LECTURE NOTES 21

Gauge Invariance of the *EM* Interaction and Electric Charge/Current Conservation

In P436 Lecture Notes # 19, pages 14-18, we saw that we could write the anti-symmetric rank-2 *EM* field tensor $F^{\mu\nu}$ in terms of covariant space-time derivatives of the 4-vector *EM* potential field A^{μ} as:

Row #

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \quad \text{where:} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad \text{and:} \quad A^{\mu} = \left(V/c, \vec{A}\right)$$

Column #

Which correctly gave the familiar relations:
$$\underset{(i.e.\ \text{LHS})}{F^{\mu\nu}} = \left\{ \begin{aligned} \vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned} \right\} = \underset{(i.e.\ \text{RHS})}{\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}}$$

We also saw that $F^{\mu\nu} = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \right)$ automatically satisfied the ***homogeneous*** Maxwell equation $\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$ or equivalently: $\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} = 0$, both of which contain the ***homogeneous*** Maxwell equations $\vec{\nabla} \cdot \vec{B} = 0$ and: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

$G^{\mu\nu}$ is the ***dual*** tensor to $F^{\mu\nu}$: $G^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$ and $\varepsilon^{\mu\nu\lambda\sigma}$ = totally anti-symmetric rank-4 tensor.

Row #

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Column #

We also saw that the relativistic 4-potential formulation $F^{\mu\nu} \equiv \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \right)$ satisfied the ***inhomogeneous*** Maxwell equation $\partial_{\nu} F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \right) = \mu_o J^{\mu}$ which contains the ***inhomogeneous*** Maxwell equations $\vec{\nabla} \cdot \vec{E} = \rho_{tot}/\varepsilon_o$ and $\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_o \vec{J}_{tot}$ ***provided*** we use the ***Lorenz gauge condition***: $\partial_\nu A^\nu = \dfrac{\partial A^\nu}{\partial x^\nu} = 0$ in the ***inhomogeneous*** Maxwell equation:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(-\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_o J^\mu$$

Lorenz Gauge Condition:

$$\partial_\nu A^\nu = \frac{\partial A^\nu}{\partial x^\nu} = 0 \quad \left(= \vec{\nabla}\bullet\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \right)$$

D'Alembertian operator:

$$\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Then: $\partial_\nu F^{\mu\nu} = \dfrac{\partial F^{\mu\nu}}{\partial x^\nu} = -\Box^2 A^\mu = \mu_o J^\mu$ or: $\Box^2 A^\mu = \dfrac{\partial}{\partial x_\nu}\dfrac{\partial}{\partial x^\nu} A^\mu = \dfrac{\partial}{\partial x^\nu}\dfrac{\partial}{\partial x_\nu} A^\mu = -\mu_o J^\mu$

The Lorenz gauge condition $\partial_\nu A^\nu = \dfrac{\partial A^\nu}{\partial x^\nu} = 0$ gives us the freedom to add to the 4-vector potential field A^μ ***any*** arbitrary constant 4-vector, usually (*i.e.* traditionally) written as the space-time / 4-gradient of an arbitrary scalar space-time point function $\lambda \left(=\lambda(\vec{r},t)\right)$:

$$A^\mu \rightarrow A^{*\mu} = A^\mu + \frac{\partial\lambda}{\partial x_\mu} \quad \text{where:} \quad \frac{\partial\lambda}{\partial x_\mu} \neq 0$$

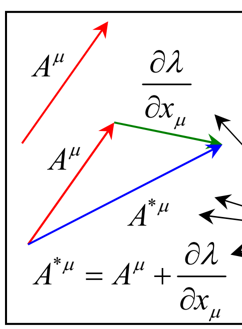
i.e. the scalar and vector potentials $V(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ respectively are ***not*** uniquely determined / defined by the *EM* fields $\left(\vec{E} \text{ and } \vec{B}\right)$ up to an arbitrary constant.

We then showed that the *EM* field tensor $F^{\mu\nu}$ is manifestly ***gauge invariant***:

$$F'^{\mu\nu} = \left(\frac{\partial A^{*\nu}}{\partial x_\mu} - \frac{\partial A^{*\mu}}{\partial x_\nu}\right) = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) = F^{\mu\nu}$$

← Several steps {See P436 Lect. Notes 19, *p.* 17}

The 4-vector A^μ in "field-space":



n.b. $A^{*\mu}(\vec{r},t) = A^\mu(\vec{r},t) + \dfrac{\partial\lambda(\vec{r},t)}{\partial x_\mu}$

i.e. The *EM* field tensor $F^{\mu\nu}$ (*i.e.* the *EM* fields $\vec{E}$ and $\vec{B}$) are ***invariant*** / unchanged by a gauge transformation (scale change/rotation in "field-space") of the *EM* 4-potential field A^μ:

$$A^\mu \rightarrow A^{*\mu} = A^\mu + \frac{\partial\lambda}{\partial x_\mu} \quad \text{where:} \quad \frac{\partial\lambda}{\partial x_\mu} \neq 0$$

Now ***local*** conservation of electric charge/current – *i.e.* conservation of charge/current at ***each*** and ***every*** point in ***space-time*** $(\vec{r},t)$ requires that $J^{\mu}_{tot} = (c\rho_{tot}, \vec{J}_{tot})$ obeys the continuity equation:

$$\partial_\mu J^\mu_{tot} = \frac{\partial J^\mu_{tot}}{\partial x^\mu} = 0 \quad i.e. \quad \vec{\nabla}\cdot\vec{J}_{tot} = -\frac{\partial \rho_{tot}}{\partial t}$$

And since: $\vec{J}_{tot} = \vec{J}_{free} + \vec{J}_{bound}$ and: $\rho_{tot} = \rho_{free} + \rho_{bound}$, thus: $J^\mu_{tot} = J^\mu_{free} + J^\mu_{bound}$

Thus, ***free*** electric charge/currents $J^\mu_{free} = (c\rho_{free}, \vec{J}_{free})$ ***separately*** obey:

$$\partial_\mu J^\mu_{free} = \frac{\partial J^\mu_{free}}{\partial x^\mu} = 0 \quad i.e. \quad \vec{\nabla}\cdot\vec{J}_{free} = -\frac{\partial \rho_{free}}{\partial t}$$

And ***bound*** electric charge/currents $J^\mu_{bound} = (c\rho_{bound}, \vec{J}_{bound})$ ***separately*** obey:

$$\partial_\mu J^\mu_{bound} = \frac{\partial J^\mu_{bound}}{\partial x^\mu} = 0 \quad i.e. \quad \vec{\nabla}\cdot\vec{J}_{bound} = -\frac{\partial \rho_{bound}}{\partial t}$$

Electric charge/current is conserved at ***each/every/all*** points $(\vec{r},t)$ in ***space-time***.

Explicitly/generically: $J^\mu(x^\sigma) = (c\rho(\vec{r},t), \vec{J}(\vec{r},t))$

But: $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\right] = \mu_o J^\mu$ = ***inhomogeneous*** Maxwell equation

$$\therefore \quad \partial_\mu \partial_\nu F^{\mu\nu} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial F^{\mu\nu}}{\partial x^\nu}\right) = \frac{\partial}{\partial x^\mu}\left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\right] = \mu_o \frac{\partial J^\mu}{\partial x^\mu} = \mu_o \partial_\mu J^\mu = 0$$

Experimentally/empirically, we ***know*** that electric charge/current ***is*** conserved, *i.e.* we know that the RHS of this equation ***is*** indeed true, that: $\partial_\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = 0$.

What about the LHS of this equation? Does: $\frac{\partial}{\partial x^\mu}\left(\frac{\partial F^{\mu\nu}}{\partial x^\nu}\right) = \frac{\partial}{\partial x^\mu}\left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\right] = 0$???

We know that: $\left[\frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\nu}{\partial x^\nu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\right] = -\frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right)$ ***provided*** that: $\frac{\partial A^\nu}{\partial x^\nu} \equiv 0$ (*i.e.* we adopt/use the Lorenz gauge condition)

But: $\boxed{\dfrac{\partial F^{\mu\nu}}{\partial x^{\nu}}=\dfrac{\partial}{\partial x^{\nu}}\left(\dfrac{\partial A^{\nu}}{\partial x_{\mu}}-\dfrac{\partial A^{\mu}}{\partial x_{\nu}}\right)}$ Thus: $\boxed{\dfrac{\partial}{\partial x^{\mu}}\left(\dfrac{\partial F^{\mu\nu}}{\partial x^{\nu}}\right)=\dfrac{\partial}{\partial x^{\mu}}\left\{-\dfrac{\partial}{\partial x_{\nu}}\left(\dfrac{\partial A^{\mu}}{\partial x^{\nu}}\right)\right\}}$.

However, we are free to interchange the order of the derivatives in this equation, *i.e.*:

$$\boxed{\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial F^{\mu\nu}}{\partial x^{\nu}}\right)=-\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\mu}}\right)} \text{ but: } \boxed{\left(\frac{\partial A^{\mu}}{\partial x^{\mu}}\right)\equiv 0} \text{ !!!}$$

This can ***also*** be equivalently written as: $\boxed{\dfrac{\partial}{\partial x^{\nu}}\left(\dfrac{\partial F^{\mu\nu}}{\partial x^{\mu}}\right)=-\dfrac{\partial}{\partial x_{\nu}}\dfrac{\partial}{\partial x^{\nu}}\left(\dfrac{\partial A^{\mu}}{\partial x^{\mu}}\right)=0}$

Then we see that the LHS of ***this*** equation ***does*** $\equiv 0$ and hence the RHS of ***this*** equation ***must*** also = 0, *i.e.*:

$$\boxed{\partial_{\mu}\partial_{\nu}F^{\mu\nu}=\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}F^{\mu\nu}=\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial A^{\nu}}{\partial x_{\mu}}-\frac{\partial A^{\mu}}{\partial x_{\nu}}\right)=\mu_o\frac{\partial J^{\mu}}{\partial x^{\mu}}=\mu_o\partial_{\mu}J^{\mu}\equiv 0}$$

provided that the Lorenz gauge condition holds, *i.e*. that: $\boxed{\partial_{\mu}A^{\mu}=\dfrac{\partial A^{\mu}}{\partial x^{\mu}}\equiv 0}$.

Stated in ***this*** manner, we realize that electric charge / electric current conservation $\boxed{\dfrac{\partial J^{\mu}}{\partial x^{\mu}}=0}$ is a ***direct*** consequence of the Lorenz gauge condition $\boxed{\dfrac{\partial A^{\mu}}{\partial x^{\mu}}\equiv 0}$!!! The Lorenz gauge condition is in fact a statement about the ***gauge*** ***invariance*** of $F^{\mu\nu}\left(\vec{E}\text{ and }\vec{B}\right)$

i.e. we have the freedom to define $\boxed{A^{\mu}\to A^{*\mu}=A^{\mu}+\dfrac{\partial\lambda}{\partial x_{\mu}}}$!!!

Clearly, the Lorenz gauge condition $\boxed{\partial_{\mu}A^{\mu}=\dfrac{\partial A^{\mu}}{\partial x^{\mu}}=0}$ ***is*** satisfied for the ***specific*** choice of A^{μ}.

But: $\boxed{\partial_{\mu}A^{*\mu}=\dfrac{\partial A^{*\mu}}{\partial x^{\mu}}=0}$ ***would*** be the ***corresponding*** Lorenz gauge condition for the 4-potential vector field $\boxed{A^{*\mu}\left(=A^{\mu}+\dfrac{\partial\lambda}{\partial x_{\mu}}\right)}$ ***provided*** (***if and only if***): $\boxed{\dfrac{\partial}{\partial x^{\mu}}\dfrac{\partial}{\partial x_{\mu}}(\lambda)\equiv 0}$, where: $\boxed{\dfrac{\partial\lambda}{\partial x_{\mu}}\neq 0}$.

$\lambda=\lambda(\vec{r},t)$ can be any/an arbitrary space-time 4-point function, as long as it satisfies:

$$\boxed{\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x_{\mu}}(\lambda)=\Box^2\lambda(\vec{r},t)=\nabla^2\lambda(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\lambda(\vec{r},t)}{\partial t^2}=0} \text{ where: } \boxed{\frac{\partial\lambda}{\partial x_{\mu}}\neq 0}.$$

$\Rightarrow \boxed{\partial_\mu A^{*\mu} = \dfrac{\partial A^{*\mu}}{\partial x^\mu} = 0}$ because: $\boxed{\dfrac{\partial}{\partial x^\mu}\dfrac{\partial}{\partial x_\mu}(\lambda) = \square^2\lambda = 0}$ and: $\boxed{\partial_\mu A^\mu = \dfrac{\partial A^\mu}{\partial x^\mu} = 0}$.

Thus: $\boxed{\dfrac{\partial A^{*\mu}}{\partial x^\mu} = \dfrac{\partial A^\mu}{\partial x^\mu} = 0}$ where: $\boxed{A^{*\mu} = A^\mu + \dfrac{\partial\lambda}{\partial x_\mu}}$ with: $\boxed{\dfrac{\partial\lambda}{\partial x_\mu} \neq 0}$ because of the gauge invariance associated with choice of A^μ leaving $F^{\mu\nu}$ (*i.e.* the *EM* fields $\vec{E}$ and $\vec{B}$) unchanged: $\boxed{F^{*\mu\nu} \equiv F^{\mu\nu}}$.

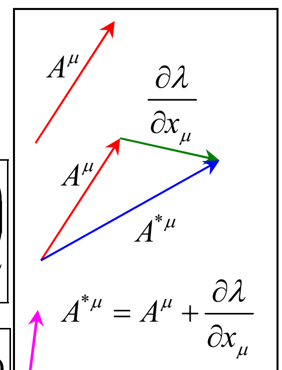
As a ***consequence*** of the gauge invariance of A^μ: $\boxed{\partial_\mu J^\mu = \dfrac{\partial J^\mu}{\partial x^\mu} = 0}$ *i.e.* electric charge / electric current is ***locally*** conserved at ***each***/***every*** point $(\vec{r},t)$ in space-time.

Put ***conversely***, if: $\boxed{A^{*\mu} \neq A^\mu + \dfrac{\partial\lambda}{\partial x_\mu}}$, with: $\boxed{\dfrac{\partial\lambda}{\partial x_\mu} \neq 0}$, then because ***this*** A^μ-field is ***not*** locally gauge invariant at ***each***/***every*** space-time point $(\vec{r},t)$, *i.e.* $\boxed{\dfrac{\partial A^{*\mu}}{\partial x^\mu} \neq \dfrac{\partial A^\mu}{\partial x^\mu} \neq 0}$ *e.g.* because $\boxed{\dfrac{\partial}{\partial x_\mu}\dfrac{\partial}{\partial x^\mu}(\lambda) = \square^2\lambda \neq 0}$ {eek!!!} Then: $\boxed{F^{*\mu\nu} \neq F^{\mu\nu}}$ (*i.e.* $\vec{E}^* \neq \vec{E}$ and $\vec{B}^* \neq \vec{B}$) which ***would*** be terrible !!! And worse yet: $\boxed{\dfrac{\partial J^\mu}{\partial x^\mu} \neq 0}$ $\Rightarrow$ Electric charge / electric current would ***not*** be conserved !!!

Let us consider what happens to the term $\boxed{A^\mu A_\mu}$ when we make a gauge transformation:

$\boxed{A^\mu \rightarrow A^{*\mu} = A^\mu + \dfrac{\partial\lambda}{\partial x_\mu}}$. Then:

$$\boxed{A^{*\mu}A_\mu^* = \left(A^\mu + \frac{\partial\lambda}{\partial x_\mu}\right)\left(A_\mu + \frac{\partial\lambda}{\partial x^\mu}\right) = A^\mu A_\mu + \underbrace{\left(\frac{\partial\lambda}{\partial x_\mu}\right)A_\mu + \left(\frac{\partial\lambda}{\partial x^\mu}\right)A^\mu + \left(\frac{\partial\lambda}{\partial x_\mu}\right)\left(\frac{\partial\lambda}{\partial x^\mu}\right)}}$$



If $A^{*\mu}A_\mu^*$ is to $= A^\mu A_\mu$, then ***these*** terms ***must*** = 0. But: $\boxed{\dfrac{\partial\lambda}{\partial x_\mu} \neq 0}$ and: $\boxed{\dfrac{\partial\lambda}{\partial x^\mu} \neq 0}$

Note that we don't ***necessarily*** expect $\boxed{A^\mu A_\mu} = \boxed{A^{*\mu}A_\mu^*}$ on ***physical*** grounds anyway, from the above "field-space" vector diagram.

Thus, ***any*** physical quantity involving $\boxed{A^\mu A_\mu}$ and/or $\boxed{A^{*\mu}A_\mu^*}$ is manifestly ***not gauge invariant***.

Note, however that ***both*** $\boxed{A^\mu A_\mu}$ and $\boxed{A^{*\mu}A_\mu^*}$ ***are*** properly ***Lorentz invariant***:

In IRF(S') $\rightarrow$ $\boxed{A'^\mu = \Lambda^\mu{}_\nu A^\nu}$ $\leftarrow$ In IRF(S) In IRF(S') $\rightarrow$ $\boxed{A'^{*\mu} = \Lambda^\mu{}_\nu A^{*\nu}}$ $\leftarrow$ In IRF(S)

$\boxed{A'_\mu = \Lambda_\mu{}^\nu A_\nu}$ $\boxed{A'^*_\mu = \Lambda_\mu{}^\nu A^*_\nu}$

e.g.: $A'^{\mu}A'_{\mu} = \left(\Lambda^{\mu}{}_{\nu}A^{\nu}\right)\left(\Lambda_{\mu}{}^{\nu}A_{\nu}\right) = \left(\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\nu}\right)\left(A^{\nu}A_{\nu}\right)$

But: $\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\nu} = 1$ {*i.e.* the Λ-boost matrices are ***unitary transformations***}

$\therefore$ $\underbrace{A'^{\mu}A'_{\mu}}_{\text{In IRF}(S')} = \underbrace{A^{\mu}A_{\mu}}_{\text{In IRF}(S)}$ Lorentz invariance ***is*** obeyed by the A^{μ}-field.

Physically, what is $A^{\mu}A_{\mu}$? The 4-vector potential: $A^{\mu} \equiv \left(V/c, \vec{A}\right)$ SI Units: *Newtons/Ampere* = "*p*/*q*" {momentum per Coulomb!}

We know what A_q^{μ} is for a point electric charge *q*, *e.g.* in its own ***rest/proper*** frame, from the {retarded} Liénard-Wiechert potentials {see P436 Lect. Notes 12, *p*. 7-8 and/or Griffiths Example 10.3, *p*. 433-434}:

$$V_q\left(\vec{r},t\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r} \quad \text{and:} \quad \vec{A}_q\left(\vec{r},t\right) = \frac{\cancel{\vec{v}\left(\vec{r},t\right)}}{c^2}V_q\left(\vec{r},t\right) = 0$$

n.b. **∃ *no* time dependence in the *rest/proper* frame of the point charged particle.**

Then: $$A_q^{\mu}\left(r\right)A_{q\mu}\left(r\right) = -\frac{1}{c^2}V_q^2\left(r\right) + \cancel{\vec{A}_q\left(r\right)\cdot\vec{A}_q\left(r\right)} = -\frac{1}{c^2}V_q^2\left(r\right) = -\frac{1}{c^2}\left(\frac{1}{4\pi\varepsilon_o}\right)^2\frac{q^2}{r^2}$$

However, recall that the {electrostatic} *EM* energy density $u_{EM}\left(r\right)$ associated with a point charged particle *q* probed by a point test charge q_T at a separation distance *r* from *q* {both at rest} is:

$$u_{EM}\left(r\right) = q_T V_q\left(r\right) = \frac{1}{4\pi\varepsilon_o}\frac{q_T q}{r}$$ referencing: $u_{EM}\left(r=\infty\right) = 0$, *i.e.* $V_q\left(r=\infty\right) = 0$

Thus, we see that: $$u_{EM}^2\left(r\right) = \left(q_T V_q\left(r\right)\right)^2 = q_T^2\left(\frac{1}{4\pi\varepsilon_o}\frac{q}{r}\right)^2 = -q_T^2c^2A_q^{\mu}\left(r\right)A_{q\mu}\left(r\right)$$

and thus: $$U_{EM}^2 = \int_v u_{EM}^2\left(r\right)d\tau = \int_v u_{EM}^2\left(r\right)d\tau = q_T^2q^2\left(\frac{1}{4\pi\varepsilon_o}\right)^2\int_v\frac{1}{r^2}d\tau = \infty$$ = (rest energy of *q*)^2

This integral has a singularity at $r = 0$, as we have discussed long ago in P435, thus it should come as no surprise here {again} that using classical and/or relativistic *EM*, the calculated rest energy (*i.e.* = rest mass $m_q c^2$) of the test charge *q* is formally infinite – this problem remains even in Quantum Electrodynamics {QED} where the technique of mass (& charge) renormalization is used to address this problem.

Physically then, we see that $-q^2c^2A^{\mu}A_{\mu} = \left(m_q c^2\right)^2$, which ***is*** a Lorentz invariant quantity, *i.e.* it has the same numerical value in any/all IRF's.

Thus: $A^{\mu}A_{\mu} = A_{\mu}A^{\mu} = -\left(mc^2/qc\right)^2$ is known as a ***mass term*** because of this.

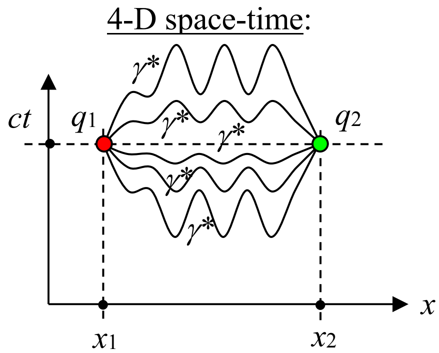
Real photons have rest mass $\boxed{m_\gamma c^2 = 0}$, but since ***real*** photons ***always*** travel at the speed of light c {in vacuum/free space} and thus have ***no rest mass*** frame, then in the ***center-of-momentum*** frame of a ***real*** photon: $\boxed{A_\gamma^\mu A_{\gamma\mu} = A_{\gamma\mu} A_\gamma^\mu = 0}$. Since this ***is*** a Lorentz invariant quantity, it ***must*** be the same numerical value in ***all*** reference frames, *e.g.* including the ***lab*** frame IRF(S).

Thus, we can now see from the point of view of ***gauge invariance***: $\boxed{A^{*\mu} = A^\mu + \partial\lambda/\partial x_\mu}$ that if $\boxed{A^{*\mu} A_\mu^* = -\left(m_* c^2/qc\right)^2} \neq \boxed{A^\mu A_\mu = -\left(mc^2/qc\right)^2}$ this ***would*** be a ***really*** bad thing!!!

Manifest gauge invariance {"eich-invarianz", auf deutsch} of the *EM* interaction as represented by the ***vector*** field A^μ is intimately connected to the {microscopic} nature of the ***force carrier*** (*a.k.a.* ***mediator***) of the *EM* interaction – the photon – an intrinsic spin-1 $\hbar$ ***vector*** particle {it carries the $\vec{E}$ and $\vec{B}$ fields – *i.e.* it carries $F^{\mu\nu}$ }.

For ***real*** photons (*i.e.* lying on the $\boxed{I = \Delta x^\mu \Delta x_\mu = 0}$ ***light cone*** somewhere in 4-D Minkowski ct *vs.* x space), the ***real*** photon mass $\boxed{m_\gamma c^2 \equiv 0}$, hence: $\boxed{E_\gamma^{tot} = p_\gamma c \quad \left(= hf_\gamma\right)}$.

Only because of the fact that $\boxed{m_\gamma c^2 = 0}$ do we have a $\equiv 1/r^2$ Coulomb force law for ***virtual*** photons { $m_\gamma c^2 \neq 0$ } exchanged between two electrically-charged particles:



i.e.

$$\boxed{\vec{F}_{Coul}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_2}{r^2}\right)\hat{r}}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)\hat{r} = -\vec{\nabla}V(\vec{r})}$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)} \leftarrow \frac{1}{r} \text{ central potential}$$

The Coulomb / *EM* force is a ***conservative*** $\left(\equiv 1/r^2\right)$ force due to the ***gauge invariant*** nature of the *EM* interaction – *i.e. EM* "news" / information propagates at c because $\boxed{m_\gamma c^2 = 0}$. The ***range*** of the *EM* force = ∞ for $\boxed{m_\gamma c^2 \equiv 0}$.

If $\boxed{m_\gamma c^2 \neq 0}$, then *EM* "news" / information would ***not*** propagate at the speed of light c.
→ The *EM* interaction would ***no longer*** be gauge invariant !!!

i.e. $\boxed{A^{*\mu} \neq A^\mu + \partial\lambda/\partial x_\mu}$ (*i.e.* $\boxed{F^{*\mu\nu} \neq F^{\mu\nu}}$!!!)

→ Electric charge q / electric currents I would ***not*** be conserved – *i.e.* $\boxed{\partial_\mu J^\mu = \dfrac{\partial J^\mu}{\partial x^\mu} \neq 0}$

If $\boxed{m_\gamma c^2 \neq 0}$, then Coulomb's force law would ***no longer*** have a ***purely*** $1/r^2$ nature
i.e. Coulomb's force law would ***no longer*** have a ***purely*** $1/r$ central ***potential*** $V(r)$.

For $\boxed{m_\gamma c^2 \neq 0}$, we would ***instead*** have:

$$\boxed{\vec{F}_{Coul}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_2}{r^2}\right)e^{-\mu r}(1+\mu r)\hat{r}} \Leftarrow \text{exponentially damped force law!}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)e^{-\mu r}(1+\mu r)\hat{r} = -\vec{\nabla}V(\vec{r})}$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)e^{-\mu r}} \leftarrow \text{General } e^{-\mu r} \text{ form is known as the } \underline{\textbf{\textit{Yukawa}}}\ \underline{\textbf{\textit{potential}}}.$$

Where: $\boxed{\mu = m_\gamma c/\hbar = m_\gamma c^2/\hbar c \equiv 1/\lambda\!\!\!^{-}_\gamma}$ and where: h = Planck's constant, and: $\hbar \equiv h/2\pi$.

$\boxed{E_\gamma^{tot} = \sqrt{p_\gamma^2 c^2 + m_\gamma^2 c^4}}$ $hc = 1240\ eV\text{-}nm$

$\boxed{v_{prop}^\gamma < c}$ $\hbar c = 197.3\ MeV\text{-}fm$

Thus for $\boxed{m_\gamma \neq 0}$, the Coulomb force would be of ***finite*** range – falling to $1/e$ of what it would have been for $\boxed{m_\gamma = 0}$ within a characteristic distance scale of $\boxed{\lambda\!\!\!^{-}_\gamma \equiv 1/\mu = \hbar/m_\gamma c = \hbar c/m_\gamma c^2}$:

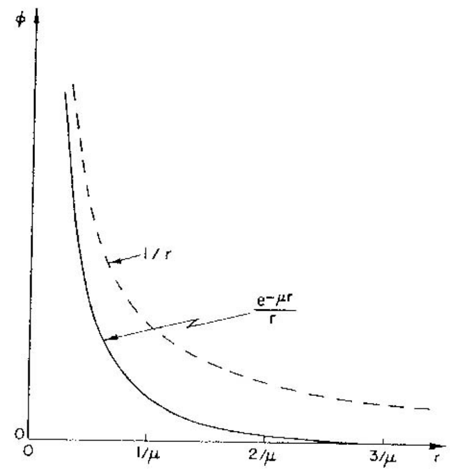


Fig. 28–6. The Yukawa potential $e^{-\mu r}/r$, compared with the Coulomb potential $1/r$.

We can also see this from another perspective:

For $\boxed{m_\gamma c^2 = 0}$, in the Lorenz gauge $\boxed{\partial_\mu A^\mu = \frac{\partial A^\mu}{\partial x^\mu} = 0}$ we ***do*** have: $\boxed{\Box^2 A^\mu = -\mu_o J^\mu}$

In the ***static limit*** (*i.e.* $\boxed{\frac{\partial \vec{A}}{\partial t} = 0}$ ← no time dependence, and $\boxed{\vec{B} = \vec{\nabla}\times\vec{A} = 0}$ ← no $\vec{B}$ -field)

this equation becomes: $\boxed{\nabla^2 V(\vec{r}) = -\mu_o c^2 \rho(\vec{r}) = -\frac{1}{\varepsilon_o}\rho(\vec{r})}$ But: $\boxed{\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}V(\vec{r})\right) = -\nabla^2 V(\vec{r})}$

$\therefore$ $\boxed{\vec{\nabla}\cdot\vec{E} = +\frac{1}{\varepsilon_o}\rho(\vec{r})}$ {*i.e.* Gauss' law}. Integrate both sides over volume τ'

We know that for a point charge q: $\boxed{\rho_q(\vec{r}) = q\delta^3(\vec{r})}$

Then: $\boxed{\vec{E}_q(\vec{r}) = -\vec{\nabla}V_q(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}}$ and: $\boxed{V_q(r) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)}$

The ***range*** $\boxed{\lambda\!\!\!^{-}_\gamma \equiv 1/\mu = \hbar/m_\gamma c = \hbar c/m_\gamma c^2}$ of the *EM* force = ∞ for $\boxed{m_\gamma c^2 \equiv 0}$.

But what if $\boxed{m_\gamma c^2 \neq 0}$?? In the Lorenz gauge: $\boxed{\partial_\mu A^\mu = \frac{\partial A^\mu}{\partial x^\mu} = 0}$

Then: $\boxed{\Box^2 A^\mu = -\mu_o J^\mu}$ is ***modified*** by adding a ***mass term*** to this relation:

$\boxed{\Box^2 A^\mu - \mu^2 A^\mu = -\mu_o J^\mu}$ where: $\boxed{\mu \equiv \frac{m_\gamma c}{\hbar c} = \frac{m_\gamma c^2}{\hbar c^2}}$ = scalar / constant

n.b.

Since: $\boxed{\Box^2 = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} = \partial^\mu \partial_\mu}$ = Lorentz invariant ***scalar*** quantity.

Then: $\boxed{\Box^2 - \mu^2}$ ***also*** = Lorentz invariant ***scalar*** quantity.

Then in the ***static limit*** (*i.e.* $\boxed{\partial\vec{A}/\partial t = 0}$ ← no time dependence, and $\boxed{\vec{B} = \vec{\nabla}\times\vec{A} = 0}$ ← no $\vec{B}$ -field)

we obtain: $\boxed{\nabla^2 V(\vec{r}) - \mu^2 V(\vec{r}) = -\mu_o c^2 \rho(\vec{r}) = -\frac{1}{\varepsilon_o}\rho(\vec{r})}$

n.b. the 4-potential A^μ (and thus $V(\vec{r})$ and $\vec{A}$) now ***do*** acquire real physical (*i.e.* observable) significance through the mass term $\boxed{\mu^2 A^\mu}$ – *i.e.* changing the gauge $\boxed{A^\mu \to A^{*\mu} = A^\mu + \frac{\partial\lambda}{\partial x_\mu}}$ now also causes ***changes*** in this mass term !!! {eeek!}

The solution for the scalar potential $V_q(\vec{r})$ for a point charge q is now: $\boxed{V_q(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}e^{-\mu r}}$

⇒ The range $\boxed{\lambda\!\!\!^{-}_\gamma \equiv 1/\mu = \hbar/m_\gamma c = \hbar c/m_\gamma c^2}$ of the *EM* force ≠ ∞ if $\boxed{m_\gamma c^2 \neq 0}$!!!

Suppose $\boxed{J^\mu = 0}$, *i.e.* ***no*** electric charges / currents present $\Rightarrow$ "free" field equation for A^μ:

For $\boxed{m_\gamma c^2 = 0}$: $\boxed{\Box^2 A^\mu = 0} \Rightarrow \boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\mu = 0}$ = ***homogeneous*** wave equation.

$\Rightarrow$ Get harmonic solutions of the form: $\boxed{\vec{A}(\vec{r},t) = \vec{A}_o e^{i(kz-\omega t)}}$ *e.g.* for an *EM* wave propagating in the $\hat{z}$ -direction. The ***dispersion relation*** for this situation is: $\boxed{\omega^2 = (ck)^2}$ *i.e.* $\boxed{\omega = ck}$.

For $\boxed{m_\gamma c^2 \neq 0}$: $\boxed{\Box^2 A^\mu - \mu^2 A^\mu = 0} \Rightarrow \boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\mu - \mu^2 A^\mu = 0}$ = ***inhomogeneous*** wave equation.

$\Rightarrow$ Has solutions of the form: $\boxed{\vec{A}(\vec{r},t) = \vec{A}_o e^{i\left(\sqrt{k^2+\mu^2}\, z - \omega t\right)}}$ $\Leftarrow$ *n.b. EM* wave propagation sort of like that in ***wave guides*** !!!

The ***dispersion relation*** for ***this*** situation becomes: $\boxed{\omega^2 = (ck)^2 + (\mu c)^2}$

Define: $\boxed{\omega_o \equiv ck} \rightarrow \boxed{\omega^2 = \omega_o^2 + \mu^2 c^2}$ where: $\boxed{\mu = m_\gamma c/\hbar = m_\gamma c^2/\hbar c \equiv 1/\lambda\!\!\!^{-}_\gamma}$.

There exist experimental ***upper limits*** on the photon ***mass*** from laboratory experiments and also from geomagnetic data – *i.e.* the earth's magnetic field $\vec{B}_\oplus(\vec{r})$!

If $\boxed{m_\gamma c^2 \neq 0}$: $\boxed{\vec{B}_\oplus(\vec{r}) = \frac{\mu_o}{4\pi}\left[\left[3\hat{r}(\hat{r}\bullet\vec{m}_\oplus) - \vec{m}_\oplus\right]\left(1 + \mu r + \frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2 \vec{m}_\oplus \frac{e^{-\mu r}}{r}\right]}$

If $\boxed{m_\gamma c^2 \neq 0}$, this formula tells us that the magnetic field at the surface of the earth $\boxed{r = R_\oplus \simeq 6370\ km}$ would have a "normal", pure magnetic dipole field component plus an added ***constant*** magnetic field – the $\boxed{\mu = m_\gamma c/\hbar = m_\gamma c^2/\hbar c \equiv 1/\lambda\!\!\!^{-}_\gamma}$ term(s) in the above formula would give rise to an apparent external $\vec{B}$ -field contribution which would be {overall} anti-parallel to the magnetic dipole moment $\vec{m}_\oplus$ of the earth.

Satellite measurements and surface observations of $\vec{B}_\oplus$ constrain this "external" constant magnetic field to be less than $(4\times10^{-3})\times$ that of a ***pure*** magnetic dipole field at the earth's ***magnetic*** equator {at 90% confidence level (CL)}, which corresponds to a 90% CL lower limit on the $\boxed{m_\gamma c^2 \neq 0}$ photon attenuation length scale of $\boxed{\lambda\!\!\!^{-}_\gamma = \mu^{-1} > 10^8 m}$, or since $\boxed{\mu = m_\gamma c^2/\hbar c}$, this corresponds to a 90% CL upper limit on the photon rest mass of $\boxed{m_\gamma c^2 < 2\times10^{-15} eV}$, or equivalently $\boxed{m_\gamma < 4\times10^{-51} kg}$.

Recently (1998), the ***absence*** of an *EM* torque on a toroid balance experiment set 90% CL upper limits on the photon rest mass of $\boxed{\lambda\!\!\!^{-}_\gamma = \mu^{-1} > 10^9 m}$, $\boxed{m_\gamma c^2 < 2\times10^{-16} eV}$, or equivalently $\boxed{m_\gamma < 4\times10^{-52} kg}$. {See Lakes, *et. al.*, PRL 80, 1826 (1998).}

There also exist upper limits on the photon rest mass $m_\gamma c^2$ from the ***absence*** of frequency shifts of the Schumann earth-earth's ionosphere resonances!

If $\boxed{m_\gamma c^2 \neq 0}$, *e.g.* the Schumann *n* = 0 resonance formula $\boxed{\omega_{0\ell} \simeq \sqrt{\ell(\ell+1)}\, c/(r_\oplus + \tfrac{1}{2}h)}$ is modified, becoming: $\boxed{\omega_\ell^2 = \omega_{0\ell}^2 + \mu^2 c^2}$. The ***absence*** of shifts in the observed Schumann *n* = 0 resonance frequencies, attributable to $\boxed{m_\gamma c^2 \neq 0}$ effects give less stringent, although still respectable 90% CL limits of $\boxed{\lambda\!\!\!^{-}_\gamma = \mu^{-1} > 10^5\, m}$, $\boxed{m_\gamma c^2 < 2\times10^{-12}\, eV}$, or $\boxed{m_\gamma < 4\times10^{-48}\, kg}$.

Note that if $\boxed{m_\gamma c^2 \neq 0}$ then {***massive***} ***real*** photons would contribute to dark matter / dark energy, affecting *e.g.* galaxy formation, large-scale structure of the universe, lead to significant changes/alterations the ~ 3K black-body/microwave radiation spectrum left over from the "Big Bang", suffer red-shifts in light emitted from stars and other gravitational bodies, *etc*…

The Nuclear / Strong Interactions:

In ~ 1935, Hideki Yukawa proposed a theoretical model for the ***nuclear*** (*i.e.* the strong) force between nucleons (*i.e.* protons and neutrons) mediated {at ***low*** energies} by massive ***scalar*** (*i.e.* spinless) particles – the **pi mesons** $\pi^\pm, \pi^0$ (intrinsic spin-$0\hbar$), with $\boxed{m_\pi \simeq 135-140\, MeV/c^2}$.

In Yukawa's model, the strong field associated with the "free"-field propagation of massive, scalar/spin-0 pi mesons {pions} in space-time was described mathematically via the equation:

$\boxed{\Box^2\varphi(\vec{r},t) - \mu^2\varphi(\vec{r},t) = 0}$ where $\varphi(\vec{r},t)$ = ***scalar*** field of the spinless pion and: $\boxed{\mu = \dfrac{m_\pi c}{\hbar} = \dfrac{m_\pi c^2}{\hbar c}}$

In nearly complete analogy to that associated with the *EM* "free-field" propagation of "***massive***" ***vector*** photons (*i.e.* intrinsic spin-$1\hbar$) in space-time, as mathematically described via the equation:

$\boxed{\Box^2 A^\mu(\vec{r},t) - \mu^2 A^\mu(\vec{r},t) = 0}$ where $A^\mu(\vec{r},t)$ = ***vector*** field of the spin-1 photon and: $\boxed{\mu = \dfrac{m_\gamma c}{\hbar} = \dfrac{m_\lambda c^2}{\hbar c}}$

In the "***static***" limit, the solution to $\boxed{\Box^2\varphi(\vec{r},t) - \mu^2\varphi(\vec{r},t) = 0}$ is the {scalar} Yukawa potential:

$\boxed{\varphi(\vec{r}) = K\dfrac{e^{-\mu r}}{r}}$ where: $\boxed{K = \dfrac{g}{4\pi}}$ = constant and *g* = strong / nuclear charge

The ***range*** of the strong / nuclear force is extremely short, due to the mass of the pion:

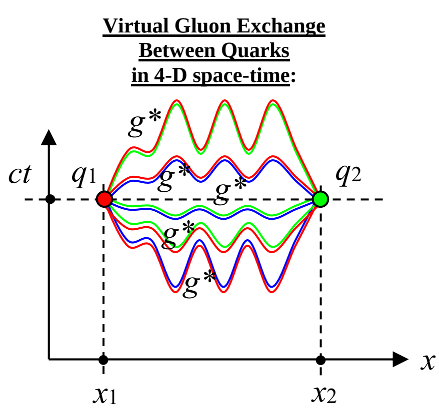
$\boxed{\lambda\!\!\!^{-}_{strong} = \mu_{strong}^{-1} = \dfrac{\hbar c}{m_\pi c^2} = \dfrac{197.3\, MeV\text{-}fm}{\sim 135\, MeV}}$ ≈ typical size of a nucleus $\sim 1.5\, fm = 1.5\times10^{-15}\, m$!!!

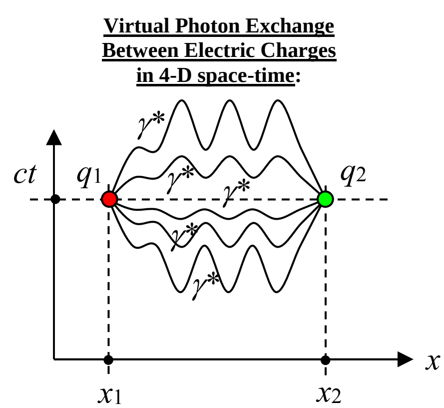
We can also repeat this for the strong interactions @ ***high energies***, in the context of QCD (Quantum Chromo-Dynamics) – the mediator of the strong force at high energies is a massless vector / spin-1 particle known as the ***gluon***, which carries {net} ***strong charge*** (*n.b.* unlike the photon, which carries ***no*** {net} electric charge) !!!

For QCD/the strong interactions, ∃ an ***octet*** of gluons – because there exist 3 distinct strong "color charges" (*r*,*g*,*b*) and 3 distinct strong "color anti-charges" $\left(\bar{r},\bar{g},\bar{b}\right)$. The octet of ***massless*** gluons thus carry {***orthogonal***} ***color-anticolor*** strong charge combinations *e.g.* $r\bar{g}$, $g\bar{b}$, $r\bar{b}$ …

In analogy to the vector field A^{μ} for the massless spin-1 photon {the mediator of the *EM* interaction} the massless spin-1 color-anticolor charged gluon {the mediator of the strong interaction at high energies} is represented by the ***vector*** field G_a^{μ}, where $\mu =$ 0:3 is the usual space-time index and the index a = 0:7 denotes which color-anticolor combination of the color octet this particular gluon has.

In analogy to $\boxed{\square^2 A^{\mu} = -\mu_o J^{\mu}}$ for the *EM* interaction, for QCD / the strong interaction at high energy we have: $\boxed{\square^2 G_a^{\mu} = -\mu_o^s J_a^{\mu}}$ where J_a^{μ} is the relativistic/space-time 4-D strong color charge / strong color current density associated with the color-anticolor index a and μ_o^s is the strong color magnetic permeability associated with the QCD vacuum, noting that the "speed of light" c is the maximum speed for ***any/all*** of the four fundamental forces / interactions of nature, thus: $\boxed{c = 1\big/\sqrt{\varepsilon_o^s \mu_o^s}}$.





The Electroweak Interactions:

In the 1970's, high-energy physicists discovered that the electromagnetic and weak interactions were in fact not unrelated to each other – they have common ***electroweak*** fields !!!

At ***very high*** energies (*e.g.* the situation in the very early universe, just moments after the "Big Bang"), the mediators (*i.e.* force carriers) of the ***electroweak*** interaction are all massless spin-$1\hbar$ ***vector*** particles:

$\exists$ a weak isospin ***triplet*** field: $\begin{pmatrix} W_{+1}^{\mu} \\ W_{0}^{\mu} \\ W_{-1}^{\mu} \end{pmatrix}$ where: $W_{-1}^{\mu} =$ antimatter field of the W_{+1}^{μ} and:

$\exists$ a weak isospin ***singlet*** field: B_0^{μ}

At ***very high*** energies, there are ***four*** inhomogeneous Maxwell-type equations – one for ***each*** of the four electroweak spin-1/vector fields:

$$\Box^2 W_{+1}^{\mu} = -\mu_o^{w^+} J_{w^+}^{\mu} \quad \text{(massless } W^+\text{)}$$

$$\Box^2 W_{0}^{\mu} = -\mu_o^{w^0} J_{w^0}^{\mu} \quad \text{(massless } W^0\text{)}$$

$$\Box^2 W_{-1}^{\mu} = -\mu_o^{w^-} J_{w^-}^{\mu} \quad \text{(massless } W^-\text{)}$$

$$\Box^2 B_{0}^{\mu} = -\mu_o^{B^0} J_{B^0}^{\mu} \quad \text{(massless } B^0\text{)}$$

The "standard model" of electroweak interactions also predicts that there exists a (complex) ***scalar*** doublet {*a.k.a.* complex Higgs field}, thought to pervade all space-time in our universe – {the new æther!} which, *e.g.* as the very early universe ***cooled***, this ***scalar field*** underwent a ***phase transition*** ~ analogous to that of the phase transition associated *e.g.* with a ferromagnetic material cooling through its Curie temperature. During this EWK phase transition, the scalar Higgs field φ is "absorbed" by the longitudinal {*i.e.* spin-0} components of the W^+, W^- spin-$1\hbar$ vector bosons, thus each acquired mass, *i.e.* becoming the ***massive*** $W^{\pm}$ bosons, whose relativistic fields (at low energies/below the phase transition) are: $W_{\pm}^{\mu} = \frac{1}{\sqrt{2}}\left(W_{+1}^{\mu} \mp W_{-1}^{\mu}\right)$.

Since there are two neutral vector bosons, ***below*** the energy associated with the EWK phase transition, the two neutral bosons will in general be ***orthogonal linear combinations*** of the two original W_0^{μ} and B_0^{μ} fields (*i.e.* they will be {***orthogonal***} mixtures of these two neutral fields). Because of the differences between weak isospin triplet *vs*. isospin singlet, only one of the orthogonal linear combinations of the W_0^{μ} and B_0^{μ} fields "absorbs" the scalar Higgs field as its longitudinal / spin-0 component, thus becoming the massive Z^0 boson: $Z^{\mu} = -B_0^{\mu}\sin\theta_W + W_0^{\mu}\cos\theta_W$. The other neutral vector boson, also as a linear (but orthogonal) combination of the original W_0^{μ} and B_0^{μ} fields remains ***massless*** – it does ***not*** couple {***directly***} to the scalar Higgs field – and becomes the photon: $A^{\mu} = B_0^{\mu}\cos\theta_W + W_0^{\mu}\sin\theta_W$!!!

Note also that, due to the intrinsic nature of the weak isospin ***triplet***, the {now} massive electrically-charged $W^{\pm}$ and electrically-neutral Z^0 spin-$1\hbar$ vector bosons ***all*** carry ***weak*** charge.

The electrically-neutral weak isospin ***singlet*** particle {the photon} does not carry any weak charge, nor does it carry electric charge.

Thus, at ***low*** energies: $\boxed{E_{CM} \ll \langle 0|\varphi|0\rangle = 246\,GeV}$ (below the energy of the EWK phase transition)

where: $\langle 0|\varphi|0\rangle = 246\,GeV$ = vacuum expectation value of the Higgs field, the ***inhomogeneous*** at

At ***low*** energies, Maxwell equations for the four spin-$1\hbar$ EWK vector boson fields become:

$$\boxed{\Box^2 W_\pm^\mu - \mu_{W^\pm}^2 W_\pm^\mu = -\mu_o^{W^+} J_{W^\pm}^\mu} \qquad \boxed{M_{W^\pm} \simeq 80.4\,GeV/c^2}, \quad \boxed{\mu_{W^\pm} = M_{W^\pm}c^2/\hbar c}$$

$$\boxed{\Box^2 Z^\mu - \mu_{Z^0}^2 Z^\mu = -\mu_o^{Z^0} J_{Z^0}^\mu} \qquad \boxed{M_{Z^0} \simeq 90.2\,GeV/c^2}, \quad \boxed{\mu_{Z^0} = M_{Z^0}c^2/\hbar c}$$

$$\boxed{\Box^2 A_\gamma^\mu = -\mu_o J_\gamma^\mu} \qquad \boxed{M_\gamma \equiv 0.0\,GeV/c^2}, \quad \boxed{\mu_\gamma = M_\gamma c^2/\hbar c \equiv 0}$$

<u>The Gravitational Interaction</u>:

We an also {attempt} to do this same thing for gravity – the massless graviton is a spin-$2\hbar$ particle {it must be spin-$2\hbar$ because the gravitational force is ***attractive***, ***only***}. Then, like the weak $W^\pm$, Z^0 vector bosons and the spin-$1\hbar$ gluon, which carry their own respective charges {thus self-coupling to themselves (!!!)}, the graviton similarly carries gravitational charge, and thus also self-couples to itself. However, because gravity is a spin-$2\hbar$ ***tensor*** field, we thus need ***two*** indices μ, ν in order to fully describe the relativistic space-time structure of the gravitational field $g^{\mu\nu}$ and its associated rank-2 tensor "matter" charges / "matter" currents $J^{\mu\nu}$.

Thus, the ***inhomogenous*** Maxwell equation for gravity is:

$$\boxed{\Box^2 g^{\mu\nu} = -\mu_o^G J^{\mu\nu}} \qquad \boxed{M_g \equiv 0.0\,GeV/c^2}, \quad \boxed{\mu_g = M_g c^2/\hbar c \equiv 0}$$

Today, a more modern approach describes the nature of fundamental interactions using relativistic versions of the classical dynamics Lagrangian: $\boxed{L \equiv T - V}$ and the Euler-Lagrange equation:

<u>Classical Dynamics Euler-Lagrange Equation</u>: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$ q_i = generalized coordinates

<u>Classical Dynamics</u>: $\boxed{L(q_i, \dot{q}_i, t)}$ $\Rightarrow$ <u>Relativistic Dynamics</u>: $\boxed{\mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial x_\mu}, x_\mu\right)}$ = Lagrangian ***density***

<u>Relativistic Dynamics Euler-Lagrange Equation</u>: $\boxed{\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0}$

SI units of Lagrangian ***density*** $\mathcal{L}$ (*Joules*/m^3) $\Rightarrow$ SI units of Lagrangian $\boxed{L = \int_v \mathcal{L}\, d\tau}$ (*Joules*).

Thus, writing the interactions associated with the four fundamental forces in terms of their relativistic Lagrangian ***densities***:

n.b. If plug $\mathcal{L}_{QED}$ into Euler-Lagrange equation can show: $\partial_\mu F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \mu_o J^\nu$ and $\partial_\mu J^\mu = 0$!!!

1.) *EM* Interaction: $\mathcal{L}_{QED} = -\frac{1}{4} \underbrace{F_{\mu\nu}F^{\mu\nu}}_{EM\ \text{fields}} - \underbrace{J^\mu A_\mu}_{\text{Current/charge interacting with the } EM \text{ field!}}$ ⇐

EM fields Current/charge interacting with the *EM* field!

2.) Strong Interaction: $\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - J^\mu_a G^a_\mu$ {and other terms}

3.) Weak Interaction: $\mathcal{L}_{Weak} = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - J^\mu_{wa} W^a_\mu - J'_z B_\mu$ {and other terms}

4.) Gravitational Interaction: $\mathcal{L}_{grav} = \frac{c^4}{16\pi G_N} \sqrt{g}\, g^{\mu\nu} R_{\mu\nu}(g)$ where: $\sqrt{g} \equiv \sqrt{-Det(g_{\mu\nu})}$

and: $R_{\mu\nu}(g) = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{\mu\nu} - \Gamma^\beta_{\nu\alpha}\Gamma^\alpha_{\mu\beta}$ and: $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu}\right)$.

OPEN QUESTIONS:

- ***Why*** are there ***four*** fundamental forces of nature? Why not just ***one***??? ∃ ***more***???
- ***Why*** are ***all*** fundamental **forces** mediated by ***integer*** spin-particles? {**bosons**}
  Why are ***all*** fundamental **matter** particles spin-1/2$\hbar$? {quarks & leptons: **fermions**}
  Why are ***no*** forces mediated by spin-1/2$\hbar$, spin-3/2$\hbar$? (⇒ ∃ SUSY {Supersymmetry}???)
- What ***precisely is charge*** (electric, strong, weak)?
- What ***precisely is intrinsic spin angular momentum*** (spin-1/2 fermions, spin-1 bosons)?
- ***Why*** are the $W^\pm, Z^0$ bosons ***massive***, while the γ is ***massless***? ⇐ "EWK symmetry-breaking"
  Carry weak charge Carries no net electrical charge
  The gluon is ***massless***, but carries strong charge (which is ***confined/screened*** beyond ~ 1 *fm*)

⇒ Relativity and the microscopic world of particle physics / fundamental forces,
and quantum mechanics ***are*** indeed at work in the "everyday" world all around us !!!
We simply weren't aware of this before !!!