LECTURE NOTES 20

RELATIVISTIC ELECTRODYNAMICS of MOVING MACROSCOPIC MEDIA

We have discussed the relativistic aspects of electrodynamics associated with moving electric charges and electric currents: $\boxed{J^{\mu} = \left(c\rho, \vec{J}\right)}$.

For ***macroscopic*** media, restricting ourselves here (for simplicity/clarity's sake) to ***linear***, ***homogeneous***, ***isotropic*** materials, such as "class A" dielectrics and magnetic materials that are described in classical electrodynamics by the auxiliary 3-D macroscopic fields:

$$\boxed{\vec{D} \equiv \varepsilon \vec{E}} \qquad \boxed{\vec{H} \equiv \frac{1}{\mu}\vec{B}}$$

n.b. $\varepsilon, \mu, K_e, K_m, \chi_e, \chi_m$ are ***not*** Lorentz invariant quantities !!!
c, ε_o, μ_o ***are*** Lorentz invariant !!!

ε = ***scalar*** quantity = macroscopic electric permittivity of the material.	μ = ***scalar*** quantity = macroscopic magnetic permeability of the material.
$\boxed{K_e \equiv \varepsilon/\varepsilon_o}$ = ***scalar*** quantity = relative electric permittivity of the material (*a.k.a.* dielectric "constant").	$\boxed{K_m \equiv \mu/\mu_o}$ = ***scalar*** quantity = relative magnetic permeability of the material.
$\boxed{\chi_e \equiv K_e - 1}$ = ***scalar*** quantity = electric susceptibility of the material. $\boxed{\chi_e = \varepsilon/\varepsilon_o - 1}$ or: $\boxed{\varepsilon = \varepsilon_o\left(1+\chi_e\right)}$	$\boxed{\chi_m \equiv K_m - 1}$ = ***scalar*** quantity = magnetic susceptibility of the material. $\boxed{\chi_m = \mu/\mu_o - 1}$ or: $\boxed{\mu = \mu_o\left(1+\chi_m\right)}$

Note that: $\boxed{\varepsilon, \mu, K_e, K_m, \chi_e, \chi_m}$ are ***all*** defined in the ***rest/proper frame*** of the ***linear*** material.

Note also that: $\boxed{c = 1/\sqrt{\varepsilon_o \mu_o}}$, ε_o μ_o and hence: $\boxed{Z_o = \sqrt{\mu_o/\varepsilon_o}}$ are ***all*** Lorentz invariant quantities:

$\boxed{c = 1/\sqrt{\varepsilon_o \mu_o}}$ = speed of light/EM waves in free space/vacuum = $3.0\times10^{-8}\ m/s$.

$\varepsilon_o \equiv$ macroscopic electric permittivity of free space/vacuum = $8.85\times10^{-12}\ F/m$.

$\mu_o \equiv$ macroscopic magnetic permeability of free space/vacuum = $4\pi\times10^{-7}\ H/m$.

$\boxed{Z_o = \sqrt{\mu_o/\varepsilon_o}}$ = macroscopic impedance of free space/vacuum = $120\pi\ \Omega \simeq 377\ \Omega$.

Thus, for ***linear*** macroscopic materials, the following 3-D vector quantities are defined in the ***rest/proper frame*** of the ***linear*** material:

$$\boxed{\vec{D}(\vec{r},t) = \varepsilon_o\left(1+\chi_e\right)\vec{E}(\vec{r},t)} \quad \text{and:} \quad \boxed{\vec{B}(\vec{r},t) = \mu_o\left(1+\chi_m\right)\vec{H}(\vec{r},t)}$$

$\boxed{\vec{\mathrm{P}}(\vec{r},t) \equiv \varepsilon_o \chi_e \vec{E}(\vec{r},t) = n\langle \vec{\mathrm{p}}(\vec{r},t)\rangle}$ = electric polarization = electric dipole moment per unit volume.

$\boxed{\vec{\mathrm{M}}(\vec{r},t) \equiv \chi_m \vec{H}(\vec{r},t) = n\langle \vec{m}(\vec{r},t)\rangle}$ = magnetization = magnetic dipole moment per unit volume.

Thus, we obtain the {usual} constitutive/auxilliary relations for linear media defined in the ***rest/proper*** ***frame*** of the linear material:

$$\boxed{\vec{D}(\vec{r},t)=\varepsilon_o\vec{E}(\vec{r},t)+\vec{\mathrm{P}}(\vec{r},t)} \quad \text{and:} \quad \boxed{\vec{H}(\vec{r},t)=\frac{1}{\mu_o}\vec{B}(\vec{r},t)-\vec{\mathrm{M}}(\vec{r},t)}$$

In order to go over to a relativistic formulation of these relations, we ***must*** be careful/precise in their physical meaning. In particular, the electric permittivity, ε and magnetic permeability, μ are defined in the rest frame of the material – *i.e.* ε is the ***proper*** electric permittivity and μ is the ***proper*** magnetic permeability.

Thus, the macroscopic fields $\vec{D},\vec{\mathrm{P}},\vec{E}$ and $\vec{H},\vec{\mathrm{M}},\vec{B}$ are all defined in the ***rest/proper*** ***frame*** of the ***linear*** material – *i.e.* they are ***proper*** macroscopic electromagnetic fields.

Because we already know/understand the Lorentz transformation properties of $\vec{E}$ and $\vec{B}$, we can write the auxiliary/constitutive relations as:

$$\boxed{\vec{E}(\vec{r},t)=\frac{1}{\varepsilon_o}\left(\vec{D}(\vec{r},t)-\vec{\mathrm{P}}(\vec{r},t)\right)} \quad \text{and:} \quad \boxed{\vec{B}(\vec{r},t)=\mu_o\left(\vec{H}(\vec{r},t)+\vec{\mathrm{M}}(\vec{r},t)\right)}$$

The relativistic *EM* field tensor $F^{\mu\nu}$ is:

$$\boxed{F^{\mu\nu}\equiv\begin{pmatrix}0 & E_x/c & E_y/c & E_z/c\\ -E_x/c & 0 & B_z & -B_y\\ -E_y/c & -B_z & 0 & B_x\\ -E_z/c & B_y & -B_x & 0\end{pmatrix}}$$

and its relativistic ***dual*** tensor $G^{\mu\nu}$ is:

$$\boxed{G^{\mu\nu}\equiv\begin{pmatrix}0 & B_x & B_y & B_z\\ -B_x & 0 & -E_z/c & E_y/c\\ -B_y & E_z/c & 0 & -E_x/c\\ -B_z & -E_y/c & E_x/c & 0\end{pmatrix}=\frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}}$$

We construct a rank-2 anti-symmetric tensor $D^{\mu\nu}$ (analogous to $F^{\mu\nu}$) for the $\vec{D}$ and $\vec{H}$ fields. It describes the (relativistic six-component) "electromagnetic displacement" field $D^{\mu\nu}$.

{*n.b.* again, comparing forms in different textbooks, variations for $D^{\mu\nu}$ will be found that depend on the choice/definition of the metric $g^{\mu\nu}$ used and conventions *r.e.* overall constants, *etc.* !!! }.

Taken together, the auxiliary/constitutive relations $\left\{\begin{array}{l}\vec{E}=\frac{1}{\varepsilon_o}\left(\vec{D}-\vec{\mathrm{P}}\right)\\ \vec{B}=\mu_o\left(\vec{H}+\vec{\mathrm{M}}\right)\end{array}\right\}$ suggest replacing $\left\{\begin{array}{l}\vec{E}\rightarrow\frac{1}{\varepsilon_o}\vec{D}\\ \vec{B}\rightarrow\mu_o H\end{array}\right\}$

However, it is a little "cleaner" if we ***also*** divide through by μ_o, *i.e.* replace $\left\{\begin{array}{l}\vec{E}\rightarrow\frac{1}{\mu_o\varepsilon_o}\vec{D}=c^2\vec{D}\\ \vec{B}\rightarrow\vec{H}\end{array}\right\}$.

Then $D^{\mu\nu}$ becomes: $$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix}$$ **SI units check:** $$D = \frac{Coul}{m^2}, \quad H = \frac{Amps}{m}, \quad cD = \frac{Coul}{m^2}\frac{m}{\sec} = \frac{Amp}{m}$$

We also construct the relativistic ***dual*** of $D^{\mu\nu}$, which is $H^{\mu\nu}$ (the analog of $G^{\mu\nu}$), defined as:

$$H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} D_{\lambda\sigma}$$

The relativistic dual tensor $H^{\mu\nu}$ may also be obtained directly from $D^{\mu\nu}$ by carrying out an appropriate ***duality*** ***transformation*** on the $\vec{D}$ and $\vec{H}$ -fields, analogous to that for the $\vec{E}$ and $\vec{B}$-fields:

$$\begin{Bmatrix} E' = E\cos\varphi_D + cB\sin\varphi_D \\ cB' = cB\cos\varphi_D - E\sin\varphi_D \end{Bmatrix} \quad \varphi_D = -90^\circ = -\frac{\pi}{2} \quad \text{or:} \quad \begin{pmatrix} E' \\ cB' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D & +\sin\varphi_D \\ -\sin\varphi_D & \cos\varphi_D \end{pmatrix} \begin{pmatrix} E \\ cB \end{pmatrix}$$

$$\begin{Bmatrix} cD' = cD\cos\varphi_D + H\sin\varphi_D \\ H' = H\cos\varphi_D - cD\sin\varphi_D \end{Bmatrix} \quad \varphi = -90^\circ = -\frac{\pi}{2} \quad \text{or:} \quad \begin{pmatrix} cD' \\ H' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D & +\sin\varphi_D \\ -\sin\varphi_D & \cos\varphi_D \end{pmatrix} \begin{pmatrix} cD \\ H \end{pmatrix}$$

Then for $\varphi_D = -90^\circ$, we replace: $\boxed{cD \to +H}$ and: $\boxed{H \to -cD}$ in $\boxed{D^{\mu\nu} \to H^{\mu\nu}}$:

$$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix} \Rightarrow H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} D_{\lambda\sigma}$$

Likewise, we may analogously define/construct a rank-2 anti-symmetric tensor $\mathrm{P}^{\mu\nu}$ for the $\vec{\mathrm{P}}$ and $\vec{\mathrm{M}}$ fields describing the (relativistic six-component) electromagnetic “polarization” field $\mathrm{P}^{\mu\nu}$. Since $\vec{\mathrm{P}}$ has the same SI units as $\vec{D}$ $\left(\frac{\text{Coulombs}}{\text{meter}}\right)$ and $\vec{\mathrm{M}}$ has the same SI units as $\vec{H}$ $\left(\frac{\text{Amps}}{\text{meter}}\right)$ we see that:

$$\mathrm{P}^{\mu\nu} \equiv \begin{pmatrix} 0 & c\mathrm{P}_x & c\mathrm{P}_y & c\mathrm{P}_z \\ -c\mathrm{P}_x & 0 & -\mathrm{M}_z & \mathrm{M}_y \\ -c\mathrm{P}_y & \mathrm{M}_z & 0 & -\mathrm{M}_x \\ -c\mathrm{P}_z & -\mathrm{M}_y & \mathrm{M}_x & 0 \end{pmatrix}$$

Note the **minus sign reversal** here for $\mathrm{M}_x, \mathrm{M}_y, \mathrm{M}_z$ relative to $\mathrm{P}_x, \mathrm{P}_y, \mathrm{P}_z$ due to $\boxed{\vec{D} = \varepsilon_o \vec{E} + \vec{\mathrm{P}}}$ *vs.* $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}}$!!!

Again, we can construct a relativistic ***dual*** tensor of $\text{P}^{\mu\nu}$ which we call $\text{M}^{\mu\nu}$ {the analog of $G^{\mu\nu}$, for $F^{\mu\nu}$ and/or $H^{\mu\nu}$, for $D^{\mu\nu}$}, defined as:

$$\text{M}^{\mu\nu} \equiv \begin{pmatrix} 0 & -\text{M}_x & -\text{M}_y & -\text{M}_z \\ \text{M}_x & 0 & -c\text{P}_z & c\text{P}_y \\ \text{M}_y & c\text{P}_z & 0 & -c\text{P}_x \\ \text{M}_z & -c\text{P}_y & c\text{P}_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\text{P}_{\lambda\sigma}$$

Again, note the **minus sign reversal** here for $\text{M}_x, \text{M}_y, \text{M}_z$ relative to $\text{P}_x, \text{P}_y, \text{P}_z$ due to $\vec{D} = \varepsilon_o \vec{E} + \vec{\text{P}}$ *vs.* $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$!!!

Again, the relativistic ***dual*** tensor $\text{M}^{\mu\nu}$ can be obtained directly from $\text{P}^{\mu\nu}$ by carrying out the same duality transformation as above, but on the $\vec{\text{P}}$ and $\vec{\text{M}}$ fields. Since $\vec{\text{P}}$ has the same SI units as $\vec{D}$, and $\vec{\text{M}}$ has the same SI units as $\vec{H}$ we see that:

$$\left\{\begin{matrix} -c\text{P}' = -c\text{P}\cos\varphi_D + \text{M}\sin\varphi_D \\ \text{M}' = +\text{M}\cos\varphi_D + c\text{P}\sin\varphi_D \end{matrix}\right\} \varphi = -90^\circ = -\frac{\pi}{2} \quad \text{or:} \quad \begin{pmatrix} -c\text{P}' \\ \text{M}' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D & +\sin\varphi_D \\ -\sin\varphi_D & \cos\varphi_D \end{pmatrix}\begin{pmatrix} -c\text{P} \\ \text{M} \end{pmatrix}$$

Then for $\varphi_D = -90^\circ$, we replace: $c\text{P} \rightarrow -\text{M}$ and: $\text{M} \rightarrow +c\text{P}$ in $\text{P}^{\mu\nu} \rightarrow \text{M}^{\mu\nu}$:

$$\text{P}^{\mu\nu} \equiv \begin{pmatrix} 0 & c\text{P}_x & c\text{P}_y & c\text{P}_z \\ -c\text{P}_x & 0 & -\text{M}_z & \text{M}_y \\ -c\text{P}_y & \text{M}_z & 0 & -\text{M}_x \\ -c\text{P}_z & -\text{M}_y & \text{M}_x & 0 \end{pmatrix} \Rightarrow \text{M}^{\mu\nu} \equiv \begin{pmatrix} 0 & -\text{M}_x & -\text{M}_y & -\text{M}_z \\ \text{M}_x & 0 & -c\text{P}_z & c\text{P}_y \\ \text{M}_y & c\text{P}_z & 0 & -c\text{P}_x \\ \text{M}_z & -c\text{P}_y & c\text{P}_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\text{P}_{\lambda\sigma}$$

From the classical electrodynamic auxiliary/constitutive relations: $\vec{D} = \varepsilon_o \vec{E} + \vec{\text{P}}$ and: $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{\text{M}}$

or: $\vec{E} = \frac{1}{\varepsilon_o}\left(\vec{D} - \vec{\text{P}}\right)$ and: $\vec{B} = \mu_o\left(\vec{H} + \vec{\text{M}}\right)$ we see that their (combined) relativistic equivalents, in terms of the *EM* field tensors and their ***duals*** are:

$$D^{\mu\nu} = \frac{1}{\mu_o}F^{\mu\nu} + \text{P}^{\mu\nu} \quad \text{and:} \quad H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + \text{M}^{\mu\nu}$$

n.b. ***must*** be + here !!!

$$\text{or:} \quad F^{\mu\nu} = \mu_o\left(D^{\mu\nu} - \text{P}^{\mu\nu}\right) \quad \text{and:} \quad G^{\mu\nu} = \mu_o\left(H^{\mu\nu} - \text{M}^{\mu\nu}\right)$$

n.b. If we ***define*** $D^{\mu\nu} \equiv \frac{1}{\mu_o}F^{\mu\nu} + \text{P}^{\mu\nu}$ then we ***must*** have the + sign in $H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}$

because the ***dual*** *EM* field tensor $G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$

connects to: $G^{\mu\nu} = \mu_o\left(H^{\mu\nu} - \text{M}^{\mu\nu}\right) = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma} = \frac{1}{2}\mu_o\varepsilon^{\mu\nu\lambda\sigma}\left(D_{\lambda\sigma} - \text{P}_{\lambda\sigma}\right)$.

Thus the macroscopic relativistic *EM* field tensors and their ***duals*** are:

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

$$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix}$$

$$H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}D_{\lambda\sigma}$$

$$\mathrm{P}^{\mu\nu} \equiv \begin{pmatrix} 0 & c\mathrm{P}_x & c\mathrm{P}_y & c\mathrm{P}_z \\ -c\mathrm{P}_x & 0 & -\mathrm{M}_z & \mathrm{M}_y \\ -c\mathrm{P}_y & \mathrm{M}_z & 0 & -\mathrm{M}_x \\ -c\mathrm{P}_z & -\mathrm{M}_y & \mathrm{M}_x & 0 \end{pmatrix}$$

$$\mathrm{M}^{\mu\nu} \equiv \begin{pmatrix} 0 & -\mathrm{M}_x & -\mathrm{M}_y & -\mathrm{M}_z \\ \mathrm{M}_x & 0 & -c\mathrm{P}_z & c\mathrm{P}_y \\ \mathrm{M}_y & c\mathrm{P}_z & 0 & -c\mathrm{P}_x \\ \mathrm{M}_z & -c\mathrm{P}_y & c\mathrm{P}_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\mathrm{P}_{\lambda\sigma}$$

Let us explicitly check the correctness/validity of these relations – assuming things are internally correct within ***each*** of the *EM* field tensors $T^{\mu\nu}$, then we only need to check two non-zero elements:

For the T^{01} components:

$$D^{01} = \frac{1}{\mu_o}F^{01} + \mathrm{P}^{01}$$
$$cD_x = \frac{1}{\mu_0}\frac{E_x}{c} + c\mathrm{P}_x$$
$$D_x = \frac{1}{\mu_o c^2}E_x + \mathrm{P}_x$$
$$D_x = \varepsilon_o E_x + \mathrm{P}_x$$
or:
$$\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$$

$$H^{01} = \frac{1}{\mu_o}G^{01} + \mathrm{M}^{01}$$
$$H_x = \frac{1}{\mu_o}B_x - \mathrm{M}_x$$
or:
$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}$$

For the T^{12} components:

$$D^{12} = \frac{1}{\mu_o}F^{12} + \mathrm{P}^{12}$$
$$H_z = \frac{1}{\mu_o}B_z - \mathrm{M}_z$$
or:
$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$$

$$H^{12} = \frac{1}{\mu_o}G^{12} + \mathrm{M}^{12}$$
$$-cD_z = -\frac{1}{\mu_o c}E_z - c\mathrm{P}_z$$
$$D_z = \frac{1}{\mu_o c^2}E_z + \mathrm{P}_z$$
$$D_z = \varepsilon_o E_z + \mathrm{P}_z$$
or:
$$\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$$

Thus, we obtain the relativistic auxiliary/constitutive relations and their ***duals***:

$$D^{\mu\nu} = \frac{1}{\mu_o}F^{\mu\nu} + \mathrm{P}^{\mu\nu} \quad \text{and:} \quad H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + \mathrm{M}^{\mu\nu}$$

or: $$F^{\mu\nu} = \mu_o\left(D^{\mu\nu} - \mathrm{P}^{\mu\nu}\right) \quad \text{and:} \quad G^{\mu\nu} = \mu_o\left(H^{\mu\nu} - \mathrm{M}^{\mu\nu}\right)$$

Lorentz Transformations of the Relativistic Macroscopic Electrodynamical Fields $\{F^{\mu\nu}, D^{\mu\nu}, \mathrm{P}^{\mu\nu}\}$ and Their Duals $\{G^{\mu\nu}, H^{\mu\nu}, \mathrm{M}^{\mu\nu}\}$

We already know/have shown that in the relativistic *EM* fields in frame IRF(*S′*) are related to those in IRF(*S*) via: $\boxed{F'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} F^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$ and $\boxed{G'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} G^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$, where $\Lambda^{\mu}{}_{\nu} =$ the Lorentz transformation tensor.

For ***linear*** macroscopic media, we have ***linear*** relations between $\{\vec{D}, \vec{E}, \vec{\mathrm{P}}\}$ and also between $\{\vec{H}, \vec{B}, \vec{\mathrm{M}}\}$, which hold relativistically between $\boxed{D^{\mu\nu} = \frac{1}{\mu_o} F^{\mu\nu} + \mathrm{P}^{\mu\nu}}$ and also $\boxed{H^{\mu\nu} = \frac{1}{\mu_o} G^{\mu\nu} + \mathrm{M}^{\mu\nu}}$.

$\therefore$ We explicitly see using the Lorentz transformations of individual field quantities defined in IRF(*S*) to the frame IRF(*S′*): $\boxed{D'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} D^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$, $\boxed{F'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} F^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$ and: $\boxed{\mathrm{P}'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} \mathrm{P}^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$

That: $\boxed{D^{\mu\nu} = \frac{1}{\mu_o} F^{\mu\nu} + \mathrm{P}^{\mu\nu}} \Rightarrow \boxed{\Lambda^{\mu}{}_{\lambda} \left[D^{\lambda\sigma} = \frac{1}{\mu_o} F^{\mu\nu} + \mathrm{P}^{\mu\nu} \right] \Lambda^{\nu}{}_{\sigma}}$

$$\Rightarrow \boxed{\Lambda^{\mu}{}_{\lambda} D^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma} = \frac{1}{\mu_o} \Lambda^{\mu}{}_{\lambda} F^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma} + \Lambda^{\mu}{}_{\lambda} \mathrm{P}^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}} \Rightarrow \boxed{D'^{\mu\nu} = \frac{1}{\mu_o} F'^{\mu\nu} + \mathrm{P}'^{\mu\nu}}$$ holds in IRF(S').

Likewise, for the ***dual*** tensors we explicitly see using the Lorentz transformations of individual field quantities defined in IRF(*S*) to the frame IRF(*S′*): $\boxed{H'^{\mu\nu} = \Lambda^{\mu}_{\lambda} H^{\lambda\sigma} \Lambda^{\nu}_{\sigma}}$, $\boxed{G'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} G^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$ and: $\boxed{\mathrm{M}'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda} \mathrm{M}^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}}$

That: $\boxed{H^{\mu\nu} = \frac{1}{\mu_o} G^{\mu\nu} + \mathrm{M}^{\mu\nu}} \Rightarrow \boxed{\Lambda^{\mu}{}_{\lambda} \left[H^{\mu\nu} = \frac{1}{\mu_o} G^{\mu\nu} + \mathrm{M}^{\mu\nu} \right] \Lambda^{\nu}{}_{\sigma}}$

$$\Rightarrow \boxed{\Lambda^{\mu}{}_{\lambda} H^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma} = \frac{1}{\mu_o} \Lambda^{\mu}{}_{\lambda} G^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma} + \Lambda^{\mu}{}_{\lambda} \mathrm{M}^{\lambda\sigma} \Lambda^{\nu}{}_{\sigma}} \Rightarrow \boxed{H'^{\mu\nu} = \frac{1}{\mu_o} G'^{\mu\nu} + \mathrm{M}'^{\mu\nu}}$$ holds in IRF(S').

Equivalently, we can ***alternatively*** express these as a ***simple*** Lorentz transformation of the macroscopic $\{\vec{E}, \vec{D}, \vec{\mathrm{P}}\}$ and $\{\vec{B}, \vec{H}, \vec{\mathrm{M}}\}$ fields, *e.g.* for a Lorentz transformation from the ***rest/proper frame*** IRF(*S*) where the macroscopic $\{\vec{E}, \vec{D}, \vec{\mathrm{P}}\}$ and $\{\vec{B}, \vec{H}, \vec{\mathrm{M}}\}$ fields are defined, to another IRF(S') moving with velocity $\vec{v} \parallel$ to one of the $\hat{x}, \hat{y}$, or $\hat{z}$ axes (*e.g.* $\vec{v} = +v\hat{x}$).

We have already shown {in P436 Lecture Notes 19 *p*. 1-3} that the components of $\vec{E}$ and $\vec{B}$ that are || and $\perp$ to the boost direction $(\vec{v})$ transform as:

IRF(S'): IRF(S): IRF(S'): IRF(S):

$$E'^{\parallel} = E^{\parallel} \qquad B'^{\parallel} = B^{\parallel}$$

$$E'^{\perp} = \gamma\left(E^{\perp} + \left(\vec{v}\times\vec{B}\right)^{\perp}\right) = \gamma\left(E^{\perp} + \left(\vec{\beta}\times c\vec{B}\right)^{\perp}\right)$$

$$B'^{\perp} = \gamma\left(B^{\perp} - \frac{1}{c^2}\left(\vec{v}\times\vec{E}\right)^{\perp}\right) = \gamma\left(B^{\perp} - \left(\vec{\beta}\times\frac{\vec{E}}{c}\right)^{\perp}\right)$$

where: $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and: $\beta = \dfrac{v}{c}$, $\vec{\beta} = \dfrac{\vec{v}}{c}$

Thus:

IRF(S'): IRF(S): IRF(S'): IRF(S):

$$D'^{\parallel} = D^{\parallel} \qquad H'^{\parallel} = H^{\parallel}$$

$$D'^{\perp} = \gamma\left(D^{\perp} + \frac{1}{c^2}\left(\vec{v}\times\vec{H}\right)^{\perp}\right) = \gamma\left(D^{\perp} + \frac{1}{2}\left(\vec{\beta}\times\vec{H}\right)^{\perp}\right)$$

$$H'^{\perp} = \gamma\left(H^{\perp} - \left(\vec{v}\times\vec{D}\right)^{\perp}\right) = \gamma\left(H^{\perp} - c\left(\vec{\beta}\times\vec{D}\right)^{\perp}\right)$$

And similarly:

IRF(S'): IRF(S): IRF(S'): IRF(S):

$$\mathrm{P}'^{\parallel} = \mathrm{P}^{\parallel} \qquad \mathrm{M}'^{\parallel} = \mathrm{M}^{\parallel}$$

$$\mathrm{P}'^{\perp} = \gamma\left(\mathrm{P}^{\perp} - \frac{1}{c^2}\left(\vec{v}\times\vec{\mathrm{M}}\right)^{\perp}\right) = \gamma\left(\mathrm{P}^{\perp} - \frac{1}{c}\left(\vec{\beta}\times\vec{\mathrm{M}}\right)^{\perp}\right)$$

$$\mathrm{M}'^{\perp} = \gamma\left(\mathrm{M}^{\perp} + \left(\vec{v}\times\vec{\mathrm{P}}\right)^{\perp}\right) = \gamma\left(\mathrm{M}^{\perp} + c\left(\vec{\beta}\times\vec{\mathrm{P}}\right)^{\perp}\right)$$

n.b. note the sign reversals here relative to the above relations !!!

n.b. We see from these two relations that in the ***rest/proper*** frame IRF(S_0) of a polarized dielectric with electric polarization $\vec{\mathrm{P}}_0$ (or a magnetized material with magnetization $\vec{\mathrm{M}}_0$) if the linear material is moving *e.g*. with velocity $\vec{v} = +v\hat{x}$ in the lab frame IRF(S), an observer in the lab frame will "see" a ***combination*** of electric polarization $\vec{\mathrm{P}}$ and magnetization $\vec{\mathrm{M}}$ in ***both*** cases!

And we ***also*** have:

In IRF(S'):

$$\vec{D}' = \varepsilon_o\vec{E}' + \vec{P}' \qquad \vec{H}' = \frac{1}{\mu_o}\vec{B}' - \vec{M}'$$

In IRF(S):

$$\vec{D} = \varepsilon_o\vec{E} + \vec{P} \qquad \vec{H} = \frac{1}{\mu_o}\vec{B} - M$$

n.b. $\varepsilon_o, \mu_o, c = \dfrac{1}{\sqrt{\varepsilon_o\mu_o}}$ are ***all*** Lorentz invariant quantities.

Relativistic Invariants Associated with Macroscopic Relativistic *EM* Fields

We have already shown/discussed that for the $\vec{E}$ and $\vec{B}$-field tensors $F^{\mu\nu}$ and $G^{\mu\nu}$, there are two and ***only*** two relativistic Lorentz invariant quantities:

1) $$F^{\mu\nu}F_{\mu\nu} = F_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}G_{\mu\nu} = -G_{\mu\nu}G^{\mu\nu} = 2\left(B^2 - \left(\frac{E}{c}\right)^2\right)$$ ⇐ *EM* field energy density ***difference*** $U_E - U_M$

2) $$F^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}G^{\mu\nu} = G^{\mu\nu}F_{\mu\nu} = G_{\mu\nu}F^{\mu\nu} = -\frac{4}{c}\left(\vec{E}\bullet\vec{B}\right)$$ ⇐ Transversality of *E* and *B*

Since the (inner) product of a ***contravariant*** tensor of rank-n with ***any other covariant*** tensor of the same rank-*n* is a Lorentz invariant quantity (*i.e.* ***same*** value in ***any*** inertial reference frame, we see that for $\{F^{\mu\nu}, D^{\mu\nu}, \mathrm{P}^{\mu\nu}\}$ and their ***duals*** $\{G^{\mu\nu}, H^{\mu\nu}, \mathrm{M}^{\mu\nu}\}$ we can literally have a "field day" / go ***wild {!!!}*** and form ***many*** additional Lorentz invariant quantities, such as:

3) $$D^{\mu\nu}D_{\mu\nu} = D_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}H_{\mu\nu} = -H_{\mu\nu}H^{\mu\nu} = 2\left(H^2 - (cD)^2\right)$$

4) $$D^{\mu\nu}H_{\mu\nu} = D_{\mu\nu}H^{\mu\nu} = H^{\mu\nu}D_{\mu\nu} = H_{\mu\nu}D^{\mu\nu} = -4c\left(\vec{D}\bullet\vec{H}\right)$$

5) $$\mathrm{P}^{\mu\nu}\mathrm{P}_{\mu\nu} = \mathrm{P}_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{M}^{\mu\nu}\mathrm{M}_{\mu\nu} = -\mathrm{M}_{\mu\nu}\mathrm{M}^{\mu\nu} = 2\left(\mathrm{M}^2 - (c\mathrm{P})^2\right)$$

6) $$\mathrm{P}^{\mu\nu}\mathrm{M}_{\mu\nu} = \mathrm{P}_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}\mathrm{P}_{\mu\nu} = \mathrm{M}_{\mu\nu}\mathrm{P}^{\mu\nu} = +4c\left(\vec{\mathrm{P}}\bullet\vec{\mathrm{M}}\right)$$ ← See these lecture notes, *p*.10 for details.

We can also form the "crossed" Lorentz invariants:

7) $$F^{\mu\nu}D_{\mu\nu} = F_{\mu\nu}D^{\mu\nu} = D^{\mu\nu}F_{\mu\nu} = D_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}H_{\mu\nu} = -G_{\mu\nu}H^{\mu\nu} = -H^{\mu\nu}G_{\mu\nu} = -H_{\mu\nu}G^{\mu\nu}$$

8) $$F^{\mu\nu}\mathrm{P}_{\mu\nu} = F_{\mu\nu}\mathrm{P}^{\mu\nu} = \mathrm{P}^{\mu\nu}F_{\mu\nu} = \mathrm{P}_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}\mathrm{M}_{\mu\nu} = -G_{\mu\nu}\mathrm{M}^{\mu\nu} = -\mathrm{M}^{\mu\nu}G_{\mu\nu} = -\mathrm{M}_{\mu\nu}G^{\mu\nu}$$

9) $$D^{\mu\nu}\mathrm{P}_{\mu\nu} = D_{\mu\nu}\mathrm{P}^{\mu\nu} = \mathrm{P}^{\mu\nu}D_{\mu\nu} = \mathrm{P}_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}\mathrm{M}_{\mu\nu} = -H_{\mu\nu}\mathrm{M}^{\mu\nu} = -\mathrm{M}^{\mu\nu}H_{\mu\nu} = -\mathrm{M}_{\mu\nu}H^{\mu\nu}$$

As well as:

10) $$F^{\mu\nu}\mathrm{M}_{\mu\nu} = F_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}F_{\mu\nu} = \mathrm{M}_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}\mathrm{P}_{\mu\nu} = -G_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{P}^{\mu\nu}G_{\mu\nu} = -\mathrm{P}_{\mu\nu}G^{\mu\nu}$$

11) $$D^{\mu\nu}G_{\mu\nu} = D_{\mu\nu}G^{\mu\nu} = G^{\mu\nu}D_{\mu\nu} = G_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}F_{\mu\nu} = -H_{\mu\nu}F^{\mu\nu} = -F^{\mu\nu}H_{\mu\nu} = -F_{\mu\nu}H^{\mu\nu}$$

12) $$D^{\mu\nu}\mathrm{M}_{\mu\nu} = D_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}D_{\mu\nu} = \mathrm{M}_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}\mathrm{P}_{\mu\nu} = -H_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{P}^{\mu\nu}H_{\mu\nu} = -\mathrm{P}_{\mu\nu}H^{\mu\nu}$$

Thus, we can form a total of 12 unique ***bi-linear*** Lorentz invariant quantities using the relativistic macroscopic *EM* field tensors $\{F^{\mu\nu}, D^{\mu\nu}, P^{\mu\nu}\}$ and their ***duals*** $\{G^{\mu\nu}, H^{\mu\nu}, M^{\mu\nu}\}$.

We explicitly work out the first ***four*** of these new Lorentz invariant quantities:

3)

$$\begin{aligned}
D^{\mu\nu}D_{\mu\nu} = &+D^{00}D^{00} - D^{01}D^{01} - D^{02}D^{02} - D^{03}D^{03} \\
&- D^{10}D^{10} + D^{11}D^{11} + D^{12}D^{12} + D^{13}D^{13} \\
&- D^{20}D^{20} + D^{21}D^{21} + D^{22}D^{22} + D^{23}D^{23} \\
&- D^{30}D^{30} + D^{31}D^{31} + D^{32}D^{32} + D^{33}D^{33} \\
= &\quad 0 \quad -c^2D_x^2 \quad -c^2D_y^2 \quad -c^2D_z^2 \\
&-c^2D_x^2 \quad +0 \quad +H_z^2 \quad +H_y^2 \\
&-c^2D_y^2 \quad +H_z^2 \quad 0 \quad +H_x^2 \\
&-c^2D_z^2 \quad +H_y^2 \quad +H_x^2 \quad +0 \\
= &\; 2\left(H^2 - c^2D^2\right)
\end{aligned}$$

4)

$$\begin{aligned}
D^{\mu\nu}H_{\mu\nu} = &+D^{00}H^{00} - D^{01}H^{01} - D^{02}H^{02} - D^{03}H^{03} \\
&- D^{10}H^{10} + D^{11}H^{11} + D^{12}H^{12} + D^{13}H^{13} \\
&- D^{20}H^{20} + D^{21}H^{21} + D^{22}H^{22} + D^{23}H^{23} \\
&- D^{30}H^{30} + D^{31}H^{31} + D^{32}H^{32} + D^{33}H^{33} \\
= &\quad 0 \quad -cD_xH_x \quad -c^2D_yH_x \quad -cD_zH_z \\
&-c^2D_x^2 \quad +0 \quad +H_z^2 \quad +H_y^2 \\
&-c^2D_y^2 \quad +H_z^2 \quad 0 \quad +H_x^2 \\
&-c^2D_z^2 \quad +H_y^2 \quad +H_x^2 \quad +0 \\
= &-4c\left(\vec{D}\bullet\vec{H}\right)
\end{aligned}$$

5)

$$\begin{aligned}
\mathrm{P}^{\mu\nu}\mathrm{P}_{\mu\nu} = &+\mathrm{P}^{00}\mathrm{P}^{00} - \mathrm{P}^{01}\mathrm{P}^{01} - \mathrm{P}^{02}\mathrm{P}^{02} - \mathrm{P}^{03}\mathrm{P}^{03} \\
&- \mathrm{P}^{10}\mathrm{P}^{10} + \mathrm{P}^{11}\mathrm{P}^{11} + \mathrm{P}^{12}\mathrm{P}^{12} + \mathrm{P}^{13}\mathrm{P}^{13} \\
&- \mathrm{P}^{20}\mathrm{P}^{20} + \mathrm{P}^{21}\mathrm{P}^{21} + \mathrm{P}^{22}\mathrm{P}^{22} + \mathrm{P}^{23}\mathrm{P}^{23} \\
&- \mathrm{P}^{30}\mathrm{P}^{30} + \mathrm{P}^{31}\mathrm{P}^{31} + \mathrm{P}^{32}\mathrm{P}^{32} + \mathrm{P}^{33}\mathrm{P}^{33} \\
= &\quad 0 \quad -c^2\mathrm{P}_x^2 \quad -c^2\mathrm{P}_y^2 \quad -c^2\mathrm{P}_z^2 \\
&-c^2\mathrm{P}_x^2 \quad +0 \quad +\mathrm{M}_z^2 \quad +\mathrm{M}_y^2 \\
&-c^2\mathrm{P}_y^2 \quad +\mathrm{M}_z^2 \quad +0 \quad +\mathrm{M}_x^2 \\
&-c^2\mathrm{P}_z^2 \quad +\mathrm{M}_y^2 \quad +\mathrm{M}_x^2 \quad +0 \\
= &\; 2\left(\mathrm{M}^2 - c^2\mathrm{P}^2\right)
\end{aligned}$$

6)

$$\begin{aligned}
\mathrm{P}^{\mu\nu}\mathrm{M}_{\mu\nu} &= +\mathrm{P}^{00}\mathrm{M}^{00} - \mathrm{P}^{01}\mathrm{M}^{01} - \mathrm{P}^{02}\mathrm{M}^{02} - \mathrm{P}^{03}\mathrm{M}^{03} \\
&\quad - \mathrm{P}^{10}\mathrm{M}^{10} + \mathrm{P}^{11}\mathrm{M}^{11} + \mathrm{P}^{12}\mathrm{M}^{12} + \mathrm{P}^{13}\mathrm{M}^{13} \\
&\quad - \mathrm{P}^{20}\mathrm{M}^{20} + \mathrm{P}^{21}\mathrm{M}^{21} + \mathrm{P}^{22}\mathrm{M}^{22} + \mathrm{P}^{23}\mathrm{M}^{23} \\
&\quad - \mathrm{P}^{30}\mathrm{M}^{30} + \mathrm{P}^{31}\mathrm{M}^{31} + \mathrm{P}^{32}\mathrm{M}^{32} + \mathrm{P}^{33}\mathrm{M}^{33} \\
&= \quad 0 \quad + \mathrm{P}_x\mathrm{M}_x \quad + c^2\mathrm{P}_y\mathrm{M}_y + c\mathrm{P}_z\mathrm{M}_z \\
&\quad + c\mathrm{P}_x\mathrm{M}_x \quad + 0 \quad + c\mathrm{P}_z M_z \quad + c\mathrm{P}_y\mathrm{M}_y \\
&\quad + c\mathrm{P}_y\mathrm{M}_y + c\mathrm{P}_z\mathrm{M}_z \quad + \ 0 \quad + c\mathrm{P}_x\mathrm{M}_x \\
&\quad + c\mathrm{P}_z\mathrm{M}_z + c\mathrm{P}_y\mathrm{M}_y \quad + c\mathrm{P}_x\mathrm{M}_x \quad + 0 \\
&= +4c\left(\vec{\mathrm{P}}\bullet\vec{\mathrm{M}}\right)
\end{aligned}$$

We can also form "higher-order" tri-linear and quadri-linear invariants from combinations of $\left\{F^{\mu\nu}, D^{\mu\nu}, P^{\mu\nu}\right\}$ and $\left\{G^{\mu\nu}, H^{\mu\nu}, M^{\mu\nu}\right\}$, *e.g.*:

$F^{\mu\nu}D_{\mu\lambda}\mathrm{P}_\nu^{\ \lambda}$ (and all allowed permutations):

Shorthand:

FFF *GGG* *FFG* ------→ *etc.*
FFD *GGH* *FFH*
FFP *GGM* *FF*M
FDD *GHH* ↓
FDD *GH*M
*F*PP *G*MM
DDD *HHH*
PPP MMM *etc.*

$F^{\mu\nu}D_{\mu\lambda}\mathrm{P}_{\nu\sigma}G^{\lambda\sigma}$ (and all allowed permutations):

Shorthand:

FFFF ------→ *etc.*
FFFD
*FFF*P
FFFG
FFFH
*FFF*M
↓
etc.

e.g. *F D* P *G* H and all allowed permutations
e.g. *F D* P *G H* M and all allowed permutations

Note: Obviously, not ***all*** of these combinations of fields will be ***truly*** unique!

e.g.

$$F^{\mu\nu}F_{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma} = \left(F^{\mu\nu}F_{\mu\nu}\right)\left(F^{\lambda\sigma}F_{\lambda\sigma}\right) = \left[2\left(B^2 - \left(\frac{E}{c}\right)^2\right)\right]^2$$

and:

$$\begin{aligned}
F^{\mu\nu}F_{\mu\nu}\mathrm{P}^{\lambda\sigma}\mathrm{P}_{\lambda\sigma} &= \left(F^{\mu\nu}F_{\mu\nu}\right)\left(\mathrm{P}^{\lambda\sigma}\mathrm{P}_{\lambda\sigma}\right) = 2\left(B^2 - \left(\frac{E}{c}\right)^2\right)\cdot 2\left(\mathrm{M}^2 - c^2\mathrm{P}^2\right) \\
&= 4\left(B^2 - \left(\frac{E}{c}\right)^2\right)\left(\mathrm{M}^2 - c^2\mathrm{P}^2\right)
\end{aligned}$$

Maxwell's Equations in Tensor Notation for Macroscopic Linear Media

The classical electrodynamics version of Maxwell's equations for macroscopic linear media are:

1.) Gauss' Law: $\boxed{\vec{\nabla}\bullet\vec{E} = \frac{1}{\varepsilon_o}\rho_{tot} = \frac{1}{\varepsilon_o}\left(\rho_{free} + \rho_{bound}\right)}$ where: $\boxed{\rho_{tot} = \rho_{free} + \rho_{bound}}$

The auxiliary/constitutive relation $\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}}$ gives: $\boxed{\vec{\nabla}\bullet\vec{D} = \varepsilon_o\vec{\nabla}\bullet\vec{E} + \vec{\nabla}\bullet\vec{P}}$

$\boxed{\vec{\nabla}\bullet\vec{E} = \frac{1}{\varepsilon_o}\rho_{tot}} \Rightarrow \boxed{\vec{\nabla}\bullet\vec{D} = \rho_{free}}$ and $\boxed{\vec{\nabla}\bullet\vec{P} = -\rho_{bound}}$

$\boxed{\vec{D} = \varepsilon\vec{E}}$, $\boxed{\vec{\mathrm{P}} = \varepsilon_o\chi_e\vec{E}} \Rightarrow \boxed{\varepsilon = \varepsilon_o\left(1+\chi_e\right)}$

2.) No magnetic monopoles: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$

3.) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

4.) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_o\vec{J}_{tot}}$ where: $\boxed{\vec{J}_{tot} = \vec{J}_{free} + \vec{J}^{M}_{bound} + \vec{J}^{P}_{bound}}$

The auxiliary/constitutive relation $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}}$ gives: $\boxed{\vec{\nabla}\times\vec{H} = \frac{1}{\mu_o}\vec{\nabla}\times\vec{B} - \vec{\nabla}\times\vec{\mathrm{M}}}$

Maxwell's displacement current term $\boxed{\vec{J}_D = \varepsilon_o\frac{\partial\vec{E}}{\partial t}}$, polarization current: $\boxed{\vec{J}^{p}_{bound} = \frac{\partial\vec{\mathrm{P}}}{\partial t}}$

$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}}$ and $\boxed{\vec{\nabla}\times\vec{M} = J^{M}_{bound}}$ where: $\boxed{\frac{\partial\vec{D}}{\partial t} = \varepsilon_o\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t} = \vec{J}_D + \vec{J}^{p}_{bound}}$

$\boxed{\vec{H} = \frac{1}{\mu}\vec{B}}$, $\boxed{\vec{\mathrm{M}} = \chi_m\vec{H}} \Rightarrow \boxed{\mu = \mu_o\left(1+\chi_m\right)}$

Relativistically, we have seen that Maxwell's four equations for free space / the vacuum {*i.e.* ***no*** matter present} are contained within the two tensor relations:

$\boxed{\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^{\mu}_{tot}}$ and: $\boxed{\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$

Given that: $\boxed{D^{\mu\nu} = \frac{1}{\mu_o}F^{\mu\nu} + \mathrm{P}^{\mu\nu}}$ and: $\boxed{H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + \mathrm{M}^{\mu\nu}}$

We see that the ***inhomogenous*** relativistic Maxwell relations are:

$$\boxed{\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu_{tot}}$$

$$\boxed{\partial_\nu D^{\mu\nu} = \frac{\partial D^{\mu\nu}}{\partial x^\nu} = J^\mu_{free} \quad = \left(c\rho_{free}, \vec{J}_{free}\right) \quad \begin{cases} \vec{\nabla}\bullet\vec{D} = \rho_{free} \\ \vec{\nabla}\bullet\vec{H} - \dfrac{\partial\vec{D}}{\partial t} = \vec{J}_{free} \end{cases}}$$

$$\boxed{\partial_\nu \mathrm{P}^{\mu\nu} = \frac{\partial \mathrm{P}^{\mu\nu}}{\partial x^\nu} = J^\mu_{bound} = \left(-c\rho_{bound}, \vec{J}_{bound}\right) \quad \begin{cases} \vec{\nabla}\bullet\vec{P} = -\rho_{bound} \\ \vec{J}_{bound} = \vec{J}^{P}_{bound} + \vec{J}^{M}_{bound} = \dfrac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{\mathrm{M}} \end{cases}}$$

Whereas the ***homogenous*** relativistic Maxwell relations are:

$$\boxed{\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \begin{cases} \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \end{cases}} \Rightarrow \boxed{\begin{matrix} \partial_\nu H^{\mu\nu} = -\partial_\nu M^{\mu\nu} \\ = \dfrac{\partial H^{\mu\nu}}{\partial x^\nu} = -\dfrac{\partial M^{\mu\nu}}{\partial x^\nu} \end{matrix} \quad \begin{cases} \vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{\mathrm{M}} \\ \vec{\nabla}\times\vec{D} = -\dfrac{1}{c^2}\dfrac{\partial\vec{H}}{\partial t} \\ \vec{\nabla}\times\vec{\mathrm{P}} = +\dfrac{1}{c^2}\dfrac{\partial\vec{\mathrm{M}}}{\partial t} \end{cases}}$$

The relativistic constitutive relations for linear, homogeneous, uniform, isotropic media {*i.e.* the relativistic analogs of $\boxed{\vec{D} = \varepsilon\vec{E}}$ and $\boxed{\vec{B} = \mu\vec{H}}$} were proposed by H. Minkowski as:

$$\boxed{D^{\mu\nu}\eta_\nu = c^2\varepsilon F^{\mu\nu}\eta_\nu} \quad \text{and:} \quad \boxed{H^{\mu\nu}\eta_\nu = \frac{1}{\mu}G^{\mu\nu}\eta_\nu}$$

<u>*n.b.*</u>

The relativistic *EM* field tensors are ***contracted*** with the ***proper*** 4-velocity η^μ .

The relativistic *EM* field tensors are defined in the ***rest/proper*** frame IRF(*S*) of the linear material.

ε is the ***proper*** electric permittivity and μ is the ***proper*** magnetic permeability.
Both quantities are defined in the ***rest/proper*** frame IRF(*S*) of the ***linear*** material.

η^μ is the (contravariant) ***proper*** 4-velocity of the material.

We can easily show that these formulae give the correct constitutive relations when the linear material is at rest, *i.e.* when $\boxed{\eta_\nu = (-c,0,0,0)}$. Here, the sum over ν collapses to a single term:

In the ***rest/proper*** frame of the linear material:

$$\begin{cases} \boxed{D^{\mu\nu}\eta_\nu = c^2\varepsilon F^{\mu\nu}\eta_\nu} \Rightarrow \boxed{D^{\mu 0}\eta_0 = c^2\varepsilon F^{\mu 0}\eta_0} \Rightarrow \boxed{-c\vec{D} = -c^2\varepsilon\left(\vec{E}/c\right)} \Rightarrow \boxed{\vec{D} = \varepsilon\vec{E}} \\ \boxed{H^{\mu\nu}\eta_\nu = \frac{1}{\mu}G^{\mu\nu}\eta_\nu} \Rightarrow \boxed{H^{\mu 0}\eta_0 = \frac{1}{\mu}G^{\mu 0}\eta_0} \Rightarrow \boxed{-\vec{H} = -\frac{1}{\mu}\vec{B}} \Rightarrow \boxed{\vec{H} = \frac{1}{\mu}\vec{B}} \end{cases}$$

If the ***linear*** medium is moving *e.g.* in the ***lab*** frame IRF(S) with ***ordinary*** velocity $\vec{u}$ then: $\boxed{\eta_\nu = (-\gamma c, \gamma\vec{u}) = \gamma(-c,\vec{u})}$. Then for $\vec{u} \neq 0$: $\boxed{D^{\mu\nu}\eta_\nu = c^2\varepsilon F^{\mu\nu}\eta_\nu}$ and: $\boxed{H^{\mu\nu}\eta_\nu = \frac{1}{\mu}G^{\mu\nu}\eta_\nu}$

For $\mu = 0$:

$$\boxed{D^{0\nu}\eta_\nu = D^{00}\eta_0 + D^{01}\eta_1 + D^{02}\eta_2 + D^{03}\eta_3 = 0 + cD_x(\gamma u_x) + cD_y(\gamma u_y) + cD_z(\gamma u_z) = \gamma c\left(\vec{D}\bullet\vec{u}\right)}$$

$$\boxed{F^{0\nu}\eta_\nu = F^{00}\eta_0 + F^{01}\eta_1 + F^{02}\eta_2 + F^{03}\eta_3 = 0 + \frac{E_x}{c}(\gamma u_x) + \frac{E_y}{c}(\gamma u_y) + \frac{E_z}{c}(\gamma u_z) = \gamma\frac{1}{c}\left(\vec{E}\bullet\vec{u}\right)}$$

$$\therefore \boxed{D^{0\nu}\eta_\nu = c^2\varepsilon F^{0\nu}\eta_\nu} \Rightarrow \boxed{\gamma c\left(\vec{D}\bullet\vec{u}\right) = c^2\varepsilon\left(\frac{\gamma}{c}\right)\left(\vec{E}\bullet\vec{u}\right)} \text{ or: } \boxed{\vec{D}\bullet\vec{u} = \varepsilon\left(\vec{E}\bullet\vec{u}\right)}$$

And:

$$\boxed{H^{0\nu}\eta_\nu = H^{00}\eta_0 + H^{01}\eta_1 + H^{02}\eta_2 + H^{03}\eta_3 = H_x(\gamma u_x) + H_y(\gamma u_y) + H_z(\gamma u_z) = \gamma\left(\vec{H}\bullet\vec{u}\right)}$$

$$\boxed{G^{0\nu}\eta_\nu = G^{00}\eta_0 + G^{01}\eta_1 + G^{02}\eta_2 + G3\eta_3 = B_x(\gamma u_x) + B_y(\gamma u_y) + B_z(\gamma u_z) = \gamma\left(\vec{B}\bullet\vec{u}\right)}$$

$$\therefore \boxed{H^{0\nu}\eta_\nu = \frac{1}{\mu}G^{0\nu}\eta_\nu} \Rightarrow \boxed{\gamma\left(\vec{H}\bullet\vec{u}\right) = \frac{1}{\mu}\gamma\left(\vec{B}\bullet\vec{u}\right)} \text{ or: } \boxed{\vec{H}\bullet\vec{u} = \frac{1}{\mu}\left(\vec{B}\bullet\vec{u}\right)}$$

Similarly for $\mu = 1$:

$$\boxed{D^{1\nu}\eta_\nu = D^{10}\eta_0 + D^{11}\eta_1 + D^{12}\eta_2 + D^{13}\eta_3 = +\gamma c^2 D_x + \gamma H_z u_y - \gamma H_y u_z = \gamma\left(c^2 D_x + \left(\vec{u}\times\vec{H}\right)_x\right)}$$

$$\boxed{F^{1\nu}\eta_\nu = F^{10}\eta_0 + F^{11}\eta_1 + F^{12}\eta_2 + F^{13}\eta_3 = \gamma E_x + \gamma B_z u_y - \gamma B_y u_z = \gamma\left(E_x + \left(\vec{u}\times\vec{B}\right)_x\right)}$$

$$\therefore \boxed{D^{1\nu}\eta_\nu = c^2\varepsilon F^{12}\eta_\nu} \Rightarrow \boxed{\gamma\left(c^2 D_x\left(\vec{u}\times\vec{H}\right)_x\right) = \gamma\left(E_x + \left(\vec{u}\times\vec{B}\right)_x\right)c} \Rightarrow \boxed{\vec{D} + \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + (\vec{u}\times\vec{B}\right]}$$

And:

$$\boxed{H^{1\nu}\eta = H^{10}\eta_0 + H^{11}\eta_1 + H^{12}\eta_2 + H^{13}\eta_3 = +\gamma c H_x - \gamma c D_z u_y + \gamma c D_y u_z = \gamma c\left(H_x - \left(\vec{u}\times\vec{D}\right)_z\right)}$$

$$\boxed{G^{1\nu}\eta = G^{10}\eta_0 + G^{11}\eta_1 + G^{12}\eta_2 + G^{13}\eta_3 = +\gamma cBx - \gamma\frac{E_z}{c}u_y + \gamma\frac{E_y}{c}u_z = \frac{\gamma}{c}\left[c^2 B_x - \left(\vec{u}\times\vec{E}\right)_x\right]}$$

$$\therefore \boxed{H^{01}\eta_\nu = \frac{1}{\mu}G^{1\nu}} \Rightarrow \boxed{\gamma c\left(H_x - \left(\vec{u}\times\vec{D}\right)_z\right) = \frac{\gamma}{c}\left[c^2 B_x - \left(\vec{u}\times\vec{E}\right)_x\right]\frac{1}{\mu}} \Rightarrow \boxed{\vec{H} - \left(\vec{u}\times\vec{D}\right) = \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$$

Thus for: $\boxed{D^{\mu\nu}\eta_\nu = c^2\varepsilon F^{\mu\nu}\eta_\nu}$ and: $\boxed{H^{\mu\nu}\eta_\nu = \frac{1}{\mu}G^{\mu\nu}\eta_\nu}$

We obtain:

$$\boxed{\vec{D} + \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right]} \quad \text{and:} \quad \boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) = \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]} \;**$$

Insert: $\boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) + \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$

Into: $\boxed{\vec{D} + \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right]}$ Solve for $\vec{D}$:

$$\boxed{\vec{D} + \frac{1}{c^2}\vec{u}\times\left(\vec{u}\times\vec{D}\right) + \frac{1}{c^2\mu}\left(\vec{u}\times\vec{B}\right) - \frac{1}{c^4\mu}\vec{u}\times\left(\vec{u}\times\vec{E}\right) = \varepsilon\vec{E} + \varepsilon\left(\vec{u}\times\vec{B}\right)}$$

Use the $\vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$ rule on the first triple cross-product:

$$\boxed{\vec{D} + \frac{1}{c^2}\left[\vec{u}\left(\vec{u}\bullet\vec{D}\right) = -u^2\vec{D}\right] = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{\mu c^2}\left(\vec{u}\times\vec{B}\right) + \frac{1}{\mu c^4}\left[\vec{u}\times\left(\vec{u}\times\vec{E}\right)\right]}$$

But: $\boxed{\left(\vec{u}\cdot\vec{D}\right) = \varepsilon\left(\vec{u}\cdot\vec{E}\right)}$

Thus:

$$\boxed{\begin{aligned}\vec{D}\left(1 - \frac{u^2}{c^2}\right) &= -\frac{\varepsilon}{c^2}\vec{u}\left(\vec{u}\bullet\vec{E}\right) + \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{\mu c^2}\left(\vec{u}\times\vec{B}\right) + \frac{1}{\mu c^4}\left[\left(\vec{E}\bullet\vec{u}\right)\vec{u} - u^2\vec{E}\right] \\ &= \varepsilon\left\{\left[1 - \frac{u^2}{\varepsilon\mu c^4}\right]\vec{E} - \frac{1}{c^2}\left[1 - \frac{1}{\varepsilon\mu c^2}\right]\left(\vec{E}\bullet\vec{u}\right)\vec{u} + \left(\vec{u}\times\vec{B}\right)\left[1 - \frac{1}{\varepsilon\mu c^2}\right]\right\}\end{aligned}}$$

Define: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1 - u^2/c^2}}}$ and: $\boxed{v \equiv \frac{1}{\sqrt{\varepsilon\mu}}}$ = speed of propagation of *EM* waves in the ***linear*** medium.

Then: $$\boxed{\vec{D} = \gamma_u^2\varepsilon\left\{\left(1 - \frac{u^2v^2}{c^4}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\left[\left(\vec{u}\times\vec{B}\right) - \frac{1}{c^2}\left(\vec{E}\bullet\vec{u}\right)\vec{u}\right]\right\}}$$

From equation ** (RHS up above):

$$\boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) = \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$$

Insert: $\boxed{\vec{D} = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right)}$ Solve for $\vec{H}$:

$$\therefore \quad \boxed{\vec{H} - \vec{u}\times\left\{-\frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) + \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right]\right\} = \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$$

$$\boxed{\vec{H} + \frac{1}{c^2}\left[\left(\vec{u}\cdot\vec{H}\right)\vec{u} - u^2\vec{H}\right] = \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right] + \varepsilon\left(\vec{u}\times\vec{E}\right) + \varepsilon\left[\vec{u}\times\left(\vec{u}\times\vec{B}\right)\right]}$$

But: $\boxed{\left(\vec{H}\cdot\vec{u}\right) = \left(\vec{u}\cdot\vec{H}\right) = \frac{1}{\mu}\left(\vec{u}\cdot\vec{B}\right)}$

$$\therefore \quad \begin{aligned} \vec{H}\left(1 - \frac{u^2}{c^2}\right) &= \frac{1}{\mu c^2}\left(\vec{B}\cdot\vec{u}\right)\vec{u} + \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right] + \varepsilon\left(\vec{u}\times\vec{E}\right) + \varepsilon\left[\left(\vec{B}\cdot\vec{u}\right)\vec{u} - u^2\vec{B}\right] \\ &= \frac{1}{\mu}\left\{\left[1 - \mu\varepsilon u^2\right]\vec{B} + \left(\varepsilon\mu - \frac{1}{c^2}\right)\left[\left(\vec{u}\times\vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\} \end{aligned}$$

But: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1-u^2/c^2}}}$ and: $\boxed{v \equiv \frac{1}{\sqrt{\varepsilon\mu}}}$

$$\therefore \quad \boxed{\vec{H} = \frac{\gamma_u^2}{\mu}\left\{\left(1 - \frac{u^2}{v^2}\right)\vec{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left[\left(\vec{u}\times\vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\}}$$

Thus, the macroscopic $\vec{D}$ and $\vec{H}$ -fields in a ***linear*** medium moving with ***ordinary*** velocity $\vec{u}$ in IRF(S) in terms of the $\vec{E}$ and $\vec{B}$ fields present ***in*** IRF(S) are:

$$\boxed{\vec{D} = \gamma_u^2\varepsilon\left\{\left(1 - \frac{u^2v^2}{c^4}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\left[\left(\vec{u}\times\vec{B}\right) - \frac{1}{c^2}\left(\vec{E}\cdot\vec{u}\right)\vec{u}\right]\right\}} \quad \text{with: } \boxed{\gamma_u \equiv \frac{1}{\sqrt{1-u^2/c^2}}}$$

$$\boxed{\vec{H} = \frac{\gamma_u^2}{\mu}\left\{\left(1 - \frac{u^2}{v^2}\right)\vec{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left[\left(\vec{u}\times\vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\}} \quad \text{and: } \boxed{v \equiv \frac{1}{\sqrt{\varepsilon\mu}}}$$

Note that ε and μ are the ***proper*** electric permittivity and the ***proper*** magnetic permeability of the linear medium – *i.e.* they are defined in the ***rest/proper*** frame of the ***linear*** medium.

When $\boxed{u = |\vec{u}| \ll c}$, then $\boxed{\gamma_u \simeq 1}$ {*i.e.* the non-relativistic limit}. Keeping only terms ***linear*** in $\vec{u}$:

$$\boxed{\vec{D} \simeq \varepsilon\vec{E} + \underbrace{\left(1-(v/c)^2\right)\left(\vec{u}\times\vec{B}\right)}_{\text{Usually very small}} \approx \varepsilon\vec{E}} \quad \text{and: } \boxed{\vec{H} \simeq \frac{1}{\mu}\vec{B} + \underbrace{\left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left(\vec{u}\times\vec{E}\right)}_{\text{Usually very small}} \approx \frac{1}{\mu}\vec{B}}$$