LECTURE NOTES 18.75

The Relativistic Version of Maxwell's Stress Tensor $T^{\mu\nu}$

Despite the fact that we know that the *EM* energy density $\boxed{u_{EM} = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2}$ and Poynting's Vector, $\boxed{\vec{S} = \frac{1}{\mu_o}\vec{E}\times\vec{B}}$ are ***not*** Lorentz invariant quantities, we ask: Is there a related entity, relativistic in nature, from which we ***can*** understand the transformation properties of u_{EM} and $\vec{S}$, in going from one IRF(S) to another IRF(S')?

The answer is ***yes*** – the 4-dimensional relativistic generalization of the 3-dimensional classical electrodynamics Maxwell's stress tensor: $\boxed{\overleftrightarrow{T}_{ij} \to T^{\mu\nu}}$!!!

Recall that the ***classical*** electrodynamics 3-dimensional Maxwell stress tensor is $\overleftrightarrow{T}$, a 9-component, rank two 3 × 3 ***symmetric*** tensor (*i.e.* a matrix) whose elements are:

n.b. T_{ij} elements are ***symmetric***: $T_{ij} = T_{ji} \Rightarrow \overleftrightarrow{T}$ is a ***symmetric*** rank-2 tensor.

$$\Rightarrow \boxed{T_{ij} \equiv \varepsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)} \quad \text{where: } \boxed{i, j = 1:3}$$

The 3-D Kroenecker δ-function: $\boxed{\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}}$

Physically, $\overleftrightarrow{T}$ is the ***force per unit area*** (or ***stress***) acting on a ***surface*** of interest.

$T_{ij} \equiv$ force per unit area in i^{th} direction acting on an element of surface in the j^{th} direction.

⇒ Thus: T_{xx}, T_{yy}, T_{zz}, physically represent ***pressures***. (SI units: N/m^2)

⇒ And: T_{xy}, T_{xz}, T_{yx}, T_{yz}, T_{zx}, T_{zy} physically represent ***shears***. (SI units: N/m^2)

n.b. SI units of $\overleftrightarrow{T}$ all same (= pressure): $\dfrac{N}{m^2} = \dfrac{kg\text{-}m/s^2}{m^2} = \dfrac{kg}{m\text{-}s^2}$

$\overleftrightarrow{T}$ has same SI units as energy ***density***: $\dfrac{J}{m^3} = \dfrac{N-m}{m^3} = \dfrac{N}{m^2}$

In ***classical*** electrodynamics, the ***force per unit volume*** (*aka* ***force density***) is:

$$\boxed{\vec{f}(\vec{r},t) = \vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t) - \frac{1}{c^2}\frac{\partial\vec{S}(\vec{r},t)}{\partial t}} \quad \text{where: } \boxed{\vec{S}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)} = \text{Poynting's Vector}$$

SI units of force ***density***: N/m^3

SI units: $Watts/m^2$

The ***total*** force is therefore: $\boxed{\vec{F}(t) = \int_v \vec{f}(\vec{r},t)\,d\tau = \int_v\left(\vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t)\right)d\tau - \frac{1}{c^2}\int_v \frac{\partial\vec{S}(\vec{r},t)}{\partial t}d\tau}$ SI units: $N = kg\text{-}m/s^2$

Use the divergence theorem on the 1st integral: $\boxed{\vec{F}(t) = \oint_S \overleftrightarrow{T}(\vec{r},t)\cdot d\vec{a} - \frac{1}{c^2}\int_v \frac{\partial\vec{S}(\vec{r},t)}{\partial t}d\tau}$

In going from the ***classical*** electrodynamics 3-D spatial version of Maxwell's stress tensor $\overleftrightarrow{T}$ a 3 × 3 = 9 element ***symmetric*** rank two tensor T_{ij}

$$T_{ij} \equiv \varepsilon_o \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_o}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) = \varepsilon_o c^2 \left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_i B_j \right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2 \right) \right] \qquad c^2 = \frac{1}{\varepsilon_o \mu_o}$$

$$i, j = 1{:}3$$

to the ***relativistic*** 4-dimensional space-time version of Maxwell's stress tensor $T^{\mu\nu}$ – a 4 × 4 = 16 element ***symmetric*** tensor $(\mu, \nu = 0{:}3)$ we expect the "new"/additional temporal components of $T^{\mu\nu}$ *i.e.* a new top ***row*** (row # $\mu = 0$ column # $\nu = 0{:}3$) and a new LHS ***column*** (row # $\mu = 0{:}3$, column # $\nu = 0$) to:

a.) be ***symmetric***, *i.e.* $T^{0\nu} = +T^{\nu 0}$
b.) have the ***same*** physical SI units (N/m^2 = pressure/energy density) as $\overleftrightarrow{T}_{ij}$
c.) have something to do with the ***temporal*** aspects of *EM* field energy flow
d.) be related to the *EM* field tensor, $F^{\mu\nu}$ (or equivalently, $G^{\mu\nu}$)

We define the ***relativistic*** version of Maxwell's stress tensor as:

$$\begin{aligned} T^{\mu\nu} &\equiv \varepsilon_o c^2 \left[\left(-g^{\mu\lambda} F_{\lambda\sigma} F^{\sigma\nu} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right] = \varepsilon_o c^2 \left[\left(-F^{\mu}{}_{\sigma} F^{\sigma\nu} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right] \\ &= \varepsilon_o c^2 \left[\left(+F^{\mu}{}_{\sigma} F^{\nu\sigma} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right] \end{aligned}$$

{*n.b.* implicit sum over ***repeated*** indices!}

Where the "flat" space-time ***metric*** ***tensor***, $g^{\mu\nu}$ is defined as:

$$g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \quad \text{where:} \quad g^{\mu\nu} = g_{\mu\nu} \equiv \begin{cases} -1 \text{ for } \mu = \nu = 0 \\ +1 \text{ for } \mu = \nu = 1,2,3 \\ \ \ 0 \text{ for } \mu \neq \nu \end{cases}$$

Note the similarities and differences in the physical appearance between definitions of the 3-D ***classical*** *EM* stress tensor $\overleftrightarrow{T}_{ij}$ and the 4-D ***relativistic*** *EM* stress tensor $T^{\mu\nu}$:

$$T_{ij} \equiv \varepsilon_o c^2 \left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_i B_j \right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2 \right) \right] \qquad \delta_{ij} \equiv \begin{pmatrix} +1 & & \\ & +1 & \\ & & +1 \end{pmatrix}$$

$$T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(F^{\mu}{}_{\sigma} F^{\nu\sigma} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right] \qquad g^{\mu\nu} \equiv \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

n.b. ***symmetric*** stress tensors, SI units:

$$N/m^2 = \frac{kg}{m-s^2}$$

The differences arise from our (& Griffiths') definition of the anti-symmetric *EM* field tensor, $F^{\mu\nu}$:

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

and: $$\left(F_{\alpha\beta}F^{\alpha\beta}\right) = 2\left(B^2 - \frac{1}{c^2}E^2\right) = -2\left(\frac{E^2}{c^2} - B^2\right)$$

(See Griffiths problem 12.50, *p*. 537)

For clarity's sake, ***here*** we present the results for $T^{\mu\nu}$ and place the (tedious!) calculational details in Appendix "A" of these lecture notes.

Recall that Poynting's Vector, $\vec{S} \equiv \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)$ SI units: $Watts/m^2 = \frac{J/s}{m^2} = \frac{J}{m^2\text{-}s} = \frac{N\text{-}m}{m^2\text{-}s} = \frac{N}{m\text{-}s}$

Recall that *EM* field linear momentum ***density***: $\vec{\wp}_{EM} \equiv \varepsilon_o\left(\vec{E}\times\vec{B}\right)$

$$\vec{\wp}_{EM} \equiv \varepsilon_o\left(\vec{E}\times\vec{B}\right) = \varepsilon_o\mu_o\left[\frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\right] = \varepsilon_o\mu_o\vec{S} = \frac{1}{c^2}\vec{S} \Leftarrow \frac{1}{c^2} = \varepsilon_o\mu_o$$

$$\therefore \quad \vec{\wp}_{EM}c = \frac{1}{c}\vec{S} = \frac{1}{c}\left[\frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\right] = \frac{1}{\mu_o}\left[\left(\frac{\vec{E}}{c}\times\vec{B}\right)\right] = \varepsilon_o c^2\left(\frac{\vec{E}}{c}\times\vec{B}\right)$$

SI units of $\vec{\wp}_{EM}c = \frac{1}{c}\vec{S}$: $\frac{kg}{m^2-s}\cdot\frac{m}{s} = \frac{kg}{m-s^2}$ but: $1N = kg - m/s^2$

$= \frac{kg-m}{m^2-s^2} = \frac{N}{m^2}$ = pressure $= \frac{N-m}{m^3} = \frac{J}{m^3}$ = energy ***density***

Recall also that the *EM* field energy ***density*** u_{EM} is defined as:

$$u_{EM} \equiv \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 = \frac{1}{2}\varepsilon_o\left(\vec{E}\bullet\vec{E}\right) + \frac{1}{2\mu_o}\vec{B}\bullet\vec{B} = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2}\right) + \frac{1}{2}\varepsilon_o c^2 B^2 = \frac{1}{2}\varepsilon_o c^2\left[\left(\frac{E^2}{c^2}\right) + B^2\right]$$

SI units: $\frac{J}{m^3} = \frac{N-m}{m^3} = \frac{N}{m^2}$ = energy ***density*** = pressure = force per unit area

= {linear} momentum flux density

Explicitly reminding the reader that the 3-D ***classical*** electrodynamics version of Maxwell's stress tensor $\overleftrightarrow{T}$ is defined as:

Column # j = 1:3

1=x 2=y 3=z Row # i = 1:3

$$\overleftrightarrow{T} = T_{ij} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{matrix} 1 = x \\ 2 = y \\ 3 = z \end{matrix} = \boxed{\varepsilon_o c^2 \left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_i B_j \right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2 \right) \right]} \quad \boxed{\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}}$$

Row # (1:3) Column # (1:3)

$$\overleftrightarrow{T} = \begin{pmatrix} \varepsilon_o c^2 \left[\left(\frac{E_x^2}{c^2} + B_x^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] & \varepsilon_o c^2 \left[\frac{E_x}{c}\frac{E_y}{c} + B_x B_y \right] & \varepsilon_o c^2 \left[\frac{E_x}{c}\frac{E_z}{c} + B_x B_z \right] \\ \varepsilon_o c^2 \left[\frac{E_y}{c}\frac{E_x}{c} + B_y B_x \right] & \varepsilon_o c^2 \left[\left(\frac{E_y^2}{c^2} + B_y^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] & \varepsilon_o c^2 \left[\frac{E_y}{c}\frac{E_z}{c} + B_y B_z \right] \\ \varepsilon_o c^2 \left[\frac{E_z}{c}\frac{E_x}{c} + B_z B_x \right] & \varepsilon_o c^2 \left[\frac{E_z}{c}\frac{E_y}{c} + B_z B_y \right] & \varepsilon_o c^2 \left[\left(\frac{E_z^2}{c^2} + B_z^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] \end{pmatrix}$$

$$\overleftrightarrow{T} = \frac{1}{2}\varepsilon_o c^2 \begin{pmatrix} \left(\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2} \right) + \left(B_x^2 - B_y^2 - B_z^2 \right) & 2\left[\frac{E_x}{c}\frac{E_y}{c} + B_x B_y \right] & 2\left[\frac{E_x}{c}\frac{E_z}{c} + B_x B_z \right] \\ 2\left[\frac{E_y}{c}\frac{E_x}{c} + B_y B_x \right] & \left(\frac{E_y^2}{c^2} + B_y^2 \right) + \left(-B_x^2 + B_y^2 - B_z^2 \right) & 2\left[\frac{E_y}{c}\frac{E_z}{c} + B_y B_z \right] \\ 2\left[\frac{E_z}{c}\frac{E_x}{c} + B_z B_x \right] & 2\left[\frac{E_z}{c}\frac{E_y}{c} + B_z B_y \right] & \left(-\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2} \right) + \left(-B_x^2 - B_y^2 + B_z^2 \right) \end{pmatrix}$$

$\Rightarrow$ Note that the T_{ij} elements ***are*** symmetric, i.e. $\boxed{T_{ji} = +T_{ij}}$

Physically:

$T_{ij}\ (i, j = 1:3)$ are pure ***space-space*** components.

$T_{ij} = i^{th}$ component of force across unit area perpendicular to j^{th} direction.

$T_{ii} = T_{xx}, T_{yy}, T_{zz}$ represent ***pressures*** on enclosing surfaces in the x, y, z directions, respectively.

$T_{ij} = \left(T_{xy} = T_{yx}\right), \left(T_{xz} = T_{zx}\right), \left(T_{yz} = T_{zy}\right)$ represent ***shear stresses*** on enclosing surfaces in the x, y, or z directions.

But note ***also*** that physically:

T_{ij} = –*ve* of the rate of ***flow*** of the i^{th} component of *EM* field linear momentum $\vec{p}$ through unit area whose normal is in the j^{th} direction, i.e. $-T_{ij}$ is the i^{th} component of the ***linear*** momentum flux ***density*** transported in the j^{th} direction by the *EM* fields.

Then, the ***relativistic*** 4-D space-time version of Maxwell's stress tensor defined as:

$$\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(F^{\mu}{}_{\sigma} F^{\nu\sigma} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right]} \quad \text{gives:}$$

Column #

Row # Column # $0=t$ $1=x$ $2=y$ $3=z$ Row #

0:3 0:3

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \begin{matrix} 0=t \\ 1=x \\ 2=y \\ 3=z \end{matrix} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix}$$

Now while we might ***hope*** that the $\{\mu,\nu = 1:3\}$ pure ***space-space*** elements of the ***relativistic*** 4-D space-time stress tensor $T^{\mu\nu}$ would be identical to that of the 3-D classical electrodynamics / Maxwell's stress tensor $\overleftrightarrow{T}_{ij}$, because of our definitions for $T^{\mu\nu}$ and $F^{\mu\nu}$ what we instead obtain is:

$$\boxed{T^{\mu\nu}\Big|_{\mu,\nu=1:3} = -\overleftrightarrow{T}_{ij}\Big|_{i,j=1:3}}$$

Thus, we see that the pure ***space-space*** $\{\mu,\nu = 1:3\}$ components of the ***relativistic*** stress tensor $T^{\mu\nu}$ <u>physically represent *EM* field linear momentum flux densities</u>, the ***negative*** of which physically corresponds to stresses/shears on bounding surfaces!

The determination of the individual 4×4 = 16 elements of the ***relativistic*** 4-D space-time stress tensor $T^{\mu\nu}$ is carried out in Appendix A at the end of these Physics 436 Lecture notes.

The ***temporal*** components of $T^{\mu\nu}$ are:

<u>$\nu = 0$ $\left(1^{\text{st}} \text{ column}\right)$:</u>

$$\boxed{\frac{1}{\mu_o} = \varepsilon_o c^2}$$

$$\boxed{T^{00} = u_{EM} = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o} B^2 = \frac{1}{2}\varepsilon_o c^2 \left(\frac{E^2}{c^2} + B^2 \right)}$$

$$\boxed{T^{10} = \wp_x c = \varepsilon_o c^2 \left(\frac{E_y}{c} B_z - \frac{E_z}{c} B_y \right) = \varepsilon_o c \left(E_y B_z - E_z B_y \right)}$$

$$\boxed{T^{20} = \wp_y c = \varepsilon_o c^2 \left(\frac{E_z}{c} B_x - \frac{E_x}{c} B_z \right) = \varepsilon_o c \left(E_z B_x - E_x B_z \right)}$$

$$\boxed{T^{30} = \wp_z c = \varepsilon_o c^2 \left(\frac{E_x}{c} B_y - \frac{E_y}{c} B_x \right) = \varepsilon_o c \left(E_x B_y - E_y B_x \right)}$$

Recalling that Poynting's Vector: $\boxed{\vec{S} \equiv \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)}$ $\therefore$ $\boxed{\frac{1}{c}\vec{S} = \frac{1}{\mu_o}\left(\frac{\vec{E}}{c}\times\vec{B}\right) = \varepsilon_o c^2 \left(\frac{\vec{E}}{c}\times\vec{B}\right) = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)}$

Thus: $\boxed{\frac{1}{c}\vec{S} = \frac{1}{c}S_x\hat{x} + \frac{1}{c}S_y\hat{y} + \frac{1}{c}S_z\hat{z} = \varepsilon_o c\left(E_y B_z - E_z B_y\right)\hat{x} + \varepsilon_o c\left(E_z B_x - E_x B_z\right)\hat{y} + \varepsilon_o c\left(E_x B_y - E_y B_x\right)\hat{z}}$

$\mu = 0 \; \left(1^{st} \text{ row}\right)$:

$$T^{01} = \frac{1}{c}S_x = \varepsilon_o c\left(E_y B_z - E_z B_y\right)$$
$$T^{02} = \frac{1}{c}S_y = \varepsilon_o c\left(E_z B_x - E_x B_z\right)$$
$$T^{03} = \frac{1}{c}S_z = \varepsilon_o c\left(E_x B_y - E_y B_x\right)$$

Thus, we (***explicitly***) see that:

$$T^{10} = T^{01} \Rightarrow \wp_x c = \frac{1}{c}S_x$$
$$T^{20} = T^{02} \Rightarrow \wp_y c = \frac{1}{c}S_y$$
$$T^{30} = T^{03} \Rightarrow \wp_z c = \frac{1}{c}S_z$$

or: $\vec{\wp} c = \frac{1}{c}\vec{S} = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)$

Thus the components of the ***relativistic*** 4-D/spacetime version of Maxwell's stress tensor $T^{\mu\nu}$ are $\{T^{\mu\nu} = T^{\nu\mu} =$ ***symmetric*** tensor$\}$: $T^{\mu\nu} \equiv \varepsilon_o c^2\left[\left(F^{\mu}{}_{\sigma}F^{\nu\sigma}\right) - \frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & -T_{xx} & -T_{xy} & -T_{xz} \\ T^{yt} & -T_{yx} & -T_{yy} & -T_{yz} \\ T^{zt} & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}$$

3-D space-space components of classical electrodynamics' Maxwell's stress tensor $\overleftrightarrow{T}_{ij}$

$$T^{\mu\nu} = \begin{pmatrix} u_{EM} = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2}+B^2\right) & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \wp_x c & -\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2}+B_x^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_x}{c}\frac{E_y}{c}\right)+\left(B_xB_y\right)\right] & -\varepsilon_0 c^2\left[\left(\frac{E_x}{c}\frac{E_z}{c}\right)+\left(B_xB_z\right)\right] \\ \wp_y c & -\varepsilon_o c^2\left[\left(\frac{E_y}{c}\frac{E_x}{c}\right)+\left(B_yB_x\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_y^2}{c^2}+B_y^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_y}{c}\frac{E_z}{c}\right)+\left(B_yB_z\right)\right] \\ \wp_z c & -\varepsilon_o c^2\left[\left(\frac{E_z}{c}\frac{E_x}{c}\right)+\left(B_zB_x\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_z}{c}\frac{E_y}{c}\right)+\left(B_zB_y\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_z^2}{c^2}+B_z^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] \end{pmatrix}$$

Physically:

$T^{\mu\nu} =$

EM field energy density, u_{EM}	*x*-flow of *EM* field energy, S_x/c	*y*-flow of *EM* field energy, S_y/c	*z*-flow of *EM* field energy, S_y/c
flux of *EM* field momentum density, $\wp_x$	x – flow of *EM* field linear momentum density, $\wp_x$	y – flow of *EM* field linear momentum density, $\wp_x$	z – flow of *EM* field linear momentum density, $\wp_x$
flux of *EM* field momentum density, $\wp_y$	x – flow of *EM* field linear momentum density, $\wp_y$	y – flow of *EM* field linear momentum density, $\wp_y$	z – flow of *EM* field linear momentum density, $\wp_y$
flux of *EM* field momentum density, $\wp_z$	x – flow of *EM* field linear momentum density, $\wp_z$	y – flow of *EM* field linear momentum density, $\wp_z$	z – flow of *EM* field linear momentum density, $\wp_z$

SI units of $T^{\mu\nu}$: energy ***density*** $= J/m^3 = N\text{-}m/m^3 = N/m^2 = Pascals =$ ***pressure***

$$T^{00} = u_{EM} = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2} + B^2\right) = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 \quad \text{using: } \quad c^2 = \frac{1}{\varepsilon_o \mu_o}$$

$$T^{10} = \wp_x c = \varepsilon_o c^2\left(\frac{E_y}{c}B_z - \frac{E_z}{c}B_y\right) = \varepsilon_o c\left(E_y B_z - E_z B_y\right) = \frac{1}{c}S_x = T^{01}$$

$$T^{20} = \wp_x c = \varepsilon_o c^2\left(\frac{E_z}{c}B_x - \frac{E_x}{c}B_z\right) = \varepsilon_o c\left(E_z B_x - E_x B_z\right) = \frac{1}{c}S_y = T^{02}$$

$$T^{30} = \wp_z c = \varepsilon_o c^2\left(\frac{E_x}{c}B_y - \frac{E_y}{c}B_x\right) = \varepsilon_o c\left(E_x B_y - E_y B_x\right) = \frac{1}{c}S_z = T^{03}$$

$$T^{01} = \frac{1}{c}S_x = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_x = \varepsilon_o c\left(E_y B_z - E_z B_y\right) = \wp_x c = T^{10}$$

$$T^{02} = \frac{1}{c}S_y = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_y = \varepsilon_o c\left(E_z B_x - E_x B_z\right) = \wp_y c = T^{20}$$

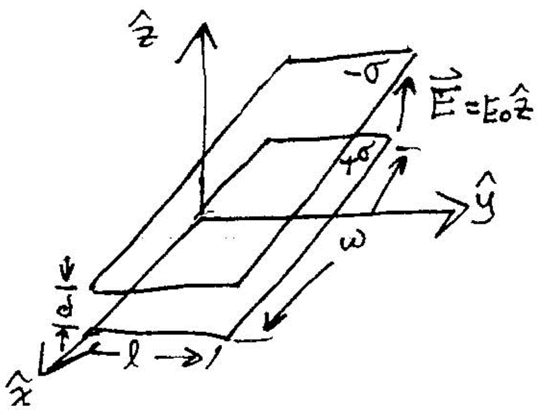
$$T^{03} = \frac{1}{c}S_z = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_z = \varepsilon_o c\left(E_x B_y - E_y B_x\right) = \wp_z c = T^{30}$$

Note that while $T^{\mu\nu}$ and/or $T_{\mu\nu}$ are ***not*** Lorentz-invariant quantities, $T^{\mu}{}_{\nu} = T^{\mu\sigma}g_{\sigma\nu}$ and/or $T_{\mu}{}^{\nu} = g_{\mu\sigma}T^{\sigma\nu}$ ***are*** indeed Lorentz-invariant quantities. Hence, additionally note the value of the ***trace*** of ***this*** form of Maxwell's ***relativistic*** stress tensor (a Lorentz-invariant quantity):

$$Tr\left\{T^{\mu}{}_{\nu}\right\} = Tr\left\{g_{\nu\lambda}T^{\mu\lambda}\right\} = -T^{00} + T^{11} + T^{22} + T^{33} = -u_{EM} - 2u_{EM} + 3u_{EM} = 0 \;\Leftarrow$$

Intimately connected to fact that the photon mass $m_\gamma \equiv 0$ – ***must*** be valid in ***any/all*** IRF's!!

As a simple example of the use of the 4-D/***relativistic*** version of Maxwell's stress tensor $T^{\mu\nu}$, let us consider the purely electrostatic problem of a parallel-plate capacitor at rest in the lab frame IRF(S), with large area plates || to the x-y plane as shown in the figure below:



Inside the || plates $\{ d \ll \omega, \ell \}$:

$\vec{E} = E_o \hat{z} \quad \left(\vec{E} = 0 \text{ elsewhere} \right)$

$\vec{B} = 0$ everywhere

Then:
$$T^{\mu\nu} = \begin{pmatrix} u_{EM} & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \wp_x c & -T_{xx} & -T_{xy} & -T_{xz} \\ \wp_y c & -T_{yx} & -T_{yy} & -T_{yz} \\ \wp_z c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\varepsilon_o E_o^2 & 0 & 0 & 0 \\ 0 & +\frac{1}{2}\varepsilon_o E_o^2 & 0 & 0 \\ 0 & 0 & +\frac{1}{2}\varepsilon_o E_o^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\varepsilon_o E_o^2 \end{pmatrix}$$

$$T^{\mu\nu} = \tfrac{1}{2}\varepsilon_o E_o^2 \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Leftarrow$$

Only the ***diagonal*** elements of $T^{\mu\nu} = T^{\mu\mu}$ are non-zero (***here***)

$T^{00} = u_{EM} = +\frac{1}{2}\varepsilon_o E_o^2 \Rightarrow$ +*ve* Energy Density (J/m^3)

$T^{11} = -T_{xx} = +\frac{1}{2}\varepsilon_o E_o^2 \Rightarrow$ +*ve EM* ***pressure*** = $+\frac{1}{2}\varepsilon_o E_o^2$ in $\hat{x}$-direction !!!

$T^{22} = -T_{yy} = +\frac{1}{2}\varepsilon_o E_o^2 \Rightarrow$ +*ve EM* ***pressure*** = $+\frac{1}{2}\varepsilon_o E_o^2$ in $\hat{y}$-direction !!!

$T^{33} = -T_{zz} = -\frac{1}{2}\varepsilon_o E_o^2 \Rightarrow$ –*ve EM* ***pressure*** = $-\frac{1}{2}\varepsilon_o E_o^2$ in $\hat{z}$-direction $\{n.b.\ \vec{E} = E_o \hat{z}\}$!!!

$\Rightarrow$ Plates of capacitor ***attracted*** to each other – net ***attractive*** force acting on bottom/top plates:

Tension: $\vec{F}_{bot} = +\sigma \ell w E_o \hat{z} = +Q E_o \hat{z}$, $\vec{F}_{top} = -\sigma \ell w E_o \hat{z} = -Q E_o \hat{z} = -\vec{F}_{bot}$

Appendix A: Calculation of the Elements of the Relativistic Version of Maxwell's Stress Tensor

$$T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(-g^{\mu\lambda} F_{\lambda\sigma} F^{\sigma\nu} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right] = \varepsilon_o c^2 \left[\left(-F^{\mu}{}_{\sigma} F^{\sigma\nu} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right]$$
$$= \varepsilon_o c^2 \left[\left(+F^{\mu}{}_{\sigma} F^{\nu\sigma} \right) - \frac{1}{4} g^{\mu\nu} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right]$$

SI units of $T^{\mu\nu}$: energy ***density*** $= J/m^3 = N\text{-}m/m^3 = N/m^2 = Pascals =$ ***pressure***

"flat" space-time metric tensor: $g^{\mu\nu} = g_{\mu\nu} \equiv \begin{cases} -1 \text{ for } \mu = \nu = 0 \\ \ \ 0 \text{ for } \mu \neq \nu \\ +1 \text{ for } \mu = \nu = 1, 2 \text{ or } 3 \end{cases}$ *i.e.* $g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$

Anti-symmetric *EM* field tensor, $F^{\mu\nu}$: $F^{\mu\nu} = -F^{\nu\mu}$

Column # ν (0, 1, 2, 3); Row # μ (0, 1, 2, 3)

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix}$$

(Column # → ν, Row # → μ)

And:

$F^{\mu}{}_{\nu} = F^{\mu\sigma} g_{\sigma\nu}$ means that we place a minus (–) sign in front of the $\nu = 0$ ***temporal*** components, *i.e.* elements in the ***first vertical column*** ($\nu = 0$) of $F^{\mu\nu}$.

$F_{\mu}{}^{\nu} = g_{\mu\sigma} F^{\sigma\nu}$ means that we place a minus (–) sign in front of the $\mu = 0$ ***temporal*** components, *i.e.* elements in the ***first horizontal row*** ($\mu = 0$) of $F^{\mu\nu}$.

$F_{\mu\nu} = g_{\mu\sigma} F^{\sigma\lambda} g_{\lambda\nu}$ means that we place a minus (–) sign in front of ***both*** of the $\nu = 0$ ***temporal*** components, elements in the ***first vertical column*** ($\nu = 0$) .**AND**. the $\mu = 0$ ***temporal*** components, elements in the ***first horizontal row*** ($\mu = 0$) of $F^{\mu\nu}$.

$F^{\mu}{}_{\nu} = g^{\mu\sigma} F_{\sigma\nu}$ means that we place a minus (–) sign in front of the $\mu = 0$ ***temporal*** components, *i.e.* elements in the ***first horizontal row*** ($\mu = 0$) of $F_{\mu\nu}$.

Thus:

$$F^{\mu}{}_{\nu} = F^{\mu\sigma} g_{\sigma\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ +E_x/c & 0 & B_z & -B_y \\ +E_y/c & -B_z & 0 & B_x \\ +E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

n.b. $F^{\mu}{}_{\nu}$ has **<u>no</u>** explicit symmetry – neither **<u>*symmetric*</u>** nor **<u>*anti-symmetric*</u>**!

$$F_{\mu}{}^{\nu} = g_{\mu\sigma} F^{\sigma\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

n.b. $F_{\mu}{}^{\nu}$ has **<u>no</u>** explicit symmetry – neither **<u>*symmetric*</u>** nor **<u>*anti-symmetric*</u>**!

$$F_{\mu\nu} = g_{\mu\sigma} F^{\sigma}{}_{\nu} = g_{\mu\sigma} F^{\sigma\lambda} g_{\lambda\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \Leftarrow \left\{ n.b. \neq F^{\mu\nu}, \neq F^{\nu\mu} = -F^{\mu\nu} \right\}$$

n.b. $F_{\mu\nu}$ is **<u>*anti-symmetric*</u>**!

$$F^{\mu}{}_{\nu} = g^{\mu\sigma} F_{\sigma\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & +E_x/c & +E_y/c & +E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} = F^{\mu\sigma} g_{\sigma\nu} = F^{\mu}{}_{\nu}$$

n.b. $F^{\mu}{}_{\nu}$ has **<u>no</u>** explicit symmetry – neither **<u>*symmetric*</u>** nor **<u>*anti-symmetric*</u>**!

First, we want to explicitly calculate: $\boxed{F_{\alpha\beta}F^{\alpha\beta} = F^{\alpha\beta}F_{\alpha\beta}}$ where: $\boxed{F^{\mu\nu} = F^{\text{row \# column \#}}}$

$$\therefore \quad \begin{aligned} F_{\alpha\beta}F^{\alpha\beta} &= F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} &&= +F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} \\ &+ F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13} &&\quad - F^{10}F^{10} + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} \\ &+ F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} &&\quad - F^{20}F^{20} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} \\ &+ F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33} &&\quad - F^{30}F^{30} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \end{aligned}$$

$$\begin{aligned} F_{\alpha\beta}F^{\alpha\beta} = &+0^2 \quad -\left(E_x/c\right)^2 - \left(E_y/c\right)^2 - \left(E_z/c\right)^2 \\ &-\left(-E_x/c\right)^2 \quad +0^2 \quad +\left(B_z\right)^2 \quad +\left(-B_y\right)^2 \\ &-\left(-E_y/c\right)^2 \quad +\left(B_z\right)^2 \quad +0^2 \quad +\left(B_x\right)^2 \\ &-\left(-E_z/c\right)^2 \quad +\left(B_y\right)^2 \quad +\left(-B_x\right)^2 \quad +0^2 \end{aligned} \Rightarrow \begin{aligned} F_{\alpha\beta}F^{\alpha\beta} = &-\left(E_x/c\right)^2 - \left(E_y/c\right)^2 - \left(E_z/c\right)^2 \\ &-\left(E_x/c\right)^2 + B_z^2 + B_y^2 \\ &-\left(E_y/c\right)^2 + B_z^2 + B_x^2 \\ &-\left(E_z/c\right)^2 + B_y^2 + B_x^2 \end{aligned}$$

$$F_{\alpha\beta}F^{\alpha\beta} = 2\left[-\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + \left(B_x^2 + B_y^2 + B_z^2\right)\right] = 2\left(-\frac{E^2}{c^2} + B^2\right) = 2\left(B^2 - \frac{E^2}{c^2}\right) = -2\left(\frac{E^2}{c^2} - B^2\right)$$

$$\therefore \quad \boxed{\left(F_{\alpha\beta}F^{\alpha\beta}\right) = \left(F^{\alpha\beta}F_{\alpha\beta}\right) = 2\left(B^2 - \left(E^2/c^2\right)\right) = -2\left(\left(E^2/c^2\right) - B^2\right)}$$

See also Griffiths problem 12.50 a.)

Then: $\boxed{g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right) = -2g^{\mu\nu}\left(\left(E^2/c^2\right) - B^2\right)}$

Next, we need to calculate $\left(-F^{\mu}{}_{\sigma}F^{\sigma\nu}\right)$ which is a 4 × 4 = 16 element ***symmetric*** tensor $\equiv S^{\mu\nu}$

Thus: since $S^{\mu\nu} = S^{\nu\mu}$, then only 10 out of the 16 elements of $S^{\mu\nu}$ are unique.

n.b. repeated indices (σ) are summed over $\sigma = 0:3$ for ***each*** element of $\boxed{S^{\mu\nu} \equiv -F^{\mu}{}_{\sigma}F^{\sigma\nu}}$

We explicitly calculate ***all*** 16 elements of the 4 × 4 tensor:

$$\begin{aligned} S^{\mu\nu} \equiv -F^{\mu}{}_{\sigma}F^{\sigma\nu} &= -\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \\ &= -\begin{pmatrix} F^0{}_0 & F^0{}_1 & F^0{}_2 & F^0{}_3 \\ F^1{}_0 & F^1{}_1 & F^1{}_2 & F^1{}_3 \\ F^2{}_0 & F^2{}_1 & F^2{}_2 & F^2{}_3 \\ F^3{}_0 & F^3{}_1 & F^3{}_2 & F^3{}_3 \end{pmatrix}\begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} \end{aligned}$$

$\nu = 0$ (1st column) of $S^{\mu\nu}$ $(\mu = 0:3)$:

$$S^{00} \equiv -F^0{}_\sigma F^{\sigma 0} = -F^0{}_0 F^{00} - F^0{}_1 F^{10} - F^0{}_2 F^{20} - F^0{}_3 F^{30} \left(= +F^{00}F^{00} - F^{01}F^{10} - F^{02}F^{20} - F^{03}F^{30}\right)$$
$$= 0^2 + \left(\frac{E_x}{c}\right)^2 + \left(\frac{E_y}{c}\right)^2 + \left(\frac{E_z}{c}\right)^2 = \frac{E^2}{c^2}$$

$$S^{10} \equiv -F^1{}_\sigma F^{\sigma 0} = -F^1{}_0 F^{00} - F^1{}_1 F^{10} - F^1{}_2 F^{20} - F^1{}_3 F^{30} \left(= +F^{10}F^{00} - F^{11}F^{10} - F^{12}F^{20} - F^{13}F^{30}\right)$$
$$= -\left(\frac{E_x}{c}\right)(0) - (0)\left(\frac{-E_x}{c}\right) - B_z\left(\frac{-E_y}{c}\right) - (-B_y)\left(\frac{-E_z}{c}\right) = \frac{1}{c}\left(E_y B_z - E_z B_y\right)$$

$$S^{20} \equiv -F^2{}_\sigma F^{\sigma 0} = -F^2{}_0 F^{00} - F^2{}_1 F^{10} - F^2{}_2 F^{20} - F^2{}_3 F^{30} \left(= +F^{20}F^{00} - F^{21}F^{10} - F^{22}F^{20} - F^{23}F^{30}\right)$$
$$= -\left(\frac{E_y}{c}\right)(0) - (-B_z)\left(\frac{-E_x}{c}\right) - (0)\left(\frac{-E_y}{c}\right) - (B_x)\left(\frac{-E_z}{c}\right) = \frac{1}{c}\left(E_z B_x - E_x B_z\right)$$

$$S^{30} \equiv -F^3{}_\sigma F^{\sigma 0} = -F^3{}_0 F^{00} - F^3{}_1 F^{10} - F^3{}_2 F^{20} - F^3{}_3 F^{30} \left(= +F^{30}F^{00} - F^{31}F^{10} - F^{32}F^{20} - F^{33}F^{30}\right)$$
$$= -\left(\frac{E_z}{c}\right)(0) - (B_y)\left(\frac{-E_x}{c}\right) - (-B_x)\left(\frac{-E_y}{c}\right) - (0)\left(\frac{E_z}{c}\right) = \frac{1}{c}\left(E_x B_y - E_y B_x\right)$$

$\nu = 1$ (2nd column) of $S^{\mu\nu}$ $(\mu = 0:3)$:

$$S^{01} \equiv -F^0{}_\sigma F^{\sigma 1} = -F^0{}_0 F^{01} - F^0{}_1 F^{11} - F^0{}_2 F^{21} - F^0{}_3 F^{31} \left(= +F^{00}F^{01} - F^{01}F^{11} - F^{02}F^{21} + F^{03}F^{31}\right)$$
$$= -(0)\left(\frac{E_x}{c}\right) - \left(\frac{E_x}{c}\right)(0) - \left(\frac{E_y}{c}\right)(-B_z) - \left(\frac{E_z}{c}\right)(B_y) = \frac{1}{c}\left(E_y B_z - E_z B_y\right)$$

$$S^{11} \equiv -F^1{}_\sigma F^{\sigma 1} = -F^1{}_0 F^{01} - F^1{}_1 F^{11} - F^1{}_2 F^{21} - F^1{}_3 F^{31} \left(= +F^{10}F^{01} - F^{11}F^{11} - F^{12}F^{21} - F^{13}F^{31}\right)$$
$$= -\left(\frac{E_x}{c}\right)\left(\frac{E_x}{c}\right) - (0)(0) - (B_z)(-B_z) - (-B_y)(B_y) = -\frac{E_x^2}{c^2} + B_y^2 + B_z^2$$

$$S^{21} \equiv -F^2{}_\sigma F^{\sigma 1} = -F^2{}_0 F^{01} - F^2{}_1 F^{11} - F^2{}_2 F^{21} - F^2{}_3 F^{31} \left(= +F^{20}F^{01} - F^{21}F^{11} - F^{22}F^{21} - F^{23}F^{31}\right)$$
$$= -\left(\frac{E_y}{c}\right)\left(\frac{E_x}{c}\right) - (-B_z)(0) - (0)(-B_z) - (B_x)(B_y) = -\left(\frac{E_x E_y}{c^2} + B_x B_y\right)$$

$$S^{31} \equiv -F^3{}_\sigma F^{\sigma 1} = -F^3{}_0 F^{01} - F^3{}_1 F^{11} - F^3{}_2 F^{21} - F^3{}_3 F^{31} \left(= +F^{30}F^{01} - F^{31}F^{11} - F^{32}F^{21} + F^{33}F^{31}\right)$$
$$= -\left(\frac{E_z}{c}\right)\left(\frac{E_x}{c}\right) - (B_y)(0) - (-B_x)(-B_z) - (0)(B_y) = -\left(\frac{E_x E_z}{c^2} + B_x B_z\right)$$

$\nu = 2$ (3rd column) of $S^{\mu\nu}$ $(\mu = 0:3)$:

$$S^{02} \equiv -F^0{}_\sigma F^{\sigma 2} = -F^0{}_0 F^{02} - F^0{}_1 F^{12} - F^0{}_2 F^{22} - F^0{}_3 F^{32} \left(= +F^{00}F^{02} - F^{01}F^{12} - F^{02}F^{22} - F^{03}F^{32}\right)$$
$$= -(0)\left(\frac{E_y}{c}\right) - \left(\frac{E_x}{c}\right)(B_z) - \left(\frac{E_y}{c}\right)(0) - \left(\frac{E_z}{c}\right)(-B_x) = \frac{1}{c}(E_z B_x - E_x B_z)$$

$$S^{12} \equiv -F^1{}_\sigma F^{\sigma 2} = -F^1{}_0 F^{02} - F^1{}_1 F^{12} - F^1{}_2 F^{22} - F^1{}_3 F^{32} \left(= +F^{10}F^{02} - F^{11}F^{12} - F^{12}F^{22} - F^{13}F^{32}\right)$$
$$= -\left(\frac{E_x}{c}\right)\left(\frac{E_y}{c}\right) - (0)(B_z) - (B_z)(0) - (-B_y)(-B_x) = -\left(\frac{E_x E_y}{c^2} + B_x B_y\right)$$

$$S^{22} \equiv -F^2{}_\sigma F^{\sigma 2} = -F^2{}_0 F^{02} - F^2{}_1 F^{12} - F^2{}_2 F^{22} - F^2{}_3 F^{32} \left(= +F^{20}F^{02} - F^{21}F^{12} - F^{22}F^{22} - F^{23}F^{32}\right)$$
$$= -\left(\frac{E_y}{c}\right)\left(\frac{E_y}{c}\right) - (-B_z)(B_z) - (0)(0) - (B_x)(-B_x) = -\frac{E_y^2}{c^2} + B_x^2 + B_z^2$$

$$S^{32} \equiv -F^3{}_\sigma F^{\sigma 2} = -F^3{}_0 F^{02} - F^3{}_1 F^{12} - F^3{}_2 F^{22} - F^3{}_3 F^{32} \left(= +F^{30}F^{02} - F^{31}F^{12} - F^{32}F^{22} - F^{33}F^{32}\right)$$
$$= -\left(\frac{E_z}{c}\right)\left(\frac{E_y}{c}\right) - (B_y)(B_z) - (-B_x)(0) - (0)(-B_x) = -\left(\frac{E_y E_z}{c^2} + B_y B_z\right)$$

$\nu = 3$ (4th column) of $S^{\mu\nu}$ $(\mu = 0:3)$:

$$S^{03} \equiv -F^0{}_\sigma F^{\sigma 3} = -F^0{}_0 F^{03} - F^0{}_1 F^{13} - F^0{}_2 F^{23} - F^0{}_3 F^{33} \left(= +F^{00}F^{03} - F^{01}F^{13} - F^{02}F^{23} - F^{03}F^{33}\right)$$
$$= -(0)\left(\frac{E_z}{c}\right) - \left(\frac{E_x}{c}\right)(-B_y) - \left(\frac{E_y}{c}\right)(B_x) - \left(\frac{E_z}{c}\right)(0) = \frac{1}{c}(E_x B_y - E_y B_x)$$

$$S^{13} \equiv -F^1{}_\sigma F^{\sigma 3} = -F^1{}_0 F^{03} - F^1{}_1 F^{13} - F^1{}_2 F^{23} - F^1{}_3 F^{33} \left(= +F^{10}F^{03} - F^{11}F^{13} - F^{12}F^{23} - F^{13}F\right)^{33}$$
$$= -\left(\frac{E_x}{c}\right)\left(\frac{E_z}{c}\right) - (0)(-B_y) - (B_z)(B_x) - (-B_y)(0) = -\left(\frac{E_x E_z}{c^2} + B_x B_z\right)$$

$$S^{23} \equiv -F^2{}_\sigma F^{\sigma 3} = -F^2{}_0 F^{03} - F^2{}_1 F^{13} - F^2{}_2 F^{23} - F^2{}_3 F^{33} \left(= +F^{20}F^{03} - F^{21}F^{13} - F^{22}F^{23} - F^{23}F^{33}\right)$$
$$= -\left(\frac{E_y}{c}\right)\left(\frac{E_z}{c}\right) - (-B_z)(-B_y) - (0)(B_x) - (B_x)(0) = -\left(\frac{E_y E_z}{c^2} + B_y B_z\right)$$

$$S^{33} \equiv -F^3{}_\sigma F^{\sigma 3} = -F^3{}_0 F^{03} - F^3{}_1 F^{13} - F^3{}_2 F^{23} - F^3{}_3 F^{33} \left(= +F^{30}F^{03} - F^{31}F^{13} - F^{32}F^{23} - F^{33}F^{33}\right)$$
$$= -\left(\frac{E_z}{c}\right)\left(\frac{E_z}{c}\right) - (B_y)(-B_y) - (-B_x)(B_x) - (0)(0) = -\frac{E_z^2}{c^2} + B_x^2 + B_y^2$$

Collecting our results for 16 elements the {intermediate} tensor:

$$S^{\mu\nu} \equiv F^{\mu}{}_{\sigma}F^{\nu\sigma} = -F^{\mu}{}_{\sigma}F^{\sigma\nu} = \begin{pmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11} & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22} & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33} \end{pmatrix}$$

$$S^{\mu\nu} \equiv F^{\mu}{}_{\sigma}F^{\nu\sigma} = \begin{pmatrix} \frac{E^2}{c^2} & \frac{1}{c}(E_yB_z - E_zB_y) & \frac{1}{c}(E_zB_x - E_xB_z) & \frac{1}{c}(E_xB_y - E_yB_x) \\ \frac{1}{c}(E_yB_z - E_zB_y) & -\frac{E_x^2}{c^2} + B_y^2 + B_z^2 & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) \\ \frac{1}{c}(E_zB_x - E_xB_z) & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\frac{E_y^2}{c^2} + B_x^2 + B_z^2 & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) \\ \frac{1}{c}(E_xB_y - E_yB_x) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) & -\frac{E_z^2}{c^2} + B_x^2 + B_y^2 \end{pmatrix}$$

Note that $S^{\mu\nu}$ ***is*** indeed a ***symmetric*** matrix, *i.e.* $S^{\mu\nu} \equiv F^{\mu}{}_{\sigma}F^{\nu\sigma} = F^{\nu\sigma}F^{\mu}{}_{\sigma} \equiv S^{\nu\mu}$

Then: $$T^{\mu\nu} \equiv \varepsilon_0 c^2\left[\underbrace{\left(F^{\mu}{}_{\sigma}F^{\nu\sigma}\right)}_{\equiv S^{\mu\nu}} - \frac{1}{4}g^{\mu\nu}\underbrace{\left(F_{\alpha\beta}F^{\alpha\beta}\right)}_{=-2\left(\frac{E^2}{c^2} - B^2\right)}\right]$$ where: $$g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$S^{\mu\nu} \equiv F^{\mu}{}_{\sigma}F^{\nu\sigma} = \begin{pmatrix} \frac{E^2}{c^2} & \frac{1}{c}(E_yB_z - E_zB_y) & \frac{1}{c}(E_zB_x - E_xB_z) & \frac{1}{c}(E_xB_y - E_yB_x) \\ \frac{1}{c}(E_yB_z - E_zB_y) & -\frac{E_x^2}{c^2} + B_y^2 + B_z^2 & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) \\ \frac{1}{c}(E_zB_x - E_xB_z) & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\frac{E_y^2}{c^2} + B_x^2 + B_z^2 & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) \\ \frac{1}{c}(E_xB_y - E_yB_x) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) & -\frac{E_z^2}{c^2} + B_x^2 + B_y^2 \end{pmatrix}$$

Note that the $g^{\mu\nu}$ term in $T^{\mu\nu}$ contributes only to the ***diagonal*** elements of $T^{\mu\nu}$, which are:

$$T^{00} = \varepsilon_o c^2\left[S^{00} + \frac{1}{4}g^{00}\cdot 2\left(\frac{E^2}{c^2} - B^2\right)\right] = \varepsilon_o c^2\left[\frac{E^2}{c^2} - \frac{1}{2}\left(\frac{E^2}{c^2} - B^2\right)\right]$$
$$= \varepsilon_o c^2\left[\frac{1}{2}\frac{E^2}{c^2} + \frac{1}{2}B^2\right] = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\varepsilon_o c^2 B^2 = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 = u_{EM}$$

$$
\begin{aligned}
T^{11} &= \varepsilon_o c^2\left[S^{11}+\frac{1}{4}g^{11}\cdot 2\left(\frac{E^2}{c^2}-B^2\right)\right]=\varepsilon_o c^2\left[-\frac{E_x^2}{c^2}+B_y^2+B_z^2+\frac{1}{2}\frac{E^2}{c^2}-\frac{1}{2}B^2\right]\\
&=\varepsilon_o c^2\left[-\frac{E_x^2}{c^2}+\frac{1}{2}\left(\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)+B_y^2+B_z^2-\frac{1}{2}\left(B_x^2+B_y^2+B_z^2\right)\right]\\
&=\varepsilon_o c^2\left[\frac{1}{2}\left(-\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)+\frac{1}{2}\left(-B_x^2+B_y^2+B_z^2\right)\right]=\frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)+\left(-B_x^2+B_y^2+B_z^2\right)\right]\\
&=-\frac{1}{2}\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2}-\frac{E_y^2}{c^2}-\frac{E_z^2}{c^2}\right)+\left(B_x^2-B_y^2-B_z^2\right)\right]\\
&=-\frac{1}{2}\varepsilon_o c^2\left[2\frac{E_x^2}{c^2}+\left(-\frac{E_x^2}{c^2}-\frac{E_y^2}{c^2}-\frac{E_z^2}{c^2}\right)+2B_x^2+\left(-B_x^2-B_y^2-B_z^2\right)\right]\\
&=-\frac{1}{2}\varepsilon_o c^2\left[2\left(\frac{E_x^2}{c^2}+B_x^2\right)-\left(\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)-\left(B_x^2+B_y^2+B_z^2\right)\right]=-\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2}+B_x^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]
\end{aligned}
$$

From the above elements S^{11}, S^{22}, S^{33} and <u>their cyclic permutation symmetries</u> we see that:

$$
T^{22}=-\frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}-\frac{E_z^2}{c^2}\right)+\left(-B_x^2+B_y^2-B_z^2\right)\right]=-\varepsilon_o\left[\left(\frac{E_y^2}{c^2}+B_y^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]
$$

and:

$$
T^{33}=-\frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2}-\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)+\left(-B_x^2-B_y^2+B_z^2\right)\right]=-\varepsilon_o\left[\left(\frac{E_z^2}{c^2}+B_z^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]
$$

Then for <u>all other remaining</u> elements of the relativistic version of Maxwell's stress tensor, $T^{\mu\nu}\big|_{\mu\neq\nu}$, we see that since: $g^{\mu\nu}\big|_{\mu\neq\nu}=0$ then: $T^{\mu\nu}\big|_{\mu\neq\nu}=\varepsilon_o c^2\left(F^{\mu}{}_{\sigma}F^{\nu\sigma}\right)\big|_{\mu\neq\nu}=\varepsilon_o c^2 S^{\mu\nu}\big|_{\mu\neq\nu}$

Thus:

$$
T^{\mu\nu}=\begin{pmatrix}
\frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2}+B^2\right)=u_{EM} & \varepsilon_o c\left(E_yB_z-E_zB_y\right) & \varepsilon_o c\left(E_zB_x-E_xB_z\right) & \varepsilon_o c\left(E_xB_y-E_yB_x\right)\\
\varepsilon_o c\left(E_yB_z-E_zB_y\right) & -\varepsilon_o c\left[\left(\frac{E_x^2}{c^2}+B_x^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right]\\
\varepsilon_o c\left(E_zB_x-E_xB_z\right) & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c\left[\left(\frac{E_y^2}{c^2}+B_y^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right]\\
\varepsilon_o c\left(E_xB_y-E_yB_x\right) & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_z^2}{c^2}+B_z^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]
\end{pmatrix}
$$

Poynting's vector is:

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) = \varepsilon_o c^2\left(\vec{E}\times\vec{B}\right) = \varepsilon_o c^2\left[\left(\vec{E}\times\vec{B}\right)_x\hat{x} + \left(\vec{E}\times\vec{B}\right)_y\hat{y} + \left(\vec{E}\times\vec{B}\right)_z\hat{z}\right]$$
$$= \varepsilon_o c^2\left[\left(E_yB_z - E_zB_y\right)\hat{x} + \left(E_zB_x - E_xB_z\right)\hat{y} + \left(E_xB_y - E_yB_x\right)\hat{z}\right]$$

Thus: $\frac{1}{c}\vec{S} = \varepsilon_o c\left[\left(E_yB_z - E_zB_y\right)\hat{x} + \left(E_zB_x - E_xB_z\right)\hat{y} + \left(E_xB_y - E_yB_x\right)\hat{z}\right]$

The 3-D relativistic ***linear*** momentum ***density*** is: $\vec{\wp} = \frac{1}{c^2}\vec{S}$ or: $\vec{\wp}c = \frac{1}{c}\vec{S}$. Hence, we see that:

$$T^{00} = u_{EM} \quad T^{01} = \frac{1}{c}S_x = \wp_x c = T^{10} \quad T^{02} = \frac{1}{c}S_y = \wp_y c = T^{20} \quad T^{03} = \frac{1}{c}S_z = \wp_z c = T^{30}$$

$$T^{10} = \wp_x c = \frac{1}{c}S_x = T^{01}$$
$$T^{20} = \wp_y c = \frac{1}{c}S_y = T^{02}$$
$$T^{30} = \wp_z c = \frac{1}{c}S_z = T^{03}$$

Thus:

$$T^{\mu\nu} \equiv \varepsilon_o c^2\left[\left(-g^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu}\right) - \frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right] = \varepsilon_o c^2\left[\left(-F^{\mu}{}_{\sigma}F^{\sigma\nu}\right) - \frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$$
$$= \varepsilon_o c^2\left[\left(+F^{\mu}{}_{\sigma}F^{\nu\sigma}\right) - \frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$$

gives:

$$T^{\mu\nu} = \begin{pmatrix} u_{EM} & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \wp_x c & -\varepsilon_o c\left[\left(\frac{E_x^2}{c^2}+B_x^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right] \\ \wp_y c & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c\left[\left(\frac{E_y^2}{c^2}+B_y^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right] \\ \wp_z c & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_z^2}{c^2}+B_z^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] \end{pmatrix}$$

The ***classical*** electrodynamics 3-D force ***density*** is: $\vec{f}=\vec{\nabla}\cdot\vec{T}-\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t}=-\frac{\partial\left(\frac{1}{c}\vec{S}\right)}{\partial(ct)}+\vec{\nabla}\cdot\vec{T}$ SI units: N/m^3

where: $\vec{f}=f_x\hat{x}+f_y\hat{y}+f_z\hat{z}$

But: $T^{\mu\nu}\big|_{\mu,\nu=1:3}=-\vec{T}_{ij}\big|_{i,j=1:3}$ and: $-\frac{\partial\left(\frac{1}{c}\vec{S}\right)}{\partial(ct)}=-\frac{\partial\left(\frac{1}{c}S_x\hat{x}+\frac{1}{c}S_y\hat{y}+\frac{1}{c}S_z\hat{z}\right)}{\partial x^0}=-\frac{\partial\left(T^{01}\hat{x}+T^{02}\hat{y}+T^{03}\hat{z}\right)}{\partial x^0}$

The ***relativistic*** electrodynamics 4-vector force ***density*** is: $f^\nu=\left(f^0,\vec{f}\right)$ (SI units: N/m^3)

The ***3-D spatial vector*** component of the force density is: $\vec{f}=\rho\vec{E}+\vec{J}\times\vec{B}$

The ***zeroth/temporal/scalar*** component of the 4-vector f^ν is: Magnetic forces do ***no*** work!

$$f^0=\vec{J}\cdot\vec{E}_{net}/c=\rho\vec{v}\cdot\left(\vec{E}+\vec{v}\times\vec{B}\right)/c=\vec{J}\cdot\vec{E}/c+\cancel{\rho\vec{v}\cdot\left(\vec{v}\times\vec{B}\right)}/c=\vec{J}\cdot\vec{E}/c=-\frac{\partial\left(T^{00}\right)}{\partial x^0}=-\frac{\partial(u_{EM})}{\partial(ct)}$$

The ***relativistic*** 4vector force density $f^\nu=F^{\nu\sigma}J_\sigma$. But we ***also*** have the relation:

$$f^\nu=-\partial_\mu T^{\mu\nu}=-\frac{\partial T^{\mu\nu}}{\partial x^\mu}=-\left(\frac{\partial T^{0\nu}}{\partial x^0}+\frac{\partial T^{1\nu}}{\partial x^1}+\frac{\partial T^{2\nu}}{\partial x^2}+\frac{\partial T^{3\nu}}{\partial x^3}\right)$$

where: $\partial_\mu\equiv\frac{\partial}{\partial x^\mu}$ is the ***covariant*** 4-D ***gradient*** **operator**.

The ***total*** relativistic 4-vector force can likewise be obtained by noting that the 3-D spatial classical electrodynamics version of the 3-D ***total*** force vector is:

$$\vec{F}=\int_v\vec{f}\,d\tau=\int_v\left(-\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t}+\vec{\nabla}\cdot\vec{T}\right)=\int_v\left(-\frac{\partial\left(\frac{1}{c}\vec{S}\right)}{\partial(ct)}+\vec{\nabla}\cdot\vec{T}\right)d\tau$$

Thus: $F^\nu=\int_{v_4}f^\nu\,d\tau_4=-\int_{v_4}\partial_\mu T^{\mu\nu}\,d\tau_4=-\int_{v_4}\left(\frac{\partial T^{\mu\nu}}{\partial x^\mu}\right)d\tau_4$ where: $F^\nu=\left(F^0,\vec{F}\right)$ and:

The 4-D space-time volume v_4 has volume element $d\tau_4=cdt\;dxdydz$ $\quad\vec{F}=F_x\hat{x}+F_y\hat{y}+F_z\hat{z}$

We explicitly show that: $f^\nu=-\partial_\mu T^{\mu\nu}=-\frac{\partial T^{\mu\nu}}{\partial x^\mu}=F^{\nu\sigma}J_\sigma$

Maxwell's relativistic stress tensor is: $T^{\mu\nu}\equiv\frac{1}{\mu_o}\left[\left(-g^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu}\right)-\frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$

Then: $-\partial_\mu T^{\mu\nu}=\frac{1}{\mu_o}\partial_\mu\left[\left(g^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu}\right)+\frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$ and: $\partial_\mu g^{\mu\lambda}=\partial^\lambda$, $\partial_\mu g^{\mu\nu}=\partial^\nu$.

Thus: $-\partial_\mu T^{\mu\nu}=\frac{1}{\mu_o}\left[\partial^\lambda\left(F_{\lambda\sigma}F^{\sigma\nu}\right)+\frac{1}{4}\partial^\nu\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$

By the chain rule of differentiation:

$$\partial^\lambda\left(F_{\lambda\sigma}F^{\sigma\nu}\right)=\left(\partial^\lambda F_{\lambda\sigma}\right)F^{\sigma\nu}+F_{\lambda\sigma}\left(\partial^\lambda F^{\sigma\nu}\right)$$ and:

$$\partial^\nu\left(F_{\alpha\beta}F^{\alpha\beta}\right)=\left(\partial^\nu F_{\alpha\beta}\right)F^{\alpha\beta}+F_{\alpha\beta}\left(\partial^\nu F^{\alpha\beta}\right)=2F_{\alpha\beta}\left(\partial^\nu F^{\alpha\beta}\right)$$

Maxwell's ***inhomogeneous*** equation is: $\partial_\nu F^{\mu\nu}=\dfrac{\partial F^{\mu\nu}}{\partial x^\nu}=\mu_o J^\mu$

But: $\partial_\nu F^{\mu\nu}=\mu_o J^\mu \Rightarrow \partial^\rho g_{\rho\nu}F^{\mu\nu}=\mu_o g^{\mu\kappa}J_\mu \Rightarrow \partial^\rho F^\mu{}_\rho=\partial^\rho\left(g^{\mu\kappa}F_{\kappa\rho}\right)=\mu_o g^{\mu\kappa}J_\kappa \Rightarrow$
$g^{\mu\kappa}\left(\partial^\rho F_{\kappa\rho}\right)=\mu_o g^{\mu\kappa}J_\kappa \Rightarrow \partial^\rho F_{\kappa\rho}=\mu_o J_\kappa$. But ρ and κ are dummy indices. Let $\rho\to\lambda$ and $\kappa\to\sigma$.

Then: $\partial^\lambda F_{\sigma\lambda}=\dfrac{\partial F_{\sigma\lambda}}{\partial x_\lambda}=\mu_o J_\sigma$.

Now: $F_{\lambda\sigma}=-F_{\sigma\lambda}$ because the *EM* field strength tensor is ***anti-symmetric***.

$\therefore$ $\partial^\lambda F_{\lambda\sigma}=-\partial^\lambda F_{\sigma\lambda}=-\mu_o J_\sigma$. Hence: $\left(\partial^\lambda F_{\lambda\sigma}\right)F^{\sigma\nu}=-\mu_o J_\sigma F^{\sigma\nu}=+\mu_o F^{\nu\sigma}J_\sigma=\mu_o f^\nu$

$$\begin{aligned}-\partial_\mu T^{\mu\nu}&=\frac{1}{\mu_o}\left(\partial^\lambda F_{\lambda\sigma}\right)F^{\sigma\nu}+\frac{1}{\mu_o}F_{\lambda\sigma}\left(\partial^\lambda F^{\sigma\nu}\right)+\frac{1}{2\mu_o}F_{\alpha\beta}\left(\partial^\nu F^{\alpha\beta}\right)\\&=f^\nu+\frac{1}{\mu_o}\left[F_{\lambda\sigma}\left(\partial^\lambda F^{\sigma\nu}\right)+\frac{1}{2}F_{\alpha\beta}\left(\partial^\nu F^{\alpha\beta}\right)\right]\end{aligned}$$

Then, noting that α and β are dummy indices, we replace $\alpha\to\lambda$ and $\beta\to\sigma$, thus re-writing the bracketed term in the above expression as:

$$\left[F_{\lambda\sigma}\left(\partial^\lambda F^{\sigma\nu}\right)+\frac{1}{2}F_{\alpha\beta}\left(\partial^\nu F^{\alpha\beta}\right)\right]=\frac{1}{2}\left[2F_{\lambda\sigma}\left(\partial^\lambda F^{\sigma\nu}\right)+F_{\lambda\sigma}\left(\partial^\nu F^{\lambda\sigma}\right)\right]=\frac{1}{2}F_{\lambda\sigma}\left[\partial^\lambda F^{\sigma\nu}+\underbrace{\partial^\lambda F^{\sigma\nu}+\partial^\nu F^{\lambda\sigma}}\right]$$

Maxwell's ***homogeneous*** equation (in ***covariant*** form) is: $\partial_\lambda F_{\sigma\nu}+\partial_\sigma F_{\nu\lambda}+\partial_\nu F_{\lambda\sigma}=0$ $\Leftarrow$ Note the cyclic permutations of λ, σ, ν

It can be shown (via more tensor manipulation) that Maxwell's ***homogeneous*** equation in its ***contravariant*** form is: $\partial^\lambda F^{\sigma\nu}+\partial^\sigma F^{\nu\lambda}+\partial^\nu F^{\lambda\sigma}=0$.

Thus, we can see that the last two terms in the RHS bracket above: $\partial^\lambda F^{\sigma\nu}+\partial^\nu F^{\lambda\sigma}=-\partial^\sigma F^{\nu\lambda}$

Hence: $-\partial_\mu T^{\mu\nu}=f^\nu+\dfrac{1}{2\mu_o}F_{\lambda\sigma}\left[\partial^\lambda F^{\sigma\nu}-\partial^\sigma F^{\nu\lambda}\right]=f^\nu+\dfrac{1}{2\mu_o}F_{\lambda\sigma}\left[\partial^\lambda F^{\sigma\nu}+\partial^\sigma F^{\lambda\nu}\right]$

But note that the term: $\left[\partial^\lambda F^{\sigma\nu}+\partial^\sigma F^{\lambda\nu}\right]$ is ***symmetric*** with respect to exchange of the indices $\lambda\leftrightarrow\sigma$, i.e. $\left[\partial^\lambda F^{\sigma\nu}+\partial^\sigma F^{\lambda\nu}\right]=\left[\partial^\sigma F^{\lambda\nu}+\partial^\lambda F^{\sigma\nu}\right]$. However, the double-contraction of an ***anti-symmetric*** tensor $F_{\lambda\sigma}=-F_{\sigma\lambda}$ with a ***symmetric*** tensor $S^{\lambda\sigma}=+S^{\sigma\lambda}$ or the ***symmetric*** portion of a rank-3 tensor $\left[\partial^\lambda F^{\sigma\nu}+\partial^\sigma F^{\lambda\nu}\right]=\left[\partial^\sigma F^{\lambda\nu}+\partial^\lambda F^{\sigma\nu}\right]$ is ***zero***!

Hence: $-\partial_\mu T^{\mu\nu}=f^\nu$ or: $f^\nu=-\partial_\mu T^{\mu\nu}=F^{\nu\sigma}J_\sigma$

If one wishes to Lorentz boost these results – defined in one IRF(S) – to another IRF(S') – moving with relative velocity $\vec{v}$, there are two equivalent methods to accomplish this task:

Method I:

First, provided at least one *EM* field component is parallel to the boost axis along $\vec{v}$, Lorentz transform the $\vec{E}$ and $\vec{B}$ fields via use of the matrix relation:

$$\begin{pmatrix}\vec{E}'\\ \vec{B}'\end{pmatrix}=\underbrace{\begin{pmatrix}\gamma\left(1-\left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) & \gamma\vec{\beta}x\\ -\gamma\vec{\beta}x & \gamma\left(1-\left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right)\end{pmatrix}}_{n.b.\ \textit{operator}\ \text{matrix}}\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}\quad \text{where:}\quad \gamma=\frac{1}{\sqrt{1-\beta^2}},\quad \vec{\beta}=\frac{\vec{v}}{c}$$

See/read P436 Lecture Notes 19 for further details of this method.

Then compute the new $F'^{\mu\nu}$, $S'^{\mu\nu}=F'^{\mu}{}_{\sigma}F'^{\sigma\nu}$ and $F'_{\alpha\beta}F'^{\alpha\beta}\ \{n.b. = F_{\alpha\beta}F^{\alpha\beta}\}$ and the new $T'^{\mu\nu}$ in IRF(S'):

$$T'^{\mu\nu}=\varepsilon_0 c^{22}\left[\left(F'^{\mu}{}_{\sigma}F'^{\sigma\nu}\right)-\frac{1}{4}g^{\mu\nu}\left(F'_{\alpha\beta}F'^{\alpha\beta}\right)\right],\quad f'^{\nu}=-\frac{\partial T'^{\mu\nu}}{\partial x'^{\mu}},\quad F'^{\nu}=\int_{v_4}f'^{\nu}d\tau_4=-\int_{v_4}\left(\frac{\partial T'^{\mu\nu}}{\partial x'^{\mu}}\right)d\tau_4$$

This method has the advantage that all quantities, *e.g.* $\vec{E}',\vec{B}',\vec{S}',\vec{\wp}'$, $F'^{\mu\nu}$, $S'^{\mu\nu}=F'^{\mu}{}_{\sigma}F'^{\sigma\nu}$, $T'^{\mu\nu}$, f'^{ν} and F'^{ν} are explicitly known/calculated in IRF(S').

Method II:

Lorentz transform $T^{\mu\nu}$ ***directly***, since the Lorentz transformation of $F^{\mu\nu}$ in IRF(S) to $F'^{\mu\nu}$ in IRF(S') moving with velocity $\vec{v}$ relative to IRF(S) is given by: $F'^{\mu\nu}=\Lambda^{\mu}{}_{\lambda}F^{\lambda\sigma}\Lambda^{\nu}{}_{\sigma}$

Then: $T'^{\mu\nu}=\Lambda^{\mu}{}_{\lambda}T^{\lambda\sigma}\Lambda^{\nu}{}_{\sigma}$ *n.b.* in matrix form: $T'=\Lambda T\Lambda^{T}=\Lambda T\Lambda$ since Λ is ***symmetric***.

Since: $T^{\mu\nu}=\varepsilon_o c^2\left[\left(F^{\mu}{}_{\rho}F^{\rho\nu}\right)-\frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]=\varepsilon_o c^2\left[S^{\mu\nu}-\frac{1}{4}g^{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$

Then: $T'^{\mu\nu}=\Lambda^{\mu}_{\lambda}T^{\lambda\sigma}\Lambda^{\nu}_{\sigma}=\varepsilon_o c^2\left[\Lambda^{\mu}_{\lambda}S^{\lambda\sigma}\Lambda^{\nu}_{\sigma}-\tfrac{1}{4}\Lambda^{\mu}_{\lambda}g^{\lambda\sigma}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\Lambda^{\nu}_{\sigma}\right]$

But: $\left(F_{\alpha\beta}F^{\alpha\beta}\right)=\left(F^{\alpha\beta}F_{\alpha\beta}\right)=-2\left(\frac{E^2}{c^2}-B^2\right)$ = relativistic invariant

i.e. $F_{\alpha\beta}F^{\alpha\beta}=F'_{\alpha\beta}F'^{\alpha\beta}$ = ***same*** **value in *any/all* IRF'S!!!**

Define: $S'^{\mu\nu}\equiv\Lambda^{\mu}_{\lambda}S^{\lambda\sigma}\Lambda^{\nu}_{\sigma}$ and: $g'^{\mu\nu}\equiv\Lambda^{\mu}_{\lambda}g^{\lambda\sigma}\Lambda^{\nu}_{\sigma}$

Also note that ε_o and c^2, μ_o are relativistically invariant ***scalar*** quantities (***same*** in ***all*** IRF's)

Then: $T'^{\mu\nu}=\varepsilon_o c^2\left[S'^{\mu\nu}-\tfrac{1}{4}g'^{\mu\nu}\left(F'_{\alpha\beta}F'^{\alpha\beta}\right)\right]$

Once $T'^{\mu\nu}$ is obtained, then calculate: $f'^{\nu}=-\dfrac{\partial T'^{\mu\nu}}{\partial x'^{\mu}}$ and: $F'^{\nu}=\int_{v_4}f'^{\nu}\,d\tau_4=-\int_{v_4}\left(\dfrac{\partial T'^{\mu\nu}}{\partial x'^{\mu}}\right)d\tau_4$