LECTURE NOTES 18

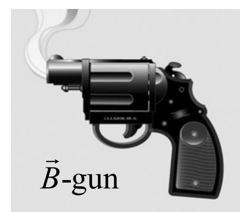
RELATIVISTIC ELECTRODYNAMICS

Classical electrodynamics (Maxwell's equations, the Lorentz force law, *etc*.) {unlike classical / Newtonian mechanics} is ***already*** consistent with special relativity – *i.e.* is valid in ***any*** IRF.

However: What one observer interprets (*e.g.*) as a ***purely electrical*** process in ***his/her*** IRF, another observer in a ***different*** IRF may interpret it (*e.g.*) as being due to ***purely magnetic*** phenomena, or a "***mix***" of ***electric*** and ***magnetic*** phenomena – however, the charged particle motion(s), viewed/seen/observed from different IRF's are related to each other via Lorentz transformations from one IRF to another (and vice versa)!

The theoretical problems/difficulties that Lorentz and others had working in late 19th Century lay entirely with their use of non-relativistic, classical / Newtonian laws of mechanics in conjunction with the laws of electrodynamics. Once this was corrected by Einstein, using relativistic mechanics with classical electrodynamics, these problems / difficulties were no longer encountered!

The phenomenon of ***magnetism*** is a "***smoking gun***" for ***relativity***!



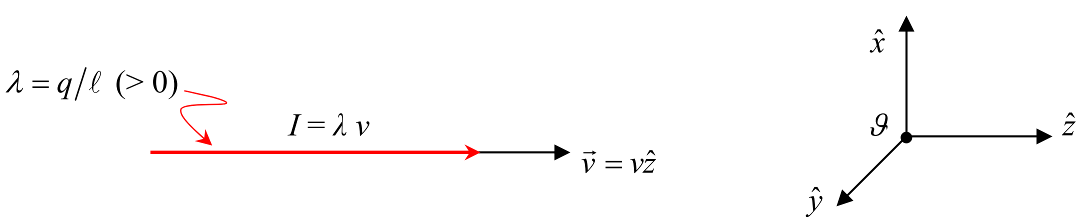
* Magnetism – arising from the ***motion*** of electric charges – the observer is ***not*** in the same IRF as that of the ***moving*** charge – thus magnetism is a consequence of the ***space-time nature*** of the universe that we live in (Lorentz contraction/time dilation and Lorentz invariance $\Delta x_\mu \Delta x^\mu = I$).

* "Magnetism" is ***not*** "just" associated with the phenomenon of ***electromagnetism***, but ***all*** four fundamental forces of nature: *EM*, strong, weak and gravity (and anything else!) – ***because*** space-time is the common "host" to all of the fundamental forces of nature – they all live / exist / co-exist in space-time, and all are subject to the laws of space-time – *i.e.* relativity!

We can *e.g.* calculate the "magnetic" force between a current-carrying "wire" and a moving (test) charge Q_T without ever invoking laws of magnetism (*e.g.* the Lorentz force law, the Biot-Savart law, or Maxwell's equations (*e.g.* Ampere's law)) – just need electrostatics and relativity!

Suppose we have an infinitely long string of positive charges moving to right at speed v in the lab frame, IRF(S). The spacing of the +*ve* charges is close enough together such that we can consider them as continuous / macroscopic ***line charge*** density $\lambda = q/\ell$ (Coulombs/meter) as shown in the figure below:

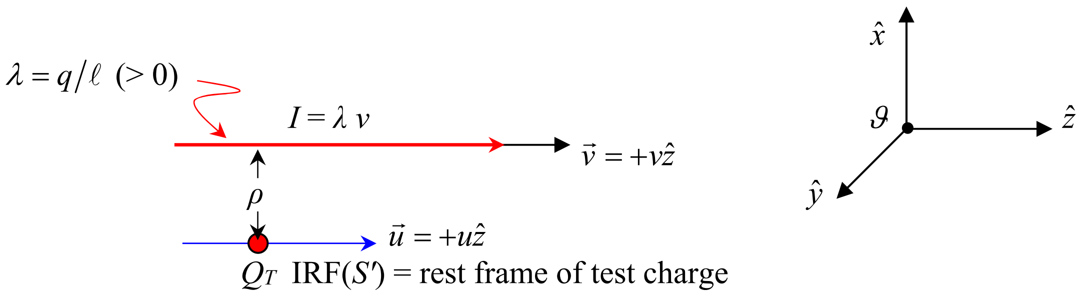
IRF(S):



Since the positive line charge density $\boxed{\lambda = q/\ell}$ is moving to right with speed v, we have a positive filamentary / line current flowing to the right of magnitude $\boxed{I = \lambda v}$ (Amps).

Now suppose we also have a point test charge Q_T moving with velocity $\vec{u} = +u\hat{z}$ (*i.e.* to the right) in IRF(S) $\{n.b.\ |\vec{u}| = u$ is ***not*** necessarily $= |\vec{v}| = v\}$. The test charge Q_T is a $\perp$ distance ρ from the moving line charge / current as shown in the figure below.

IRF(S):



Let's examine this situation as viewed by an observer in the ***rest frame*** of the ***test charge*** Q_T = the ***proper*** frame of the ***test charge*** Q_T. Call this ***rest/proper*** frame = IRF(S').

By Einstein's "***ordinary***" velocity addition rule, the speed of $+ve$ charges in the ***right***-moving line charge density / filamentary line current as viewed by an observer in the ***rest*** frame IRF(S') of the ***test charge*** Q_T {which is moving with velocity $\vec{v} = +v\hat{z}$ in the ***lab*** frame IRF(S)} is:

$$\boxed{v' = \frac{v-u}{1-vu/c^2}} \quad \text{with: } \boxed{\vec{v} = +v\hat{z}} \text{ and: } \boxed{\vec{u} = +u\hat{z}}$$

However, in IRF(S'), due to Lorentz contraction the {infinitesimal} ***spacing*** between positive charges in the ***right***-moving line charge / filamentary line current is also ***changed***, which therefore changes the line charge density as observed in IRF(S'), relative to the lab IRF(S)!

In IRF(S'): $\boxed{\lambda' = \gamma'\lambda_0}$ where: $\boxed{\gamma' \equiv \dfrac{1}{\sqrt{1-\beta'^2}} = \dfrac{1}{\sqrt{1-(v'/c)^2}}}$ and: $\boxed{\lambda_0 = q/\ell_0}$, $\boxed{\lambda' = q/\ell'} \Rightarrow \boxed{\ell' = \ell_0/\gamma'}$

where $\boxed{\lambda_0 = q/\ell_0} \equiv$ linear charge density as observed in its ***own rest*** frame IRF(S_0).

Once the line charge density λ_0 starts moving at speed v in IRF(S), then: $\boxed{\ell_0 \to \ell}$ and $\boxed{\lambda_0 \to \lambda}$.

In IRF(S): $\boxed{\lambda = \gamma\lambda_0}$ where: $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-(v/c)^2}}}$ and: $\boxed{\lambda_0 = q/\ell_0}$, $\boxed{\lambda = q/\ell} \Rightarrow \boxed{\ell = \ell_0/\gamma}$

But: $\boxed{\gamma' \equiv \dfrac{1}{\sqrt{1-(v'/c)^2}}}$ and: $\boxed{v' = \dfrac{v-u}{1-vu/c^2}}$ where: $\boxed{\vec{v} = v\hat{z}}$ and: $\boxed{\vec{u} = u\hat{z}}$ in IRF(S).

$$\therefore \boxed{\gamma' = \frac{1}{\sqrt{1-\dfrac{1}{c^2}\dfrac{(v-u)^2}{\left(1-\dfrac{vu}{c^2}\right)^2}}} = \frac{1}{\sqrt{1-\dfrac{c^2(v-u)^2}{(c^2-vu)^2}}} = \frac{(c^2-vu)}{\sqrt{(c^2-vu)^2-c^2(v-u)^2}}}$$

Or: $$\gamma' = \frac{(c^2 - uv)}{\sqrt{(c^2-v^2)-(c^2-u^2)}} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \bullet \frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = \gamma\gamma_u\left(1-\frac{uv}{c^2}\right)$$ with: $$\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}} \qquad \gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}$$

Thus: $\gamma' = \gamma\gamma_u\left(1-\frac{uv}{c^2}\right)$ with: $\vec{v} = v\hat{z}$ and: $\vec{u} = u\hat{z}$ in IRF(S).

Thus the line charge density λ' as observed in the ***rest frame*** of the ***test charge*** Q_T, *i.e.* in IRF(S') is:

$$\lambda' = \gamma'\lambda_0 = \gamma\gamma_u\left(1-\frac{uv}{c^2}\right)\lambda_0 = \gamma_u\left(1-\frac{uv}{c^2}\right)\underbrace{\gamma\lambda_0}_{\equiv\lambda} = \gamma_u\left(1-\frac{uv}{c^2}\right)\lambda$$

Check: If $\vec{u} = \vec{v}$, does $\lambda' = \lambda_0$?

When $\vec{u} = \vec{v}$, the test charge Q_T is moving with the ***same*** velocity as the line charge, thus the test charge Q_T is in the ***rest frame*** of the line charge, *i.e.* IRF(S') ***coincides*** with IRF(S_0)!

If $\vec{u} = \vec{v}$ then: $u = v$ and: $\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}} = \gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}$

Then: $$\lambda' = \gamma\gamma_u\left(1-\frac{uv}{c^2}\right)\lambda_0 = \left[\frac{\left(1-\left(\frac{u}{c}\right)^2\right)}{\left(1-\left(\frac{u}{c}\right)^2\right)}\right]\lambda_0 = \lambda_0$$ YES! $\lambda' = \lambda_0$.

Note that the line charge density λ as observed in the ***lab frame*** {*i.e.* in IRF(S)}, in terms of the line charge density λ_0 in the ***rest frame*** of the line charge itself (*i.e.* IRF(S_0)) is:

$$\lambda = \gamma\lambda_0 = \frac{1}{\sqrt{1-(v/c)^2}}\lambda_0$$ since: $$\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}$$

Check: If $\vec{u} = 0$, does $\lambda' = \lambda$?

When $\vec{u} = 0$, the test charge Q_T is not moving in the ***lab*** frame IRF(S), thus IRF(S') ***coincides*** with IRF(S)!

If $\vec{u} = 0$ then: $u = 0$ and: $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = 1$ and thus: $\lambda' = \left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\lambda = \lambda$ Yes!

An observer in the ***proper/rest frame*** IRF(S') of the test charge Q_T sees a radial $(\hat{\rho})$ ***electrostatic*** field in IRF(S') associated with the infinitely long line charge density $\boxed{\lambda' = q/\ell'}$ of:

$$\boxed{\vec{E}'(\rho) = \frac{\lambda'}{2\pi\varepsilon_o \rho}\hat{\rho}} \text{ with } \boxed{\lambda' = \left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\lambda} \text{ and } \boxed{\lambda = \gamma\lambda_0}$$

n.b. $\hat{\rho}$ is the radial unit vector $\perp$ to $\vec{v} = v\hat{z}$ {and $\vec{u} = u\hat{z}$}.

$\Rightarrow$ ρ and $\hat{\rho}$ are unaltered / unaffected by Lorentz boosts along the $\hat{z}$-direction.

$\therefore$ In IRF(S'):

$$\boxed{E'(\rho) = \frac{1}{2\pi\varepsilon_o\rho}\lambda' = \frac{1}{2\pi\varepsilon_o\rho}\left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\lambda = \frac{1}{2\pi\varepsilon_o\rho}\left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\gamma\lambda_0}$$

$\boxed{\lambda = q/\ell}$ in IRF(S) $\boxed{\lambda_0 \equiv q/\ell_0}$ in IRF(S_0)

In the special case when $\boxed{\vec{u} = \vec{v}}$ when IRF(S') $\equiv$ IRF(S_0) coincide $\rightarrow$ the test charge Q_T and the line charge λ_0 are both at rest/in the same rest frame/same IRF:

Then: $$\boxed{E'(\rho)\Big|_{\vec{u}=\vec{v}} = \frac{1}{2\pi\varepsilon_o\rho}\lambda' = \frac{1}{2\pi\varepsilon_o\rho}\lambda_0 \equiv E_0(\rho)}$$ $\leftarrow$ Purely electrostatic field, $\boxed{\lambda_0 = q/\ell_0}$

n.b. Notice that when IRF(S') $\equiv$ IRF(S_0) coincide, that $\vec{F}_0 = Q_T\vec{E}_0 \perp (\vec{u} = \vec{v})$:

$$\boxed{\vec{F}_0(\rho) = Q_T\vec{E}_0(\rho) = \frac{Q_T}{2\pi\varepsilon_o\rho}\lambda_0\hat{\rho} = \frac{Q_T}{2\pi\varepsilon_o\rho}\left(\frac{q}{\ell_0}\right)\hat{\rho}} \text{ where: } \boxed{\lambda_0 = \frac{q}{\ell_0}}$$

For the more general case where $\boxed{\vec{u} \neq \vec{v}}$, the force acting on the test charge Q_T in its own rest frame IRF(S') is:

$$\boxed{F'_{tot}(\rho) = Q_T E'_{tot}(\rho) = \frac{Q_T}{2\pi\varepsilon_o\rho}\lambda' = \frac{Q_T}{2\pi\varepsilon_o\rho}\left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\lambda} \text{ where: } \boxed{\lambda = \frac{q}{\ell}} \text{ In IRF}(S)$$

Or: $$\boxed{F'_{tot}(\rho) = \frac{Q_T}{2\pi\varepsilon_o\rho}\gamma_u\lambda - \frac{Q_T}{2\pi\varepsilon_o\rho}\gamma_u\lambda\left(\frac{uv}{c^2}\right)} \text{ where: } \boxed{\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}}$$

Or: $$\boxed{F'_{tot}(\rho) = \frac{Q_T}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}} - \frac{\lambda v}{2\pi\varepsilon_o c^2}\left(\frac{1}{\rho}\right)\frac{Q_T u}{\sqrt{1-(u/c)^2}}}$$

But: $\boxed{I \equiv \lambda v}$ in IRF(S) {the ***lab*** frame} and: $\boxed{1/c^2 = \varepsilon_o\mu_o}$:

Thus: $$\boxed{F'_{tot}(\rho) = \frac{Q_T}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}} - \left(\frac{\mu_o I}{2\pi\rho}\right)\frac{Q_T u}{\sqrt{1-(u/c)^2}}} \text{ in IRF}(S')$$

Or: $$\boxed{F'_{tot}(\rho) = Q_T E'(\rho) - Q_T uB'(\rho) = Q_T E'_{tot}(\rho)} \text{ and: } \boxed{E'_{tot}(\rho) = E'(\rho) - uB'(\rho)}$$

Where: $\boxed{E'(\rho)=\frac{1}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}}}$ and: $\boxed{B'(\rho)=\left(\frac{\mu_o}{2\pi\rho}\right)\frac{I}{\sqrt{1-(u/c)^2}}}$ in IRF(S') !!!

Vectorially, in the general IRF(S') / rest frame of the test charge Q_T, for $\boxed{\vec{u}\neq\vec{v}}$ {necessarily}

$\boxed{\vec{F}'_{tot}(\rho)=Q_T\vec{E}'_{tot}(\rho)}$ ← *n.b.* in the radial / $\hat{\rho}$ direction ***only***.

$$\therefore \quad \boxed{\vec{F}'_{tot}(\rho)=Q_T E'(\rho)\hat{\rho}-Q_T uB'(\rho)\hat{\rho}}$$

<u>Very Useful Table</u>
<u>Cylindrical Coordinates</u>:

$\hat{\rho}\times\hat{\varphi}=\hat{z}$	$\hat{\varphi}\times\hat{\rho}=-\hat{z}$
$\hat{\varphi}\times\hat{z}=\hat{\rho}$	$\hat{z}\times\hat{\varphi}=-\hat{\rho}$
$\hat{z}\times\hat{\rho}=\hat{\varphi}$	$\hat{\rho}\times\hat{z}=-\hat{\varphi}$

But: $\boxed{\vec{u}=u\hat{z}}$ $\therefore$ $\boxed{\vec{u}\times\vec{B}'(\rho)=-uB'(\rho)\hat{\rho}}$ $\Rightarrow$ $\boxed{\vec{B}'=B'\hat{\varphi}}$ then: $\boxed{uB'\underbrace{(\hat{z}\times\hat{\varphi})}_{=-\hat{\rho}}=-uB'\hat{\rho}}$

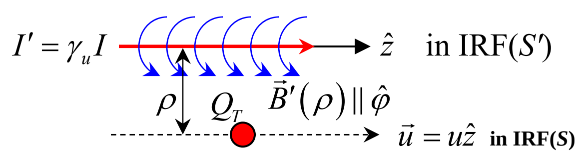
$$\boxed{\vec{F}'_{tot}(\rho)=Q_T\vec{E}'_{tot}(\rho)=Q_T\vec{E}'(\rho)+Q_T\left(\vec{u}\times\vec{B}'(\rho)\right)}$$ ← Lorentz Force Law in IRF(S') !!!

Where:

$$\boxed{\vec{E}'(\rho)=\frac{1}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}}\hat{\rho}=\frac{\gamma_u\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}=\frac{\gamma\gamma_u\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}$$

$$\boxed{\vec{B}'(\rho)=\left(\frac{\mu_o}{2\pi\rho}\right)\frac{I}{\sqrt{1-(u/c)^2}}\hat{\varphi}=\left(\frac{\mu_o}{2\pi\rho}\right)\gamma_u I\hat{\varphi}=\left(\frac{\mu_o}{2\pi\rho}\right)\gamma\gamma_u I_0\hat{\varphi}}$$ in IRF(S')

$\boxed{I_0\equiv\lambda_0 v}$ $\boxed{I\equiv\lambda v=\gamma\lambda_0 v=\gamma I_0}$ $\boxed{I'\equiv\gamma_u\lambda v=\gamma\gamma_u\lambda_0 v=\gamma\gamma_u I_0}$



$$\vec{F}'_{tot}(\rho)=\frac{Q_T}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}}\hat{\rho}-Q_T u\left(\frac{\mu_o I}{2\pi\rho}\right)\frac{1}{\sqrt{1-(u/c)^2}}\hat{\rho}$$

$$=\underbrace{Q_T\vec{E}'(\rho)}_{\text{repulsive force}}+\underbrace{Q_T\vec{u}\times\vec{B}'(\rho)}_{\text{attractive force}}$$ in IRF(S')

***n.b.* <u>Parallel currents attract</u> each other!!! {2nd current is test charge Q_T !!!}**

For ***<u>like</u>*** charges $q=\lambda\ell$ and Q_T If $\vec{u}\parallel\vec{v}$ {remember: $\boxed{\vec{I}=\lambda\vec{v}}$}

Next, we Lorentz transform the IRF(S') results (defined in the rest / proper frame of Q_T) to the IRF(S) (lab frame), using the rule(s) for Lorentz transformation of forces:

$\boxed{\vec{E}'(\rho)=\frac{1}{2\pi\varepsilon_o\rho}\frac{\lambda}{\sqrt{1-(u/c)^2}}\hat{\rho}}$ and: $\boxed{\vec{B}'(\rho)=\frac{\lambda v}{2\pi\varepsilon_o c^2\rho}\frac{I}{\sqrt{1-(u/c)^2}}\hat{\varphi}}$

$$\boxed{\vec{F}'_{tot}(\rho)=Q_T\vec{E}'_{tot}(\rho)=Q_T\vec{E}'(\rho)+Q_T\vec{u}\times\vec{B}'(\rho)}$$

$\boxed{\vec{F}'_{tot}(\rho)=\frac{Q_T}{2\pi\varepsilon_o\rho}\lambda'\hat{\rho}=\frac{Q_T}{2\pi\varepsilon_o\rho}\left[\gamma_u\left(1-\frac{uv}{c^2}\right)\right]\lambda\hat{\rho}}$ where: $\boxed{\gamma_u=\frac{1}{\sqrt{1-(u/c)^2}}}$

The test charge Q_T is moving with velocity $\vec{u}=u\hat{z}$ in the lab frame, IRF(S).

Note that {here}: $\vec{u} = u\hat{z} = u_z\hat{z}$ $\{i.e.\ u = u_z,\ \vec{u} \parallel \hat{z}\}$, note also that: $\vec{F}'_{tot} \perp \vec{u}$ and $F'_{\parallel} = F'_z = 0$.

Then the Lorentz transformation of the forces from IRF(S') to IRF(S):

In IRF(S): $F_{\perp} = \frac{1}{\gamma'_u} F'_{\perp} = \sqrt{1-(u/c)^2}\, F'_{\perp}$ where: $\gamma'_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}$ and: $F_{\parallel} = F'_{\parallel}$ $(=0)$

$\therefore$ In IRF(S):

Note the cancellation of γ_u factors !!!

$$\vec{F}_{tot}(\rho) = \frac{1}{\gamma_u}\vec{F}'_{tot}(\rho) = \frac{1}{\gamma_u}\frac{Q_T}{2\pi\varepsilon_o}\rho\lambda'\hat{\rho} = \frac{1}{\not{\gamma}_u}\frac{Q_T}{2\pi\varepsilon_o\rho}\left[\not{\gamma}_u\left(1-\frac{uv}{c^2}\right)\right]\lambda\hat{\rho}$$

$$= \frac{Q_T}{2\pi\varepsilon_o\rho}\left(1-\frac{uv}{c^2}\right)\lambda\hat{\rho} = \frac{Q_T\lambda}{2\pi\varepsilon_o\rho}\hat{\rho} - \frac{Q_T\lambda v}{2\pi\varepsilon_o c^2} * u\hat{\rho} \quad \text{but: } I \equiv \lambda v \text{ and } \frac{1}{c^2} = \varepsilon_o\mu_o$$

$$= Q_T\left(\frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}\right) - Q_T\left(u * \frac{\mu_o I}{2\pi\rho}\hat{\rho}\right)$$

Again: $\vec{u} = u\hat{z}$, $\hat{z}\times\hat{\varphi} = -\hat{\rho}$ thus: $\vec{B}(\rho) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi}$

$\therefore$ In IRF(S):

$$\vec{F}_{tot}(\rho) = Q_T\underbrace{\left(\frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}\right)}_{\equiv\vec{E}(\rho)} + Q_T\vec{u}\times\underbrace{\left(\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right)}_{\equiv\vec{B}(\rho)}$$

$$\vec{F}_{tot}(\rho) = Q_T\vec{E}_{tot}(\rho) = Q_T\vec{E}(\rho) + Q_T\vec{u}\times\vec{B}(\rho)$$ ← Lorentz Force Law in IRF(S) !!!

In IRF(S):

$$\vec{E}(\rho) = \frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho} = \frac{(q/\ell)}{2\pi\varepsilon_o\rho}\hat{\rho}$$ where: $\lambda = q/\ell = \gamma\lambda_0 = \gamma(q/\ell_0)$

$$\vec{B}(\rho) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi} = \frac{\mu_o\lambda v}{2\pi\rho}\hat{\varphi} = \frac{\mu_o(q/\ell)v}{2\pi\rho}\hat{\varphi}$$ where: $I = \lambda v = (q/\ell)v = \gamma\lambda_0 v = \gamma(q/\ell_0)v$

Thus, an observer at rest in either the ***lab*** frame IRF(S) or the ***rest*** frame of the ***test charge*** IRF(S') will see ***both*** a static ***electric*** field {different in each IRF} ***and*** a static (but velocity-dependent) magnetic field {different in each IRF} due to the {infinitely long} filamentary line charge density $\lambda = q/\ell$ that is moving with velocity $\vec{v} = v\hat{z}$ in IRF(S) = filamentary line current $I = \lambda v$ in IRF(S).

The magnetic field arises simply from the relativistic effect(s) of electric charge in {relative} motion!

For an observer in the ***rest*** frame IRF(S_0) of the filamentary line charge density $\lambda = q/\ell$, he/she will see ***only*** a static, radial electric field!

Let's summarize these results by inertial reference frame:

IRF(S) Laboratory Frame	IRF(S') Rest Frame of Test Charge	IRF(S_0) Rest Frame of Line Charge
	Moving with $\vec{u} = u\hat{z} = u_z\hat{z}$ in lab	Moving with $\vec{v} = v\hat{z} = v_z\hat{z}$ in lab
	$v' = \dfrac{v-u}{1-\left(uv/c^2\right)}$ **Speed of line charge in IRF(*S'*)**	
$\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}$	$\gamma' = \gamma\gamma_u\left(1-\dfrac{uv}{c^2}\right)$ $\gamma_u = \dfrac{1}{\sqrt{1-(u/c)^2}}$	
$\lambda = q/\ell = \gamma\lambda_0 = \gamma\left(q/\ell_0\right)$	$\lambda' = q/\ell' = \gamma'\lambda_0 = \gamma\gamma_u\left[1-\left(\dfrac{uv}{c^2}\right)\right]\lambda_0$	$\lambda_0 = q/\ell_0$
$\ell = \ell_0/\gamma$	$\ell' = \ell_0/\gamma' = \ell_0 \Big/ \gamma\gamma_u\left[1-\left(\dfrac{uv}{c^2}\right)\right]$	ℓ_0
$\vec{E}(\rho) = \dfrac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho} = \dfrac{\gamma\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}$	$\vec{E}'(\rho) = \dfrac{\gamma_u\lambda}{2\pi\varepsilon_o\rho}\hat{\rho} = \dfrac{\gamma\gamma_u\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}$	$\vec{E}_0(\rho) = \dfrac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}$
$I \equiv \lambda v = \gamma\lambda_0 v = \gamma I_0$ $I_0 \equiv \lambda_0 v$	$I' \equiv \gamma_u\lambda v = \gamma_u I = \gamma\gamma_u\lambda_0 v = \gamma\gamma_u I_0$	No current in IRF(S_0)
$\vec{B}(\rho) = \dfrac{\mu_o I}{2\pi\rho}\hat{\varphi} = \dfrac{\mu_o\gamma I_0}{2\pi\rho}\hat{\varphi}$	$\vec{B}'(\rho) = \dfrac{\mu_o I'}{2\pi\rho}\hat{\varphi} = \dfrac{\mu_o\gamma_u I}{2\pi\rho}\hat{\varphi} = \dfrac{\mu_o\gamma\gamma_u I_0}{2\pi\rho}\hat{\varphi}$	No B-field in IRF(S_0)
$\vec{F}_{tot} = Q_T\vec{E} + Q_T\vec{u}\times\vec{B}$ $\neq$	$\vec{F}'_{tot} = Q_T\vec{E}' + Q_T\vec{u}\times\vec{B}'$ $\neq$	$\vec{F}'_{0_{tot}}(\rho) = Q_T\vec{E}_0(\rho)$

n.b. **In the rest frame IRF(*S'*) of the test charge Q_T, the Lorentz force $\vec{F}'_{TOT}$ uses the velocity $\vec{u}$ of the test charge as observed in the lab frame IRF(S).**

We see that the observed line charge densities λ and λ' as seen in the lab frame IRF(S) and the test charge rest frame IRF(S'), respectively are larger by factors of γ and γ' respectively compared to the line charge density as observed in the rest frame IRF(S_0) of the line charge density itself. This difference arises due to the effect of the {longitudinal} Lorentz contraction of the moving line charge density λ_0, as viewed from the lab frame IRF(S) and the rest frame IRF(S') of the test charge, respectively.

Because of this, the electric fields as seen in the lab frame IRF(S) and rest frame of the test charge IRF(S') are ***larger*** by factors of γ and $\gamma\gamma_u$, respectively than that observed in the rest frame IRF(S_0) of the line charge density itself, hence the magnitude of the electrostatic forces are larger by these same amounts in their respective IRF's, and are thus {in general} not equal.

An important point here is that in all 3 inertial reference frames, what we ***call*** the electric field in each IRF is such that a.) they are ***all*** oriented in the ***same*** direction {here, the radial direction and b.) they all have the same functional dependence (here, $\sim 1/\rho$), differing only by γ-factors from each other.

In the ***rest*** frame IRF(S_0) of the line charge density λ_0 the electromagnetic field seen there is purely electro***static***, oriented in the radial ($\hat{\rho}$) direction, whereas in the lab frame IRF(S) and the rest frame of the test charge IRF(S'), the electromagnetic field observed in each of these two reference frames is a ***combination*** of a static, ***radial*** electric field ***and*** a static, ***azimuthal*** magnetic field.

The "appearance" of azimuthal magnetic fields in the lab frame IRF(S) and the rest frame of the test charge IRF(S') is due to the relativistic effects associated with the motion of the line charge density relative to an observer in the lab frame IRF(S) and/or the rest frame of the test charge, IRF(S').

We ***say*** that the relative motion of the electric line charge density $\boxed{\lambda = \gamma\lambda_0}$ {as viewed by an observer in the lab frame IRF(S)} constitutes an ***electric current*** $\boxed{I \equiv \lambda v = \gamma\lambda_0 v}$ {as viewed by that same observer in the lab frame IRF(S)}.

We then connect / associate the "appearance" of azimuthal magnetic fields $\vec{B}$ and $\vec{B}'$ in the lab frame IRF(S) and the rest frame of the test charge IRF(S'), respectively with the existence of the electric currents I and I' as observed in their respective inertial reference frames.
The $\vec{B}$-field in each IRF is linearly proportional to {the magnitude of} the electric current $|\vec{I}|$ as observed in that IRF, *i.e.* $\boxed{|\vec{B}| \sim |\vec{I}| = |\lambda\vec{v}|}$.

Another interesting/important aspect of the magnetic fields $\vec{B}$ that "appear" in IRF(S) and/or IRF(S') is that they are ***mutually*** $\perp$ to both $\vec{E}$ ***.and.*** $\vec{I} = \lambda\vec{v}$ in ***that*** IRF.

Note that we could instead refer to electric currents I alternatively and equivalently, exclusively and explicitly as to what they are truly are – the {relative} motion(s) of charges $q\vec{v}$, line charge densities $\lambda\vec{v}$, surface charge densities $\sigma\vec{v}$ and/or volume charge densities $\rho\vec{v}$.

Then we also wouldn't have to explicitly use the descriptor "magnetic" field to describe the resulting component of the electromagnetic field that ***does*** arise from the relative motion(s) of electric charge(s) as viewed by an observer who is ***not*** in the rest frame of these electric charge(s). We could call it something else instead – *e.g.* "the relativity field".

We humans call this field "the magnetic field" largely for ***historical "inertia"*** reasons. The phenomenon of magnetism/magnetic fields was discovered ***centuries*** before relativity and space-time were finally understood; we humans simply keep calling this field "the magnetic field". The magnetic field is truly and simply ***one*** component of the ***overall*** **electromagnetic field** that is associated with a physical situation, and one which ***only*** arises whenever that physical situation is viewed by an observer whose IRF(S) is ***not*** coincident with the ***rest frame*** IRF(S_0) of the electric charge(s) that are present in that particular physical situation.

The "traditional" way of equivalently saying the above is: "**Magnetic fields are only produced when electrical currents are present**".

***Physical* Electric Currents**:

It is important to understand that there exist **different** **kinds** of ***physical*** electric currents.

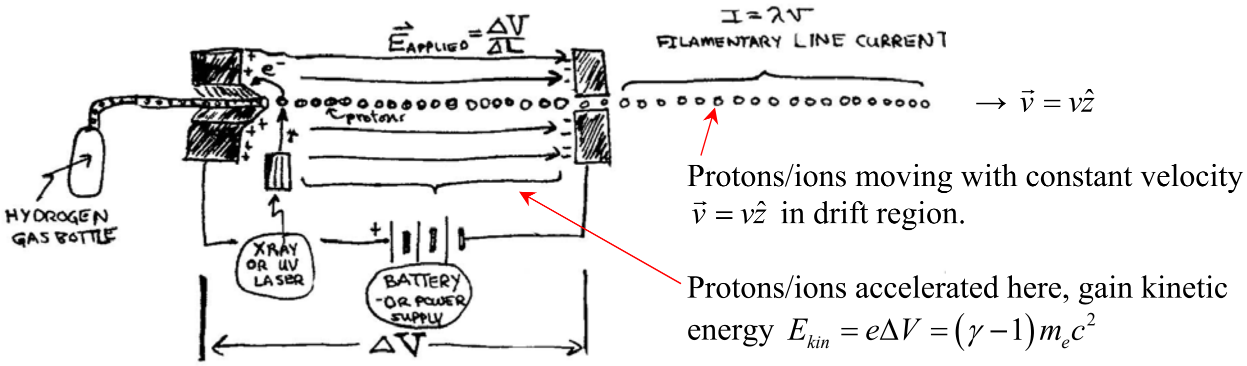
- A "bare" filamentary line charge density $\boxed{\lambda = q/\ell}$ *e.g.* moving with uniform velocity $\boxed{\vec{v} = v\hat{z}}$ with respect to the lab frame IRF(S), creates a filamentary line current $\boxed{\vec{I} = \lambda v\hat{z}}$ in the lab frame IRF(S). This filamentary line current is ***not*** equivalent to a ***physical*** electrical current flowing *e.g.* in an "infinitesimally-thin" ***physical*** wire at rest in the lab frame IRF(S). For an observer in {***any***} IRF the "bare" filamentary line charge density has a ***net***/***overall*** electric charge. An observer in the lab frame IRF(S) sees both a static, non-zero radial electric field and a static, non-zero azimuthal magnetic field arising from the "bare" filamentary line charge density $\boxed{\lambda = q/\ell}$ and "bare" filamentary line current $\boxed{\vec{I} = \lambda v\hat{z}}$ respectively, whereas an observer in the rest frame IRF(S_0) of the filamentary line charge density $\boxed{\lambda_0 = q/\ell_0}$ sees ***no*** magnetic field – only a static, radial electric field!

- In a physical wire (*e.g.* a copper wire, made up of copper atoms with "free" conduction electrons), the "free" negatively-charged electrons move / drift through the macroscopic volume of the copper wire *e.g.* with {mean} drift velocity $\boxed{\vec{v}_D = -v_D\hat{z}}$ and constitute a ***physical*** electric current $\boxed{I_{phys} = \vec{J}_{e^-} \bullet \vec{A}_\perp^{wire} = -n_{e^-} e\vec{v}_D \bullet \vec{A}_\perp^{wire}}$ as viewed by an observer in the ***lab*** frame IRF(S). Microscopically, the copper wire is a 3-D "matrix" (or lattice) of bound / fixed copper atoms with a "gas" of "free" conduction electrons drifting through it. In the ***lab*** frame IRF(S), the copper atoms are at ***rest***, but the electrons are not. Note importantly {also} that in the ***lab*** IRF(S), the physical current-carrying copper wire has no ***net*** electric charge – because there is ***one*** "free" conduction electron associated with ***each*** copper atom of the copper wire. Thus, an observer in the lab frame IRF(S) sees no ***net*** electric field, but ***does*** see a ***static***, non-zero azimuthal magnetic field arising from the "free" conduction electron volume current density $\boxed{\vec{J}_{e^-} = -n_{e^-} e\vec{v}_D}$, whereas an observer in the rest frame IRF(S_0) of the "free" conduction electron charge density $\boxed{\rho_{e^-}^0 = n_{e^-}^0 e}$ sees no magnetic field associated with the "free" conduction electrons, but ***does*** see {the same!} non-zero azimuthal magnetic field that is associated with volume current density $\boxed{\vec{J}_{Cu} = +n_{Cu} e\vec{v}_D}$ of the 3-D lattice of copper atoms that are moving with {relative} velocity $\boxed{\vec{v}_D = +v_D\hat{z}}$ to an observer in rest frame IRF(S_0) !!!

- In semiconducting materials (*e.g.* silicon, germanium, graphite, diamond, SiC, gallium, …) electrical conduction occurs either by mobile "drift" electrons and/or "holes" {= the absence of an electron). The number densities of electrons and/or "holes" are both typically $\ll$ number density of semiconductor atoms and depend on details associated with the condensed matter physics of the semiconductor. In general $n_{e^-} \neq n_{hole}$ and both are strong (exponential) functions of {absolute} temperature. The drift velocities of electrons and holes are not in general the same. Thus, in the lab frame IRF(S), an observer will, in general see static electric field contributions arising from both electron and hole charge density distributions as well as magnetic field contributions from both electron and hole current densities. An observer at rest either in IRF(S_0) of the electrons or at rest IRF(S_0^*) of the holes will again see static electric field contributions from both electrons and holes, but a B-field contribution only from holes (electrons), respectively.

- The situation of a "bare" filamentary line charge $\boxed{\lambda = q/\ell}$ moving with {relative} velocity $\boxed{\vec{v} = v\hat{z}}$ in IRF(S), producing a filamentary line current $\boxed{I = \lambda v}$ in IRF(S) can be physically realised *e.g.* as "beam" of $+ve$ current of protons ($+q$) {or *e.g.* $+ve$ ions, or *e.g.* $-ve$ electrons} flowing in a vacuum (*e.g.* made via laser photo-ionized hydrogen, argon, or thermionic emission of electrons, respectively):

Vacuum Chamber (Lab IRF(S))

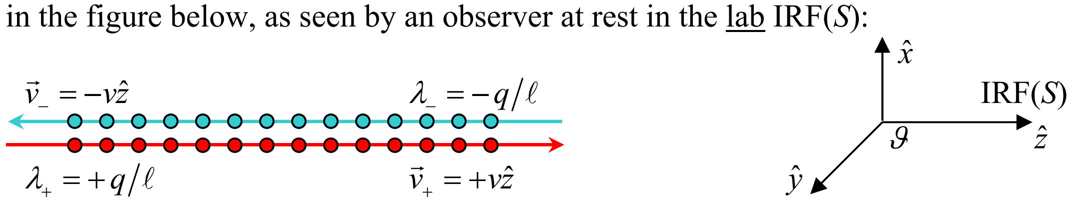


Having discussed the *EM* field(s) and *EM* force(s) acting on a test charge Q_T associated with a single filamentary line charge / filamentary line current as observed in different IRF's, we now discuss the problem of two counter-moving, opposite-charged filamentary line charges / filamentary line currents superimposed on top of each other.

Consider two opposite-charged filamentary line charges (both infinitely long) that are initially stationary in the lab frame IRF(S). One initially stationary filamentary line charge has negative charge per unit length $\boxed{\lambda_{0_-} \equiv -q/\ell_0}$ and the other initially stationary filamentary line charge has positive charge per unit length $\boxed{\lambda_{0_+} = +q/\ell_0}$. The two line charges are then set in motion parallel to / along their axes (in the $\hat{z}$-direction). The negative line charge moves to the left ($-\hat{z}$ direction) with velocity $\boxed{\vec{v}_- = -v\hat{z}}$ in the lab frame IRF(S), and the positive line charge moves to the right ($+\hat{z}$ direction) with velocity $\boxed{\vec{v}_+ = +v\hat{z}}$ in the lab frame IRF(S) {*i.e.* it has the same exact speed, but moves in the opposite direction to that of the first line charge}.

The two counter-moving filamentary line charges are superimposed on top of each other / coaxial with each other, but we draw them as slightly displaced (transverse to their motion) for clarity's sake in the figure below, as seen by an observer at rest in the lab IRF(S):

In IRF(S):



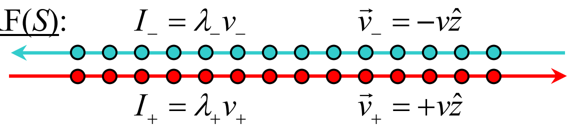
In IRF(S), the moving filamentary line charges have charge per unit length $\boxed{\lambda_\pm = \pm q/\ell}$, whereas in the respective ***rest*** frame(s) IRF($S_\pm$) of the filamentary line charges, we have $\boxed{\lambda_{0_\pm} = \pm q/\ell_0 \equiv \pm\lambda_0}$.

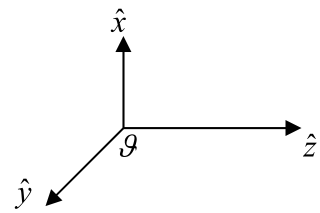
Because of the respective motions of the line charge densities: $\boxed{\vec{v}_\pm = \pm v\hat{z}}$

Then: $\boxed{\lambda_\pm = \pm\gamma\lambda_0}$ where: $\boxed{\gamma = \dfrac{1}{\sqrt{1-(v_\pm/c)^2}} = \dfrac{1}{\sqrt{1-(v/c)^2}}}$

In the lab frame IRF(S): A negative current $I_- = \lambda_- v_-$ flowing to the left is superimposed on a positive current $I_+ = \lambda_+ v_+$ flowing to the right, as shown in the figure below:

In IRF(S):



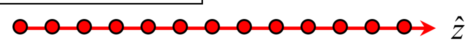


Using the ***principle of linear superposition***, the ***net/total*** current {as observed in the ***lab*** frame IRF(S)} is:

$\boxed{I_{tot} = I_+ + I_- = \lambda_+ v_+ + \lambda_- v_-}$ but: $\boxed{\lambda_- = -\lambda_+}$ and: $\boxed{v_- = -v_+}$

$\therefore$ $\boxed{I_{tot} = \lambda_+ v_+ + (-\lambda_+)(-v_+) = \lambda_+ v_+ + \lambda_+ v_+ = 2\lambda_+ v_+}$ flowing to the ***right*** (*i.e.* in $+\hat{z}$ direction)

$\Rightarrow$ $\boxed{I_{tot} = 2\lambda_+ v_+ = 2\lambda v}$ flowing in the $\hat{z}$ -direction: with: $\boxed{\lambda_+ \equiv +\lambda = +q/\ell}$, $\boxed{\vec{v}_+ = +v\hat{z}}$



and: $\boxed{\lambda_- \equiv -\lambda = -q/\ell}$, $\boxed{\vec{v}_- = -v\hat{z}}$

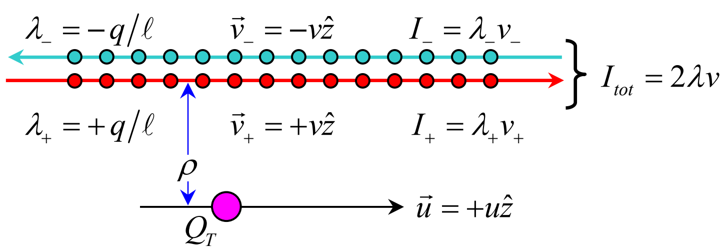
Note that because we have superimposed these two counter-moving, filamentary oppositely-charged line-charges / counter-moving, filamentary line currents, the ***net*** electric charge Q_{TOT} {as observed in the lab frame IRF(S)} is ***zero*** because:

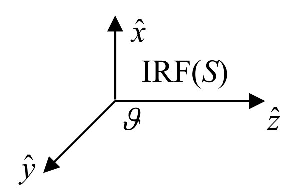
$\boxed{\lambda_{tot} = \lambda_+ + \lambda_- = +\lambda - \lambda = 0}$ in the ***lab*** frame IRF(S).

If $Q_{TOT} = 0$ in IRF(S), then we also know that the ***net*** electric field $\vec{E}_{tot}(\vec{r}) = 0$ in the ***lab*** frame IRF(S) due to these two counter-moving, superimposed oppositely-charged filamentary line charges/line currents in IRF(S).

Now additionally suppose that we also have a test charge Q_T moving with velocity $\vec{u} = u\hat{z}$ (*i.e.* to the right) in IRF(S). As before, $\vec{u}$ is ***not*** necessarily $= \vec{v} = v\hat{z}$, the velocity of the ***right*** moving line charge. The test charge Q_T is a $\perp$ distance ρ from the superimposed opposite-charged, opposite-moving filamentary line charges λ_+ and λ_-:

In IRF(S):



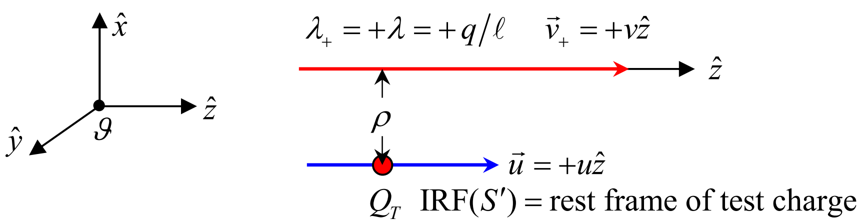


Let's examine the situation as viewed by an observer in IRF(S') – *i.e.* the rest frame of the test charge Q_T. There are ***four*** distinct cases to consider for the 1-D Einstein velocity addition rule:

---

a.) In the ***lab*** frame IRF(S), the test charge Q_T is moving with velocity $\boxed{\vec{u} = +u\hat{z}}$, the $+ve$ filamentary line charge density $\lambda_+ = +\lambda$ is moving with velocity $\boxed{\vec{v}_+ = +v\hat{z}}$.

Lab Frame IRF(S):

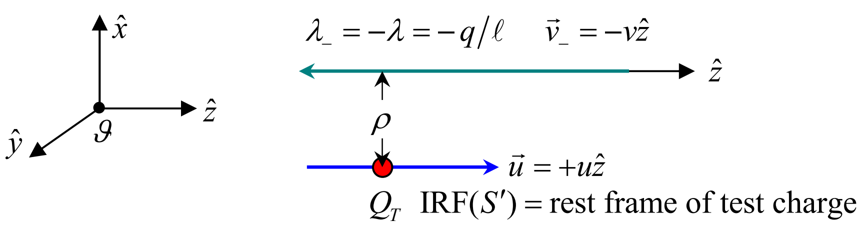


$$\boxed{v'_+ = \frac{v-u}{1-\frac{vu}{c^2}}} = \boxed{\text{Relative speed of } +\lambda \text{ viewed from IRF}(S')}$$

---

b.) In the ***lab*** frame IRF(S), the test charge Q_T is moving with velocity $\boxed{\vec{u} = +u\hat{z}}$, the $-ve$ filamentary line charge density $\lambda_- = -\lambda$ is moving with velocity $\boxed{\vec{v}_- = -v\hat{z}}$.

Lab Frame IRF(S):



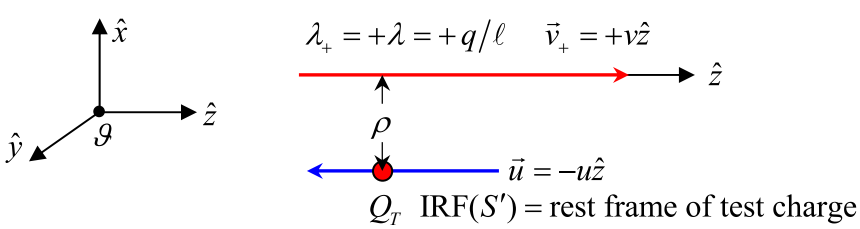
$$\boxed{v'_- = \frac{-v-u}{1+\frac{vu}{c^2}}} = \boxed{\text{Relative speed of } -\lambda \text{ viewed from IRF}(S')}$$

n.b. only $\vec{v}$ reversed relative to case a.) above

---

c.) In the ***lab*** frame IRF(S), the test charge Q_T is moving with velocity $\boxed{\vec{u} = -u\hat{z}}$, the $+ve$ filamentary line charge density $\lambda_+ = +\lambda$ is moving with velocity $\boxed{\vec{v}_+ = +v\hat{z}}$.

Lab Frame IRF(S):



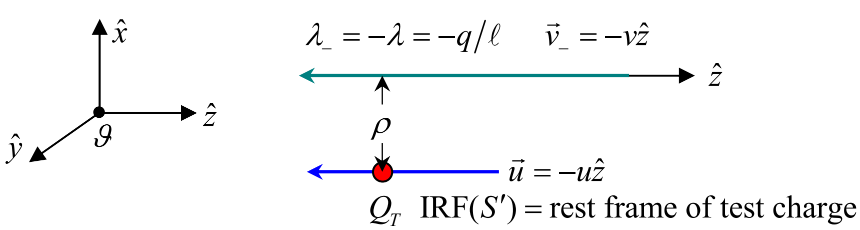
$$\boxed{v'_+ = \frac{v+u}{1+\frac{vu}{c^2}}} = \boxed{\text{Relative speed of } +\lambda \text{ viewed from IRF}(S')}$$

n.b. only $\vec{u}$ reversed relative to case a.) above

---

d.) In the ***lab*** frame IRF(S), the test charge Q_T is moving with velocity $\boxed{\vec{u} = -u\hat{z}}$, the $-ve$ filamentary line charge density $\lambda_- = -\lambda$ is moving with velocity $\boxed{\vec{v}_- = -v\hat{z}}$.

Lab Frame IRF(S):



$$\boxed{v'_- = \frac{-v+u}{1-\frac{vu}{c^2}}} = \boxed{\text{Relative speed of } -\lambda \text{ viewed from IRF}(S')}$$

n.b. both $\vec{u}$ and $\vec{v}$ reversed relative to case a.) above

---

The above ***four*** relative 1-D speed formulae can be more compactly written as ***two*** specific cases:

i.) For $\boxed{\vec{u}=+u\hat{z}}$: $\boxed{v'_{\pm}=\dfrac{\pm v-u}{1\mp\dfrac{vu}{c^2}}}$

ii.) For $\boxed{\vec{u}=-u\hat{z}}$: $\boxed{v'_{\pm}=\dfrac{\pm v+u}{1\pm\dfrac{vu}{c^2}}}$

n.b. Equation 12.76, *p.* 523 in Griffith's book ***is*** correct, however the ***proper*** use of his equation ***explicitly*** requires placing a – (minus) sign in front of the formula for the v'_- case. Note that (obviously) u must ***also*** be ***explicitly*** signed in ***his*** formula for the $\vec{u}=-u\hat{z}$ case. Then ***his*** formula ***agrees*** with the 4 that are explicitly given ***here***.

$$\boxed{v'_{\pm}=\frac{v\mp u}{1\mp\dfrac{vu}{c^2}}}$$

1-D general: $\boxed{\vec{v}'=\dfrac{\vec{v}-\vec{u}}{1-\dfrac{\vec{v}\bullet\vec{u}}{c^2}}}$

Thus, for an observer in IRF(S') (= ***rest*** frame of Q_T) moving to the ***right*** with velocity $\boxed{\vec{u}=+u\hat{z}}$ in IRF(S) we see that $\boxed{v'_->v'_+}$.

Because $\boxed{v'_->v'_+}$ for an observer in IRF(S'), the Lorentz contraction of the –*ve* filamentary line charge density $\lambda_-=-q/\ell$ will be more "severe" than that associated with the +*ve* filamentary line charge density $\lambda_+=+q/\ell$.

In IRF(S'): $\boxed{\lambda'_{\pm}=\pm\gamma'_{\pm}\lambda_0}$ where: $\boxed{\gamma'_{\pm}\equiv\dfrac{1}{\sqrt{1-(v'_{\pm}/c)^2}}}$

And: $\boxed{\pm\lambda_0=q/\ell_0}$ = filamentary line charge densities in their ***own*** ***rest*** frames.

But: $\boxed{v'_{\pm}=\dfrac{\pm v-u}{1\mp\dfrac{vu}{c^2}}}$ for: $\boxed{\begin{matrix}\vec{v}=+v\hat{z}\\ \vec{u}=+u\hat{z}\end{matrix}}$ in IRF(S)

Thus:

$$\gamma'_{\pm}=\frac{1}{\sqrt{1-\left(\dfrac{v'_{\pm}}{c}\right)^2}}=\frac{1}{\sqrt{1-\dfrac{1}{c^2}\dfrac{(\pm v-u)^2}{\left(1\mp\dfrac{vu}{c^2}\right)^2}}}=\frac{1}{\sqrt{1-\dfrac{c^2(\pm v-u)^2}{(c^2\mp vu)^2}}}=\frac{(c^2\mp vu)}{\sqrt{(c^2\mp vu)^2-c^2(\pm v-u)^2}}$$

$$=\frac{(c^2\mp vu)}{\sqrt{c^4\mp\cancel{2vuc^2}+(vu)^2-c^2v^2\pm\cancel{2vuc^2}-c^2u^2}}=\frac{(c^2\mp vu)}{\sqrt{c^4-c^2v^2-c^2u^2+(vu)^2}}$$

$$=\frac{(c^2\mp vu)}{\sqrt{(c^2-v^2)(c^2-u^2)}}=\frac{1}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}\bullet\frac{\left(1\mp\dfrac{vu}{c^2}\right)}{\sqrt{1-\left(\dfrac{u}{c}\right)^2}}=\gamma\gamma_u\left(1\mp\frac{uv}{c^2}\right)$$

Or: $\boxed{\gamma'_{\pm}=\gamma\gamma_u\left(1\mp\dfrac{uv}{c^2}\right)}$ where: $\boxed{\gamma\equiv\dfrac{1}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}}$ and: $\boxed{\gamma_u\equiv\dfrac{1}{\sqrt{1-\left(\dfrac{u}{c}\right)^2}}}$

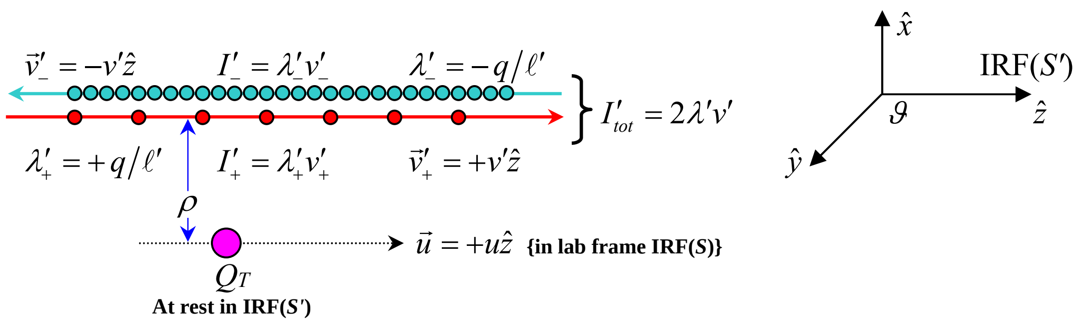
<u>Then in IRF(S')</u>: $$\boxed{\lambda'_{\pm} = \pm\gamma'_{\pm}\lambda_0 = \pm\gamma\gamma_u\lambda_0\left(1\mp\frac{uv}{c^2}\right) = \pm\gamma_u\left(\gamma\lambda_0\right)\left(1\mp\frac{uv}{c^2}\right) = \pm\gamma_u\lambda\left(1\mp\frac{uv}{c^2}\right)}$$

where: $\boxed{\gamma_u = \dfrac{1}{\sqrt{1-(u/c)^2}}}$ and: $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-(v/c)^2}}}$

But: $\boxed{\pm\lambda = \pm\gamma\lambda_0}$ = charge per unit length in the lab frame, IRF(S).

$\therefore$ <u>In IRF(S')</u>: $$\boxed{\lambda'_{+} = +\gamma_u\lambda\left(1-\frac{uv}{c^2}\right) = \lambda\left[\frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right]}$$ and: $$\boxed{\lambda'_{-} = -\gamma_u\lambda\left(1+\frac{uv}{c^2}\right) = -\lambda\left[\frac{\left(1+\frac{uv}{c^2}\right)}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\right]}$$

<u>In IRF(S')</u>:



$\therefore$ In the rest frame IRF(S') of the test charge Q_T, the total/net line charge density is: $\boxed{\lambda'_{tot} = \lambda'_{+} + \lambda'_{-}}$.

<u>In IRF(S')</u>: $$\boxed{\lambda'_{tot} = \lambda'_{+} + \lambda'_{-} = \gamma_u\lambda\left(1-\frac{uv}{c^2}\right) - \gamma_u\lambda\left(1+\frac{uv}{c^2}\right) = \cancel{\gamma_u\lambda} - \gamma_u\lambda\left(\frac{uv}{c^2}\right) - \cancel{\gamma_u\lambda} - \gamma_u\lambda\left(\frac{uv}{c^2}\right)}$$

$$\boxed{\lambda'_{tot} = -2\gamma_u\lambda\left(\frac{uv}{c^2}\right) = -2\lambda\frac{\left(uv/c^2\right)}{\sqrt{1-(u/c)^2}} \neq 0!!!}$$

$\Rightarrow$ In IRF(S') {= <u>***rest***</u> frame of the <u>***test***</u> <u>***charge***</u> Q_T (which moves with velocity $\vec{u} = u\hat{z}$ in IRF(S))}

$\exists$ a <u>***net***</u> $-ve$ line charge density $\boxed{\lambda'_{tot} = -2\lambda\dfrac{\left(uv/c^2\right)}{\sqrt{1-(u/c)^2}}}$!!!

Whereas in the <u>***lab***</u> frame IRF(S), $\exists$ <u>***no***</u> net line charge, *i.e.* $\boxed{\lambda_{tot} = 0}$ in IRF(S) !!!

$\Rightarrow$ The non-zero λ'_{tot} observed in IRF(S') (= rest frame of Q_T) is due to / arises from the <u>unequal</u> Lorentz contraction of the $+ve$ *vs.* $-ve$ filamentary line charge densities, as observed in IRF(S') (= rest frame of Q_T).

$\Rightarrow$ A current-carrying "wire" that is <u>***electrically***</u> <u>***neutral***</u> ($\lambda_{TOT} = 0$) in one IRF(S) will NOT be so in another IRF(S') !!! It will have a <u>***net***</u> <u>***electrical***</u> <u>***charge***</u> in IRF(S') $\neq$ IRF(S) !!!

Thus in IRF(S'), where there exists a ***net*** –*ve* line charge density of: $\lambda'_{tot} = -2\lambda \dfrac{(uv/c^2)}{\sqrt{1-(u/c)^2}}$

a corresponding (radial-inward) ***electric*** field exists: $\vec{E}'(\rho) = \dfrac{\lambda'_{tot}}{2\pi\varepsilon_o\rho}\hat{\rho} = -\dfrac{\lambda}{\pi\varepsilon_o\rho}\dfrac{(uv/c^2)}{\sqrt{1-(u/c)^2}}\hat{\rho}$.

Thus an observer in the ***rest*** frame IRF(S') of the ***test charge*** Q_T "sees" a radial-inward (*i.e.* attractive) electrostatic force acting on the test charge Q_T (for $Q_T > 0$) of:

$$\vec{F}'(\rho) = Q_T\vec{E}'(\rho) = Q_T\frac{\lambda'_{tot}}{2\pi\varepsilon_o\rho}\hat{\rho}$$

_n.b._ Lorentz-invariant !!! Valid in any/all IRF's

But: $\vec{E}'(\rho) = \dfrac{\lambda'_{tot}}{2\pi\varepsilon_o\rho}\hat{\rho} = -\dfrac{2\lambda(uv/c^2)}{2\pi\varepsilon_o\rho}\dfrac{1}{\sqrt{1-(u/c)^2}}\hat{\rho} = -\dfrac{\lambda(uv/c^2)}{\pi\varepsilon_o\rho}\dfrac{1}{\sqrt{1-(u/c)^2}}\hat{\rho}$ and: $\dfrac{1}{c^2} = \varepsilon_o\mu_o$

$\therefore$ $\vec{E}'(\rho) = -\dfrac{\lambda\varepsilon_o\mu_0 uv}{\pi\varepsilon_o\rho}\dfrac{1}{\sqrt{1-(u/c)^2}}\hat{\rho} = -\dfrac{\mu_o\lambda v}{\pi\rho}\dfrac{u}{\sqrt{1-(u/c)^2}}\hat{\rho}$ But: $I = 2\lambda v$ in ***lab*** IRF(S).

$\therefore$ In IRF(S') (= rest frame of Q_T): $\vec{E}'(\rho) = -\dfrac{\mu_o I}{2\pi\rho}\dfrac{u}{\sqrt{1-(u/c)^2}}\hat{\rho}$ $\Leftarrow$ *n.b.* points radially ***inward***!

Therefore equivalently, the force $\vec{F}'(\rho) = Q_T\vec{E}'(\rho)$ acting on Q_T in its own rest frame IRF(S') is:

$$\vec{F}'(\rho) = Q_T\vec{E}'(\rho) = -\frac{\mu_o Q_T I}{2\pi\rho}\frac{u}{\sqrt{1-(u/c)^2}}\hat{\rho}$$ $\Leftarrow$

***n.b.* Q_T is attracted <u>towards</u> wire if $Q_T > 0$.**

<u>Parallel</u> <u>currents</u> <u>attract</u> each other !!! {The test charge Q_T is the 2nd current !!!}

This force is none other than the magnetic Lorentz force acting on Q_T:

In IRF(S') (= rest frame of Q_T): $\vec{F}'(\rho) = Q_T\left(\vec{u}\times\vec{B}'(\rho)\right)$

Where $\vec{u} = +u\hat{z}$ = velocity of test charge Q_T in IRF(S)

$\vec{E}'(\rho) = \vec{u}\times\vec{B}'(\rho)$ $\{\hat{z}\times\hat{\varphi} = -\hat{\rho}\}$

$\vec{B}'(\rho) = \dfrac{\mu_o I}{2\pi\rho}\dfrac{1}{\sqrt{1-(u/c)^2}}\hat{\varphi} = \gamma_u\dfrac{\mu_o I}{2\pi\rho}\hat{\varphi}$ <u>where:</u> $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}}$

If $\exists$ a force $\vec{F}'$ in IRF(S') (where Q_T is at rest), then there ***must*** also be a force $\vec{F}$ in the lab frame IRF(S) {the laws of physics are the same in all inertial reference frames…}.

We can Lorentz transform the force in IRF(S') to obtain the force $\vec{F}$ in the lab frame IRF(S), where we already know that $\lambda_{tot} = 0$ in the lab frame IRF(S).

Again, since Q_T is at rest in IRF(S') and $\vec{F}'(\rho)\sim\hat{\rho}$ $\{i.e. \perp \vec{u} = u\hat{z}$ in IRF(S)$\}$

Then in IRF(S): $\boxed{F_\perp = \frac{1}{\gamma'_u} F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)   ⊥ and || refer to ⊥ and || to $\vec{u}$ - the Lorentz boost direction

where: $\boxed{\gamma'_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}}$ = Lorentz factor to transform from IRF(S') (Q_T at rest) to lab frame IRF(S). IRF(S) moves with velocity $-\vec{u}$ with respect to IRF(S').

Then in IRF(S): $\boxed{F_\perp = \frac{1}{\gamma'_u} F'_\perp = \sqrt{1-(u/c)^2}\, F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)

∴ In the lab frame IRF(S):

$$\vec{F}(\rho) = Q_T \vec{E}(\rho) = \sqrt{1-(u/c)^2} * \left[-\frac{\mu_o Q_T I}{2\pi\rho} \frac{u}{\sqrt{1-(u/c)^2}} \hat{\rho} \right]$$

$$= -\frac{\mu_o Q_T I}{2\pi\rho} u\hat{\rho} = Q_T \vec{E}(\rho)$$

← Radial E-field in lab frame IRF(S)

In the lab frame IRF(S): The test charge Q_T is moving with velocity $\vec{u} = +u\hat{z}$ in IRF(S)

An observer in lab frame IRF(S) "sees" a force $\vec{F}(\rho) = Q_T \vec{E}(\rho)$ acting on moving test charge Q_T. The "effective" electric field in lab frame IRF(S) is:

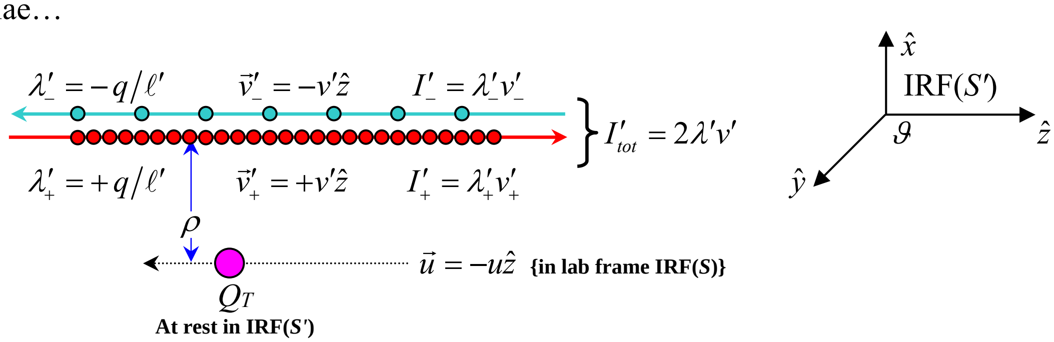
$$\boxed{\vec{E}(\rho) = -\frac{\mu_o I}{2\pi\rho} u\hat{\rho} = \vec{u} \times \left[\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right] = \vec{u}\times\vec{B}(\rho)} \quad \text{where: } \boxed{\vec{B}(\rho) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi}}$$

From the perspective of a stationary observer in the lab frame IRF(S), where the net linear charge density $\lambda_{TOT} = 0$, no ***true electrostatic*** field exists. However, a "magnetic", ***velocity-dependent*** attractive force $\vec{F}(\rho)$ ***does*** indeed exist, acting radially inward for a +*ve* test charge Q_T, when it is moving with velocity $\vec{u} = +u\hat{z}$ in IRF(S).

∴ In the lab frame IRF(S): $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho) = -Q_T * u\left[\frac{\mu_o I}{2\pi\rho}\right]\hat{\rho} = Q_T \vec{u}\times\vec{B}(\rho)}$ where: $\boxed{I = 2\lambda v}$

Suppose the test charge Q_T was instead moving with velocity $\boxed{\vec{u} = -u\hat{z}}$ in IRF(S). What would the resulting force $\vec{F}(\rho)$ be in the lab frame IRF(S)? One can explicitly go through all of the above for this case; one will discover that one {simply} needs to change $\boxed{u \rightarrow -u}$ in all of the above formulae…

In IRF(S'):



An observer in the rest frame IRF(S') of the test charge Q_T "sees" a ***net*** $+ve$ line charge density $\boxed{\lambda'_{tot} = +2\gamma_u \lambda \left(\frac{uv}{c^2}\right) = +2\lambda \frac{\left(uv/c^2\right)}{\sqrt{1-\left(u/c\right)^2}}}$ when the test charge Q_T is moving with velocity $\boxed{\vec{u} = -u\hat{z}}$ in the ***lab*** frame IRF(S).

A corresponding (radial-outward) ***electric*** field thus exists in IRF(S'): $\boxed{\vec{E}'(\rho) = \frac{\lambda'_{tot}}{2\pi\varepsilon_o \rho}\hat{\rho}}$.

The observer in IRF(S') also "sees" a radial-outward electrostatic force acting on the test charge Q_T of: $\boxed{\vec{F}'(\rho) = Q_T \vec{E}'(\rho) = Q_T \frac{\lambda'_{tot}}{2\pi\varepsilon_o \rho}\hat{\rho}}$

Transforming these results to the lab frame IRF(S) in the same manner as we have already done once {see above}, an observer in lab frame IRF(S) "sees" a net force $\vec{F}(\rho) = Q_T \vec{E}(\rho)$ acting on the moving test charge Q_T. The "effective" electric field in the lab frame IRF(S) is:

$$\boxed{\vec{E}(\rho) = +\frac{\mu_o I}{2\pi\rho} u\hat{\rho} = \vec{u} \times \left[\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right] = \vec{u} \times \vec{B}(\rho)} \quad \text{where: } \boxed{\vec{B}(\rho) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi}}$$

which corresponds to a lab-frame force acting on the test charge Q_T of:

$$\boxed{\vec{F}(\rho) = Q_T \vec{E}(\rho) = Q_T \vec{u} \times \vec{B}(\rho) = Q_T \vec{u} \times \left[\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right]} \quad \text{where: } \boxed{I = 2\lambda v}$$

There are two limiting cases that are of special / particular interest to us:

a.) When the ***lab*** velocity $\boxed{\vec{u} = +u\hat{z}}$ of the test charge Q_T is equal to the ***lab*** velocity $\boxed{\vec{v}_+ = +v\hat{z}}$ of the $+ve$ filamentary line charge density, *i.e.* $\boxed{\vec{u} = +u\hat{z}} = \boxed{\vec{v}_+ = +v\hat{z}}$, then the ***rest*** ***frame*** IRF(S') of the test charge Q_T ***coincides*** with the ***rest*** ***frame*** IRF(S_+) of the $+ve$ filamentary line charge density $\boxed{\lambda_{0_+} = +q/\ell_0}$. Note that this corresponds to the ***true*** lab frame {*i.e.* the ***rest*** ***frame*** of copper atoms} of a ***physical*** copper wire carrying a steady {conventional} current I !!!

b.) When the ***lab*** velocity $\boxed{\vec{u} = -u\hat{z}}$ of the test charge Q_T is equal to the ***lab*** velocity $\boxed{\vec{v}_- = -v\hat{z}}$ of the $-ve$ filamentary line charge density, *i.e.* $\boxed{\vec{u} = -u\hat{z}} = \boxed{\vec{v}_- = -v\hat{z}}$, then the ***rest*** ***frame*** IRF(S') of the test charge Q_T ***coincides*** with the ***rest*** ***frame*** IRF(S_-) of the $-ve$ filamentary line charge density $\boxed{\lambda_{0_-} = -q/\ell_0}$. Note that this corresponds to the ***rest*** ***frame*** of the ***electrons*** flowing in a ***physical*** copper wire carrying a steady {conventional} current I !!!

For situation a.), when the test charge Q_T's ***lab*** velocity $\boxed{\vec{u}=+u\hat{z}}=\boxed{\vec{v}_+=+v\hat{z}}$ ***lab*** velocity of the $+ve$ filamentary line charge density in IRF(S), then an observer in IRF(S') = IRF(S_+) will "see" a linear superposition of two electrostatic fields: a pure, radial-outward electrostatic field $\vec{E}_0'(\rho)$ associated with the stationary/non-moving $+ve$ filamentary line charge density $\boxed{\lambda_{0_+}=+\lambda_0=+q/\ell_0}$ and a {***lab*** velocity-dependent} radial-inward electric field $\vec{E}_v'(\rho)$ {*i.e.* an azimuthal magnetic field} associated with the $\boxed{v_-'=-2v/(1+\beta^2)}$ ***left***-moving $-ve$ filamentary line charge ***density*** of $\boxed{\lambda_-'=\gamma_-'\lambda_-=-\gamma(1+\beta^2)\lambda=-\gamma^2(1+\beta^2)\lambda_0}$, which in turn corresponds to a filamentary line ***current*** of $\boxed{I_-'=\lambda_-'v_-'=+\left[\gamma\cancel{(1+\beta^2)}\lambda\right]\left[2v/\cancel{(1+\beta^2)}\right]=+2\gamma\lambda v=+2\gamma^2\lambda_0 v}$

Thus in IRF(S') = IRF(S_+) with $\boxed{\vec{u}=+u\hat{z}}=\boxed{\vec{v}_+=+v\hat{z}}$:

$$\boxed{\vec{E}_0'(\rho)=+\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}\quad\text{and:}\quad\boxed{\vec{E}_v'(\rho)=\frac{\lambda_-'}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma(1+\beta^2)\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma^2(1+\beta^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}$$

The net/total electrostatic field observed in IRF(S') = IRF(S_+) is then:

$$\boxed{\begin{aligned}\vec{E}_{tot}'(\rho)&=\vec{E}_0'(\rho)+\vec{E}_v'(\rho)=+\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2(1+\beta^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=\frac{\lambda_0\left[1-\gamma^2(1+\beta^2)\right]}{2\pi\varepsilon_o\rho}\hat{\rho}\\&=\frac{(1-\gamma^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{2\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}\end{aligned}}$$

Notice the (amazing!) ***partial*** cancellation of the pure radial-outward electric field $\vec{E}_0'(\rho)$ (due to the static $+ve$ filamentary line charge density) with a ***portion*** of the velocity-dependent radial inward electric field $\vec{E}_v'(\rho)$ (due to the $-ve$ left-moving filamentary line current density) that is associated with the terms in the ***numerator*** of this equation:

$$\boxed{\begin{aligned}&1-\gamma^2(1+\beta^2)=1-(\gamma^2+\gamma^2\beta^2)=(1-\gamma^2)-\gamma^2\beta^2=\left(1-\frac{1}{1-\beta^2}\right)-\gamma^2\beta^2\\&=\left(\frac{\cancel{1}-\beta^2+\cancel{1}}{1-\beta^2}\right)-\gamma^2\beta^2=-\frac{\beta^2}{1-\beta^2}-\gamma^2\beta^2=-\gamma^2\beta^2-\gamma^2\beta^2=-2\gamma^2\beta^2\end{aligned}}$$

The net ***electric*** field is thus: $\boxed{\vec{E}'(\rho)=\frac{\lambda_{tot}'}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\cancel{2}\gamma\beta^2\lambda}{\cancel{2}\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma\beta^2\lambda}{\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma v^2\lambda}{\pi\varepsilon_o c^2\rho}\hat{\rho}}$

Thus an observer in the ***rest*** frame IRF(S') = IRF(S_+) of the test charge Q_T / rest frame of the $+ve$ filamentary line charge density "sees" a radial-inward/attractive electrostatic force (for $Q_T>0$) acting on the test charge Q_T of:

$$\boxed{\vec{F}'(\rho)=Q_T\vec{E}'(\rho)=Q_T\frac{\lambda_{tot}'}{2\pi\varepsilon_o\rho}\hat{\rho}=-Q_T\frac{\gamma\beta^2\lambda}{\pi\varepsilon_o\rho}\hat{\rho}=-Q_T\frac{\gamma v^2\lambda}{\pi\varepsilon_o c^2\rho}\hat{\rho}}\quad\text{but:}\quad\boxed{\frac{1}{c^2}=\varepsilon_o\mu_o}$$

$\therefore$ In IRF(S') = IRF(S_+): $\boxed{\vec{E}'(\rho) = -\dfrac{\mu_o \gamma \lambda v^2}{\pi\rho}\hat{\rho}}$ But: $\boxed{I = 2\lambda v}$ in the lab IRF(S).

$\therefore$ In IRF(S') = IRF(S_+): $\boxed{\vec{E}'(\rho) = -\dfrac{\mu_o \gamma I}{2\pi\rho} v\hat{\rho}}$ $\Leftarrow$ *n.b.* points radially ***inward***!

Therefore equivalently, the force $\boxed{\vec{F}'(\rho) = Q_T \vec{E}'(\rho)}$ acting on Q_T in its own rest frame IRF(S') is:

$$\boxed{\vec{F}'(\rho) = Q_T \vec{E}'(\rho) = -\frac{\mu_o Q_T \gamma I}{2\pi\rho} v\hat{\rho}} \Leftarrow$$

***n.b.* Q_T is attracted *towards* wire if $Q_T > 0$.**

Parallel currents attract each other !!! {The test charge Q_T is the 2nd current !!!}

Again, this force is none other than the magnetic Lorentz force acting on Q_T:

In IRF(S') = IRF(S_+): $\boxed{\vec{F}'(\rho) = Q_T\left(\vec{v}\times\vec{B}'(\rho)\right)}$

Where $\boxed{\vec{u} = +u\hat{z}} = \boxed{\vec{v} = +v\hat{z}}$ = velocity of test charge Q_T and +***ve*** filamentary line charge density in IRF(S)

$$\boxed{\vec{E}'(\rho) = \vec{v}\times\vec{B}'(\rho) \quad \{\hat{z}\times\hat{\varphi} = -\hat{\rho}\}} \quad \boxed{\vec{B}'(\rho) = \frac{\mu_o \gamma I}{2\pi\rho}\hat{\varphi} = \gamma\frac{\mu_o I}{2\pi\rho}\hat{\varphi}} \text{ where: } \boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(v/c)^2}}}$$

If $\exists$ a force $\vec{F}'$ in IRF(S') = IRF(S_+) (where Q_T and +*ve* filamentary line charge density are at rest), then there ***must*** also be a force $\vec{F}$ in the lab frame IRF(S) {the laws of physics are the ***same*** in ***all*** inertial reference frames…}.

We again Lorentz transform the force $\vec{F}'$ in IRF(S') = IRF(S_+) to obtain the force $\vec{F}$ in the ***lab*** frame IRF(S), where we already know that $\lambda_{TOT} = 0$ in the ***lab*** frame IRF(S). Again, since Q_T is at rest in IRF(S') and $\vec{F}'(\rho) \sim \hat{\rho}$ {*i.e.* $\perp$ $\vec{u} = u\hat{z}$ in IRF(S)}

Then in IRF(S): $\boxed{F_\perp = \dfrac{1}{\gamma'}F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)

$\perp$ and $\parallel$ refer to $\perp$ and $\parallel$ to $\vec{u}$ - the Lorentz boost direction

where: $\boxed{\gamma' \equiv \dfrac{1}{\sqrt{1-(v/c)^2}} = \gamma}$ = Lorentz factor to transform from IRF(S') (Q_T at rest) to lab frame IRF(S). IRF(S) moves with velocity $-\vec{u}$ with respect to IRF(S').

Then in IRF(S): $\boxed{F_\perp = \dfrac{1}{\gamma}F'_\perp = \sqrt{1-(v/c)^2}\,F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)

$\therefore$ In the lab frame IRF(S): $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho) = -\dfrac{\mu_o I}{2\pi\rho}v\hat{\rho}}$ $\leftarrow$ Radial E-field in lab frame IRF(S)

In the lab frame IRF(S): The test charge Q_T is moving with velocity $\boxed{\vec{u} = +u\hat{z}} = \boxed{\vec{v} = +v\hat{z}}$ in IRF(S)

An observer in lab frame IRF(S) "sees" a force $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho)}$ acting on moving test charge Q_T.

The "effective" electric field seen by a test charge Q_T moving with velocity $\boxed{\vec{u}=+u\hat{z}}=\boxed{\vec{v}=+v\hat{z}}$ in the lab frame IRF(*S*) is:

$$\boxed{\vec{E}(\rho)=-\frac{\mu_o I}{2\pi\rho}v\hat{\rho}=\vec{v}\times\left[\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right]=\vec{v}\times\vec{B}(\rho)} \quad \text{where: } \boxed{\vec{B}(\rho)=\frac{\mu_o I}{2\pi\rho}\hat{\varphi}} \text{ and: } \boxed{I=2\lambda v}.$$

From the perspective of a stationary observer in the lab frame IRF(*S*), where the net linear charge density $\lambda_{TOT}=0$, no ***true*** ***electrostatic*** field exists. However, a "magnetic", ***velocity-dependent*** attractive force $\vec{F}(\rho)$ ***does*** indeed exist, acting radially inward for a +*ve* test charge Q_T, when it is moving with velocity $\boxed{\vec{u}=+u\hat{z}}=\boxed{\vec{v}=+v\hat{z}}$ in IRF(*S*).

$\therefore$ In the lab frame IRF(*S*): $\boxed{\vec{F}(\rho)=Q_T\vec{E}(\rho)=-Q_T * v\left[\frac{\mu_o I}{2\pi\rho}\right]\hat{\rho}=Q_T\vec{v}\times\vec{B}(\rho)}$ where: $\boxed{I=2\lambda v}$

For situation b.), when the test charge Q_T ***lab*** velocity $\boxed{\vec{u}=-u\hat{z}}=\boxed{\vec{v}_-=-v\hat{z}}$ ***lab*** velocity of the –*ve* filamentary line charge density in IRF(*S*), then an observer in IRF(*S′*) = IRF(S_-) will "see" a linear superposition of two electrostatic fields: a pure, radial-inward electrostatic field $\vec{E}_0'(\rho)$ associated with the stationary/non-moving –*ve* filamentary line charge density $\boxed{\lambda_{0_-}=-\lambda_0=-q/\ell_0}$ and a {***lab*** velocity-dependent} radial-outward electric field $\vec{E}_v'(\rho)$ {*i.e.* an azimuthal magnetic field} associated with the $\boxed{v_+'=+2v/(1+\beta^2)}$ ***right***-moving +*ve* filamentary line charge ***density*** of $\boxed{\lambda_+'=\gamma_+'\lambda_+=+\gamma(1+\beta^2)\lambda=+\gamma^2(1+\beta^2)\lambda_0}$, which in turn corresponds to a filamentary line ***current*** of $\boxed{I_+'=\lambda_+'v_+'=+\left[\gamma\cancel{(1+\beta^2)}\lambda\right]\left[2v/\cancel{(1+\beta^2)}\right]=+2\gamma\lambda v=+2\gamma^2\lambda_0 v}$

Thus in IRF(*S′*) = IRF(S_-) with $\boxed{\vec{u}=-u\hat{z}}=\boxed{\vec{v}_-=-v\hat{z}}$:

$$\boxed{\vec{E}_0'(\rho)=-\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}} \quad \text{and: } \boxed{\vec{E}_v'(\rho)=\frac{\lambda_+'}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{\gamma(1+\beta^2)\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{\gamma^2(1+\beta^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}$$

The net/total electrostatic field observed in IRF(*S′*) = IRF(S_-) is then:

$$\boxed{\begin{aligned}\vec{E}_{tot}'(\rho)&=\vec{E}_0'(\rho)+\vec{E}_v'(\rho)=-\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2(1+\beta^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\lambda_0\left[1-\gamma^2(1+\beta^2)\right]}{2\pi\varepsilon_o\rho}\hat{\rho}\\&=-\frac{(1-\gamma^2)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{2\gamma^2\beta^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}\end{aligned}}$$

Notice again the (amazing!) ***partial*** cancellation of the pure radial-outward electric field $\vec{E}_0'(\rho)$ (due to the static –*ve* filamentary line charge density) with a ***portion*** of the velocity-dependent radial inward electric field $\vec{E}_v'(\rho)$ (due to the +*ve* right-moving filamentary line current density) that is associated with the terms in the ***numerator*** of this equation:

$$-1+\gamma^2\left(1+\beta^2\right)=-1+\left(\gamma^2+\gamma^2\beta^2\right)=-\left(1-\gamma^2\right)+\gamma^2\beta^2==\left(1-\frac{1}{1-\beta^2}\right)+\gamma^2\beta^2$$
$$=-\left(\frac{\cancel{1}-\beta^2+\cancel{1}}{1-\beta^2}\right)+\gamma^2\beta^2=+\frac{\beta^2}{1-\beta^2}+\gamma^2\beta^2=+\gamma^2\beta^2+\gamma^2\beta^2=+2\gamma^2\beta^2$$

The net ***electric*** field is thus: $\vec{E}'(\rho)=\dfrac{\lambda'_{tot}}{2\pi\varepsilon_o\rho}\hat{\rho}=+\dfrac{\cancel{2}\gamma\beta^2\lambda}{\cancel{2}\pi\varepsilon_o\rho}\hat{\rho}=+\dfrac{\gamma\beta^2\lambda}{\pi\varepsilon_o\rho}\hat{\rho}=+\dfrac{\gamma v^2\lambda}{\pi\varepsilon_o c^2\rho}\hat{\rho}$

Thus an observer in the ***rest*** frame IRF(S') = IRF(S_-) of the test charge Q_T / rest frame of the $-ve$ filamentary line charge density "sees" a radial-outward/repulsive electrostatic force (for $Q_T > 0$) acting on the test charge Q_T of:

$$\vec{F}'(\rho)=Q_T\vec{E}'(\rho)=Q_T\frac{\lambda'_{tot}}{2\pi\varepsilon_o\rho}\hat{\rho}=+Q_T\frac{\gamma\beta^2\lambda}{\pi\varepsilon_o\rho}\hat{\rho}=+Q_T\frac{\gamma v^2\lambda}{\pi\varepsilon_o c^2\rho}\hat{\rho}$$ but: $\dfrac{1}{c^2}=\varepsilon_o\mu_o$

$\therefore$ In IRF(S') = IRF(S_-): $\vec{E}'(\rho)=+\dfrac{\mu_o\gamma\lambda v^2}{\pi\rho}\hat{\rho}$ But: $I=2\lambda v$ in the ***lab*** IRF(S).

$\therefore$ In IRF(S') = IRF(S_-): $\vec{E}'(\rho)=+\dfrac{\mu_o\gamma I}{2\pi\rho}v\hat{\rho}$ $\Leftarrow$ *n.b.* points radially ***outward***!

Therefore equivalently, the force $\vec{F}'(\rho)=Q_T\vec{E}'(\rho)$ acting on Q_T in its ***own rest*** frame IRF(S') is:

$$\vec{F}'(\rho)=Q_T\vec{E}'(\rho)=+\frac{\mu_o Q_T\gamma I}{2\pi\rho}v\hat{\rho}\quad\Leftarrow$$

_n.b._ Q_T is _repelled_ from wire if $Q_T > 0$.

Opposite currents repell each other !!! {The test charge Q_T is the 2nd current !!!}

Again, this force is none other than the magnetic Lorentz force acting on Q_T:

In IRF(S') = IRF(S_-): $\vec{F}'(\rho)=Q_T\left(\vec{v}\times\vec{B}'(\rho)\right)$

Where $\vec{u}=-u\hat{z}$ = $\vec{v}=-v\hat{z}$ = velocity of test charge Q_T and ***−ve*** filamentary line charge density in IRF(S)

$\vec{E}'(\rho)=\vec{v}\times\vec{B}'(\rho)$ $\{\hat{z}\times\hat{\varphi}=-\hat{\rho}\}$

$\vec{B}'(\rho)=\dfrac{\mu_o\gamma I}{2\pi\rho}\hat{\varphi}=\gamma\dfrac{\mu_o I}{2\pi\rho}\hat{\varphi}$ where: $\gamma\equiv\dfrac{1}{\sqrt{1-\beta^2}}=\dfrac{1}{\sqrt{1-(v/c)^2}}$

If $\exists$ a force $\vec{F}'$ in IRF(S') = IRF(S_-) (where Q_T and +ve filamentary line charge density are at rest), then there ***must*** also be a force $\vec{F}$ in the lab frame IRF(S) {the laws of physics are the ***same*** in ***all*** inertial reference frames…}.

We again Lorentz transform the force $\vec{F}'$ in IRF(S') = IRF(S_-) to obtain the force $\vec{F}$ in the ***lab*** frame IRF(S), where we already know that $\lambda_{TOT}=0$ in the ***lab*** frame IRF(S). Again, since Q_T is at rest in IRF(S') and $\vec{F}'(\rho)\sim\hat{\rho}$ {*i.e.* $\perp$ $\vec{u}=u\hat{z}$ in IRF(S)}

Then in IRF(S): $\boxed{F_\perp = \dfrac{1}{\gamma'}F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)

⊥ and || refer to ⊥ and || to $\vec{u}$ - the Lorentz boost direction

where: $\boxed{\gamma' \equiv \dfrac{1}{\sqrt{1-(v/c)^2}} = \gamma} =$ Lorentz factor to transform from IRF(S') (Q_T at rest) to lab frame IRF(S). IRF(S) moves with velocity $-\vec{u}$ with respect to IRF(S').

Then in IRF(S): $\boxed{F_\perp = \dfrac{1}{\gamma}F'_\perp = \sqrt{1-(v/c)^2}\,F'_\perp}$ and: $\boxed{F_\parallel = F'_\parallel}$ (= 0 ***here***)

∴ In the lab frame IRF(S): $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho) = +\dfrac{\mu_o I}{2\pi\rho}v\hat{\rho}}$ ← Radial E-field in lab frame IRF(S)

In the lab frame IRF(S): The test charge Q_T is moving with velocity $\boxed{\vec{u} = -u\hat{z}} = \boxed{\vec{v} = -v\hat{z}}$ in IRF(S)

An observer in ***lab*** frame IRF(S) "sees" a force $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho)}$ acting on moving test charge Q_T. The "effective" electric field in lab frame IRF(S) is:

$$\boxed{\vec{E}(\rho) = +\frac{\mu_o I}{2\pi\rho}v\hat{\rho} = \vec{v}\times\left[\frac{\mu_o I}{2\pi\rho}\hat{\varphi}\right] = \vec{v}\times\vec{B}(\rho)} \quad \text{where: } \boxed{\vec{B}(\rho) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi}} \text{ and: } \boxed{I = 2\lambda v}.$$

From the perspective of a stationary observer in the lab frame IRF(S), where the net linear charge density $\lambda_{TOT} = 0$, no ***net*** ***electrostatic*** field exists. However, a "magnetic", ***velocity-dependent*** repulsive force $\vec{F}(\rho)$ ***does*** indeed exist, acting radially ***outward*** for a +*ve* test charge Q_T, when it is moving with velocity $\boxed{\vec{u} = -u\hat{z}} = \boxed{\vec{v} = -v\hat{z}}$ in IRF(S).

∴ In the lab frame IRF(S): $\boxed{\vec{F}(\rho) = Q_T\vec{E}(\rho) = +Q_T * v\left[\dfrac{\mu_o I}{2\pi\rho}\right]\hat{\rho} = Q_T\vec{v}\times\vec{B}(\rho)}$ where: $\boxed{I = 2\lambda v}$

Before leaving this subject, we wish to point out some additional fascinating aspects of the physics:

As mentioned above, situation a.) corresponds to the ***true*** lab frame of a ***physical*** wire carrying steady {conventional} current I where the lattice of {*e.g.*} copper atoms of the physical wire are at rest in IRF(S_+), whereas situation b.) corresponds to the ***rest*** frame IRF(S_-) of the ***drift electrons*** in the ***physical*** wire. What we have been calling the "***lab***" frame IRF(S) is the inertial reference frame which is intermediate/"splits-the-difference" between these two "extremes", with right- (left-) moving +*ve* (−*ve*) filamentary line charge densities $\lambda_+\,(\lambda_-)$ moving with velocities (in IRF(S)) of $\vec{v}_+ = +v\hat{z}$ $(\vec{v}_- = -v\hat{z})$ respectively.

In situation a.), the rest frame IRF(S_+) of the *e.g.* ***copper atoms*** of a ***physical*** filamentary wire, an observer in IRF(S_+) "sees" both a static, radial-outward electric field (due to the static $+\lambda_0$) and a velocity-dependent radial-inward electric field (due to the moving λ'_-). In IRF(S_+):

$$\begin{aligned}\vec{E}'_{tot}(\rho) &= \vec{E}'_0(\rho)+\vec{E}'_v(\rho) = +\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2\left(1+\beta^2\right)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}\\ &= +\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}-\frac{\gamma^2v^2\lambda_0}{2\pi\varepsilon_o c^2\rho}\hat{\rho}\\ &= \underbrace{-\frac{\left(\gamma^2-1\right)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}\; + \;\underbrace{\vec{v}\times\frac{\mu_o}{2\pi\rho}\left(\tfrac{1}{2}I'_-\right)\hat{\varphi}}\\ &= \quad \vec{E}'_{S_+}(\rho) \quad + \quad \vec{v}\times\vec{B}'_{S_+}(\rho)\end{aligned}$$

with:

$$\begin{aligned}\mu_o &= \frac{1}{\varepsilon_o c^2} \text{ and:}\\ \lambda'_- &= -\gamma\left(1+\beta^2\right)\lambda\\ &= -\gamma^2\left(1+\beta^2\right)\lambda_0\\ I'_- &= \lambda'_- v'_- = +2\gamma\lambda v\\ &= +2\gamma^2\lambda_0 v = \gamma I\\ I &= 2\lambda v = I'_-/\gamma\end{aligned}$$

The *EM* field energy density, Poynting's vector, linear momentum density and angular momentum density as seen by an observer in IRF(S_+) respectively are:

$$u_{\mathrm{IRF}(S_+)}(\rho)=\frac{1}{2}\varepsilon_o\vec{E}'_{S_+}(\rho)\bullet\vec{E}'_{S_+}(\rho)+\frac{1}{2\mu_o}\vec{B}'_{S_+}(\rho)\bullet\vec{B}'_{S_+}(\rho)=\frac{\gamma^4\beta^4\lambda_0^2}{8\pi^2\varepsilon_o\rho^2}=\frac{\mu_o I'^2_-}{32\pi^2\rho^2}\frac{v^2}{c^2} \quad (Joules/m^3)$$

$$\vec{S}_{\mathrm{IRF}(S_+)}(\rho)=\frac{1}{\mu_o}\vec{E}'_{S_+}(\rho)\times\vec{B}'_{S_+}(\rho)=\frac{\left(\gamma^2-1\right)\lambda_0 I'_-}{8\pi^2\varepsilon_o\rho^2}\underbrace{\left(-\hat{\rho}\times\hat{\varphi}\right)}_{=-\hat{z}}=-\frac{\left(\gamma^2-1\right)\lambda_0 I'_-}{8\pi^2\varepsilon_o\rho^2}\hat{z} \quad (Watts/m^2)$$

$$\vec{\wp}^{EM}_{\mathrm{IRF}(S_+)}(\rho)=\varepsilon_o\mu_o\vec{S}_{\mathrm{IRF}(S_+)}(\rho)=-\mu_o\frac{\left(\gamma^2-1\right)\lambda_0 I'_-}{8\pi^2\rho^2}\hat{z} \quad (kg/m^2\text{-}s)$$

$$\vec{\ell}^{EM}_{\mathrm{IRF}(S_+)}(\rho)=\vec{\rho}\times\vec{\wp}^{EM}_{\mathrm{IRF}(S_+)}(\rho)=\mu_o\frac{\left(\gamma^2-1\right)\lambda_0 I'_-}{8\pi^2\rho}\underbrace{\left(\hat{\rho}\times-\hat{z}\right)}_{+\hat{\varphi}}=+\mu_o\frac{\left(\gamma^2-1\right)\lambda_0 I'_-}{8\pi^2\rho}\hat{\varphi} \quad (kg/m\text{-}s)$$

In situation b.), the rest frame IRF(S_-) of the ***drift electrons*** in a ***physical*** filamentary wire, an observer in IRF(S_-) also "sees" both a static, radial-outward electric field (due to the static $-\lambda_0$) and a velocity-dependent radial-outward electric field (due to the moving λ'_+). In IRF(S_-):

$$\begin{aligned}\vec{E}'_{tot}(\rho) &= \vec{E}'_0(\rho)+\vec{E}'_v(\rho) = -\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2\left(1+\beta^2\right)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}\\ &= -\frac{\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}+\frac{\gamma^2v^2\lambda_0}{2\pi\varepsilon_o c^2\rho}\hat{\rho}\\ &= \underbrace{+\frac{\left(\gamma^2-1\right)\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}}\; + \;\underbrace{\vec{v}\times\frac{\mu_o}{2\pi\rho}\left(\tfrac{1}{2}I'_+\right)\hat{\varphi}}\\ &= \quad \vec{E}'_{S_-}(\rho) \quad + \quad \vec{v}\times\vec{B}'_{S_-}(\rho)\end{aligned}$$

with:

$$\begin{aligned}\mu_o &= \frac{1}{\varepsilon_o c^2} \text{ and:}\\ \lambda'_+ &= +\gamma\left(1+\beta^2\right)\lambda\\ &= +\gamma^2\left(1+\beta^2\right)\lambda_0\\ I'_+ &= \lambda'_+ v'_+ = +2\gamma\lambda v\\ &= +2\gamma^2\lambda_0 v = \gamma I\\ I &= 2\lambda v = I'_-/\gamma\end{aligned}$$

The *EM* field energy density, Poynting's vector, linear momentum density and angular momentum density as seen by an observer in IRF(S_-) respectively are:

$$u_{\text{IRF}(S_-)}(\rho) = \frac{1}{2}\varepsilon_o \vec{E}'_{S_-}(\rho)\bullet\vec{E}'_{S_-}(\rho) + \frac{1}{2\mu_o}\vec{B}'_{S_-}(\rho)\bullet\vec{B}'_{S_-}(\rho) = \frac{\gamma^4\beta^4\lambda_0^2}{8\pi^2\varepsilon_o\rho^2} = \frac{\mu_o I_+'^2}{32\pi^2\rho^2}\frac{v^2}{c^2} \quad (Joules/m^3)$$

$$\vec{S}_{\text{IRF}(S_-)}(\rho) = \frac{1}{\mu_o}\vec{E}'_{S_-}(\rho)\times\vec{B}'_{S_-}(\rho) = \frac{(\gamma^2-1)\lambda_0 I'_+}{8\pi^2\varepsilon_o\rho^2}\underbrace{(+\hat{\rho}\times\hat{\varphi})}_{=+\hat{z}} = +\frac{(\gamma^2-1)\lambda_0 I'_+}{8\pi^2\varepsilon_o\rho^2}\hat{z} \quad (Watts/m^2)$$

$$\vec{\wp}^{\,EM}_{\text{IRF}(S_-)}(\rho) = \varepsilon_o\mu_o\vec{S}_{\text{IRF}(S_-)}(\rho) = +\mu_o\frac{(\gamma^2-1)\lambda_0 I'_+}{8\pi^2\rho^2}\hat{z} \quad (kg/m^2\text{-}s)$$

$$\vec{\ell}^{\,EM}_{\text{IRF}(S_-)}(\rho) = \vec{\rho}\times\vec{\wp}^{\,EM}_{\text{IRF}(S_-)}(\rho) = \mu_o\frac{(\gamma^2-1)\lambda_0 I'_+}{8\pi^2\rho}\underbrace{(\hat{\rho}\times+\hat{z})}_{-\hat{\varphi}} = -\mu_o\frac{(\gamma^2-1)\lambda_0 I'_+}{8\pi^2\rho}\hat{\varphi} \quad (kg/m\text{-}s)$$

We see that observers in IRF(S_+) *vs*. IRF(S_-) "see" the *same* energy densities. Observers in IRF(S_+) *vs*. IRF(S_-) "see" the respective ***magnitudes*** of Poynting's vector, the *EM* linear momentum and angular momentum densities as being the same, however the ***directions*** of these 3 vector quantities in IRF(S_-) are ***opposite*** to what they are to an observer in IRF(S_+) !!!

An observer in IRF(S_+) "sees" that both the *EM* energy flow and *EM* linear momentum density are pointing in the $-\hat{z}$ direction, which physically makes sense because the negative electrons {moving in the $-\hat{z}$ direction} are the ***only*** objects in motion in IRF(S_+). Thus, an observer in IRF(S_+) concludes that the *EM* power/energy present in the *EM* fields associated with the infinitely long pair of filamentary wires in IRF(S_+) is supplied from the ***negative*** terminal of the battery (or power supply) driving the circuit. In IRF(S_+), an observer "sees" the *EM* field angular momentum density pointing in the $+\hat{\varphi}$ direction.

Contrast this with an observer in IRF(S_-) who "sees" that both the *EM* energy flow and *EM* linear momentum density are pointing in the $+\hat{z}$ direction, which physically makes sense because the positive-charged copper atoms {moving in the $+\hat{z}$ direction} are the ***only*** objects in motion in IRF(S_-). Thus, an observer in IRF(S_-) concludes that the *EM* power/energy present in the *EM* fields associated with the infinitely long pair of filamentary wires in IRF(S_-) is supplied from the ***positive*** terminal of the battery (or power supply) driving the circuit. In IRF(S_-), an observer "sees" the *EM* field angular momentum density pointing in the $-\hat{\varphi}$ direction.

Let's now compare these two sets of results for IRF(S_-) and IRF(S_-) with those obtained in our "original" rest frame, IRF(S), where both filamentary line current densities are in motion. In our "original" lab frame IRF(S), the net line charge density is $\boxed{\lambda_{tot} = \lambda_+ + \lambda_- = +\lambda - \lambda = 0}$ where $\boxed{\lambda_+ \equiv +\lambda = +q/\ell = +\gamma\lambda_0}$, $\boxed{\lambda_- \equiv -\lambda = -q/\ell = -\gamma\lambda_0}$ and $\boxed{\vec{v}_+ = +v\hat{z}}$, $\boxed{\vec{v}_- = -v\hat{z}}$, however the net current in IRF(S) is non-zero: $\boxed{I_{tot} = \lambda_+ v_+ + \lambda_- v_- = \lambda v + \lambda v = 2\lambda v = 2\gamma\lambda_0 v}$ flowing in the $+\hat{z}$ -direction. Thus, to an observer in IRF(S) there is no ***net*** electrostatic field, only a non-zero static magnetic field.

In IRF(S):

$$\vec{E}_+(\rho)=\frac{\lambda_+}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}=+\frac{\gamma\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}$$ and: $$\vec{E}_-(\rho)=\frac{\lambda_-}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}=-\frac{\gamma\lambda_0}{2\pi\varepsilon_o\rho}\hat{\rho}$$

The filamentary line currents in IRF(S) are: $I_+\equiv\lambda_+v_+=+\gamma\lambda_0 v$ and: $I_-\equiv\lambda_-v_-=+\gamma\lambda_0 v$, thus: $I_+=I_-=I$ and: $I_{tot}=I_++I_-=2I=2\lambda v=2\gamma\lambda_0 v$.

The magnetic fields associated with the currents I_+ and I_- are equal, and both point in the $+\hat{\varphi}$-direction: $\vec{B}_+(\rho)=+\frac{\mu_o I_+}{2\pi\rho}\hat{\varphi}$ and: $\vec{B}_-(\rho)=+\frac{\mu_o I_-}{2\pi\rho}\hat{\varphi}$

Then:

$$\begin{aligned}\vec{E}_{tot}^{\text{IRF}(S)}(\rho)&=\underbrace{\vec{E}_+(\rho)+\vec{E}_-(\rho)}_{=0}+\vec{v}\times\vec{B}_-(\rho)+\vec{v}\times\vec{B}_+(\rho)=\vec{v}\times\vec{B}_{TOT}^{\text{IRF}(S)}(\rho)\\&=\vec{v}\times\frac{\mu_o I_{TOT}}{2\pi\rho}\hat{\varphi}=\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}(\hat{z}\times\hat{\varphi})=-\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}\hat{\rho}\end{aligned}$$

Thus, an observer in IRF(S) “sees” a non-zero static magnetic field:

$$\vec{B}_{tot}^{\text{IRF}(S)}(\rho)=\vec{B}_+(\rho)+\vec{B}_-(\rho)=+\frac{\mu_o I_+}{2\pi\rho}\hat{\varphi}+\frac{\mu_o I_-}{2\pi\rho}\hat{\varphi}=+\frac{2\mu_o I}{2\pi\rho}\hat{\varphi}=+\frac{\mu_o I_{TOT}}{2\pi\rho}\hat{\varphi}$$

which is equivalent to an electric field seen by a test charge Q_T moving with velocity $\vec{v}$ in IRF(S) of:

$$\vec{E}_{tot}^{\text{IRF}(S)}(\rho)=\vec{v}\times\vec{B}_+(\rho)+\vec{v}\times\vec{B}_-(\rho)=\vec{v}\times\vec{B}_{TOT}(\rho)=\vec{v}\times\frac{\mu_o I_{TOT}}{2\pi\rho}\hat{\varphi}=\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}(\hat{z}\times\hat{\varphi})=-\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}\hat{\rho}$$

which gives rise to an attractive, radial-inward force acting on the test charge Q_T (for $Q_T>0$) of:

$$\vec{F}_{tot}^{\text{IRF}(S)}(\rho)=Q_T\vec{E}_{tot}^{\text{IRF}(S)}=Q_T\vec{v}\times\vec{B}_{tot}(\rho)=Q_T\vec{v}\times\frac{\mu_o I_{tot}}{2\pi\rho}\hat{\varphi}=Q_T\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}(\hat{z}\times\hat{\varphi})=-Q_T\frac{\lambda v^2}{\pi\varepsilon_o c^2\rho}\hat{\rho}$$

Thus, in IRF(S), even though there is no ***net*** λ_{tot}, a non-zero current $I_{tot}=2\lambda v\neq 0$ exists.

If $Q_T>0$ and $\vec{v}=+v\hat{z}$ {or $Q_T<0$ and $\vec{v}=-v\hat{z}$} the {radial-inward} force acting on the test charge Q_T is attractive – parallel currents attract!

If $Q_T>0$ and $\vec{v}=-v\hat{z}$ {or $Q_T<0$ and $\vec{v}=+v\hat{z}$} the {radial-outward} force acting on the test charge Q_T is repulsive – opposite currents repell!

It is also interesting to note that the two superimposed, oppositely-charged, counter-moving filamentary line charge densities / line current densities ***are attracted to each other*** {parallel currents attract!!!}, because in IRF(*S*) the force $\vec{F}_+(\rho)$ $\left(\vec{F}_-(\rho)\right)$ seen by {any} one of the +*ve* (−*ve*) "test charges" $+q_T$ ($-q_T$) associated with the moving positive (negative) filamentary line charge density λ_+ (λ_-) is, respectively:

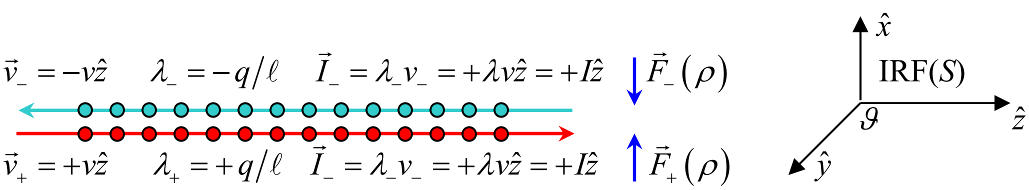
$$\boxed{\vec{F}_+(\rho) = +q_T\vec{v}_+\times\vec{B}_-(\rho) = +q_T v\frac{\mu_o I}{2\pi\rho}(\hat{z}\times\hat{\varphi}) = q_T\frac{\lambda v^2}{2\pi\varepsilon_o c^2\rho}(\hat{z}\times\hat{\varphi}) = -q_T\frac{\lambda v^2}{2\pi\varepsilon_o c^2\rho}\hat{\rho}}$$

And:

$$\boxed{\vec{F}_-(\rho) = -q_T\vec{v}_-\times\vec{B}_+(\rho) = +q_T v\frac{\mu_o I}{2\pi\rho}(\hat{z}\times\hat{\varphi}) = q_T\frac{\lambda v^2}{2\pi\varepsilon_o c^2\rho}(\hat{z}\times\hat{\varphi}) = -q_T\frac{\lambda v^2}{2\pi\varepsilon_o c^2\rho}\hat{\rho}}$$

***n.b.* Both radially-inward pointing forces!!!**

In IRF(*S*): $\vec{v}_- = -v\hat{z}$ $\lambda_- = -q/\ell$ $\vec{I}_- = \lambda_- v_- = +\lambda v\hat{z} = +I\hat{z}$ $\downarrow\vec{F}_-(\rho)$



$\vec{v}_+ = +v\hat{z}$ $\lambda_+ = +q/\ell$ $\vec{I}_- = \lambda_- v_- = +\lambda v\hat{z} = +I\hat{z}$ $\uparrow\vec{F}_+(\rho)$

Since this mutually-attractive, radial-inward force between opposite-moving line charges λ_+ & λ_- exists in IRF(*S*), this must also be true in ***all*** other inertial reference frames, *e.g.* IRF(S_+), IRF(S_-), *etc.* – the laws of physics are the same in all IRF's… we leave this as an exercise for the interested reader!

Obviously, since we have ***infinite***-length line charge densities λ_+ & λ_-, the ***net*** attractive force in ***each*** case is ***infinite***, even for slightly transversely-displaced line charge densities.

In ***lab*** frame IRF(*S*), the *EM* field energy density is ***non-zero***, finite positive (except at $\rho = 0$):

$$u^{EM}_{\text{IRF}(S)}(\rho) = u^{E_{tot}}_{\text{IRF}(S)}(\rho) + = u^{B_{tot}}_{\text{IRF}(S)}(\rho) = \frac{1}{2}\varepsilon_o\underbrace{\vec{E}^{net}_{\text{IRF}(S)}(\rho)}_{=0}\cdot\underbrace{\vec{E}^{net}_{\text{IRF}(S)}(\rho)}_{=0} + \frac{1}{2\mu_o}\vec{B}^{\text{IRF}(S)}_{tot}(\rho)\cdot\vec{B}^{\text{IRF}(S)}_{tot}(\rho)$$

$$= \frac{1}{2}\varepsilon_o\left[\vec{E}_+(\rho)+\vec{E}_-(\rho)\right]\cdot\left[\vec{E}_+(\rho)+\vec{E}_-(\rho)\right] + \frac{1}{2\mu_o}\left[\vec{B}_+(\rho)+\vec{B}_-(\rho)\right]\cdot\left[\vec{B}_+(\rho)+\vec{B}_-(\rho)\right]$$

$$= \frac{1}{2}\varepsilon_o\left\{E_+^2(\rho)+2\vec{E}_+(\rho)\cdot\vec{E}_-(\rho)+E_-^2(\rho)\right\} + \frac{1}{2\mu_o}\left\{B_+^2(\rho)+2\vec{B}_+(\rho)\cdot\vec{B}_-(\rho)+B_-^2(\rho)\right\}$$

$$= \frac{1}{2}\varepsilon_o\underbrace{\left\{E_+^2(\rho)-2E_+^2(\rho)+E_+^2(\rho)\right\}}_{=0} + \frac{1}{2\mu_o}\underbrace{\left\{B_+^2(\rho)+2B_+^2(\rho)+B_+^2(\rho)\right\}}_{=B^{2\,\text{IRF}(S)}_{tot}(\rho)}$$

$$= \frac{\mu_o I_{tot}^2}{8\pi^2\rho^2}\hat{\varphi} = \frac{\lambda^2}{2\pi^2\varepsilon_o\rho^2}\frac{v^2}{c^2}\quad\left(Joules/m^3\right)$$

with: $\boxed{I_{tot} = 2I = 2\lambda v = 2\gamma\lambda_0 v}$

The ***net*** Poynting's vector $\vec{S}_{\text{IRF}(S)}(\rho)$, ***net*** *EM* field linear momentum density $\vec{\wp}^{EM}_{\text{IRF}(S)}(\rho)$ and ***net*** *EM* field angular momentum density $\vec{\ell}^{EM}_{\text{IRF}(S)}(\rho)$ as seen by an observer in IRF(*S*) are all ***zero***, because $\boxed{\vec{E}^{net}_{\text{IRF}(S)}(\rho)=0}$.

However, these ***net*** physical quantities are all zero because of the ***superposition principle*** – each are sums of two ***counter-propagating*** contributions that ***cancel*** each other!

$$\boxed{\begin{aligned}\vec{S}_{\text{IRF}(S)}(\rho)&=\vec{S}^{+-}_{\text{IRF}(S)}(\rho)+\vec{S}^{-+}_{\text{IRF}(S)}(\rho)=\frac{1}{\mu_o}\left\{\vec{E}_+(\rho)\times\vec{B}_-(\rho)\right\}+\frac{1}{\mu_o}\left\{\vec{E}_-(\rho)\times\vec{B}_+(\rho)\right\}\\&=\frac{\lambda_+I_-}{4\pi^2\varepsilon_o\rho^2}(\hat{\rho}\times\hat{\varphi})+\frac{\lambda_-I_+}{4\pi^2\varepsilon_o\rho^2}(\hat{\rho}\times\hat{\varphi})=+\frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\hat{z}-\frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\hat{z}=0\end{aligned}}\quad (Watts/m^2)$$

Thus, we explicitly see that: $\boxed{\vec{S}^{-+}_{\text{IRF}(S)}(\rho)=-\vec{S}^{+-}_{\text{IRF}(S)}(\rho)=-\frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\hat{z}}$.

Consequently/similarly:
$$\boxed{\begin{aligned}\vec{\wp}^{EM}_{\text{IRF}(S)}(\rho)&=\varepsilon_o\mu_o\vec{S}_{\text{IRF}(S)}(\rho)=\varepsilon_o\mu_o\vec{S}^{+-}_{\text{IRF}(S)}(\rho)+\varepsilon_o\mu_o\vec{S}^{-+}_{\text{IRF}(S)}(\rho)\\&=\vec{\wp}^{+-}_{\text{IRF}(S)}(\rho)+\vec{\wp}^{-+}_{\text{IRF}(S)}(\rho)=+\frac{\mu_o\lambda I}{4\pi^2\rho^2}\hat{z}-\frac{\mu_o\lambda I}{4\pi^2\rho^2}\hat{z}=0\end{aligned}}\quad (kg/m^2\text{-}s)$$

Thus, we explicitly see that: $\boxed{\vec{\wp}^{-+}_{\text{IRF}(S)}(\rho)=-\vec{\wp}^{+-}_{\text{IRF}(S)}(\rho)=-\frac{\mu_o\lambda I}{4\pi^2\rho^2}\hat{z}}$.

Similarly:
$$\boxed{\begin{aligned}\vec{\ell}^{EM}_{\text{IRF}(S)}(\rho)&=\vec{\rho}\times\vec{\wp}^{EM}_{\text{IRF}(S)}(\rho)=\vec{\rho}\times\vec{\wp}^{+-}_{\text{IRF}(S)}(\rho)+\vec{\rho}\times\vec{\wp}^{-+}_{\text{IRF}(S)}(\rho)=\vec{\ell}^{+-}_{\text{IRF}(S)}(\rho)+\vec{\ell}^{-+}_{\text{IRF}(S)}(\rho)\\&=+\frac{\mu_o\lambda I}{4\pi^2\rho}(\hat{\rho}\times\hat{z})-\frac{\mu_o\lambda I}{4\pi^2\rho}(\hat{\rho}\times\hat{z})=-\frac{\mu_o\lambda I}{4\pi^2\rho}\hat{\varphi}+\frac{\mu_o\lambda I}{4\pi^2\rho}\hat{\varphi}=0\end{aligned}}\quad (kg/m\text{-}s)$$

Thus, we explicitly see that: $\boxed{\vec{\ell}^{-+}_{\text{IRF}(S)}(\rho)=-\vec{\ell}^{+-}_{\text{IRF}(S)}(\rho)=+\frac{\mu_o\lambda I}{4\pi^2\rho}\hat{\varphi}}$.

Thus, an observer in IRF(*S*) "sees" ***two counter-propagating*** fluxes of *EM* energy, linear momentum density and angular momentum density, which respectively cancel each other out such that the ***net*** fluxes of *EM* energy, linear momentum density and angular momentum density are all zero in IRF(*S*)!

An observer in IRF(*S*) concludes that the *EM* power/energy present in the *EM* fields associated with the infinitely long pair of oppositely-charged, opposite-moving filamentary line charge densities λ_+ & λ_- in IRF(*S*) is supplied ***equally*** from ***both*** the ***positive*** and ***negative*** terminals of the battery (or power supply) driving the circuit!

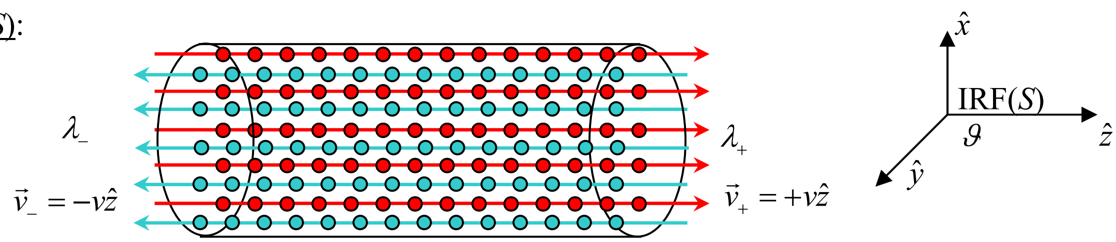
Thus, we finally understand how electrical power is transported down a physical wire – it is a manifestly relativistic effect; electrical power in a wire is transported by the combination of the radial *E*-field and the azimuthal *B*-field associated with a current flowing in the wire!

Because we have an ***infinitely-long*** filamentary 1-D physical wire (*i.e.* zero radius), consisting of an infinitely long pair of oppositely-charged, opposite-moving filamentary line charge densities λ_+ & λ_-, in ***any*** IRF the *EM* field energy $\boxed{U_{EM} = \int_{all\ space} u_{EM}(\rho)\,d\tau = \infty}$. Similarly, the *EM* power transported down such a wire $\boxed{P_{EM} = \int_{all\ space} \vec{S}(\rho)\cdot d\vec{a}_\perp = \infty}$, the *EM* field linear momentum $\boxed{\vec{p}_{EM} = \int_{all\ space} \vec{\wp}_{EM}(\rho)\,d\tau = \infty}$ and *EM* field angular momentum $\boxed{\vec{\mathcal{L}}_{EM} = \int_{all\ space} \vec{\ell}_{EM}(\rho)\,d\tau = \infty}$ except in IRF(S), where the latter two quantities are zero.

For a ***real***, ***finite-length*** physical wire of ***finite*** radius a, these four quantities are all ***finite***, as long as λ_+ & λ_- are both finite and v_+ & v_- are both $< c$.

Using the superposition principle, a ***real***, ***finite-length*** physical wire of ***finite*** radius a can be thought of as a collection of 2N parallel filamentary "infinitesimal" 1-D line charge densities. In IRF(S), the N right-moving λ_+ lines represent 1-D parallel strings of {*e.g.* copper} atoms and the N left-moving λ_- lines represent 1-D parallel strings of drift electrons, as shown schematically in the figure below:

In IRF(S):



Even though the net volume charge density in IRF(S) for a real physical wire of radius a is $\boxed{\rho_{tot} = \rho_+ + \rho_- = N\lambda_+/A_\perp + N\lambda_-/A_\perp = N\lambda/A_\perp - N\lambda/A_\perp = 0}$, while there is no ***net pure*** electrostatic field in IRF(S) (the ***net*** charge on the wire is ***zero***), there is again a non-zero azimuthal magnetic field $\vec{B}_{tot}^{\text{IRF}(S)}(\rho)$, which has two contributions – one from the N ***right-moving*** λ_+ lines (copper atoms) and another, equal contribution from the N ***left-moving*** λ_- lines (drift electrons). For an ***infinitely*** long real physical wire of radius a, we know that:

$$\boxed{\vec{B}_{tot}^{\text{IRF}(S)}(\rho \le a) = \frac{\mu_o I}{2\pi}\frac{\rho}{a^2}\hat{\varphi}} \quad \underline{\text{and}}: \quad \boxed{\vec{B}_{tot}^{\text{IRF}(S)}(\rho \ge a) = \frac{\mu_o I}{2\pi\rho}\hat{\varphi}}$$

An interesting phenomenon occurs in a real physical wire, due to the fact that parallel currents attract each other. The radial-inward Lorentz force $\boxed{\vec{F}_-(\rho) = -q_T\vec{v}_- \times \vec{B}_+(\rho)}$ acting on the "gas" of left-moving drift electrons exerts a radial-inward pressure on the "free" electron gas, and ***compresses*** it (slightly)! The radial-inward Lorentz force $\boxed{\vec{F}_+(\rho) = +q_T\vec{v}_+ \times \vec{B}_-(\rho)}$ acting on the 3-D lattice of right-moving copper atoms exerts a radial-inward pressure on the copper atoms, but because they are bound together in the 3-D lattice, they undergo very little compression, if any!

This manifest ***asymmetry*** between the "free" electron "gas" and the 3-D lattice of copper atoms thus gives rise to a {slight} ***differential*** compression between electrons and copper atoms – resulting in a {very thin} "skin" of positive charge {of thickness δ } on the surface of the wire {*n.b*. the skin thickness δ is much thinner than the diameter of an atom, for "normal"/everyday currents!}. Inside this "skin" of positive charge on the outer surface of the wire, there exists a slightly higher negative volume charge density $\rho_-(\rho < a-\delta)$ than positive volume charge density $\rho_+(\rho < a-\delta)$. The net charge on the wire still remains zero.

The compression of the "free" electron "gas" is only a slight, but non-negligible amount. The radial-inward Lorentz force $\boxed{\vec{F}_-(\rho) = -q_T\vec{v}_- \times \vec{B}_+(\rho)}$ is countered by the repulsive, radial-outward force associated with (local) electric charge neutrality of electrons & copper atoms, and also by a quantum effect – since electrons are fermions {no two electrons can simultaneously occupy the same quantum state}, there also exists a radial-outward ***quantum*** ***pressure*** on the electrons preventing them from becoming too dense!

From the above discussion(s), while it can be seen that gaining an insight of the underlying physics associated with electrical power transport, *etc*. in a wire via use of special relativity may be somewhat more tedious than using the "standard" *E&M* approach, special relativity makes it ***profoundly*** clear what the underlying physics actually is, whereas the "standard" *E&M* approach does not do a very good job in elucidating the actual physics…