

LECTURE NOTES 19

LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS (SLIGHT RETURN)

Before continuing on with our onslaught of the development of relativistic electrodynamics via tensor analysis, I want to briefly discuss an equivalent, simpler method of Lorentz transforming the *EM* fields \vec{E} and \vec{B} from one IRF(*S*) to another IRF(*S'*), which also sheds some light (by contrast) on how the *EM* fields Lorentz transform vs. “normal” 4-vectors.

In P436 Lecture Notes 18.5 {p. 18-22} we discussed the tensor algebra method for Lorentz transformation of the electromagnetic field *e.g.* in the lab frame IRF(*S*), represented by the *EM* field strength tensor $F^{\mu\nu}$ to another frame IRF(*S'*), represented by the *EM* field strength tensor $F'^{\mu\nu}$ via the relation:

$$\boxed{F'^{\mu\nu} = \Lambda^\mu_\lambda F^{\lambda\sigma} \Lambda^\nu_\sigma} \quad \text{n.b. in matrix form: } \boxed{F' = \Lambda F \Lambda^T = \Lambda F \Lambda} \quad \text{since } \Lambda \text{ is } \underline{\text{symmetric}}.$$

Analytically carrying out this tensor calculation by hand can be tedious and time-consuming. If such calculations are to be carried out repeatedly/frequently, we encourage people to code this up and simply let the computer do the repetitive work, which it excels at.

For 1-dimensional Lorentz transformations (only) there is a simpler, less complicated, perhaps somewhat more intuitive method. Starting with the algebraic rules for Lorentz-transforming $\{\vec{E} \text{ and } \vec{B}\}$ in one IRF(*S*) to $\{E' \text{ and } B'\}$ in another IRF(*S'*) *e.g.* moving with relative velocity $\vec{v} = +v\hat{x}$ with respect to IRF(*S*):

component(s):	$E'_x = E_x$	$B'_x = B_x$	$\gamma \equiv 1/\sqrt{1-\beta^2}$
⊥ components:	$E'_y = \gamma(E_y - \beta c B_z)$	$B'_y = \gamma(B_y + \beta E_z/c)$	$\beta \equiv v/c$
	$E'_z = \gamma(E_z + \beta c B_y)$	$B'_z = \gamma(B_z + \beta E_y/c)$	

We can write these relations more compactly and elegantly by resolving them into their || and ⊥ components relative to the boost direction: here, || is along $\vec{v} = +v\hat{x}$ and ⊥ is perpendicular to \vec{v} , defined as follows {n.b. in general, \vec{v} could be || *e.g.* to $\hat{x}, \hat{y}, \hat{z}$ or \hat{r} }:

$E'^{\parallel} = E^{\parallel}$	$\vec{\beta} \equiv \vec{v}/c$
$E'^{\perp} = \gamma(E^{\perp} + \vec{v} \times B^{\perp}) = \gamma(E^{\perp} + \vec{\beta} c \times B^{\perp})$	$\gamma \equiv 1/\sqrt{1-\beta^2}$
$B'^{\parallel} = B^{\parallel}$	
$B'^{\perp} = \gamma\left(B^{\perp} - \frac{1}{c^2} \vec{v} \times E^{\perp}\right) = \gamma\left(B^{\perp} - \frac{1}{c} \vec{\beta} \times E^{\perp}\right)$	

Now since $\vec{v} = +v\hat{x}$ {here} then: $E^{\parallel} \equiv E_x$, $B^{\parallel} \equiv B_x$ and: $B^{\perp} \equiv B_y\hat{y} + B_z\hat{z}$, $E^{\perp} \equiv E_y\hat{y} + E_z\hat{z}$ {and similarly for corresponding quantities in IRF(*S'*)}.

Since $\vec{v} = +v\hat{x}$ and $E^{\parallel} \equiv E_x$ then: $\vec{v} \times E^{\perp} = \vec{v} \times E_x \hat{x} = 0$
 And likewise, since $B^{\parallel} \equiv B_x$ then: $\vec{v} \times B^{\parallel} = \vec{v} \times B_x \hat{x} = 0$.

Thus, we can **{safely}** write: $\vec{v} \times E^{\perp} = \vec{v} \times \vec{E}$ and $\vec{v} \times B^{\perp} = \vec{v} \times \vec{B}$, as long as \vec{v} is **always** || to **one** of the components of \vec{E} and \vec{B} - e.g. \hat{x} or \hat{y} or \hat{z} .

Then we **can** write the Lorentz transformation of *EM* fields as:

$$\begin{array}{l} E'^{\parallel} = E^{\parallel} \\ E'^{\perp} = \gamma(E^{\perp} + \vec{v} \times \vec{B}) = \gamma(E^{\perp} + \vec{\beta}c \times \vec{B}) \\ B'^{\parallel} = B^{\parallel} \\ B'^{\perp} = \gamma\left(B^{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}\right) = \gamma\left(B^{\perp} - \frac{1}{c}\vec{\beta} \times \vec{E}\right) \end{array} \quad \begin{array}{l} \beta \equiv v/c \\ \vec{\beta} \equiv \vec{v}/c \\ \gamma = 1/\sqrt{1-\beta^2} \end{array}$$

This can be written more compactly in 2-D matrix form as:

$$\begin{array}{ccc} \begin{array}{c} \text{EM Fields:} \\ \left(\begin{array}{c} E'^{\parallel} \\ cB'^{\parallel} \end{array} \right) = \left(\begin{array}{c} E^{\parallel} \\ cB^{\parallel} \end{array} \right) \end{array} & \rightarrow & \begin{array}{c} \left(E'^{\parallel} = E^{\parallel} \text{ and } B'^{\parallel} = B^{\parallel} \right) \end{array} \leftarrow \text{compare to} \rightarrow \begin{array}{c} \text{“Normal” 4-Vector:} \\ \left(x'^{\perp} = x^{\perp} \right) \end{array} \\ \\ \begin{array}{c} \left(\begin{array}{c} E'^{\parallel} \\ cB'^{\parallel} \end{array} \right) = \underbrace{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)}_{\text{Unit Matrix}} \left(\begin{array}{c} E^{\parallel} \\ cB^{\parallel} \end{array} \right) \end{array} & & \begin{array}{c} \left(\begin{array}{c} E'^{\perp} \\ cB'^{\perp} \end{array} \right) = \underbrace{\left(\begin{array}{cc} \gamma & +\gamma\vec{\beta} \times \\ -\gamma\vec{\beta} \times & \gamma \end{array} \right)}_{\text{Operator Matrix}} \left(\begin{array}{c} E^{\perp} \\ cB^{\perp} \end{array} \right) \leftrightarrow \begin{array}{c} \left(\begin{array}{c} x'^{\parallel} \\ ct' \end{array} \right) = \underbrace{\left(\begin{array}{cc} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{array} \right)}_{\text{Scalar Matrix}} \left(\begin{array}{c} x^{\parallel} \\ ct \end{array} \right) \end{array} \end{array} \end{array}$$

Thus, we see that for the *EM* fields vs. the 3-D space-part of a “normal” 4-vector, the || vs. ⊥ components are **switched**, \vec{B} transforms “sort of” like time t , but 2×2 Lorentz boost matrices for (\vec{E} and \vec{B}) vs. 4-vectors are **not** the same (they are **similar**, but they are **not** identical).

We can also write compact **inverse** Lorentz transformations (e.g. from IRF(S') rest frame \rightarrow IRF(S) lab frame):

$$\begin{array}{ccc} \begin{array}{c} \text{EM Fields:} \\ \left(\begin{array}{c} E^{\parallel} \\ cB^{\parallel} \end{array} \right) = \left(\begin{array}{c} E'^{\parallel} \\ cB'^{\parallel} \end{array} \right) \end{array} & \rightarrow & \begin{array}{c} \left(E^{\parallel} = E'^{\parallel} \text{ and } B^{\parallel} = B'^{\parallel} \right) \end{array} \leftarrow \text{compare to} \rightarrow \begin{array}{c} \text{“Normal” 4-Vector:} \\ \left(x^{\perp} = x'^{\perp} \right) \end{array} \\ \\ \begin{array}{c} \left(\begin{array}{c} E^{\parallel} \\ cB^{\parallel} \end{array} \right) = \underbrace{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)}_{\text{Unit Matrix}} \left(\begin{array}{c} E'^{\parallel} \\ cB'^{\parallel} \end{array} \right) \end{array} & & \begin{array}{c} \left(\begin{array}{c} E^{\perp} \\ cB^{\perp} \end{array} \right) = \underbrace{\left(\begin{array}{cc} \gamma & -\gamma\vec{\beta} \times \\ +\gamma\vec{\beta} \times & \gamma \end{array} \right)}_{\text{Operator Matrix}} \left(\begin{array}{c} E'^{\perp} \\ cB'^{\perp} \end{array} \right) \leftrightarrow \begin{array}{c} \left(\begin{array}{c} x^{\parallel} \\ ct \end{array} \right) = \underbrace{\left(\begin{array}{cc} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{array} \right)}_{\text{Scalar Matrix}} \left(\begin{array}{c} x'^{\parallel} \\ ct' \end{array} \right) \end{array} \end{array} \end{array}$$

For a **general** Lorentz transformation (*i.e.* **no** restriction on the orientation of \vec{v} {arbitrary}):

A.) Lorentz transformation from IRF(S) \rightarrow IRF(S'):

$$\begin{aligned} \vec{E}' &= \gamma \left(\vec{E} + \vec{\beta} c \times \vec{B} \right) - \frac{\gamma^2}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{E}) & \vec{\beta} &= \vec{v}/c & \beta &= v/c \\ \vec{B}' &= \gamma \left(\vec{B} - \frac{1}{c} \vec{\beta} \times \vec{E} \right) - \frac{\gamma^2}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{B}) & \gamma &= 1/\sqrt{1-\beta^2} \end{aligned}$$

Or:

$$\begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma \left(1 - \left(\frac{\gamma}{\gamma+1} \right) \vec{\beta} \vec{\beta} \cdot \right) & +\gamma \vec{\beta} \times \\ -\gamma \vec{\beta} \times & \gamma \left(1 - \left(\frac{\gamma}{\gamma+1} \right) \vec{\beta} \vec{\beta} \cdot \right) \end{pmatrix}}_{\text{operator matrix}} \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}$$

B.) Inverse Lorentz transformation from IRF(S') \rightarrow IRF(S):

$$\begin{aligned} \vec{E} &= \gamma \left(\vec{E}' - \vec{\beta} c \times \vec{B}' \right) - \frac{\gamma^2}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{E}') & \text{Switch } \vec{\beta} \rightarrow -\vec{\beta}, \vec{E} \rightarrow \vec{E}' \text{ and} \\ \vec{B} &= \gamma \left(\vec{B}' + \frac{1}{c} \vec{\beta} \times \vec{E}' \right) - \frac{\gamma^2}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{B}') & \vec{B} \rightarrow \vec{B}' \text{ in above relations} \end{aligned}$$

Or:

$$\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma \left(1 - \left(\frac{\gamma}{\gamma+1} \right) \vec{\beta} \vec{\beta} \cdot \right) & -\gamma \vec{\beta} \times \\ +\gamma \vec{\beta} \times & \gamma \left(1 - \left(\frac{\gamma}{\gamma+1} \right) \vec{\beta} \vec{\beta} \cdot \right) \end{pmatrix}}_{\text{operator matrix}} \begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix}$$

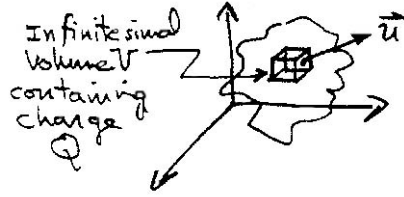
Electrodynamics in Tensor Notation

So now that we know how to represent the *EM* field in relativistic tensor notation (as $F^{\mu\nu}$ or $G^{\mu\nu}$), we can also reformulate **all** laws of electrodynamics (*e.g.* Maxwell's equations, the Lorentz force law, the continuity equation {expressing electric charge conservation}, *etc.* . . .) in the mathematical language of tensors.

In order to begin this task, we must first determine how the sources of the *EM* fields – the electric charge density ρ (a **scalar** quantity) and the electric current density \vec{J} (a 3-D **vector** quantity) Lorentz transform.

The electric charge density $\rho = Q/V$ = charge per unit volume (*Coulombs/m³*)

Imagine a cloud of electric charge drifting by. Concentrate on an infinitesimal volume V containing charge Q moving at (ordinary) velocity \vec{u} :



Then: $\rho = Q/V$ = charge density (Coulombs/m³)

And: $\vec{J} = \rho\vec{u}$ = current density (Amps/m²).

A subtle, but important detail:

If there is only one species (*i.e.* kind / type) of charge carrier contained within the infinitesimal volume V , then all charge carriers travel at the same (average / mean) speed \vec{u} .

However, if there are multiple species (kinds / types) of charge carriers (*e.g.* with different masses) and/or different signs of charge carriers contained within the infinitesimal volume V (*e.g.* electrons e^- with rest masses $m_e c^2$ and protons p with rest masses $m_p c^2$) then the different constituents / species must be treated separately in the following:

If $\exists N$ species:

Current density $\vec{J}_i = \rho_i \vec{u}_i$ for the i^{th} species ($i = 1, \dots, N$), the electric charge density $\rho_i = Q_i/V$

And:
$$\vec{J} = \sum_{i=1}^N \vec{J}_i = \sum_{i=1}^N \rho_i \vec{u}_i$$

We also need to express ρ and \vec{J} in terms of the proper charge density ρ_0 = volume charge density defined in the rest frame of the charge Q , IRF(S_0).

The infinitesimal rest volume / proper volume = V_0 {defined in the rest/proper frame IRF(S_0)}

The proper charge density: $\rho_0 = Q/V_0$ ← Recall that electric charge Q (like c) is a Lorentz invariant scalar quantity

Because the longitudinal direction of motion undergoes Lorentz contraction from the rest frame IRF(S_0) in the Lorentz transformation → another reference frame, *e.g.* lab frame IRF(S)

Then: $V = \frac{1}{\gamma_u} V_0$ where: $V_0 = \ell_0 w_0 d_0$ and: $V = \ell w d$, where: $\gamma_u = \frac{1}{\sqrt{1 - \beta_u^2}}$ and: $\beta_u = \frac{u}{c}$

If the Lorentz transformation is along (*i.e.* || to) the length ℓ, ℓ_0 of the infinitesimal volumes

Then: $\ell = \frac{1}{\gamma_u} \ell_0$ and the \perp components of the volumes are unchanged: $w_0 = w, d_0 = d$.

Then if: $V = \frac{1}{\gamma_u} V_0 \rightarrow \rho = \frac{Q}{V} = \gamma_u \left(\frac{Q}{V_0} \right) = \gamma_u \rho_0 \therefore \vec{J} = \rho \vec{u} = \gamma_u \rho_0 \vec{u} = \rho_0 (\gamma_u \vec{u})$

Recall that the 3-D vector associated with the proper velocity is: $\vec{\eta} = \gamma_u \vec{u} \left(\equiv \frac{d\vec{\ell}}{d\tau} \right) \therefore \vec{J} = \rho_0 \vec{\eta}$

The **zeroth** (i.e. **temporal/scalar**) component of the **proper** 4-velocity is: $\eta^0 \equiv \frac{dx^0}{d\tau} = c \frac{dt}{d\tau} = \gamma_u c$

The corresponding **zeroth** (i.e. **temporal/scalar**) component of the current density 4-vector J^μ is:

$$J^0 \equiv \rho_0 \eta^0 = \rho_0 \gamma_u c = (\gamma_u \rho_0) c = \rho c = c \rho$$

The current density 4-vector is: $J^\mu = (J^0, \vec{J}) = (c\rho, \vec{J}) = (c\rho, J_x, J_y, J_z)$ (SI units: *Amps/m²*)

Then: $J^\mu = \rho_0 \eta^\mu = (\rho_0 \eta^0, \rho_0 \vec{\eta})$ where: $\eta^\mu = (\eta^0, \vec{\eta}) = (\gamma_u c, \gamma_u \vec{u}) = \gamma_u (c, \vec{u}) = \gamma_u (c, u_x, u_y, u_z)$
 constant **scalar** quantity

∴ J^μ is a **proper** four vector, i.e. $J^\mu = \text{proper}$ current density 4-vector.

Thus: $J^\mu J_\mu = J_\mu J^\mu$ is a Lorentz invariant quantity. Is it ???

$$J^\mu J_\mu = J_\mu J^\mu = \rho_0^2 \eta^\mu \eta_\mu = \rho_0^2 \gamma_u^2 \left(-c^2 + \underbrace{u_x^2 + u_y^2 + u_z^2}_{=u^2} \right) = \rho_0^2 \left(\frac{-c^2 + u^2}{1 - u^2/c^2} \right) = -(\rho_0 c)^2 \left(\frac{1 - u^2/c^2}{1 - u^2/c^2} \right) = -(\rho_0 c)^2$$

$$J^\mu J_\mu = J_\mu J^\mu = -(\rho_0 c)^2 = \rho_0^2 \eta^\mu \eta_\mu = \rho_0^2 \eta_\mu \eta^\mu = -\rho_0^2 c^2 \quad \text{Since: } \eta^\mu \eta_\mu = \eta_\mu \eta^\mu = -c^2$$

Yes, $J^\mu J_\mu = J_\mu J^\mu$ **is** a Lorentz invariant quantity!

The 3-D continuity equation mathematically expresses **local** conservation of electric charge (using differential vector calculus):

$$\vec{\nabla} \cdot \vec{J}(\vec{r}, t) = -\frac{\partial \rho(\vec{r}, t)}{\partial t} \quad \rho(\vec{r}, t) = \text{scalar point function, } \vec{J} = \vec{J}(\vec{r}, t) = \text{3-D vector point function}$$

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \quad (\text{in Cartesian coordinates})$$

We can also express the continuity equation in 4-vector tensor notation:

$$\vec{\nabla} \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^3 \frac{\partial J^i}{\partial x^i} \quad \text{and:} \quad \frac{\partial \rho}{\partial t} = \frac{1}{c} \frac{\partial (c\rho)}{\partial t} = \frac{1}{c} \frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0} \quad (J^0 = c\rho)$$

n.b.
repeated indices implies summation!

$$\text{Then: } \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \sum_{i=1}^3 \frac{\partial J^i}{\partial x^i} + \frac{\partial J^0}{\partial x^0} = \frac{\partial J^0}{\partial x^0} + \sum_{i=1}^3 \frac{\partial J^i}{\partial x^i} = 0 = \sum_{\mu=0}^3 \frac{\partial J^\mu}{\partial x^\mu} = \frac{\partial J^\mu}{\partial x^\mu} = 0$$

$$\text{Thus: } \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{or} \quad \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial J^\mu}{\partial x^\mu} = 0 \quad \text{Continuity equation (local charge conservation)}$$

Physically, note that $\frac{\partial J^\mu}{\partial x^\mu}$ is the 4-dimensional space-time **divergence** of the current density 4-vector $J^\mu = (c\rho, \vec{J})$. The 4-current density $J^\mu = (c\rho, \vec{J})$ is divergenceless because $\frac{\partial J^\mu}{\partial x^\mu} = 0$.

The 4-vector operator $\frac{\partial}{\partial x^\mu}$ is called the 4-D gradient operator, (*a.k.a* the quad operator \square_μ or “quad” for short). However, because the 4-D gradient operator $\frac{\partial}{\partial x_\mu}$ functions like a **covariant** 4-vector, *e.g.* when it operates on **contravariant** J^μ (or any other **contravariant** 4-vectors), it is often alternatively given the shorthand notation $\frac{\partial}{\partial x^\mu} \equiv \square_\mu = \partial_\mu$.

Similarly, because the 4-D gradient operator $\frac{\partial}{\partial x^\mu}$ functions like a **contravariant** 4-vector, *e.g.* when it operates on **covariant** J_μ (or any other **covariant** 4-vectors), it is given the shorthand notation $\frac{\partial}{\partial x_\mu} \equiv \square^\mu = \partial^\mu$.
 {See/work thru Griffiths Problem 12.55 (*p.* 543) for more details.}

The **contravariant** quad/gradient 4-vector operator: $\frac{\partial}{\partial x^\mu} = \left(+\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) \equiv \square_\mu = \partial_\mu$

The **covariant** quad/gradient 4-vector operator: $\frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) \equiv \square^\mu = \partial^\mu$

Then: $\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} = \frac{\partial^2}{\partial x_\mu \partial x^\mu} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \equiv \square^\mu \square_\mu = \square_\mu \square^\mu = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu \equiv \square^2$
 = D’Alembertian 4-vector operator = 4-D Laplacian operator = **Lorentz-invariant quantity!**

The 4-vector product of **any** two (bona-fide) relativistic 4-vectors is a Lorentz invariant quantity (*i.e.* the **same** value in **any/all** IRF’s)!

We can equivalently write the relativistic 4-D continuity equation as:

$$\frac{\partial J^\mu}{\partial x^\mu} \equiv \square_\mu J^\mu = \partial_\mu J^\mu = 0 \quad \leftarrow n.b. \text{ A } \underline{\text{Lorentz-invariant quantity}}!!!$$

\Rightarrow Electric charge is (locally) conserved in **any/all** IRF’s (as it **must** be!!!)

Maxwell's Equations in Tensor Notation

Maxwell's Equations:

- 1) Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$
- 2) No Magnetic Monopoles: $\vec{\nabla} \cdot \vec{B} = 0$
- 3) Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- 4) Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ with Maxwell's Displacement Current

Can be written as 4-derivatives of the relativistic EM field strength tensors $F^{\mu\nu}$ and $G^{\mu\nu}$:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad \text{and} \quad \partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

n.b. summation over $\nu = 0:3$ is implied

- 1) Gauss' law $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$. If $\mu = 0$ in: $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$, *i.e.* $\partial_\nu F^{0\nu} = \frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0$, $\nu = 0:3$

Physically, $\mu = 0$ is the **temporal/scalar** component of any space-time 4-vector.

Then: $\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$ $\leftarrow \mu = 0$ (first row of $F^{\mu\nu}$)

Row #

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Column #

$$\therefore \frac{\partial F^{0\nu}}{\partial x^0} = 0 + \frac{1}{c} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \frac{1}{c} \vec{\nabla} \cdot \vec{E} \quad \text{and:} \quad \mu_0 J^0 = \mu_0 (c\rho)$$

$$\therefore \frac{1}{c} \vec{\nabla} \cdot \vec{E} = \mu_0 c \rho \quad \text{or:} \quad \vec{\nabla} \cdot \vec{E} = \mu_0 c^2 \rho \quad \text{but:} \quad \mu_0 c^2 = \frac{1}{\epsilon_0} \quad \therefore \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

Gauss' law arises from the $\mu = 0$ (scalar / temporal) component of the 4-vector relation:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$

4) Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ If $\mu = 1$ in $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$

Then: $\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$ $\mu = 1$ (second row of $F^{\mu\nu}$)

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + 0 + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} \right)_x$$
 and: $\mu_0 J^1 = \mu_0 J_x$

$\therefore \frac{\partial F^{1\nu}}{\partial x^\nu} = \left(-\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} \right)_x = \mu_0 J_x$

Then for $\mu = 2$ and $\mu = 3$ (third and fourth rows of $F^{\mu\nu}$), likewise we find that:

$$\frac{\partial F^{2\nu}}{\partial x^\nu} = \left(-\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} \right)_y = \mu_0 J_y$$
 and: $\frac{\partial F^{3\nu}}{\partial x^\nu} = \left(-\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} \right)_z = \mu_0 J_z$

$\therefore \frac{\partial F^{\mu\nu}}{\partial x^\nu} \Big|_{\mu=1:3} = \mu_0 J^\mu \Big|_{\mu=1:3} \Rightarrow \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

3-D spatial components of 4-vector J^μ

Or: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ Ampere's law with Maxwell's displacement current term !!!

\Rightarrow Ampere's law arises from the $\mu = 1:3$ (3-D spatial / vector component) of 4-vector relation:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ ($\mu = 0$ temporal / scalar component)

$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$ ($\mu = 1:3$ 3-D spatial / vector component)

Thus, Gauss' law and Ampere's law form a 4-vector:

$$\mu_0 J^\mu = \left(\underbrace{\frac{1}{c} \vec{\nabla} \cdot \vec{E} = \frac{1}{c \epsilon_0} \rho}_{= \mu_0 J^0}, \underbrace{\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}} \right) = \left(\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \partial_\nu F^{\mu\nu} \right)$$

Gauss' law temporal / scalar component $\mu_0 J^0$ of 4-vector $\mu_0 J^\mu$

Ampere's law 3-D spatial / vector component $\mu_0 \vec{J}$ of 4-vector $\mu_0 J^\mu$

And: $J^\mu J_\mu = J_\mu J^\mu = -\rho_0^2 c^2 = \text{Lorentz invariant quantity \{from above, page 5\}}$

2) $\vec{\nabla} \cdot \vec{B} = 0$ no magnetic monopoles / no magnetic charges. If $\mu = 0$ in $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$

$\mu = 0$ is the temporal (scalar) component of space-time “null” 4-vector $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0^\mu = (0, \vec{0}) = 0$

Then: $\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = 0$ $\leftarrow \mu = 0$ (First row of $G^{\mu\nu}$)

Row #

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

Column #

$$\therefore \frac{\partial G^{0\nu}}{\partial x^\nu} = 0 + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 = \vec{\nabla} \cdot \vec{B} = 0 \quad \therefore \vec{\nabla} \cdot \vec{B} = 0 \iff \frac{\partial G^{0\nu}}{\partial x^\nu} = 0$$

3) Faraday’s law: $\vec{\nabla} \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ If $\mu = 1$ in $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$

Then: $\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = 0$ $\leftarrow \mu = 1$ (Second row of $G^{\mu\nu}$)

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = -\frac{1}{c} \frac{\partial B_x}{\partial t} + 0 - \frac{1}{c} \frac{\partial E_z}{\partial y} + \frac{1}{c} \frac{\partial E_y}{\partial z} = -\frac{1}{c} \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right)_x = 0$$

$$\therefore \frac{\partial G^{1\nu}}{\partial x^\nu} = 0 \text{ gives } \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right)_x = 0$$

Likewise, for $\mu = 2$ and $\mu = 3$ (third and fourth rows of $G^{\mu\nu}$)

$$\frac{\partial G^{2\nu}}{\partial x^\nu} = 0 \text{ gives } \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right)_y = 0 \text{ and: } \frac{\partial G^{3\nu}}{\partial x^\nu} = 0 \text{ gives } \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right)_z = 0$$

$$\therefore \frac{\partial G^{\mu\nu}}{\partial x^\nu} \Big|_{\mu=1:3} = 0 \text{ gives } \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \text{ or: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Thus: $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ $\leftarrow \vec{\nabla} \cdot \vec{B} = 0$ ($\mu = 0$ temporal / scalar component)
 $\leftarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ($\mu = 1:3$ 3-D spatial / vector component)

Arise from temporal ($\mu = 0$) and spatial ($\mu = 1:3$) component of the “null” 4-vector $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$

Thus, in relativistic 4-vector / tensor notation, Maxwell's 4 equations {written in language of 3-D differential vector calculus}:

Maxwell's Equations:

- 1) Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$
- 2) No Magnetic Monopoles: $\vec{\nabla} \cdot \vec{B} = 0$
- 3) Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- 4) Ampere's law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ with Maxwell's displacement current

are elegantly represented by two simple 4-vector equations:

$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$

$\mu = 0$ temporal/scalar component:

$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$

1) Gauss' law

$\mu = 1:3$ spatial/vector component:

$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

4) Ampere's law

With Maxwell's displacement current

$\mu = 0$ temporal/scalar component:

$\vec{\nabla} \cdot \vec{B} = 0$

2) No Magnetic Charges

$\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$

$\mu = 1:3$ spatial/vector component:

$\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$

3) Faraday's law

Griffiths Problem 12.53:

We can show that Maxwell's two equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$ that are contained in

$\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ can also be obtained from (the more cumbersome/inelegant relation):

$$\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \iff \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \iff$$

n.b. note the cyclic permutations of indices λ, μ and ν .

Since there are 3 indices in the latter equation $\mu = 0:3, \nu = 0:3, \lambda = 0:3$, there are actually 64 (= 4³) separate equations!!! However, many of these 64 equations are either trivial or redundant.

Suppose two indices are the same (e.g. $\mu = \nu$)

Then: $\frac{\partial F_{\mu\mu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\mu} = 0$. But the EM field strength tensor $F_{\mu\nu}$ (like $F^{\mu\nu}$) is **anti-symmetric**.

$\therefore F_{\mu\mu} = 0$ and: $F_{\mu\lambda} = -F_{\lambda\mu}$. Thus, ≥ 2 indices the same gives the trivial relation: $0 = 0$.

Thus, in order to obtain a / any non-trivial result, μ , ν , and λ must **all** be different from each other.

1) The indices μ , ν , and λ could all be **spatial** indices, such as: $\mu = 1$ (x), $\nu = 2$ (y), $\lambda = 3$ (z) (or permutations thereof).

Or:

2) One index could be **temporal**, and two indices could be **spatial**, such as:

$\mu = 0$, $\nu = 1$, $\lambda = 2$ (or permutations thereof), or: $\mu = 0$, $\nu = 1$, $\lambda = 3$

1) For the case(s) of all **spatial** indices, e.g. $\mu = 1$, $\nu = 2$, $\lambda = 3$:

$$\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0 = \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 = \vec{\nabla} \cdot \vec{B} = 0$$

All other permutations involving the all-spatial indices $\{1, 2, 3\}$ yield the **same** relation $\vec{\nabla} \cdot \vec{B} = 0$ or minus it: i.e. $-\vec{\nabla} \cdot \vec{B} = 0$.

2) For the case of one **temporal** and two **spatial** indices, e.g. $\mu = 0$, $\nu = 1$, $\lambda = 2$:

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0 = -\frac{1}{c} \frac{\partial E_x}{\partial y} + \frac{1}{c} \frac{\partial B_z}{\partial t} + \frac{1}{c} \frac{\partial E_y}{\partial x} = 0 = \frac{1}{c} \frac{\partial B_z}{\partial t} + \frac{1}{c} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0$$

$$= \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} \right)_z = 0$$

Other Permutations:

For $\nu = 0$, μ & $\lambda = 1:3$ and $\lambda = 0$, μ & $\nu = 1:3$ get **redundant** results (same as above).

If $\mu = 0$, $\nu = 1$, $\lambda = 3$ get y – component of above relation!

If $\mu = 0$, $\nu = 2$, $\lambda = 3$ get x – component of above relation!

$$\therefore \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \text{ contains: } \vec{\nabla} \cdot \vec{B} = 0 \text{ and: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Duality Transformation of the Relativistic EM Field Strength Tensors $F^{\mu\nu}$ and $G^{\mu\nu}$:

The **duality transformation** for the specific case of space-time “rotating” $\vec{E} \rightarrow c\vec{B}$ and $c\vec{B} \rightarrow -\vec{E}$ ($\varphi_{\text{duality}} = 90^\circ$) takes $F^{\mu\nu} \rightarrow G^{\mu\nu}$, and can be mathematically represented in tensor notation as:

$$G^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$$

where: $F_{\lambda\sigma}$ is the **{doubly} covariant** form of the **contravariant** tensor $F^{\lambda\sigma}$.

and: $\varepsilon^{\mu\nu\lambda\sigma}$ is the **totally anti-symmetric rank-four tensor**.

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{for all **even** permutations of } \mu = 0, \nu = 1, \lambda = 2, \sigma = 3 \\ 0 & \text{if **any** two indices are equal/identical/the same.} \\ -1 & \text{for all **odd** permutations of } \mu = 0, \nu = 1, \lambda = 2, \sigma = 3 \end{cases}$$

Since $\varepsilon^{\mu\nu\lambda\sigma}$ is a rank-four tensor (= 4-dimensional “matrix”) we can’t write it down on 2-D paper all at once! $\varepsilon^{\mu\nu\lambda\sigma}$ has $(\mu, \nu, \lambda, \sigma = 0:3) \rightarrow 4^4$ elements = 256 elements!!!

We could write out 16 {4x4} matrices – e.g. one μ - ν matrix for each unique combination of λ and σ :

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{pmatrix} \begin{matrix} \lambda=0 & \nu \rightarrow \mu \\ \sigma=0 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=0 & \nu \rightarrow \mu \\ \sigma=1 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=0 & \nu \rightarrow \mu \\ \sigma=2 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=0 & \nu \rightarrow \mu \\ \sigma=3 & (4 \times 4) \downarrow \end{matrix} \\ \begin{matrix} \lambda=1 & \nu \rightarrow \mu \\ \sigma=0 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=1 & \nu \rightarrow \mu \\ \sigma=1 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=1 & \nu \rightarrow \mu \\ \sigma=2 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=1 & \nu \rightarrow \mu \\ \sigma=3 & (4 \times 4) \downarrow \end{matrix} \\ \begin{matrix} \lambda=2 & \nu \rightarrow \mu \\ \sigma=0 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=2 & \nu \rightarrow \mu \\ \sigma=1 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=2 & \nu \rightarrow \mu \\ \sigma=2 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=2 & \nu \rightarrow \mu \\ \sigma=3 & (4 \times 4) \downarrow \end{matrix} \\ \begin{matrix} \lambda=3 & \nu \rightarrow \mu \\ \sigma=0 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=3 & \nu \rightarrow \mu \\ \sigma=1 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=3 & \nu \rightarrow \mu \\ \sigma=2 & (4 \times 4) \downarrow \end{matrix} & \begin{matrix} \lambda=3 & \nu \rightarrow \mu \\ \sigma=3 & (4 \times 4) \downarrow \end{matrix} \end{pmatrix}$$

Define $\varepsilon^{\lambda\sigma} = 4 \times 4$ totally anti-symmetric rank-two tensor (aka the 4x4 Levi-Civita symbol):

$$\varepsilon^{\lambda\sigma} = \begin{pmatrix} 0 & +1 & +1 & +1 \\ -1 & 0 & +1 & +1 \\ -1 & -1 & 0 & +1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

Then, we can define $\varepsilon^{\mu\nu\lambda\sigma}$ in terms of the product of two $\varepsilon^{\lambda\sigma}$ ’s: $\varepsilon^{\mu\nu\lambda\sigma} = \varepsilon^{\mu\nu} \varepsilon^{\lambda\sigma}$

The Minkowski / Proper Force on a Point Electric Charge

The Minkowski force (*a.k.a.* **proper** force) K^μ acting on a point electric charge q can be written in 4-vector / tensor notation in terms of the EM field strength tensor $F^{\mu\nu}$ and the **proper** 4-velocity η^μ . Recall that:

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = \gamma_u \frac{dp^\mu}{dt} = \gamma_u F^\mu \quad \text{where the ordinary force: } F^\mu = \frac{dp^\mu}{dt} \quad \text{and: } \gamma_u \equiv 1/\sqrt{1-\beta_u^2}$$

However, we can **equivalently** write the Minkowski/**proper** force as: $K^\mu = q\eta_\nu F^{\mu\nu}$

where η_ν is the **covariant** form of the **contravariant proper** 4-velocity η^ν .

i.e. we **contract** the EM field strength tensor $F^{\mu\nu}$ with the **covariant proper** 4-velocity η_ν .

Since: $K^\mu = \gamma_u F^\mu$ and: $\eta_\nu = \gamma_u u_\nu$, where: $\gamma_u \equiv 1/\sqrt{1-\beta_u^2}$ and: $\beta_u = u/c$

$\therefore K^\mu = q\eta_\nu F^{\mu\nu} \Rightarrow \gamma_u F^\mu = q\gamma_u u_\nu F^{\mu\nu}$ or: $F^\mu = qu_\nu F^{\mu\nu} \leftarrow$ **ordinary** force on point charge

If $\mu = 1$ (*i.e.* row #1): $K^1 = q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13})$

$u_\nu \equiv (c, \vec{u}) \leftrightarrow \eta_\nu \equiv (\gamma_u c, \gamma_u \vec{u}) = \gamma_u u_\nu$ where: $\gamma_u = 1/\sqrt{1-\beta_u^2}$ and: $\beta = u/c$

Row #

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Column #

$$\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \end{pmatrix} = q \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$\therefore K^1 = q\gamma_u [-c(-E_x/c) + u_x \cdot 0 + u_y B_z + u_z (-B_y)] = q\gamma_u [E_x + u_y B_z - u_z B_y] = q\gamma_u (\vec{E} + \vec{u} \times \vec{B})_x$

$K^1 = q\gamma_u (\vec{E} + \vec{u} \times \vec{B})_x$

$\vec{K} = q\gamma_u (\vec{E} + \vec{u} \times \vec{B}) \leftarrow$ Minkowski 3-D Force law

Similarly, for $\mu = 2, \mu = 3$:

$K^2 = q\gamma_u (\vec{E} + \vec{u} \times \vec{B})_y$

But: $\vec{K} = \gamma_u \vec{F}$

$K^3 = q\gamma_u (\vec{E} + \vec{u} \times \vec{B})_z$

$\therefore \vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) \leftarrow$ Lorentz 3-D Force law

For $\mu = 0$ (the temporal / scalar component) {see/work Griffiths Problem 12.54, page 541}:

$$\begin{aligned} K^0 &= q\gamma_u (-c \cdot 0 + u_x (E_x/c) + u_y (E_x/c) + u_z (E_x/c)) = q\gamma_u \vec{u} \cdot \vec{E}/c \\ &= -\eta^0 F^{00} + \eta^1 F^{01} + \eta^2 F^{02} + \eta^3 F^{03} = \eta_\nu F^{0\nu} \end{aligned}$$

n.b. this relation explicitly shows that E_x, E_y, E_z are **temporal-spatial** (or **spatial-temporal**) components of $F^{\mu\nu}$, whereas B_x, B_y, B_z are pure **spatial-spatial** components of $F^{\mu\nu}$!!!

We also know that: $K^0 \equiv \frac{dp^0}{d\tau} = \frac{1}{c} \frac{dE}{d\tau} = \frac{1}{c} \frac{dt}{d\tau} \frac{dE}{dt} = \frac{1}{c} \gamma_u \frac{dE}{dt} \quad \therefore \quad K^0 = q\gamma_u \vec{u} \cdot \vec{E}/c = \frac{1}{c} \gamma_u \frac{dE}{dt}$

or: $P_q = \frac{dE}{dt} = q(\vec{u} \cdot \vec{E}) = \{\text{ordinary}\}$ relativistic power delivered to point charged particle (> 0)

$$P_q = \frac{dE}{dt} = q(\vec{u} \cdot \vec{E}) = (q\vec{E}) \cdot \vec{u} = \vec{F} \cdot \vec{u} = \frac{dW}{dt} = \text{Time rate of change of work done on charged particle by EM field}$$

Note: The **ordinary** Lorentz force: $\vec{F} = q\vec{E} + q(\vec{u} \times \vec{B})$

$$\therefore \vec{F} \cdot \vec{u} = q(\vec{u} \cdot \vec{E}) + q[\vec{u} \cdot (\vec{u} \times \vec{B})] = q(\vec{u} \cdot \vec{E})$$

But: $(\vec{u} \times \vec{B}) \perp \vec{u} \quad \therefore \quad \vec{u} \cdot (\vec{u} \times \vec{B}) \equiv 0 \quad \leftarrow \text{magnetic forces do no work !!!}$

Thus we have the relations: $K^\mu = \gamma_u F^\mu = K^\mu = q\eta_\nu F^{\mu\nu}$ and also: $F^\mu = qu_\nu F^{\mu\nu}$ with $\eta_\nu = \gamma_u u_\nu$.

The Relativistic 4-Vector Potential A^μ

We know that the electric and magnetic fields \vec{E} and \vec{B} can be expressed in terms of a scalar potential V and a vector potential \vec{A} as:

$$\vec{E}(\vec{r}, t) = -\vec{\nabla}V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad \text{and:} \quad \vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t)$$

Thus, it should not be surprising to realize that the scalar potential V and the vector potential \vec{A} form the temporal and spatial components (respectively) of the relativistic 4-vector potential A^μ :

The 4-Vector Potential: $A^\mu \equiv (V/c, \vec{A}) = (V/c, A_x, A_y, A_z)$ SI Units: *Newtons/Ampere* = "*p/q*"
{= momentum per Coulomb!}

n.b. SI units of V : $\text{Volts} = \frac{\text{Newton-meters}}{\text{Coulomb}}$ then: $\frac{V}{c} = \frac{N \cdot m}{Coul} \bigg/ \frac{m}{sec} = \frac{N \cdot sec}{Coul} = \frac{\text{Newtons}}{\text{Amp}}$

The EM field strength tensor $F^{\mu\nu}$ can be written in terms of **covariant** space-time derivatives of the 4-vector potential field A^μ as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \quad \Leftarrow \text{n.b. } \underline{\text{covariant}} \text{ differentiation here!!}$$

n.b. We **must** change the **sign** of the **temporal/scalar component** of the **covariant** derivatives

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \text{ and } \partial^\nu \equiv \frac{\partial}{\partial x_\nu} \quad \underline{\text{relative}} \text{ to that of the } \underline{\text{contravariant}} \text{ derivatives } \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \text{ and } \partial_\nu \equiv \frac{\partial}{\partial x^\nu}.$$

Explicitly evaluate a few terms: $A^\mu = (V/c, \vec{A})$

For $\mu = 0$ and $\nu = 1$:

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial(ct)} - \frac{\partial(V/c)}{\partial x} = -\frac{1}{c} \left(\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} V \right)_x = \frac{1}{c} \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right)_x = \frac{E_x}{c}$$

Likewise, for $(\mu = 0, \nu = 2)$ and $(\mu = 0, \nu = 3)$ we obtain:

$$F^{02} = \frac{1}{c} \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right)_y = \frac{E_y}{c} \quad \text{and:} \quad F^{03} = \frac{1}{c} \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right)_z = \frac{E_z}{c}$$

For $\mu = 1$ and $\nu = 2$:

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = (\vec{\nabla} \times \vec{A})_z = B_z$$

Likewise, for $(\mu = 1, \nu = 3)$ and $(\mu = 2, \nu = 3)$ we obtain:

$$F^{13} = -(\vec{\nabla} \times \vec{A})_y = -B_y \quad \text{and:} \quad F^{23} = (\vec{\nabla} \times \vec{A})_x = B_x$$

Note that the relativistic 4-potential formulation **automatically** takes care of the

homogeneous Maxwell equation $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ {it gives $\vec{\nabla} \cdot \vec{B} = 0$ and: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ }

because $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ is equivalent to $\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$.

{See/read pages 10-11 of these lecture notes – also see/work Griffiths Problem 12.53, page 541}.

And since: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)$

Thus:

$$\frac{\partial F^{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$$

$$= \frac{\partial}{\partial x^\lambda} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) + \frac{\partial}{\partial x^\mu} \left(\frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\lambda} \right) + \frac{\partial}{\partial x^\nu} \left(\frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\mu} \right) = 0$$

$$= \frac{\partial}{\partial x^\lambda} \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial}{\partial x^\mu} \frac{\partial A_\nu}{\partial x^\lambda} + \frac{\partial}{\partial x^\nu} \frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial}{\partial x^\nu} \frac{\partial A_\lambda}{\partial x^\mu} = 0$$

$$= \left(\frac{\partial}{\partial x^\lambda} \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial A_\nu}{\partial x^\lambda} \right) + \left(\frac{\partial}{\partial x^\nu} \frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial}{\partial x^\lambda} \frac{\partial A_\mu}{\partial x^\nu} \right) + \left(\frac{\partial}{\partial x^\mu} \frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial}{\partial x^\nu} \frac{\partial A_\lambda}{\partial x^\mu} \right) = 0$$

$$= \underbrace{\left(\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \right)}_{=0} A_\nu + \underbrace{\left(\frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\lambda} - \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\nu} \right)}_{=0} A_\mu + \underbrace{\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \right)}_{=0} A_\lambda = 0$$

But:

$$\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \quad \text{i.e. can change the **order** of differentiation – has **no** effect!}$$

$$= \frac{\partial^2}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2}{\partial x^\mu \partial x^\lambda} \quad \therefore \quad \boxed{0=0}$$

∴ The relativistic 4-potential formulation $F^{\mu\nu} = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right)$ does indeed automatically satisfy

$$\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \text{because} \quad \frac{\partial F^{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \quad \left\{ \text{shown to be equivalent to} \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \right\}$$

is satisfied / obeyed for $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right)$.

Does the relativistic 4-potential formulation $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right)$

satisfy the **inhomogeneous** Maxwell relation $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu$???

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) = \frac{\partial^2 A^\nu}{\partial x^\nu \partial x_\mu} - \frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = \mu_o J^\mu$$

Switching the order of derivatives:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x_\mu} \left(\frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x_\nu} \left(\frac{\partial A^\mu}{\partial x^\nu} \right) = \mu_o J^\mu$$

This is an **intractable** equation, as it stands now...

However, from our formulation of $F^{\mu\nu}$ in terms of (differences) in space-time derivatives of the 4-vector potential A^μ : $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right)$ it is clear that we can add to the

4-vector potential A^μ the space-time gradient of any scalar function λ : $A^\mu \rightarrow A^{*\mu} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}$!!!

The scalar and vector potentials V and \vec{A} are not uniquely determined by the EM fields \vec{E} and \vec{B} . Thus:

$$F^{*\mu\nu} = \left(\frac{\partial A^{*\nu}}{\partial x_\mu} - \frac{\partial A^{*\mu}}{\partial x_\nu} \right) = \left(\frac{\partial A^\nu}{\partial x_\mu} + \frac{\partial}{\partial x_\mu} \frac{\partial \lambda}{\partial x_\nu} \right) - \left(\frac{\partial A^\mu}{\partial x_\nu} + \frac{\partial}{\partial x_\nu} \frac{\partial \lambda}{\partial x_\mu} \right) = \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) + \underbrace{\left(\frac{\partial^2 \lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 \lambda}{\partial x_\nu \partial x_\mu} \right)}_{=0}$$

$$= \left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \right) = F^{\mu\nu} \quad !!!$$

$\therefore F^{*\mu\nu} = F^{\mu\nu}$ by $A^\mu \rightarrow A^{*\mu} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}$ is gauge invariance associated with the EM field $F^{\mu\nu}$!!!

We can exploit the gauge invariant properties of $F^{\mu\nu}$ to simplify the seemingly intractable relation:

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x_\mu} \left(\frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x_\nu} \left(\frac{\partial A^\mu}{\partial x^\nu} \right) = \mu_o J^\mu$$

Using the Lorenz gauge condition: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Rightarrow \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \Rightarrow \partial_\mu A^\mu = \frac{\partial A^\mu}{\partial x^\mu} = 0$

Thus: $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x_\mu} \left(\frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x_\nu} \left(\frac{\partial A^\mu}{\partial x^\nu} \right) = \mu_o J^\mu = -\frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = \mu_o J^\mu$ or: $\partial^\nu \partial_\nu A^\mu = \frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = -\mu_o J^\mu$

But: $\square_\nu = \partial_\nu \equiv \frac{\partial}{\partial x^\nu}$, $\square^\nu = \partial^\nu \equiv \frac{\partial}{\partial x_\nu}$ and: $\square^2 \equiv \square^\nu \square_\nu = \square_\nu \square^\nu = \partial^\nu \partial_\nu = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial x_\nu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x_\nu} = \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right)$

D'Alembertian operator (4-dimensional Laplacian operator)

$\therefore \square^2 A^\mu = \partial^\nu \partial_\nu A^\mu = -\mu_o J^\mu = \frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = -\mu_o J^\mu \leftarrow$ Single 4-vector equation!
= the most elegant and simple formulation of Maxwell's equations – it contains all four of Maxwell's equations!!!

Taken together with the continuity equation (charge conservation): $\square_\mu J^\mu = \partial_\mu J^\mu = 0$, these two relations compactly describe virtually all of {non-matter/free space} EM phenomena!!!

Note that the choice of the (*instantaneous*) Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ is a **bad** one for use in **relativistic electrodynamics**, because $\vec{\nabla} \cdot \vec{A}$ {alone} is **not** a Lorentz invariant quantity!

However: $\vec{\nabla} \cdot \vec{A}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial V(\vec{r}, t)}{\partial t} = \partial_\mu A^\mu(\vec{r}, t) = \frac{\partial A^\mu(\vec{r}, t)}{\partial x^\mu} = 0$ **is** a Lorentz invariant quantity

because it is the product of two relativistic 4-vectors: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and A^μ .

n.b. $\vec{\nabla} \cdot \vec{A} = 0$ is “destroyed” by **any** Lorentz transformation from one IRF(*S*) to another IRF(*S'*) !!!

\Rightarrow In order to **restore** $\vec{\nabla} \cdot \vec{A} = 0$, one must perform an appropriate gauge transformation for **each** new inertial system entered, in **addition** to carrying out the Lorentz transformation itself !!!

In the Coulomb gauge A^μ is **not** a “true” relativistic 4-vector, because $A^\mu A_\mu = A_\mu A^\mu$ is actually **not** a Lorentz invariant quantity in the Coulomb gauge !!!

n.b. The Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ **is** useful when $v \ll c$, *i.e.* for **non-relativistic** problems.