:

\[
\nabla^2 V(x,y,z) = 0 \Rightarrow \frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0
\]

But:

\[
V(x,y,z) = X(x)Y(y)Z(z)
\]
Thus:
\[
\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0
\]
\[
= Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0
\]

Now divide both sides of the above equation by \(X(x)Y(y)Z(z)\):

Then:
\[
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = 0
\]

But:
\[
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = 0 \quad \text{i.e.} \quad C_1 + C_2 + C_3 = 0
\]

True for all points \((x, y, z)\) in volume \(v\) of problem.

The only way the above equation can be true for all points \((x, y, z)\) in volume \(v\) is if:

\[
\begin{align*}
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} &= \text{constant } C_1 \Rightarrow \frac{d^2 X(x)}{dx^2} - C_1 X(x) = 0 \quad \#1 \\
\frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} &= \text{constant } C_2 \Rightarrow \frac{d^2 Y(y)}{dy^2} - C_2 Y(y) = 0 \quad \#2 \\
\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} &= \text{constant } C_3 \Rightarrow \frac{d^2 Z(z)}{dz^2} - C_3 Z(z) = 0 \quad \#3
\end{align*}
\]

Subject to the constraint: \(C_1 + C_2 + C_3 = 0\)

Can now solve 3 ORDINARY 1-D differential equations, \#1–3, which are subject to \(C_1 + C_2 + C_3 = 0\), PLUS the specific Dirichlet / Neumann boundary conditions for the problem on either \(V(x, y, z)\) or \(\vec{V}(x, y, z)\) at surfaces for this 3-D problem.

Essentially, we have replaced the 3-D problem with three 1-D problems, and the constraint: \(C_1 + C_2 + C_3 = 0\).
* If one has a 2-D rectangular coordinate problem \( \nabla^2 V(x, y) = 0 \), then: \( V(x, y) = X(x)Y(y) \) (only).

\[
\begin{align*}
\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} &= C_1 \Rightarrow \frac{d^2 X(x)}{dx^2} - C_1 X(x) = 0 \\
\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} &= C_2 \Rightarrow \frac{d^2 Y(y)}{dy^2} - C_2 Y(y) = 0
\end{align*}
\]

Subject to the constraint: \( C_1 + C_2 = 0 \), i.e. \( C_1 = -C_2 \).

Plus BC’s: either on \( V(x, y) \) or \( \nabla V(x, y) \cdot \hat{n} \) for the 2-D problem.

* If one has a 1-D rectangular coordinate problem \( \nabla^2 V(x) = 0 \), then: \( V(x) = X(x) \) (only).

\[
\frac{d^2 Y(x)}{dx^2} = 0 \Rightarrow \frac{d^2 X(x)}{dx^2} = 0 \Rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = 0 = C_1
\]

\[
\frac{d^2 X(x)}{dx^2} = 0 \Rightarrow X(x) = V(x) = ax + b \text{ is the 1-D general solution.}
\]

For 1-D problem \( \nabla^2 V(x) = 0 \), only need to solve one ordinary differential equation subject to the constraint \( C_1 = 0 \) and BC’s on either \( V(x) \) or \( \frac{dV(x)}{dx} \).
The General Solution $V(x, y, z) = X(x)Y(y)Z(z)$ for $\nabla^2 V(x, y, z) = 0$

in Rectangular Coordinates

Since we have the constraint $C_1 + C_2 + C_3 = 0$, at least one of the $C_i$'s ($i = 1, 2$ or $3$) must be less than zero.

Let us “choose” $C_1 = -\alpha^2$, $C_2 = -\beta^2$, $C_3 = \gamma^2$

Then:

$C_1 + C_2 + C_3 = 0$

$-\alpha^2 - \beta^2 + \gamma^2 = 0$ \quad or: \quad $\alpha^2 + \beta^2 = \gamma^2$

The boundary conditions on the surfaces will define $\alpha$ and $\beta$, and hence define $\gamma$.

IMPORTANT NOTE:

The geometry ($x – y – z$) of the problem and the boundary conditions dictate whether:

$C_1 > 0$ or $C_1 < 0$

$C_2 > 0$ or $C_2 < 0$

$C_3 > 0$ or $C_3 < 0$

i.e. have sine / cosine type solutions vs. sinh / cosh (or $e^x$, $e^{-x}$) type solutions for $x, y, z$.

Then the General Solution is (for above choice of $C_1 = -\alpha^2$, $C_2 = -\beta^2$, $C_3 = \gamma^2$):

$V(x, y, z) = \sum_{m,n=0}^{\infty} A_{mn} \cos(\alpha_n x) \cos(\beta_n y) \sinh(\gamma_n z) + \sum_{m,n=0}^{\infty} B_{mn} \sin(\alpha_n x) \cos(\beta_n y) \sinh(\gamma_n z) + \sum_{m,n=0}^{\infty} C_{mn} \sin(\alpha_n x) \sin(\beta_n y) \cosh(\gamma_n z) + \sum_{m,n=0}^{\infty} D_{mn} \cos(\alpha_n x) \sin(\beta_n y) \cosh(\gamma_n z)$

so we also have the additional series solutions:

$n.b.$ $\cosh(x) = \frac{1}{2} (e^x + e^{-x})$ \quad $\sinh(x) = \frac{1}{2} (e^x - e^{-x})$

$n.b.$ $\sin(x) = \frac{1}{2i} (e^{ix} - e^{-ix})$ \quad $\cos(x) = \frac{1}{2} (e^{ix} + e^{-ix})$ \quad $i \equiv \sqrt{-1}$

$n.b.$ $e^{ix} = \cos(x) + i \sin(x)$ \quad $e^{-ix} = \cos(x) - i \sin(x)$

$n.b.$ $e^x = \cosh(x) + \sinh(x)$ \quad $e^{-x} = \cosh(x) - \sinh(x)$

The BC’s and symmetries will determine which of the coefficients $A_{mn}$, $B_{mn}$, $C_{mn}$, $D_{mn} = 0$. 
We solve for the non-zero coefficients \( A_{pq}, B_{pq}, C_{pq} \) and \( D_{pq} \) by taking inner products. i.e. we multiply \( V(x,y,z) = \sum (\text{stuff}) \) by e.g. \( \sin(\alpha_p x) \sin(\beta_q y) \) to project out the \( p-q \)th component (i.e. we use the orthogonality properties of the individual terms in \( \sin(\ ) \) and \( \cos(\ ) \) Fourier Series.) and then integrate over the relevant intervals in \( x \) and \( y \): 

\[
\int_0^{x_v} \int_0^{y_v} V(x,y) \sin(\alpha_p x) \sin(\beta_q y) dx dy = \int_0^{x_v} \int_0^{y_v} \left( \sum_{m,n=0}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{mn} z) * \sin(\alpha_p x) \sin(\beta_q y) \right) + \left( \sum_{m,n=0}^{\infty} B_{mn} \cos(\alpha_n x) \cos(\beta_m y) \sinh(\gamma_{mn} z) * \sin(\alpha_p x) \sin(\beta_q y) \right) + \left( \sum_{m,n=0}^{\infty} C_{mn} \sin(\alpha_n x) \sin(\beta_m y) \cosh(\gamma_{mn} z) * \sin(\alpha_p x) \sin(\beta_q y) \right) + \left( \sum_{m,n=0}^{\infty} D_{mn} \cos(\alpha_n x) \cos(\beta_m y) \cosh(\gamma_{mn} z) * \sin(\alpha_p x) \sin(\beta_q y) \right) dx dy
\]

Fourier Functions: orthonormality properties of \( \sin(\ ) \) and \( \cos(\ ) \):

\[
\int_0^{x_v} \sin(\alpha_n x) \sin(\alpha_p x) dx = \begin{cases} \delta_{np} & \text{for } n \neq p \\ \text{some constant} & \text{for } n = p \end{cases} \\
\int_0^{x_v} \cos(\alpha_n x) \sin(\alpha_p x) dx = 0
\]

Kronecker \( \delta \) -function: \( \delta_{np} = \begin{cases} -1 & \text{for } n \neq p \\ 0 & \text{for } n = p \end{cases} \)

So all terms in above \( \Sigma \)'s vanish, except for a single term (in each sum) – that for the \( A_{pq} / B_{pq} / C_{pq} / D_{pq} \) coefficient!!! The BC's will e.g. kill off 3 out of remaining 4 non-zero terms, thus only one term survives…

Suppose only the \( A_{pq} \) coefficient survives. Its analytic form is now known for all integers \( p \) and \( q \).

Then the analytic form of 3-D potential \( V(x,y,z) \) is now known – it is an infinite series solution of the form:

\[
V(x,y,z) = \sum_{m,n=0}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{mn} z) = \sum_{m,n=0}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \frac{\sinh(\gamma_{mn} z)}{\sqrt{\alpha_n^2 + \beta_m^2}}
\]
Laplace’s Equation \( \nabla^2 V(\rho, \varphi, z) = 0 \)

And Potential Problems with Cylindrical Symmetry (Cylindrical Coordinates)

\[
\begin{align*}
\hat{\mathbf{r}} &= \hat{\rho} + \hat{z} = \rho\hat{\rho} + z\hat{\mathbf{z}} \quad r = \sqrt{\rho^2 + z^2} \\
\end{align*}
\]

Again, we use the separation of variables technique:

\[
V(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z) \quad \Rightarrow \quad \nabla^2 V = 0 \quad \Rightarrow \quad \text{yields 3 ordinary differential equations:}
\]

\[
\begin{aligned}
\frac{d^2Z(z)}{dz^2} - k^2Z(z) &= 0 \quad \Rightarrow \quad Z(z) = e^{\pm kz} \\
\frac{d^2Q(\varphi)}{d\varphi^2} + \nu^2Q(\varphi) &= 0 \quad \Rightarrow \quad Q(\varphi) = e^{\pm i\nu \varphi} \\
\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{dR(\rho)}{d\rho} + \left( k^2 - \frac{\nu^2}{\rho^2} \right)R(\rho) &= 0
\end{aligned}
\]

Note(s):  
1.) \( k \) is arbitrary without imposing boundary conditions.  
2.) \( k \) appears in both \( Z(z) \) and \( R(\rho) \) equations.  
3.) In order for \( Q(\varphi) \) to be single-valued (i.e. \( Q(\varphi) = Q(\varphi + 2\pi) \)), \( \nu \) must be an integer!

Let \( x \equiv kp \)  
Then: \( \frac{d^2R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} + \left( 1 - \frac{\nu^2}{x^2} \right)R(x) = 0 \quad \Leftarrow \quad \text{Bessel’s Equation}  

\[
R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j \quad \Leftarrow \quad \text{Power Series Solution} \quad \alpha = \pm \nu
\]

\[
a_{2j} = -\frac{1}{4j(j+\alpha)} a_{2j-2} \quad \text{for } j = 0, 1, 2, 3, \ldots
\]

All odd powers of \( x_j \) have vanishing coefficients, i.e. \( a_1 = a_3 = a_5 = a_{2j+1} = 0 \)
Coefficients $a_{2j}$ expressed in terms of $a_0$:

$$ a_{2j} = \left[ \frac{(-1)^j \Gamma(\alpha + 1)}{2^{2j} j! \Gamma(j + \alpha + 1)} \right] a_0 = \frac{(-1)^j}{2^{3j+\alpha} j! \Gamma(j + \alpha + 1)} $$

where

$$ a_0 = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \quad \Gamma(x) = \text{Gamma Function} $$

There exist TWO solutions of the Radial Equation (i.e. Bessel’s Equation):

They are:

Bessel Functions of 1st kind, of order $\pm \nu$:

$$ J_{+\nu}(x) = \left( \frac{x}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left( \frac{x}{2} \right)^{2j} $$

These series converge for all values of $x$.

$$ J_{-\nu}(x) = \left( \frac{x}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j - \nu + 1)} \left( \frac{x}{2} \right)^{2j} $$

If $\nu$ is not an integer (which is not the case here), then the $J_{\pm\nu}(x)$ form a pair of linearly independent solutions to the 2nd order Bessel’s Equation:

$$ R(x) = A_x J_{\nu}(x) + A_{-\nu} J_{-\nu}(x) \quad \text{for } \nu \neq \text{integer} $$

However, note that if $\nu = \text{integer}$ (which is the case for us here) then the Bessel functions $J_{\nu}(x)$ and $J_{-\nu}(x)$ are NOT linearly independent!!

If $\nu = m = \text{integer}$ (0, 1, 2, 3, …), then $J_{-m}(x) = (-1)^m J_m(x)$

∴ We must find another linearly independent solution for $R(x)$ when $\nu = m = \text{integer}$

It is “customary” to replace $J_{\pm\nu}(x)$ by just $J_{\nu}(x)$ and another function $N_\nu(x)$ (called Neumann Functions)

Where:

$$ N_\nu(x) = \text{Bessel Function of 2nd kind} = \frac{J_{\nu}(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)} $$

NOTE: $N_\nu(x)$ is divergent (i.e singular) at $x \rightarrow 0$

Complex Bessel Functions = Bessel Functions of 3rd kind = Hankel Functions

Hankel Functions are complex linear combinations of $J_{\nu}(x)$ and $N_{\nu}(x)$ (Bessel Functions of 1st and 2nd kind respectively). They are defined as follows:

$$ H_{\nu}^{(1)}(x) \equiv J_{\nu}(x) + iN_{\nu}(x) \quad \text{The Hankel Functions } H_{\nu}^{(1)}(x) \text{ and } H_{\nu}^{(2)}(x) \text{ also form a fundamental set/basis of solutions to the Bessel equation.} $$
The General Solution for $\nabla^2 V(\rho, \varphi, z) = 0$ in Cylindrical Coordinates:

$$V(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z)$$

$cosh(k_{mn}z)$ is also allowed

$$V(\rho, \varphi, z) = \sum_{m,n=0}^{\infty} J_m(k_{mn}\rho) sinh(k_{mn}z) \left[ A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi) \right]$$
$cosh(k_{mn}z)$ is also allowed

$$+ \sum_{m,n=0}^{\infty} N_m(k_{mn}\rho) sinh(k_{mn}z) \left[ C_{mn} \sin(m\varphi) + D_{mn} \cos(m\varphi) \right]$$

Apply ALL boundary conditions on surfaces (and also impose for $r = \infty$, that $V(r = \infty) = \text{finite!}$) {If $r = \infty$ is part of the problem!}

Note that sometimes we want $V(\tilde{r})$ only inside some finite region of space, e.g. coaxial capacitor – if so, then don’t have to worry about $r = \infty$ solutions being finite – an example – the Coaxial Capacitor:

*End View of a Coaxial Capacitor*

If the $\tilde{r} = 0$ region is an excluded region in the problem, then must include (i.e. allow) the $N_r(x)$ solutions (singular at $x = k\rho = 0$)!!!

If $\tilde{r} = 0$ region is included in problem, then ALL coefficients $C_{mn} = D_{mn} \equiv 0$ (for all $m, n$), if $V(\rho, \varphi, z)$ is finite @ $\tilde{r} = 0$.

Using/imposing BC’s on surfaces, orthogonality conditions on sines, cosines, $J_r(x)$, $N_r(x)$, etc. can find / determine values for all $A_{mn}, B_{mn}, C_{mn}, D_{mn}$ coefficients!!
### 2-Dimensional Circular Symmetry

#### Laplace’s Equation in (Circular) Cylindrical Coordinates

\[ \nabla^2 V(\rho, \varphi) = 0 \]

Potential, \( V(\rho, \varphi) \) is independent of \( z \)
(e.g. infinitely long coaxial cable)

\[ \nabla^2 V(\rho, \varphi) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \varphi^2} = 0 \]

Get:

\[ \frac{\rho}{R(\rho)} \frac{d}{d\rho} \left( \rho \frac{dR(\rho)}{d\rho} \right) = C_1 = -\frac{1}{Q(\varphi)} \frac{d^2 Q(\varphi)}{d\varphi^2} \]

Let \( C_1 = k^2 \)

Then:

\[ \rho \frac{d}{d\rho} \left( \rho \frac{dR(\rho)}{d\rho} \right) - k^2 R(\rho) = 0 \quad \text{and} \quad \frac{d^2 Q(\varphi)}{d\varphi^2} + k^2 Q(\varphi) = 0 \]

Require all solutions \( Q(\varphi) \) to be single-valued, i.e. \( Q(\varphi) = Q(\varphi + 2\pi) \)

because must have \( V(\varphi) = V(\varphi + 2\pi) \).

Solutions for \( Q(\varphi) \) are of the form:

\[ Q(\varphi) = A \cos k\varphi + B \sin k\varphi \]

\[ Q(\varphi) = Q(\varphi + 2k\pi) \]

requires \( k = \text{integer} = 0, \pm 1, \pm 2, \pm 3, \ldots \pm n \ldots \)

\[ \frac{d^2 Q(\varphi)}{d\varphi^2} + n^2 Q(\varphi) = 0 \quad \Rightarrow \quad Q_n(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi) \]

singular @ \( \rho \to \infty \)

singular @ \( \rho = 0 \)

\[ \rho \frac{d}{d\rho} \left( \rho \frac{dR(\rho)}{d\rho} \right) - n^2 R(\rho) = 0 \Rightarrow R_n(\rho) = C_n \rho^n + D_n \rho^{-n} \quad \text{for } n \geq 1 \quad (i.e. n = 1, 2, 3, \ldots) \]

\[ R_o(\rho) = C_o + D_o \ln(\rho) \quad \text{for } n = 0 \quad \text{only} \]
**General Solution for** $\nabla^2 V(\rho, \phi) = 0$ **in Two Dimensions:**

**Cylindrical (a.k.a. Zonal) Harmonics**

$$V(\rho, \phi) = V_0 + V_1 \ln(\rho) + \sum_{n=1}^{\infty} \left[ a_n \rho^n \cos(n\phi) + b_n \rho^{-n} \cos(n\phi) + c_n \rho^n \sin(n\phi) + d_n \rho^{-n} \sin(n\phi) \right]$$

Again, apply BC’s on all relevant surfaces, impose $V(r \to \infty) = \text{finite}$, etc. – these will dictate / determine all coefficients, $V_0$, $V_1$, $a_n$, $b_n$, $c_n$ and $d_n$.

i.e. Solve for $V_0$, $V_1$, $a_n$, $b_n$, $c_n$ and $d_n$ by applying all boundary conditions, $V(r \to \infty) = \text{finite}$, and using orthogonality conditions / properties:

$$a_n = "A" \int_0^{2\pi} d\phi \int_0^{\rho_o} d\rho \rho V(\rho, \phi) \rho^n \cos(n\phi) \quad dA = \rho d\rho d\phi$$

$$b_n = "B" \int_0^{2\pi} d\phi \int_0^{\rho_o} d\rho \rho V(\rho, \phi) \rho^{-n} \cos(n\phi)$$

$$c_n = "C" \int_0^{2\pi} d\phi \int_0^{\rho_o} d\rho \rho V(\rho, \phi) \rho^n \sin(n\phi)$$

$$d_n = "D" \int_0^{2\pi} d\phi \int_0^{\rho_o} d\rho \rho V(\rho, \phi) \rho^{-n} \sin(n\phi)$$

“A”, “B”, “C”, “D” are appropriate normalization factors (we will discuss later).

**Laplace’s Equation** $\nabla^2 V(r, \theta, \phi) = 0$ **In Spherical Coordinates**

$$\nabla^2 V(r, \theta, \phi) = 0$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$
Again, try separation of variables / try product solution:

\[ V(r, \theta, \phi) = \frac{U(r)}{r} P(\theta) Q(\phi) \leftrightarrow \text{of this form!!} \]

\[
P(\theta)Q(\phi) \frac{d^2U(r)}{dr^2} + \frac{U(r)Q(\phi)}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{U(r)P(\theta)}{r^2 \sin^2 \theta} \frac{d^2Q(\phi)}{d\phi^2} = 0
\]

Multiply by \( r^2 \sin^2 \theta / U(r) P(\theta) Q(\phi) \):

\[
r^2 \sin^2 \theta \left[ \frac{1}{U(r)} \frac{d^2U(r)}{dr^2} + \frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) \right] + \frac{1}{Q(\phi)} \frac{d^2Q(\phi)}{d\phi^2} = 0
\]

Now:

\[
\frac{1}{Q(\phi)} \frac{d^2Q(\phi)}{d\phi^2} = -m^2 \Rightarrow \frac{d^2Q(\phi)}{d\phi^2} + m^2 Q(\phi) = 0
\]

Solutions are of the form: \( Q(\phi) = e^{\pm im\phi} \) where \( m \) = integer = 0, 1, 2, 3, …

Since \( V(r, \theta, \phi) = V(r, \theta, \phi \pm 2\pi) \) i.e. \( Q(\phi) = Q(\phi \pm 2\pi) \)

Then: \( Q(\phi) \) must be single-valued!

Thus:

\[
r^2 \sin^2 \theta \left[ \frac{1}{U(r)} \frac{d^2U(r)}{dr^2} + \frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) \right] = +m^2
\]

\[
\frac{1}{U(r)} \frac{d^2U(r)}{dr^2} = - \frac{1}{r^2 \sin \theta} \frac{1}{P(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \frac{m^2}{r^2 \sin^2 \theta}
\]

multiply above equation by \( r^2 \):

\[
\frac{d^2U(r)}{dr^2} - \frac{\alpha}{r} U(r) = 0 \quad \text{and} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[ \frac{\alpha}{\sin^2 \theta} - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0
\]

\[\therefore \frac{d^2U(r)}{dr^2} - \frac{\alpha}{r} U(r) = 0 \quad \text{and} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[ \frac{\alpha - \frac{m^2}{\sin^2 \theta}}{\sin^2 \theta} \right] P(\theta) = 0\]

let / define \( \alpha \equiv \ell(\ell + 1) \) where \( \ell = \text{integer} = 0, 1, 2, 3, \ldots \)

(Trust me, \( \therefore \) I know the answer . . .)

\[\therefore \frac{d^2U(r)}{dr^2} - \frac{\ell(\ell + 1)}{r^2} U(r) = 0 \quad \text{and} \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[ \frac{\ell(\ell + 1)}{\sin^2 \theta} - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0\]
Now let \( x = \cos \theta \)  
\[ x^2 = \cos^2 \theta = 1 - \sin^2 \theta \]  
\[ \therefore \sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2 \]  
\[ \sin \theta = \sqrt{1 - x^2} \]

Then:  
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \frac{\sin^2 \theta}{\sin \theta} \frac{dP(\theta)}{d\theta} \right) + \left[ \ell (\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0 \]

Becomes:  
\[ \frac{d}{dx} \left( (1-x^2) \frac{dP(x)}{dx} \right) + \left[ \ell (\ell + 1) - \frac{m^2}{(1-x^2)} \right] P(x) = 0 \quad \Leftarrow \quad \text{Generalized Legendre' Equation} \]

General Solutions of the radial equation,  
\[ \frac{d^2U(r)}{dr^2} - \frac{\ell (\ell + 1)}{r^2} U(r) = 0 \]  
are of the form:  
\[ U(r) = Ar^\ell + Br^{- (\ell + 1)} \]  
\((l + A + B)\) are determined by boundary conditions...

For \( m = 0 \) (azimuthally-symmetric problems – no \( \phi \)-dependence) the general solution for azimuthally-symmetric potential \( V(r, \theta) \) is of the form:  
\[ V(r, \theta) = \sum_{\ell = 0}^{\infty} \left[ A_\ell r^\ell + B_\ell r^{- (\ell + 1)} \right] P_\ell (\cos \theta) \]

The coefficients \( A_\ell \) and \( B_\ell \) are determined by the boundary conditions

\textit{n.b.} If \( \exists \) no charges at \( r = 0 \), then \( B_\ell = 0 \ \forall \ \ell \) !!

Rodrigues’ Formula is useful for “ordinary” Legendre' Polynomials:  
\[ P_\ell (x) = \frac{1}{2^{\ell} \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \]

The coefficients \( A_\ell \) and \( B_\ell \) can be found / determined by evaluating \( V(r, \theta) \) on the conducting surfaces in the problem, \textit{e.g.} suppose we want to determine \( V(\bar{r}) \) inside a conducting sphere of radius \( r = a \). Then on the surface of the conducting sphere at radius \( r = a \) (an equipotential!!):  
\[ V(r = a, \theta) = \sum A_\ell a^\ell P_\ell (\cos \theta) = \text{constant} \quad \Leftarrow \quad \text{Legendre' Series} \]

\textit{n.b.} inside conducting sphere, \textit{e.g.} there are no charges at \( r = 0 \)  
\[ \therefore \quad B_\ell = 0 \ \forall \ \ell \]

In order to determine coefficients, take \textit{inner product}:  
\[ A_\ell = \left( \frac{2\ell + 1}{2a^\ell} \right) \int_0^\pi V(r = a, \theta) P_\ell (\cos \theta) \sin \theta d\theta \]
Orthogonality condition on $P_{\ell}(x)$'s:

$$\int_{-1}^{1} P_{\ell}(x) P_{\ell'}(x) \, dx = \frac{2}{(2\ell+1) \delta_{\ell\ell'}} \delta_{\ell\ell'} \begin{cases} 
\delta_{\ell\ell'} = 0 & \text{for } \ell' \neq \ell \\
\delta_{\ell\ell'} = 1 & \text{for } \ell' = \ell 
\end{cases}$$

_n.b._ The $P_{\ell}(\cos \theta)$ functions form a complete orthonormal basis set on the unit circle ($r = 1$) for $-1 \leq \cos \theta \leq 1$ or $0 \leq \theta \leq \pi$

“Ordinary” Legendre’ Polynomials $P_{\ell}(x)$ ($x = \cos \theta$) defined on the interval $-1 \leq x \leq 1$:

$$P_{0}(x) = 1$$
$$P_{1}(x) = x$$
$$P_{2}(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_{3}(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_{4}(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_{5}(x) = \frac{1}{8}(63x^5 + 70x^3 + 15x)$$

... 

Note: All $P_{\ell=even}(x)$ functions are _even_ functions of $x$: $P_{\ell=even}(-x) = +P_{\ell=even}(x)$

All $P_{\ell=odd}(x)$ functions are _odd_ functions of $x$: $P_{\ell=odd}(-x) = -P_{\ell=odd}(x)$

under $x \to -x$ reflection. Generally speaking, $P_{\ell}(-x) = (-1)^\ell P_{\ell}(x)$.

If 3-D spherical coordinate problem DOES have azimuthal / $\varphi$-dependence, then $m^2 \neq 0$

in Associated Legendre’ Equation (A.L.E.):

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP(\theta)}{d\theta} \right) + \left[ \ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] P(\theta) = 0$$

$x = \cos \theta$

Solutions to A.L.E. are Associated Legendre' Polynomials (A.L.P.’s)

Associated Legendre’ Polynomials: $P_{\ell}^m(x) \equiv (-1)^m \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_{\ell}(x)$

$m = \pm \text{integer} \neq 0$

_i.e._ $m = \pm 1, \pm 2, \pm 3, \ldots$ but have a constraint on $m$ !!!

\[ -\ell \leq m \leq +\ell \]

_i.e._ $m = -\ell, -\ell + 1, -\ell + 2, \ldots -2, -1, 0, +1, +2, \ell - 2, \ell - 1, \ell$
Also: \[ P_{\ell}^m(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_{\ell}^m(x) \]

Orthogonality condition for \( P_{\ell}^m(x) \) for fixed \( m \):

\[
\int_{-1}^{1} P_{\ell}^m(x) P_{\ell}^m(x) \, dx = \frac{2}{(2\ell + 1)(\ell - m)!} \delta_{\ell\ell}'^m.
\]

We now define normalized \( P(\theta)Q(\phi) \) functions known as Spherical Harmonics:

\[
Y_{\ell m}(\theta, \phi) \equiv \sqrt{\frac{(2\ell + 1)\ell!}{4\pi(\ell + m)!}} P_{\ell}^m(\cos \theta) e^{i m \phi} = \Omega_{\ell m}(\phi)
\]

The Spherical Harmonics \( Y_{\ell m}(\theta, \phi) \) form a complete orthonormal set of basis “vectors” on the surface of the unit sphere \( (r = 1) \)

Note that \( Y_{\ell - m}(\theta, \phi) = (-1)^m Y_{\ell m}^{*}(\theta, \phi) \) complex conjugate

\[ i.e. \ i \rightarrow -i \ \text{where} \ i \equiv \sqrt{-1} \]

\( Y_{\ell m}(\theta, \phi) \) Normalization and Orthogonality Condition:

\[
\int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta Y_{\ell m}^{*}(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell\ell}' \delta_{m'm}^m
\]

\[ i.e. \ \int_{\Omega=0}^{\Omega=4\pi} d\Omega Y_{\ell m}^{*}(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell\ell}' \delta_{m'm}^m \]

\[ d\Omega = \sin \theta d\theta d\phi \]

Completeness’ Relation:

\[
\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')
\]

\[ \text{DIRAC } \delta \text{-functions} \]
\[ Y_{\ell m}(\theta, \varphi) \text{ Spherical Harmonics} \]

\[ \ell = 0 \quad \left\{ \begin{array}{l}
Y_{00} = \frac{1}{\sqrt{4\pi}} \\
\text{Use } Y_{-m}(\theta, \varphi) = (-1)^m Y_{m}^*(\theta, \varphi) \\
in \text{order to obtain } Y_{-2}, Y_{-1}, Y_{1}, Y_{2}, Y_{3}, \text{etc.}
\end{array} \right. \]

\[ \ell = 1 \quad \left\{ \begin{array}{l}
Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \\
Y_{10} = -\sqrt{\frac{3}{4\pi}} \cos \theta \\
\text{Note:} \\
Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{(2\ell+1)(\ell - m)!}{4\pi(\ell + m)!}} P_{\ell}^m(\cos \theta) e^{im\varphi} \\
Y_{00}(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}} P_{\ell}(\cos \theta)
\end{array} \right. \]

\[ \ell = 2 \quad \left\{ \begin{array}{l}
Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\varphi} \\
Y_{21} = -\sqrt{\frac{5}{8\pi}} \sin \theta \cos \theta e^{i\varphi} \\
Y_{20} = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)
\end{array} \right. \]

\[ \ell = 3 \quad \left\{ \begin{array}{l}
Y_{33} = -\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{3i\varphi} \\
Y_{32} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\varphi} \\
Y_{31} = -\sqrt{\frac{1}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\varphi} \\
Y_{30} = \sqrt{\frac{7}{4\pi}} \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right)
\end{array} \right. \]

\[ \ldots \ldots \text{etc.} \]
General Solution for Laplace’s Equation \( \nabla^2 V(r, \theta, \varphi) \) in Spherical Polar Coordinates

\[
V(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \varphi)
\]

Coefficients \( A_{lm} \) and \( B_{lm} \) are determined by / from Boundary Conditions on spherical surface(s)

If \( V = V(\theta, \varphi) \) on surface (e.g. at \( r = a \))

\( i.e. \) no charge at \( r = 0 \) in problem \( \rightarrow B_{lm} = 0 \ \forall_{l,m} \)

Then: \( V(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \varphi) \) on surface \( (r = a). \)

And: \( A_{lm} = \int_{0}^{\frac{4\pi}{\text{north pole}}} d\Omega Y_{lm}^*(\theta, \varphi) V(\theta, \varphi) \) on surface \( (r = a). \)

Note: \( \frac{V(\theta = 0, \varphi)}{V(\theta = \frac{\pi}{2}, \varphi)} = \sum_{l=0}^{\infty} \sqrt{\frac{(2l+1)}{4\pi}} A_{l0} \)

\[
A_{l0} = \sqrt{\frac{(2l+1)}{4\pi}} \int_{0}^{\frac{4\pi}{\text{north pole}}} d\Omega \cos(\theta) V(\theta, \varphi)
\]

General Comments: The method of separation of variables used in Laplace’s equation \( \nabla^2 V = 0 \) in rectangular, cylindrical and spherical coordinates shows up again in Poisson’s Equation

\[
\nabla^2 V = \frac{-\rho_{\text{free}}}{\varepsilon_0}
\]

and also in the wave equation (valid for all classical wave phenomena)

\[
\nabla^2 \psi(\vec{r}, t) + \frac{1}{c^2} \frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = 0 \]

and in Schrödinger’s wave equation \( H\psi = E\psi \) in Quantum Mechanics problems. These equations will appear again and again, in one form or another for \( E&M \), Classical Mechanics, Quantum Mechanics courses as well as for Classical / Newtonian Gravity problems…

For more detailed information \( e.g. \) on separation of variables and solutions to 3-D Wave Equation \( \nabla^2 \psi = -\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \) in rectangular, cylindrical and spherical coordinates see Prof. S. Errede lecture notes (Lecture IV – parts 1 & 2) on (sound) waves in 1-D, 2-D, 3D Physics 406 Acoustical Physics of Music website:

http://online.physics.uiuc.edu/courses/phys406/406_lectures.html

and also see/read his Fourier Analysis Lectures on this website, if interested.