LECTURE NOTES 17

Proper Time and ***Proper*** Velocity

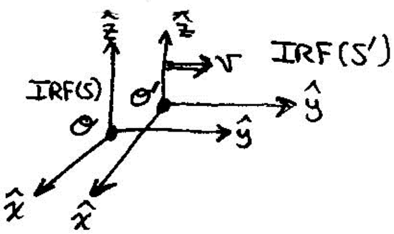
- As ***you*** progress along ***your*** world line {moving with "***ordinary***" velocity $\vec{u}$ in ***lab*** frame IRF(*S*)} on the ct *vs*. x Minkowski/space-time diagram, ***your*** watch runs ***slow*** {in ***your*** ***rest*** frame IRF(*S′*)} in comparison to clocks on the wall in the ***lab*** frame IRF(*S*).

- The clocks on the wall in the ***lab*** frame IRF(*S*) tick off a time interval dt, whereas in ***your*** ***rest*** frame IRF(S') the time interval is: $\boxed{dt' = dt/\gamma_u = \sqrt{1-\beta_u^2}\,dt}$

- *n.b.* this is the ***exact*** ***same*** time dilation formula that we obtained earlier, with:

$$\boxed{\gamma_u \equiv 1\Big/\sqrt{1-(u/c)^2} = 1\Big/\sqrt{1-\beta_u^2}} \quad \text{and:} \quad \boxed{\beta_u \equiv (u/c)}$$

- We use $u = |\vec{u}| =$ relative speed of an ***object*** as observed in an inertial reference frame {***here***, u = speed of ***you***, as observed in the ***lab*** IRF(*S*)}.



- We will henceforth use $v = |\vec{v}| =$ relative speed between two inertial systems – *e.g.* IRF(S') relative to IRF(*S*):

- Because the time ***interval*** dt' occurs in ***your*** ***rest*** frame IRF(S'), we give it a ***special*** name: $\boxed{d\tau' = dt'}$ = ***proper*** ***time*** ***interval*** (in ***your*** ***rest*** frame), and: $\boxed{\tau' = t'}$ = ***proper*** time (in ***your*** ***rest*** frame).

- The name "***proper***" is due to a mis-translation of the French word "***propre***", meaning "***own***".

- ***Proper*** time τ' is ***different*** than "***ordinary***" time, *t*.

  Proper time τ' is a ***Lorentz-invariant*** quantity, whereas "***ordinary***" time *t* depends on the choice of IRF - *i.e.* "***ordinary***" time is ***not*** a ***Lorentz-invariant*** quantity.

  The Lorentz-invariant *interval*: $\boxed{dI \equiv dx'_\mu dx'^\mu = dx'^\mu dx'_\mu = ds'^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2}$

  ***Proper* time *interval*:** $\boxed{d\tau' \equiv \sqrt{-dI/c^2} = \sqrt{-ds'^2/c^2} = \sqrt{dt'^2 - \left(\underbrace{dx'^2 + dy'^2 + dz'^2}_{=0 \text{ in rest frame IRF}(S')}\right)\Big/c^2} = \sqrt{dt'^2} = dt'}$

  ***Proper* time:** $\boxed{\tau' \equiv \tau'_2 - \tau'_1 \equiv \int_{\tau'_1}^{\tau'_2} d\tau' = \int_{t'_1}^{t'_2} dt' = t'_2 - t'_1 = \Delta t'}$

- Because $d\tau'$ and τ' are ***Lorentz-invariant*** quantities: $\boxed{d\tau' = d\tau}$ and: $\boxed{\tau' = \tau}$ {*i.e.* drop primes}.

- In terms of 4-D space-time, ***proper*** time is analogous to ***arc*** ***length*** *S* in 3-D Euclidean space.

- **Special** **designation** is given to being in the ***rest*** frame of an object.

- The ***rest*** frame of an object = the ***proper*** frame.

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Consider a situation where ***you*** are on an airplane flight from NYC to LA. The pilot comes on the loudspeaker and announces in mid-flight that the jet stream is flowing backwards today, and that the plane's present velocity is $u = 0.8c\ (\beta_u = 0.8!!)$, due ***west***.

What the pilot means by "velocity" is the spatial displacement $d\vec{\ell}$ per unit time interval dt.

The pilot is referring to the plane's velocity ***relative to the ground*** (*n.b.* ***here***, we make the simplifying assumption that the earth is ***non-rotating/non-moving***, so that we ***can*** use IRF's…)

Thus, $d\vec{\ell}$ and dt are quantities as ***measured by an observer on the ground*** (*e.g.* an airplane flight controller, using RADAR) in the ***ground-based*** (***lab***) IRF(*S*).

Thus: $\vec{u} = \dfrac{d\vec{\ell}}{dt}$ = "***ordinary***" velocity in the ***lab*** IRF(*S*)

$d\vec{\ell}$ and dt are measured in the ***ground-based*** (***lab***) IRF(*S*)

You, on the other hand are in ***your*** own ***rest*** frame IRF(*S′*) in the airplane, sitting in your seat.

You know that the distance from NYC to LA is $L = 2763$ miles (as measured on the ***ground***, referring to your trusty Rand-McNally Road Atlas {back pages} that you brought along with you).

So ***you***, from ***your*** perspective, might be more interested in the quantity known as ***your proper*** velocity $\vec{\eta}$, defined as:

Proper 3-Velocity: $\vec{\eta} \equiv \dfrac{d\vec{\ell}}{d\tau}$ = ***hybrid*** measurement =

Spatial displacement, as measured on the ***ground*** (in ***lab*** IRF(*S*)) ***per unit time interval***, as measured in ***your*** (or an ***object's***) ***rest*** frame (in IRF(*S′*)).

Since: $d\tau = dt' = \dfrac{1}{\gamma_u}dt = \sqrt{1-\beta_u^2}\,dt = \sqrt{1-(u/c)^2}\,dt$ and: $\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}$, $\beta_u \equiv (u/c)$

Then: $\vec{\eta} \equiv \dfrac{d\vec{\ell}}{d\tau} = \dfrac{d\vec{\ell}}{\frac{1}{\gamma_u}dt} = \gamma_u \dfrac{d\vec{\ell}}{dt}$, but: $\vec{u} \equiv \dfrac{d\vec{\ell}}{dt}$ $\therefore$ $\vec{\eta} = \gamma_u \vec{u} = \dfrac{1}{\sqrt{1-\beta_u^2}}\vec{u} = \dfrac{1}{\sqrt{1-(u/c)^2}}\vec{u}$

n.b. $0 \le \gamma_u \le \infty$

If $u = 0.8c\ (\beta_u = 0.8)$, then: $\gamma_u = 1\big/\sqrt{1-\beta_u^2} = 1\big/\sqrt{1-0.8^2} = \frac{5}{3}$, hence: $|\vec{\eta}| = \gamma_u|\vec{u}| = \frac{5}{3}0.8c = \frac{\cancel{5}}{3}\frac{4}{\cancel{5}}c = \frac{4}{3}c\,!!!$

Of course, for ***non-relativistic*** speeds $u \ll c$, then: $\vec{\eta} \simeq \vec{u}$ to a high degree.

From a theoretical perspective, an appealing aspect of ***proper*** 3-velocity $\vec{\eta}$ is that it Lorentz-transforms ***simply*** from one IRF to another IRF.

$\vec{\eta}$ = 3-D ***spatial*** component(s) of a ***relativistic 4-vector***, η^μ

The {contravariant} ***proper 4-velocity*** is: $\eta^\mu \equiv \dfrac{dx^\mu}{d\tau}$ whose ***zeroth/temporal/scalar*** component is:

$$\eta^0 \equiv \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\frac{dt}{\frac{1}{\gamma_u}dt} = \gamma_u c\frac{\cancel{dt}}{\cancel{dt}} = \gamma_u c = \frac{c}{\sqrt{1-\beta_u^2}} = \frac{c}{\sqrt{1-(u/c)^2}}$$

with: $\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}$, $\beta_u \equiv (u/c)$

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The ***proper*** 4-velocity vector is:

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \left(\eta^0, \vec{\eta}\right) = \left(\gamma_u c, \vec{\eta}\right) \quad \text{or:} \quad \eta^\mu = \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{pmatrix}$$

The ***numerator*** of the ***proper*** 4-velocity dx^μ is the ***displacement*** 4-vector (as measured in the ***ground-based*** (***lab***) IRF(*S*). The ***denominator*** of the ***proper*** 4-velocity $d\tau$ = ***proper*** time interval (as measured in ***your*** (or an object's) ***rest*** frame IRF(*S′*).

The Lorentz Transformation of a Proper 4-Velocity η^μ :

Suppose we want to Lorentz transform ***your*** proper 4-velocity from the ***lab*** IRF(*S*) to ***another*** (different) IRF(*S″*) along a common $\hat{x}$ -axis, in which IRF(*S″*) is moving with relative velocity $\vec{v} = v\hat{x}$ with respect to ***lab*** IRF(*S*):

Most generally, in tensor notation: $\eta''^\mu = \Lambda^\mu{}_\nu \eta^\nu$ with $\Lambda^\mu{}_\nu$ = Lorentz boost tensor. Thus:

$$\eta''^\mu = \Lambda^\mu{}_\nu \eta^\nu \Rightarrow \begin{pmatrix} \eta''^0 \\ \eta''^1 \\ \eta''^2 \\ \eta''^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} \Rightarrow \begin{pmatrix} \eta''^0 \\ \eta''^1 \\ \eta''^2 \\ \eta''^3 \end{pmatrix} = \begin{pmatrix} \gamma\left(\eta^0 - \beta\eta^1\right) \\ \gamma\left(\eta^1 - \beta\eta^0\right) \\ \eta^2 \\ \eta^3 \end{pmatrix} \quad \text{with:} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \quad \beta \equiv \frac{v}{c}$$

Where: $\eta''^\mu \equiv \dfrac{dx''^\mu}{d\tau}$ and: $\eta^\mu \equiv \dfrac{dx^\mu}{d\tau}$

Compare this result to the same Lorentz transformation of "***ordinary***" 3-velocities, along a common $\hat{x}$ -axis. We use the Einstein velocity addition rule:

$$\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z} \qquad u''_x = \frac{dx''}{dt''} = \frac{u_x - v}{1-\left(u_x v/c^2\right)}$$

$$\vec{u}'' = u''_x\hat{x} + u''_y\hat{y} + u''_z\hat{z} \qquad u''_y = \frac{dy''}{dt''} = \frac{u_y}{\gamma\left(1-\left(u_x v/c^2\right)\right)} \quad \text{with:} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \quad \text{and:} \quad \beta \equiv \frac{v}{c}$$

$$u''_z = \frac{dz''}{dt''} \quad \frac{u_z}{\gamma\left(1-\left(u_x v/c^2\right)\right)}$$

{See Griffiths Example 12.6 (*p*. 497-98) and Griffiths Problem 12.14 (*p*. 498)}

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Now we can see why Lorentz transformation of "***ordinary***" velocities is more cumbersome than Lorentz transformation of ***proper*** 4-velocities:

- For "***ordinary***" 3-velocities $\boxed{\vec{u} = \frac{d\vec{\ell}}{dt}}$, we must Lorentz transform ***both*** $\left\{\begin{matrix}\text{numerator, } d\vec{\ell} \\ \text{denominator, } dt\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} d\vec{\ell}'' \\ dt'' \end{matrix}\right\}$
- For ***proper*** 4-velocities $\boxed{\eta^\mu \equiv \frac{dx^\mu}{d\tau}}$ we only need to transform the ***numerator***, $d\vec{\ell} \Rightarrow d\vec{\ell}''$.

Relativistic Energy and Momentum - Relativistic 4-Momentum:

In ***classical*** mechanics, the 3-D vector linear momentum $\vec{p} = \text{mass} \times \text{velocity } \vec{v}$, *i.e.* $\boxed{\vec{p} = m\vec{v}}$. How do we extend this to ***relativistic*** mechanics?

Should we use the "***ordinary***" velocity $\boxed{\vec{u} \equiv \frac{d\vec{\ell}}{dt}}$ for $\vec{v}$,

or should we use the ***proper*** velocity $\boxed{\vec{\eta} \equiv \frac{d\vec{\ell}}{d\tau}}$ for $\vec{v}$??

In ***classical*** mechanics, $\vec{\eta}$ and $\vec{u}$ are identical.
In ***relativistic*** mechanics, $\vec{\eta}$ and $\vec{u}$ are ***not*** identical.

$\Rightarrow$ We ***must*** use the ***proper*** velocity $\vec{\eta}$ in ***relativistic*** mechanics, because otherwise, the ***law of conservation of momentum*** would be ***inconsistent*** with the ***principle of relativity*** {the laws of physics are the ***same*** in all IRF's} if we ***were*** to define ***relativistic 3-momentum*** as: $\vec{p} = m\vec{u}$. ***No***!!

Thus, we define the ***relativistically-correct 3-momentum*** as:

$$\boxed{\vec{p} \equiv m\vec{\eta} = \gamma_u m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\beta_u^2}} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}} \quad \text{with: } \boxed{\gamma_\mu \equiv \frac{1}{\sqrt{1-\beta_u^2}}} \quad \text{and: } \boxed{\beta_u \equiv \left(\frac{u}{c}\right)}$$

Relativistic 3-momentum: $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u}}$ is the ***spatial*** part of a relativistic 4-momentum vector: $\boxed{p^\mu \equiv m\eta^\mu}$, *i.e.* $\boxed{p^\mu = (p^0, \vec{p})}$.

The ***temporal/zeroth/scalar*** component of the relativistic 4-momentum vector is: $\boxed{p^0 = E/c}$

But: $$\boxed{p^0 \equiv m\eta^0 = \gamma_u mc = \frac{mc}{\sqrt{1-\beta_u^2}} = \frac{mc}{\sqrt{1-(u/c)^2}}} \quad \text{with: } \boxed{\gamma_\mu \equiv \frac{1}{\sqrt{1-\beta_u^2}}} \quad \text{and: } \boxed{\beta_u \equiv (u/c)}$$

Thus: $\boxed{p^0 = E/c = m\eta^0 = \gamma_u mc}$ where: $\boxed{\eta^0 = \gamma_u c = \frac{c}{\sqrt{1-\beta_u^2}} = \frac{c}{\sqrt{1-(u/c)^2}}}$

Since: $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u}}$, then: $\boxed{p = |\vec{p}| = \gamma_u m|\vec{u}| = \gamma_u mu = \gamma_u \beta_u mc = \beta_u(\gamma_u mc) = \beta_u E/c}$.

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Relativistic Energy: $E \equiv \gamma_u mc^2 = \dfrac{mc^2}{\sqrt{1-\beta_u^2}} = \dfrac{mc^2}{\sqrt{1-(u/c)^2}}$ with: $\gamma_\mu \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}$ and: $\beta_u \equiv (u/c)$

Therefore, the components of the relativistic 4-momentum are: $p^\mu = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \equiv \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$

The 4-vector dot/scalar product $p_\mu p^\mu$ is a ***Lorentz-invariant*** quantity (***same*** in all IRF's):

$$p_\mu p^\mu = -(E/c)^2 + p_x^2 + p_y^2 + p_z^2 = -(E/c)^2 + p^2 = -(mc)^2$$

This can be rewritten in the more familiar form as: $E^2 = (pc)^2 + (mc^2)^2$ or: $E = \sqrt{(pc)^2 + (mc^2)^2}$.

Since: $E \equiv \gamma_u mc^2$ then: $\gamma_u^2 (mc^2)^2 = (pc)^2 + (mc^2)^2$ or: $(pc)^2 = (\gamma_u^2 - 1)(mc^2)^2$. But: $\gamma_\mu \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}$

hence: $(pc)^2 = (\gamma_u^2 - 1)(mc^2)^2 = \left(\dfrac{1}{1-\beta_u^2} - 1\right)(mc^2)^2 = \left(\dfrac{1-1+\beta_u^2}{1-\beta_u^2}\right)(mc^2)^2 = \left(\dfrac{\beta_u^2}{1-\beta_u^2}\right)(mc^2)^2$

or: $pc = \gamma_u \beta_u mc^2$. However: $E \equiv \gamma_u mc^2$ Thus, we {again} also see that: $p = |\vec{p}| = \beta_u E/c$.

Note that the relativistic energy E of a ***massive*** object is ***non-zero*** even when that object is ***stationary*** - *i.e.* in its own ***rest*** frame – when: $p = 0$, $\beta_u = 0$ and: $\gamma_u = 1/\sqrt{1-\beta_u^2} = 1$.

Then: $E_{rest} = mc^2$ = rest energy = rest mass * c^2. ⇐ Einstein's famous formula!

If $\beta_u \neq 0$, then the ***remainder*** of the relativistic energy E is attributable to the ***motion*** of the particle – *i.e.* it is relativistic ***kinetic*** energy, E_{kin}.

Total Relativistic Energy: $E \equiv E_{tot} = E_{kin} + E_{rest} = \gamma_u mc^2$ but: $E_{rest} = mc^2$

$$\therefore \quad E_{kin} = E_{tot} - E_{rest} = \gamma_u mc^2 - mc^2 = (\gamma_u - 1) mc^2$$

Relativistic *Kinetic* Energy: $E_{kin} = (\gamma_u - 1) mc^2 = \left(\dfrac{1}{\sqrt{1-\beta_u^2}} - 1\right) mc^2 = \left(\dfrac{1}{\sqrt{1-(u/c)^2}} - 1\right) mc^2$

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In the ***non-relativistic*** regime $u \ll c$, then: $\boxed{E_{kin} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \ldots \simeq \frac{1}{2}mu^2}$ (***classical*** formula).

However, for $u \ll c$ then: $p \simeq mu$ and thus: $\boxed{E_{kin} \simeq \frac{p^2}{2m}}$ (***classical*** formula).

Note that ***total*** relativistic energy, E_{tot} and ***total*** relativistic 3-momentum, $p_{tot} = |\vec{p}_{tot}|$ are ***separately*** ***conserved*** in a ***closed*** system.

If the system is ***not*** closed, (*e.g.* ***external*** forces are present) then E_{tot} and $\vec{p}_{tot}$ will ***not*** {necessarily} be conserved. $\Rightarrow$ Simply expand/enlarge the definition of the "system" until it ***is*** closed {*e.g.* include what's producing the external forces}, then the (new) E_{tot} and $\vec{p}_{tot}$ ***will*** be conserved.

Note the distinction between a ***Lorentz-invariant*** quantity and a ***conserved*** quantity.

Same in ***all*** inertial reference frames — Same before *vs.* after a process/an "event"

Rest mass m is a ***Lorentz-invariant*** quantity, but it is ***not*** {necessarily} a ***conserved*** quantity.

Example: The {unstable} charged pi-meson decays (via weak charged-current interaction, with mean/proper lifetime $\tau_{\pi^+} = 26.0\,ns$) to a muon and muon neutrino: $\pi^+ \to \mu^+ \nu_\mu$. The charged pion ***mass*** m_{π^+} is ***not*** conserved in the decay $\{m_{\pi^+} > (m_{\mu^+} + m_{\nu_\mu})\}$, however the relativistic ***energy*** of the charged pion $E_{\pi^+} = \sqrt{p_{\pi^+}^2 c^2 + m_{\pi^+}^2 c^4}$ ***is*** a ***conserved*** quantity: $E_{\pi^+} = E_{\mu^+} + E_{\nu_\mu}$, but E_{π^+} is ***not*** a ***Lorentz-invariant*** quantity.

Since the scalar product of ***any*** relativistic 4-vector a^μ with itself ***is*** a ***Lorentz-invariant*** quantity (*i.e.* = same numerical value in ***any*** IRF): then ***here***, for $\pi^+ \to \mu^+ \nu_\mu$ decay:

$$\boxed{p_\mu p^\mu = p^\mu p_\mu = -(p^0)^2 + (\vec{p}\bullet\vec{p}) = -(E_{\pi^+}/c)^2 + p_{\pi^+}^2} \quad \text{But: } \boxed{(E_{\pi^+}/c)^2 = p_{\pi^+}^2 + (m_{\pi^+}c)^2}$$

Thus: $$\boxed{p_\mu p^\mu = -\cancel{p_{\pi^+}^2} - (m_{\pi^+}c)^2 + \cancel{p_{\pi^+}^2} = -(m_{\pi^+}c)^2}$$

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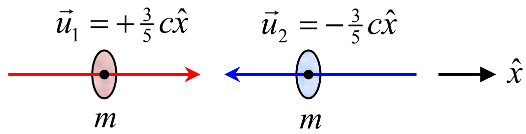
<u>Griffiths Example 12.7</u>: <u>Relativistic Kinematics</u>

Two ***<u>relativistic</u>*** lumps of clay {each of rest mass m} collide head-on with each other. Each lump of clay is traveling at relativistic speed $u = \tfrac{3}{5}c$ as shown in the figure below:



The two relativistic lumps of clay ***<u>stick</u>*** ***<u>together</u>*** (*i.e.* this is an ***<u>inelastic</u>*** collision). What is the ***<u>total</u>*** mass M of the composite lump of clay ***<u>after</u>*** the collision?

<u>Conservation of momentum - before *vs.* after</u>:

Since the two lumps of clay have identical rest masses and equal, but opposite velocities:

$\boxed{\vec{p}_{tot}^{\,before} = \vec{p}_1 + \vec{p}_2}$ but: $\boxed{\vec{p}_1 = -\vec{p}_2 = \gamma_u m\vec{u}_1}$ where: $\boxed{\gamma_u = \dfrac{1}{\sqrt{1-\beta_u^2}}}$ $\therefore$ $\boxed{\vec{p}_{tot}^{\,before} = 0}$

<u>Conservation of energy - before *vs.* after</u>:

<u>Before</u>: ***<u>Each</u>*** lump of clay has total energy: $\boxed{E = \gamma_u mc^2 = \dfrac{mc^2}{\sqrt{1-\beta_u^2}} = \dfrac{mc^2}{\sqrt{1-(u/c)^2}} = \gamma_u mc^2}$

$$\therefore \boxed{E = \frac{mc^2}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{mc^2}{\sqrt{1-\frac{9}{25}}} = \frac{mc^2}{\sqrt{\frac{16}{25}}} = \frac{5}{4}mc^2}$$

Thus: $\boxed{E_{tot}^{before} = E_{tot_1} + E_{tot_2} = 2\gamma_u mc^2 = 2\cdot\frac{5}{4}mc^2 = \frac{5}{2}mc^2}$

However, E_{tot} is {***<u>always</u>***} conserved in a ***<u>closed</u>*** system. $\Rightarrow$ $\boxed{E_{tot}^{after} = E_{tot}^{before} = \frac{5}{2}mc^2}$

And $\vec{p}_{tot}$ is also {***<u>always</u>***} ***<u>separately</u>*** conserved in a ***<u>closed</u>*** system. $\Rightarrow$ $\boxed{\vec{p}_{tot}^{\,after} = \vec{p}_{tot}^{\,before} = 0}$

$\Rightarrow \boxed{\vec{u}^{\,after} = 0}$ since: $\boxed{\vec{p}_{tot}^{\,after} = \gamma_{u_{after}} M\vec{u}^{\,after} = 0}$. *n.b.* $\Rightarrow$ $\boxed{\gamma_{u_{after}} = \dfrac{1}{\sqrt{1-\beta_{u_{after}}^2}} = \dfrac{1}{\sqrt{1-(u_{after}/c)^2}} = 1}$

Then: $\boxed{E_{tot}^{after} = \gamma_{u_{after}} Mc^2 = Mc^2 = \frac{5}{2}mc^2\left(= E_{ToT}^{before}\right)}$ $\therefore$ $\boxed{M = \frac{5}{2}m \neq 2m}$!!! Does this sound ***<u>crazy</u>***??

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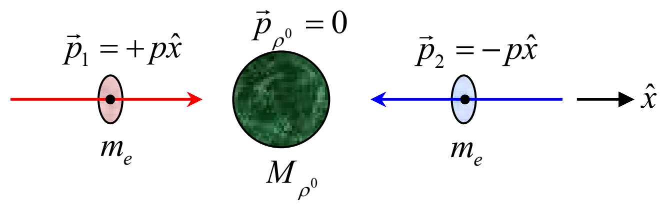
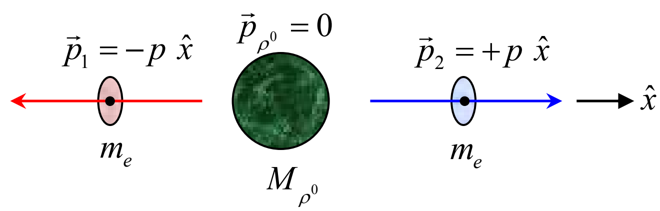
This ***is*** what happens in the "everyday" world of particle physics! It's ***perfectly*** OK !!!

e.g. The production of a neutral rho meson in electron-positron collisions: $\boxed{e^+ + e^- \to \rho^0}$.
The rest mass of the neutral rho meson is: $\boxed{M_{\rho^0} = 770\, MeV/c^2}$ Electron rest mass: $\boxed{m_e = 0.511 MeV/c^2}$



Run the collision process ***backwards*** in time, *e.g.* the ***decay*** of a neutral rho meson: $\boxed{\rho^0 \to e^+ + e^-}$



The production of a neutral rho meson $\boxed{e^+ + e^- \to \rho^0}$ manifestly involves the *EM* interaction. Similarly, the ***time-reversed*** situation: the decay of a neutral rho meson $\boxed{\rho^0 \to e^+ + e^-}$ manifestly ***also*** involves the *EM* interaction.

The *EM* interaction is ***invariant*** under time-reversal, *i.e.* $t \to -t$, thus {in the ***rest*** frame of the neutral rho meson} the ***transition*** rate $\boxed{\Gamma\left(e^+ + e^- \to \rho^0\right)}$ (#/sec) *vs*. the ***decay*** rate $\boxed{\Gamma\left(\rho^0 \to e^+ + e^-\right)}$ (#/sec) are ***identical*** {for the same/identical electron / positron momenta in neutral rho meson production *vs*. decay}. Experimentally: $\boxed{\Gamma\left(\rho^0 \to e^+ + e^-\right) = 7.02 \text{ KeV} = 1.70\times 10^{18} \text{ sec}^{-1}}$.

For our above ***macroscopic*** inelastic collision problem, ***microscopically*** what would the ***new*** matter of the ***macroscopic*** mass M be made up of, since $\boxed{\Delta M = M - 2m = \tfrac{5}{2}m - 2m = \tfrac{1}{2}m}$???

In a ***classical*** analysis of the ***inelastic*** collision of two relativistic ***macroscopic*** lumps of clay {each of mass m} the composite / stuck-together single lump of clay of mass $\boxed{M = \tfrac{5}{2}m > 2m}$ would be very ***hot*** – it would have a ***great deal*** of ***thermal energy*** in fact !!!

$$\boxed{Mc^2 = \frac{5}{2}mc^2 = \underbrace{2mc^2}_{\substack{\text{classical mass}\\ \text{of composite}\\ \text{lump}}} + E_{thermal}} \Rightarrow E_{thermal} = 0.5mc^2\text{!!!} \quad E = mc^2 = \text{Einstein's energy-mass formula}$$

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Conserved Quantities *vs*. ***Lorentz-Invariant*** Quantities in Collisions/Scattering Processes:

Before: $p^{\mu}_{before} = \left(E_{before}/c\,,\vec{p}_{before}\right)$ **After**: $p^{\mu}_{after} = \left(E_{after}/c\,,\vec{p}_{after}\right)$. Neither is a Lorentz invariant quantity. However, total relativistic energy E and total relativistic momentum $\vec{p}$ are ***separately conserved*** quantities: $E_{after} = E_{before} = Mc^2$ and: $\vec{p}_{after} = \vec{p}_{before} = 0$. The scalar 4-vector dot-product is a Lorentz invariant quantity, which is also a conserved quantity – i.e. its value is the ***same*** before *vs*. after the collision/scattering process:

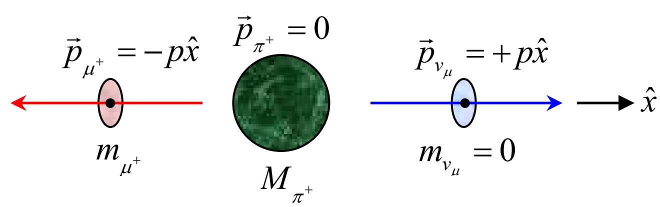
$$p_{\mu}p^{\mu} = p^{\mu}p_{\mu} = -\left(p^0\right)^2 + \left(\vec{p}\bullet\vec{p}\right) = -\left(E/c\right)^2 + p^2 = -M^2c^2 + 0 = -M^2c^2$$

Griffiths Example 12.8: Relativistic Kinematics Associated with $\pi^+ \to \mu^+ \nu_\mu$ Decay.

Pion rest mass: $m_{\pi^+} = 139.57\,MeV/c^2$ Pion mean lifetime: $\tau_{\pi^+} = 26.033\text{ nsec} = 26.033\times10^{-9}\text{ sec}$

Muon rest mass: $m_{\mu^+} = 105.66\,MeV/c^2$ Muon neutrino rest mass: $m_{\nu_\mu} = 0$ (assumed).

In the *rest* frame of the π^+ meson:



Energy Conservation: **Momentum Conservation:**

Before: $E^{before}_{tot} = m_{\pi^+}c^2$ $\vec{p}^{\,before}_{tot} = 0$

After: $E^{after}_{tot} = E_{\mu^+} + E_{\nu_\mu} = m_{\pi^+}c^2$ $\vec{p}^{\,after}_{tot} = \vec{p}_{\mu^+} + \vec{p}_{\nu_\mu} = 0 \Rightarrow \vec{p}_{\mu^+} = -\vec{p}_{\nu_\mu} = -p\hat{x}$

But: $E_{\nu_\mu} = p_{\nu_\mu}c = \left|\vec{p}_{\nu_\mu}\right|c$ since: $m_{\nu_\mu} = 0$. $\Rightarrow p_{\mu^+} = \left|\vec{p}_{\mu^+}\right| = p_{\nu_\mu} = \left|\vec{p}_{\nu_\mu}\right|$

And: $E^2_{\mu^+} = p^2_{\mu^+}c^2 + m^2_{\mu^+}c^4$ or: $p^2_{\mu^+}c^2 = E^2_{\mu^+} - m^2_{\mu^+}c^4 \Rightarrow p_{\mu^+}c = \sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4}$

$\therefore$ $p_{\mu^+} = \left|\vec{p}_{\mu^+}\right| = p_{\nu_\mu} = \left|\vec{p}_{\nu_\mu}\right| = \sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4}\,/c$

Then: $E^{after}_{tot} = E_{\mu^+} + E_{\nu_\mu} = E_{\mu^+} + p_{\nu_\mu}c$ but: $p_{\nu_\mu} = p_{\mu^+} = \dfrac{\sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4}}{c}$

$\therefore$ $E^{after}_{tot} = E_{\mu^+} + E_{\nu_\mu} = E_{\mu^+} + p_{\nu_\mu}c = E_{\mu^+} + \sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4} = E^{before}_{tot} = m_{\pi^+}c^2$

$\therefore$ $E_{\mu^+} + \sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4} = m_{\pi^+}c^2$ Solve for E_{μ^+} :

$E^2_{\mu^+} - m^2_{\mu^+}c^4 = \left(m_{\pi^+}c^2 - E_{\mu^+}\right)^2 = m^2_{\pi^+}c^4 - 2\left(m_{\pi^+}c^2E_{\mu^+}\right) + E^2_{\mu^+}$ or: $2m_{\pi^+}c^2E_{\mu^+} = m^2_{\pi^+}c^4 - m^2_{\mu^+}c^4$

Thus: $E_{\mu^+} = \dfrac{m^2_{\pi^+}c^4 - m^2_{\mu^+}c^4}{2m_{\pi^+}c^2} = \dfrac{\left(m^2_{\pi^+} - m^2_{\mu^+}\right)c^2}{2m_{\pi^+}}$ and: $p_{\nu_\mu} = p_{\mu^+} = \dfrac{\sqrt{E^2_{\mu^+} - m^2_{\mu^+}c^4}}{c}$ with: $\vec{p}_{\mu^+} = -\vec{p}_{\nu_\mu}$

as viewed from the ***rest*** frame of the π^+ meson.

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- In ***classical*** collisions, total ***3-momentum*** $\vec{p}_{tot}$ and total ***mass***, m_{tot} are ***always*** ***conserved***:
  $\boxed{\vec{p}_{tot}^{\,before} = \vec{p}_{tot}^{\,after}}$, $\boxed{m_{tot}^{before} = m_{tot}^{after}}$.

- In ***classical*** collisions, if total ***kinetic*** energy E_{kin}^{tot} is ***not*** conserved $\Rightarrow$ ***inelastic*** collision.

- An ***inelastic*** (*i.e.* a "sticky") collision generates ***heat*** at the expense of ***kinetic*** energy.

- An ***inelastic*** collision of an electron (e^-) with an atom {initially in its ***ground*** state} may leave the atom in an ***excited*** state, or even ***ionized***, kicking out a once-bound atomic electron!
  $\Rightarrow$ Internal {quantum} degrees of freedom can be excited in ***inelastic*** e^- - atom collisions.

- An "***explosive***" collision generates ***kinetic*** energy at the expense of ***chemical*** (*i.e. EM*) energy, or ***nuclear*** (*i.e.* ***strong***-force) energy, or ***weak***-force energy. . . .

- If ***kinetic*** energy ***is*** conserved (***classically***), $\Rightarrow$ ***elastic*** (*i.e.* billiard-ball) collision.

- In ***relativistic*** collisions, total ***3-momentum*** and total ***energy*** are ***always*** conserved (in a ***closed*** system) but total ***mass*** and total ***kinetic*** energy are ***not*** in general conserved.

  * Once again, in ***relativistic*** collisions, a process is called ***elastic*** if the total ***kinetic*** energy ***is*** conserved $\Rightarrow$ total mass is ***also*** conserved in relativistic ***elastic*** collisions.

  * A relativistic collision is called ***inelastic*** if the total kinetic energy is ***not*** conserved.
  $\Rightarrow$ Total mass is ***not*** conserved in a relativistic ***inelastic*** collision.

Griffiths Example 12.9:

Compton Scattering = Relativistic *Elastic* Scattering of Photons with Electrons.

An incident photon of energy $E_\gamma^0 = p_\gamma^0 c$ ***elastically*** scatters (*i.e.* "bounces" off of/recoils) from an electron, which is initially at ***rest*** in the ***lab*** frame. Determine the final energy E_γ of the outgoing ***scattered*** photon as a function of the scattering angle θ of the photon:

Consider conservation of relativistic momentum in the ***transverse*** ($\perp$) (*i.e.* $\hat{y}$ -axis) direction:

$$\boxed{p_{\perp_{tot}}^{before} = 0 = p_{\perp_{tot}}^{after}}$$

$$\boxed{p_{\perp_{tot}}^{before} = p_{\perp_\gamma}^{before} + p_{\perp_{e^-}}^{before} = 0 + 0 = 0}$$

$$\boxed{p_{\perp_{tot}}^{after} = p_{\perp_\gamma}^{after} + p_{\perp_{e^-}}^{after} = 0} \Rightarrow \boxed{\underbrace{p_{\perp_\gamma}^{after}}_{+\hat{y}\text{ direction}} = -\underbrace{p_{\perp_{e^-}}^{after}}_{-\hat{y}\text{ direction}}}$$

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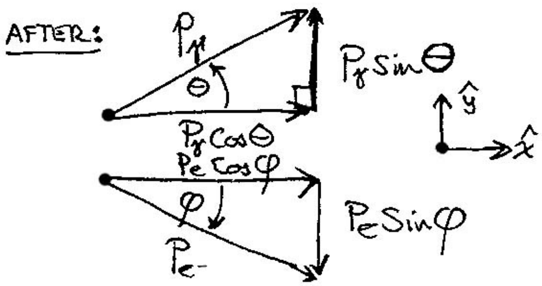

Since: $\overbrace{p_{\perp_\gamma}^{after}}^{+\hat{y}\text{ direction}} = -\overbrace{p_{\perp_{e^-}}^{after}}^{-\hat{y}\text{ direction}}$

Or: $\left|p_{\perp_\gamma}^{after}\right| = \left|p_{\perp_{e^-}}^{after}\right|$

Or: $p_\gamma \sin\theta = p_{e^-} \sin\varphi$

But: $p_\gamma = E_\gamma / c$

$\therefore$ $\dfrac{E_\gamma}{c} \sin\theta = p_{e^-} \sin\varphi$

Solve for $\sin\varphi$: $\sin\varphi = \left(\dfrac{E_\gamma}{p_{e^-} c}\right) \sin\theta$

Conservation of relativistic momentum in the ***longitudinal*** (*i.e.* $\hat{x}$) direction gives:

$p_{\|_{tot}}^{before} = \dfrac{E_\gamma^0}{c}$ (*n.b.* $p_{e^-}^{before} = 0$, since e^- initially at rest, hence $p_{\|e^-}^{before} = 0$)

$$p_{\|_{tot}}^{after} = p_{\|_\gamma}^{after} + p_{\|_{e^-}}^{after} = p_\gamma \cos\theta + p_{e^-} \cos\varphi$$

Since: $p_{\|_{TOT}}^{before} = p_{\|_{TOT}}^{after}$ then: $E_\gamma^0 / c = p_\gamma \cos\theta + p_{e^-} \cos\varphi$

But: $\sin\varphi = \left(\dfrac{E_\gamma}{p_{e^-} c}\right) \cos\theta$ thus: $\cos\varphi = \sqrt{1-\sin^2\varphi} = \sqrt{1-\left(\dfrac{\mathrm{E}_\gamma}{\mathrm{P}_{e^-} c}\right)^2 \sin^2\theta}$

$\therefore$ $$\frac{E_\gamma^0}{c} = p_\gamma \cos\theta + p_{e^-} \sqrt{1-\left(\frac{E_\gamma}{p_{e^-} c}\right)^2 \sin^2\theta}$$

Or: $$p_{e^-}^2 c^2 = \left(E_\gamma^0 - E_\gamma \cos\theta\right)^2 + E_\gamma^2 \sin^2\theta = E_\gamma^{0^2} - 2E_\gamma^0 E_\gamma \cos\theta + E_\gamma^2$$

Conservation of Energy: $E_{tot}^{before} = E_{tot}^{after} \Rightarrow \overbrace{E_\gamma^0 + m_e c^2}^{E_{tot}^{before}} = \overbrace{E_\gamma + E_{e^-}}^{E_{tot}^{after}} = E_\gamma + \sqrt{p_e^2 c^2 + m_e^2 c^4}$

$\therefore$ $$E_\gamma^0 + m_e c^2 = E_\gamma + \sqrt{E_\gamma^{0^2} - 2E_\gamma^0 E_\gamma \cos\theta + E_\gamma^2 + m_e^2 c^4}$$

Solve for E_γ (after some algebra): $$E_\gamma = \frac{1}{\left[\left(1-\cos\theta\right)/m_e c^2 + 1/E_\gamma^0\right]}$$

E_γ = energy of recoil photon in terms of initial photon energy E_γ^0, scattering angle of photon θ and rest energy/mass of electron, $m_e c^2$.

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We can alternatively express this relation in terms of photon ***wavelengths***:

Before: $E_\gamma^0 = hf_\gamma^0 = hc/\lambda_\gamma^0$

After: $E_\gamma = hf_\gamma = hc/\lambda_\gamma$

Get: $$\lambda_\gamma = \lambda_0 + \left(\frac{hc}{m_e c^2}\right)(1-\cos\theta)$$

Useful constants:

$hc = 1239.841\, eV\text{-}nm \simeq 1240\, eV\text{-}nm$

$m_e c^2 \simeq 0.511\, MeV = 0.511\times 10^6\ eV$

Define the so-called ***Compton wavelength of the electron***: $\lambda_e \equiv \left(\frac{hc}{m_e c^2}\right) = 2.426\times 10^{-12}\, m$

Then: $$\lambda_\gamma = \lambda_0 + \lambda_e(1-\cos\theta)$$

The Compton Differential Scattering Cross Section:

As we learned in P436 Lecture Notes 14.5 (*p*. 9-22) ***non-relativistic*** photon-free electron scattering $\left(E_\gamma^0 \ll m_e c^2\right)$ is adequately described by the ***classical*** *EM* physics-derived {unpolarized} differential Thomson scattering cross section:

$$\frac{d\sigma_{T_{e^-}}^{unpol}(\theta,\varphi)}{d\Omega} \simeq \frac{1}{2} r_e^2 \left(1+\cos^2\theta\right) \quad \text{where: } r_e \equiv \frac{e^2}{4\pi\varepsilon_o m_e c^2} \simeq 2.82\times 10^{-15}\, m$$

Classical electron radius

However, when $E_\gamma^0 \geq m_e c^2$ from the above discussion of the ***relativistic*** kinematics of photon-free electron scattering, it is obvious that the ***classical*** theory is ***not*** valid in this regime. The fully-relativistic quantum mechanical theory – that of quantum electrodynamics (*QED*) – is required to get it right... Without going into the gory details, the results of the *QED* calculation associated with the two Feynman graphs {the so-called *s*- and *u*-channel diagrams} shown on *p*. 5 of P436 Lect. Notes 14.5 for the Compton differential scattering cross section – known as the ***Klein-Nishina formula*** for ***relativistic*** {unpolarized} photon-free electron scattering is:

$$\frac{d\sigma_{C_{e^-}}^{unpol}(\theta,\varphi)}{d\Omega} = \frac{1}{2} r_e^2 \left(1+\cos^2\theta\right) \frac{1}{\left[1+x_\gamma(1-\cos\theta)\right]^2}\left[1+\frac{x_\gamma^2(1-\cos\theta)^2}{\left(1+\cos^2\theta\right)\left[1+x_\gamma(1-\cos\theta)\right]}\right]$$

where: $x_\gamma \equiv E_\gamma^0/m_e c^2 = hf_\gamma^0/m_e c^2$. In the ***non-relativistic*** limit $\left(x_\gamma \to 0\right)$, the ***relativistic*** Compton scattering cross section agrees with the ***classical*** Thomson scattering cross section, as shown in the figure below of the ***normalized*** differential scattering cross section $\frac{1}{r_e^2} d\sigma_{C_{e^-}}^{unpol}(\theta)/d\cos\theta\ \ vs.\ \theta$.

Note that as $x_\gamma \to \infty$ the ***relativistic*** Compton differential scattering cross section becomes increasingly sharply peaked in the forward direction, $\theta \to 0$.

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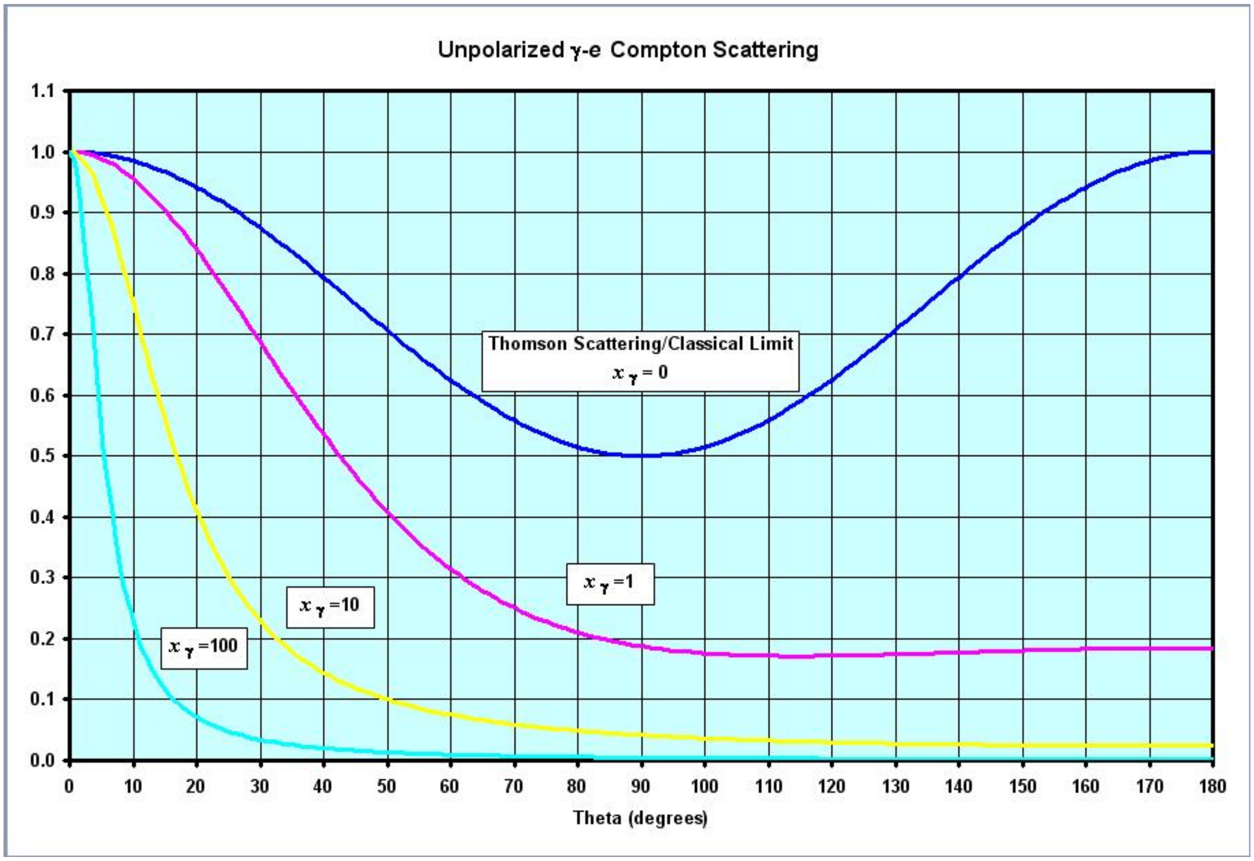
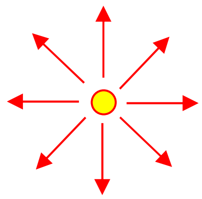


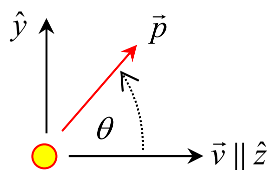
The Relativistic Doppler Shift – for Photons/Light:

A rapidly moving atom isotropically emits monochromatic light (photons of frequency f') in its own ***rest*** frame IRF′. What is the frequency f of the emitted photons as observed in the ***lab*** frame IRF as a function of the ***lab*** angle θ between the atom's velocity $\vec{v} \parallel +\hat{z}$ and the direction of observation $\{ =$ photon's momentum vector $\vec{p}\,\}$ in the ***lab*** frame?

Rest frame of atom, IRF′:



Lab frame, IRF:



Without any loss of generality, we can choose the ***lab*** velocity $\vec{v}$ of the atom to be along the $+\hat{z}$ axis in the ***lab*** frame IRF {note that: $\hat{z}' \parallel \hat{z} \parallel \vec{v}$ }.

The energy of the photon in the atom's ***rest*** frame IRF′ is: $\boxed{E' = p'c = hf'}$ where: $\boxed{p' = |\vec{p}'| = \frac{hf'}{c}}$ is the magnitude of the photon's momentum in the atom's ***rest*** frame IRF′. We can ***also*** assume without loss of generality that the emitted photon's momentum vector $\vec{p}'$ lies in the y'-z' plane of the atom's ***rest*** frame IRF′. In the atom's ***rest*** frame IRF′, the emitted photon makes an angle θ' with respect to the $+\hat{z}'$ axis.

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Hence: $\boxed{p'_z = p'\cos\theta' = (E'/c)\cos\theta'}$ and: $\boxed{p'_y = p'\sin\theta' = (E'/c)\sin\theta'}$ and: $\boxed{p'_x = 0}$.

The 4-momentum vector of the emitted photon in the atom's ***rest*** frame IRF′ is thus:

$$\boxed{p'_\mu = (E'/c, p'_x, p'_y, p'_z) = \left(\tfrac{hf'}{c}, 0, \tfrac{hf'}{c}\sin\theta', \tfrac{hf'}{c}\cos\theta'\right) = \tfrac{hf'}{c}(1, 0, \sin\theta', \cos\theta')}$$

We then carry out a 1-D Lorentz transformation from the atom's ***rest*** frame IRF′ to the ***lab*** frame IRF, boosted along the $+\hat{z}'\{\|\hat{z}\|\vec{v}\}$ axis (see *e.g.* Physics 436 Lect. Notes 16, *p*. 11), where:

$\boxed{\beta \equiv v/c}$ and: $\boxed{\gamma \equiv 1/\sqrt{1-\beta^2}}$:

$$\boxed{\begin{aligned} p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \Lambda'^\mu{}_\nu p'^\nu = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \frac{hf'}{c} \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \sin\theta' \\ \cos\theta' \end{pmatrix} \\ = \frac{hf'}{c} \begin{pmatrix} \gamma + \gamma\beta\cos\theta' \\ 0 \\ \sin\theta' \\ \gamma\beta + \gamma\cos\theta' \end{pmatrix} = \frac{hf'}{c} \begin{pmatrix} \gamma(1+\beta\cos\theta') \\ 0 \\ \sin\theta' \\ \gamma(\beta+\cos\theta') \end{pmatrix} \end{aligned}}$$

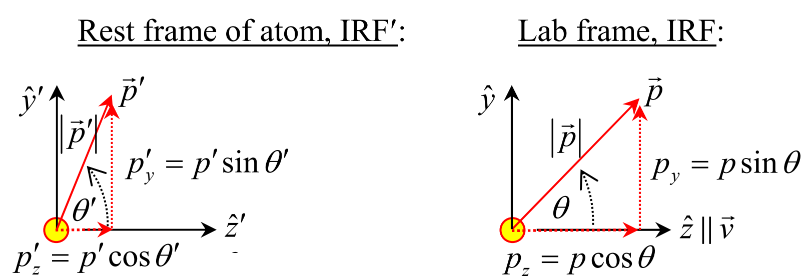
Thus, in the ***lab*** IRF, the emitted photon's 4-momentum vector is:

$$\boxed{p_\mu = (E/c, p_x, p_y, p_z) = \frac{hf'}{c}\left(\gamma(1+\beta\cos\theta'), 0, \sin\theta', \gamma(\beta+\cos\theta')\right)}$$

The emitted photon's energy as observed in the ***lab*** IRF is: $\boxed{E = hf = \gamma hf'(1+\beta\cos\theta')}$.

The frequency of the emitted photon observed in the ***lab*** IRF is: $\boxed{f = \gamma f'(1+\beta\cos\theta')}$.

Experimentally, the atom's ***rest*** frame photon emission angle θ' is {often} ***not*** measureable; the ***lab*** frame photon emission angle θ ***is*** what is measured experimentally. Hence, in order for this formula to be ***useful***, we must re-write this expression in terms of the ***lab*** frame photon emission angle θ. The relationship between the atom's ***rest*** frame photon emission angle θ' and the ***lab*** frame photon emission angle θ can be obtained by analyzing the 3-momentum components of the photon in the atom's ***rest*** frame $\vec{p}'$ *vs*. the ***lab*** frame $\vec{p}$, as shown in the figures below:



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In the ***lab*** frame IRF, the 4-vector momentum components of the emitted photon are:

$$\begin{aligned} E &= hf = \gamma h f'(1+\beta\cos\theta') = \gamma E'(1+\beta\cos\theta') \\ p_x &= 0 = p'_x \\ p_y &= p\sin\theta = p'\sin\theta' = p'_y \\ p_z &= p\cos\theta = \gamma p'(\beta+\cos\theta') \end{aligned}$$

We see that: $p_z = p\cos\theta = \gamma p'(\beta+\cos\theta')$. But for photons: $p = E/c$ and: $p' = E'/c$

Thus: $p_z = (E/c)\cos\theta = \gamma(E'/c)(\beta+\cos\theta')$. But: $E = \gamma E'(1+\beta\cos\theta')$

Hence: $p_z = (\gamma E'(1+\beta\cos\theta')/c)\cos\theta = \gamma(E'/c)(\beta+\cos\theta')$

Or: $\cancel{\gamma E'}(1+\beta\cos\theta')\cos\theta = \cancel{\gamma E'}(\beta+\cos\theta') \Rightarrow (1+\beta\cos\theta')\cos\theta = (\beta+\cos\theta')$.

Thus: $$\cos\theta = \frac{\cos\theta'+\beta}{1+\beta\cos\theta'} = \frac{\beta+\cos\theta'}{1+\beta\cos\theta'}.$$

However, we need an expression for $\cos\theta'$ in terms of $\cos\theta$. Solve for $\cos\theta'$:

$$(1+\beta\cos\theta')\cos\theta = (\beta+\cos\theta') \Rightarrow \cos\theta + \beta\cos\theta'\cos\theta = \beta + \cos\theta'$$

$$\Rightarrow \beta\cos\theta'\cos\theta - \cos\theta' = \beta - \cos\theta \Rightarrow (\beta\cos\theta - 1)\cos\theta' = \beta - \cos\theta$$

$$\Rightarrow \cos\theta' = \frac{\beta-\cos\theta}{\beta\cos\theta - 1} = \frac{\cos\theta-\beta}{1-\beta\cos\theta}$$

Thus: $$f = \gamma f'(1+\beta\cos\theta') = \gamma f'\left(1+\beta\left[\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right]\right)$$ where: $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$ and: $\beta \equiv \dfrac{v}{c}$

Or: $$f' = \frac{1}{\gamma}\left(1+\beta\left[\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right]\right)^{-1} f = \frac{1}{\gamma}\left(\frac{1-\beta\cos\theta}{1-\beta^2}\right) f = \gamma(1-\beta\cos\theta) f \Leftarrow$$

n.b. RHS expressed entirely in ***lab*** frame IRF variables – *i.e.* experimentally measured quantities

Similarly, we can ***also*** obtain a relation for $\sin\theta$ using:

$p_y = p\sin\theta = p'\sin\theta' = p'_y$. But for photons: $p = E/c$ and: $p' = E'/c$

Thus: $p_y = (E/c)\sin\theta = (E'/c)\sin\theta'$ But: $E = \gamma E'(1+\beta\cos\theta')$

Hence: $p_y = \gamma\cancel{(E'/c)}(1+\beta\cos\theta')\sin\theta = \cancel{(E'/c)}\sin\theta' \Rightarrow \gamma(1+\beta\cos\theta')\sin\theta = \sin\theta'$

Thus: $$\sin\theta = \frac{\sin\theta'}{\gamma(1+\beta\cos\theta')}$$

And: $$\cos\theta = \frac{\cos\theta'+\beta}{(1+\beta\cos\theta')}$$

Hence: $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta'}{\gamma(\cos\theta'+\beta)}$$

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Since: $\boxed{\cos\theta' = \dfrac{\cos\theta - \beta}{1-\beta\cos\theta}}$

Then:

$$\sin\theta = \frac{\sin\theta'}{\gamma(1+\beta\cos\theta')} = \frac{\sin\theta'}{\gamma\left(1+\beta\left(\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\right)\right)} = \frac{\sin\theta'}{\gamma\left(\dfrac{1-\cancel{\beta\cos\theta}+\cancel{\beta\cos\theta}-\beta^2}{1-\beta\cos\theta}\right)}$$

$$= \frac{\sin\theta'}{\gamma\left(\dfrac{1-\beta^2}{1-\beta\cos\theta}\right)} = \frac{\gamma^{\cancel{2}}(1-\beta\cos\theta)\sin\theta'}{\cancel{\gamma}} = \gamma(1-\beta\cos\theta)\sin\theta'$$

Thus: $\boxed{\sin\theta' = \dfrac{\sin\theta}{\gamma(1-\beta\cos\theta)}}$

And: $\boxed{\cos\theta' = \dfrac{\cos\theta-\beta}{(1-\beta\cos\theta)}}$

Hence: $\boxed{\tan\theta' = \dfrac{\sin\theta'}{\cos\theta'} = \dfrac{\sin\theta}{\gamma(\cos\theta-\beta)}}$

Comments:

From: $\boxed{\cos\theta = \dfrac{\cos\theta'+\beta}{(1+\beta\cos\theta')}}$ and: $\boxed{f = \gamma f'(1+\beta\cos\theta') = \gamma f'\left(1+\beta\left[\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\right]\right)}$

1.) The photon will be emitted in the ***forward*** direction in the ***lab*** frame IRF $\{0° \le \theta \le 90°\}$ when the numerator: $\boxed{\cos\theta'+\beta \ge 0}$, *i.e.* when: $\boxed{\beta \ge -\cos\theta'}$ $\{n.b.\ \cos\theta' < 0 \text{ for } 90° < \theta' \le 180°\}$.

2.) The photon will be emitted in the ***backward*** direction in the ***lab*** frame IRF $\{90° < \theta \le 180°\}$ when the numerator: $\boxed{\cos\theta'+\beta < 0}$, *i.e.* when: $\boxed{\beta < -\cos\theta'}$.

3.) When $\beta \to 1\,(v \to c)$, ***all*** photons are emitted in the ***forward*** direction in the ***lab*** frame IRF.

4.) When: $\theta' = 0°$, then: $\theta = 0° = \theta'$, and: $\boxed{f = \gamma f'(1+\beta)}$.

5.) When: $\theta' = 90°$, then: $\boxed{f = \gamma f'}$. When: $\theta = 90°$, then: $\boxed{f = \gamma f'(1-\beta^2) = f'/\gamma}$. $\Leftarrow$ *n.b.* so-called ***transverse*** Doppler shift(s)

6.) When: $\theta' = 180°$, then: $\theta = 180° = \theta'$, and: $\boxed{f = \gamma f'(1-\beta)}$.

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Special Relativity and Stellar Luminosity:

In its own ***rest*** frame IRF′, a star ***isotropically*** radiates {thermal/black-body} photons. For simplicity's sake {***here***}, we will assume that ***all*** radiated photons in the star's ***rest*** frame IRF′ have the ***same*** energy E'. The ***total rate*** of emission of such photons into 4π steradians in the ***rest*** frame IRF′ of the star {assumed to be constant} is: $\boxed{R' = dN/dt' \; (\#/sec)}$. Note that in the star's ***rest*** frame IRF′, the temporal interval $dt' = d\tau' =$ the ***proper*** time interval. The ***total luminosity*** of the star in its ***rest*** frame IRF′ is: $\boxed{L' = E'R' = E'(dN/dt') \; (Joules/sec = Watts)}$.

The ***differential rate*** of emission of such photons into solid angle element $\boxed{d\Omega' = d\cos\theta' d\varphi'}$ in the star's ***rest*** frame IRF′ is:

$$\boxed{\frac{dR'}{d\Omega'} = \frac{d}{d\Omega'}\left(\frac{dN}{dt'}\right) = \frac{d(dN/dt')}{d\Omega'} \; (\#/sec/sr)}$$

The ***differential luminosity*** of the star as measured in its ***rest*** frame IRF′ is:

$$\boxed{\frac{dL'}{d\Omega'} = \frac{d(E'R')}{d\Omega'} = E'\frac{dR'}{d\Omega'} = E'\frac{d}{d\Omega'}\left(\frac{dN}{dt'}\right) = E'\frac{d(dN/dt')}{d\Omega'} \; (Joules/sec/sr = Watts/sr)}$$

Suppose that ***you*** are an astronomer on earth, observing this star through a telescope. ***Your*** inertial reference frame is the ***lab*** frame IRF. {For simplicity's sake ***here***, we neglect/ignore the motion of the earth}. If the star is moving with velocity $\vec{v}\,\{\parallel \hat{z} \parallel \hat{z}'\}$, then what is the ***differential*** luminosity $dL/d\Omega$ in the earth's ***lab*** frame IRF – *i.e.* how is $dL/d\Omega$ related to $dL'/d\Omega'$?

There are ***three*** inertial reference frame effects that must be taken into account ***here***:

- Time dilation: $dt \neq dt'$ ← Rate of emission in ***rest*** frame IRF′ ≠ rate of emission in ***lab*** frame IRF
- Angle transformation: $d\Omega \neq d\Omega'$
- Doppler effect: $E \neq E'$

(Angle transformation and Doppler effect:) ← Lorentz transformation from ***rest*** frame IRF′ of star to ***lab*** frame IRF.

The time dilation effect is: $\boxed{dt = \gamma dt'}$ or: $\boxed{\frac{dt'}{dt} = \frac{1}{\gamma}}$, where: $\boxed{\gamma \equiv 1/\sqrt{1-\beta^2}}$ and: $\boxed{\beta \equiv v/c}$.

The 1-D Lorentz transformation from the star's ***rest*** frame IRF′ to the ***lab*** frame IRF on earth, for $\vec{v} \parallel \hat{z}\,\{\parallel \hat{z}'\}$ is the same as that for the above relativistic Doppler shift example:

The 4-momentum vector associated with a photon emitted from the surface of the star in the ***rest*** frame IRF′ of the star is: $\boxed{p'_\mu = (E'/c, p'_x, p'_y, p'_z) = \left(\tfrac{E'}{c}, 0, \tfrac{E'}{c}\sin\theta', \tfrac{E'}{c}\cos\theta'\right) = \tfrac{E'}{c}(1, 0, \sin\theta', \cos\theta')}$

The Lorentz transformation is:

$$\boxed{p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \Lambda'^{\mu}{}_{\nu} p'^{\nu} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \frac{E'}{c}\begin{pmatrix} \gamma(1+\beta\cos\theta') \\ 0 \\ \sin\theta' \\ \gamma(\beta+\cos\theta') \end{pmatrix}}$$

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Thus, the photon's 4-momentum vector as seen by an astronomer in the ***lab*** frame IRF is:

$$\boxed{p_\mu = \left(E/c, p_x, p_y, p_z\right) = \frac{E'}{c}\left(\gamma\left(1+\beta\cos\theta'\right), 0, \sin\theta', \gamma\left(\beta+\cos\theta'\right)\right)}$$

Here {again}, we need to express this result in terms of ***lab*** frame IRF measured variables {only}:

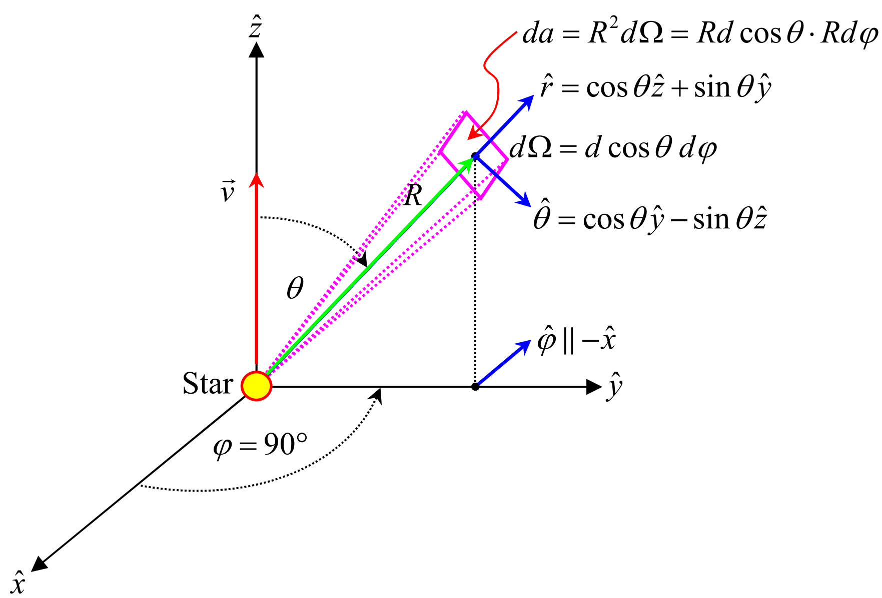
$$\boxed{\cos\theta' = \frac{\cos\theta-\beta}{\left(1-\beta\cos\theta\right)}},\quad \boxed{\sin\theta' = \frac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)}} \text{ and: } \boxed{E' = \gamma\left(1-\beta\cos\theta\right)E}$$

Now, the ***lab*** frame IRF *vs*. the star's ***rest*** frame IRF′ ***solid angle elements*** are, respectively:

$$\boxed{d\Omega = d\cos\theta\, d\varphi} \text{ and: } \boxed{d\Omega' = d\cos\theta'\, d\varphi'}.$$

The infinitesimal ***solid angle element*** $d\Omega = d\cos\theta\, d\varphi$ and the infinitesimal ***area element*** $da = R^2 d\Omega = Rd\cos\theta \cdot Rd\varphi$ {where the infinitesimal θ and φ ***arc lengths*** $S_\theta = Rd\cos\theta$ and $S_\varphi = Rd\varphi$, respectively} associated with the ***lab*** frame IRF are shown in the figure below.

As per the above discussion of the relativistic Doppler shift, for $\vec{v} \parallel \hat{z}\{\parallel \hat{z}'\}$, without any loss of generality we can choose the observation point $P\left(\vec{r} = R\hat{r}\right)$ to lie in the *y-z* plane ($\varphi = 90°$):



For the choice of observation point $P\left(\vec{r} = R\hat{r}\right)$ lying in the *y-z* plane ($\varphi = 90°$), note that $\hat{\varphi} \parallel -\hat{x}$ is $\perp$ to $\hat{r} = \cos\theta\hat{z} + \sin\theta\hat{y}$. Thus, since ***transverse*** components of a 4-vector are ***unaffected*** by a Lorentz transformation *e.g.* from the ***rest*** frame IRF′ to the ***lab*** frame IRF, then $\hat{\varphi} = \hat{\varphi}'$, hence $\varphi = \varphi'$ and $d\varphi = d\varphi'$. Thus, in order to determine the relationship between solid angle element $d\Omega = d\cos\theta\, d\varphi$ and $d\Omega' = d\cos\theta'\, d\varphi'$, we only need to determine how $d\cos\theta$ is related to $d\cos\theta'$. From above, we already have the relation:

$$\boxed{\cos\theta' = \frac{\cos\theta-\beta}{\left(1-\beta\cos\theta\right)}}$$

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Thus, using the chain rule of differentiation:

$$\frac{d\cos\theta'}{d\cos\theta} = \frac{d}{d\cos\theta}\left\{\frac{\cos\theta-\beta}{(1-\beta\cos\theta)}\right\} = \frac{1}{(1-\beta\cos\theta)} - \frac{-\beta(\cos\theta-\beta)}{(1-\beta\cos\theta)^2}$$
$$= \frac{1-\cancel{\beta\cos\theta}+\cancel{\beta\cos\theta}-\beta^2}{(1-\beta\cos\theta)^2} = \frac{1-\beta^2}{(1-\beta\cos\theta)^2} = \frac{1}{\gamma^2}\frac{1}{(1-\beta\cos\theta)^2}$$

Hence:

$$\frac{d\Omega'}{d\Omega} = \frac{d\cos\theta'}{d\cos\theta}\cdot\underbrace{\frac{d\varphi'}{d\varphi}}_{=1} = \frac{d\cos\theta'}{d\cos\theta} = \frac{1}{\gamma^2}\frac{1}{(1-\beta\cos\theta)^2}$$

We also already have the relationship between E and E' from the Lorentz transformation result:

$$E' = \gamma(1-\beta\cos\theta)E \quad \text{or:} \quad \frac{E}{E'} = \frac{1}{\gamma(1-\beta\cos\theta)}$$

The ***lab*** frame *vs*. ***rest*** frame ***differential luminosity*** of the star are related to each other by:

$$\frac{dL}{d\Omega} = E\frac{dR}{d\Omega} = E\frac{d}{d\Omega}\left(\frac{dN}{dt}\right) = \left(\frac{dt'}{dt}\right)\left(\frac{d\Omega'}{d\Omega}\right)\left(\frac{E}{E'}\right)E'\frac{d}{d\Omega'}\left(\frac{dN}{dt'}\right)$$
$$= \left(\frac{dt'}{dt}\right)\left(\frac{d\Omega'}{d\Omega}\right)\left(\frac{E}{E'}\right)E'\frac{dR'}{d\Omega'} = \left(\frac{dt'}{dt}\right)\left(\frac{d\Omega'}{d\Omega}\right)\left(\frac{E}{E'}\right)\frac{dL'}{d\Omega'}\left(\frac{Watts}{sr}\right)$$

Thus:

$$\frac{dL}{d\Omega} = \left(\frac{dt'}{dt}\right)\left(\frac{d\Omega'}{d\Omega}\right)\left(\frac{E}{E'}\right)\frac{dL'}{d\Omega'} = \left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma^2}\frac{1}{(1-\beta\cos\theta)^2}\right)\left(\frac{1}{\gamma(1-\beta\cos\theta)}\right)\frac{dL'}{d\Omega'}\left(\frac{Watts}{sr}\right)$$

Or:

$$\frac{dL}{d\Omega} = \frac{1}{\gamma^4}\frac{1}{(1-\beta\cos\theta)^3}\frac{dL'}{d\Omega'}\left(\frac{Watts}{sr}\right) \Leftarrow$$

n.b. peaks sharply in $\theta = 0$ (forward) direction, and $\to 0$ in $\theta = \pi$ (backward) direction as $\beta \to 1$ ($\gamma \to \infty$).

The ***differential luminosity*** of the star in its own ***rest*** frame IRF′ is thus:

$$\frac{dL'}{d\Omega'} = \gamma^4(1-\beta\cos\theta)^3\frac{dL}{d\Omega}\left(\frac{Watts}{sr}\right) \Leftarrow$$

n.b. RHS expressed entirely in ***lab*** frame IRF variables – *i.e.* experimentally measured quantities

The ***total*** luminosity of the star in its own ***rest*** frame IRF′ is: $L' = \int \frac{dL'}{d\Omega'}d\Omega' = 4\pi\frac{dL'}{d\Omega'}\ (Watts)$

since the emission of photons in the star's own ***rest*** frame is ***isotropic***.

Hence the ***total*** luminosity of the star in its own ***rest*** frame IRF′ is:

$$L' = 4\pi\frac{dL'}{d\Omega'} = 4\pi\gamma^4(1-\beta\cos\theta)^3\frac{dL}{d\Omega}\ (Watts) \Leftarrow$$

n.b. RHS expressed entirely in ***lab*** frame IRF variables – *i.e.* experimentally measured quantities

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Relativistic Dynamics

Newton's 1st Law of Motion: "An object at rest remains at rest, an object moving with speed v continues to move at speed v, unless acted upon by a net/non-zero/unbalanced force"
– the Law of Inertia – is built/incorporated into in the **Principle of Relativity**.

Newton's 2nd law of motion {***classical*** mechanics} retains its validity in ***relativistic*** mechanics, provided that ***relativistic*** momentum is used:

$$\vec{F}(\vec{r},t) = \frac{d\vec{p}(\vec{r},t)}{dt} \quad (= m\vec{a}(\vec{r},t))$$

Griffiths Example 12.10: 1-D Relativistic Motion Under a *Constant* Force.

A particle of (rest) mass m is subject to a ***constant*** force: $\vec{F}(\vec{r},t) = \vec{F} = F\hat{x} = constant$ vector.

If the particle starts from ***rest*** at the origin at time $t = 0$, find its position $x(t)$ as a function of t.

Since the relativistic motion is 1-D, then: $F = \frac{dp(t)}{dt}$ = constant, or: $\frac{dp(t)}{dt} = F$ = constant.

$\Rightarrow$ $p(t) = Ft$ + constant of integration. The particle starts from ***rest*** at $t = 0$. $\therefore$ $p(t=0) = 0$

$\Rightarrow$ constant of integration = 0. $\therefore$ $p(t) = Ft$ {***here***}

Relativistically: $p(t) = \gamma_u(t)\, m u(t) = \frac{mu(t)}{\sqrt{1-(u(t)/c)^2}} = Ft$ where: $\gamma_u(t) = \frac{1}{\sqrt{1-(u(t)/c)^2}}$

Solve for $u(t)$: $m^2u^2 = F^2t^2\left(1-(u^2/c^2)\right) = F^2t^2 - (F^2t^2/c^2)u^2$ $\Rightarrow$ $\left(m^2 + (F^2t^2/c^2)\right)u^2 = F^2t^2$

Or: $u^2 = \frac{F^2t^2}{m^2 + (F^2t^2/c^2)} = \frac{(Ft/m)^2}{1+(Ft/mc)^2}$ $\Rightarrow$ $u(t) = \frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}$ = Relativistic particle velocity for ***constant*** applied force $\vec{F}$

n.b. when: $(Ft/mc) \ll 1$ *i.e.* $(Ft/m) \ll c$ then: $u(t) \simeq Ft/m$ $\Leftarrow$ Classical dynamics answer.

Note also that as $t \to \infty$: $u(t\to\infty) \to c$!!! (Relativistic denominator ***ensures*** this!)

Since: $u(t) = \frac{Ft/m}{\sqrt{1+(Ft/mc)^2}} = \frac{dx(t)}{dt}$ Then: $x(t) = \int_0^t u(t')\,dt' = (F/m)\int_0^t \frac{t'}{\sqrt{1+(F/mc)^2 t'^2}}\,dt'$

The motion is ***hyperbolic***: $x(t) = \left(\frac{F}{m}\right)\left(\frac{mc}{F}\right)^2 \sqrt{1+(F/mc)^2 t'^2}\,\Big|_0^t = \left(\frac{mc^2}{F}\right)\left[\sqrt{1+(F/mc)^2 t^2} - 1\right]$

n.b. Had we done this in ***classical*** dynamics, the result would have been ***parabolic*** motion:

$$x(t) = \left(\frac{F}{2m}\right)t^2$$

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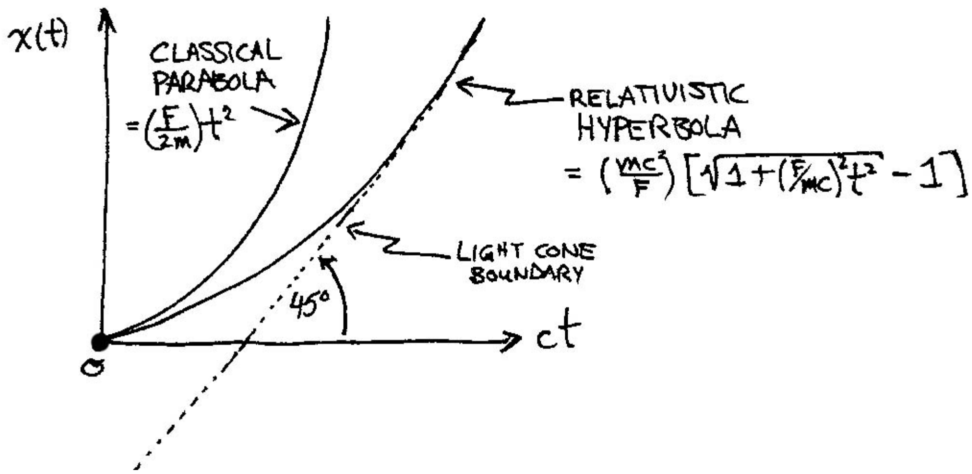
Thus, in ***relativistic*** dynamics – *e.g.* a charged particle placed in a ***uniform*** electric field $\vec{E}$, the resulting motion under a constant force $\boxed{\vec{F} = q\vec{E}}$ is ***hyperbolic*** motion (not ***parabolic*** motion, as in ***classical*** dynamics) – see/compare two cases, as shown in figure below:



Relativistic Work:

Relativistic work is defined the same as ***classical*** work: $\boxed{W \equiv \int \vec{F}\bullet d\vec{\ell}}$

The **Work-Energy** **Theorem** (the net work done ***on*** a particle = ***increase*** in particle's kinetic energy) also holds relativistically:

$$\boxed{W = \int \vec{F}\bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt}\bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt}\bullet\frac{d\vec{\ell}}{dt}dt = \int \frac{d\vec{p}}{dt}\bullet\vec{u}\,dt = \Delta E_{kin}} \quad \text{since: } \boxed{\vec{u} = \frac{d\vec{\ell}}{dt}}$$

But: $$\boxed{\left(\frac{d\vec{p}}{dt}\bullet\vec{u}\right) = \frac{d}{dt}\frac{m\vec{u}}{\sqrt{1-(u/c)^2}}\bullet\vec{u}} \quad \text{since: } \boxed{\vec{p} = \gamma_u m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\beta_u^2}} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}}$$

Thus:

$$\begin{aligned}
\left(\frac{d\vec{p}}{dt}\bullet\vec{u}\right) &= \frac{m}{\sqrt{1-(u/c)^2}}\left(\frac{d\vec{u}}{dt}\right)\bullet\vec{u} + \frac{\left(m\vec{u}/c^2\right)u}{\left[1-(u/c)^2\right]^{3/2}}\left(\frac{du}{dt}\right)\bullet\vec{u} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}\bullet\frac{d\vec{u}}{dt} + \frac{m\left(u/c\right)^2\vec{u}}{\left[1-(u/c)^2\right]^{3/2}}\bullet\frac{d\vec{u}}{dt} \\
&= \left\{\frac{1}{\sqrt{1-(u/c)^2}} + \frac{(u/c)^2}{\left[1-(u/c)^2\right]^{3/2}}\right\}m\vec{u}\bullet\frac{d\vec{u}}{dt} = \left\{\frac{1-\cancel{(u/c)^2}+\cancel{(u/c)^2}}{\left[1-(u/c)^2\right]^{3/2}}\right\}m\vec{u}\bullet\frac{d\vec{u}}{dt} = \frac{m\vec{u}}{\left[1-(u/c)^2\right]^{3/2}}\bullet\frac{d\vec{u}}{dt} \\
&= \frac{mu}{\left[1-(u/c)^2\right]^{3/2}}\frac{du}{dt} = \frac{d}{dt}\left\{\frac{mc^2}{\sqrt{1-(u/c)^2}}\right\}
\end{aligned}$$

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But: $\gamma_u \equiv \dfrac{1}{\sqrt{1-(u/c)^2}}$ $\therefore$ $\left(\dfrac{d\vec{p}}{dt}\bullet\vec{u}\right) = \dfrac{d}{dt}\{\gamma_u mc^2\} = \dfrac{dE_{tot}}{dt}$

Thus:
$$W = \int \vec{F}\bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt}\bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt}\bullet\frac{d\vec{\ell}}{dt}dt = \int \frac{d\vec{p}}{dt}\bullet\vec{u}dt = \Delta E_{kin}$$
$$= \int \frac{dE_{tot}}{dt}\bullet dt = E_{tot}^{final} - E_{tot}^{initial} = \Delta E_{tot}$$

But: $E_{tot} = E_{kin} + E_{rest} = E_{kin} + mc^2$ *n.b.* $E_{tot} = \gamma_u mc^2 = \underbrace{(\gamma_u - 1)mc^2}_{E_{kin}} + mc^2$, $E_{kin} = (\gamma_u - 1)mc^2$

$$\therefore \quad \underbrace{E_{tot}^{final} - E_{tot}^{initial}}_{=\Delta E_{tot}} = \left(E_{kin}^{final} + \cancel{mc^2}\right) - \left(E_{kin}^{initial} + \cancel{mc^2}\right) = \underbrace{E_{kin}^{final} - E_{kin}^{initial}}_{=\Delta E_{kin}}$$

i.e. $W = \Delta E_{tot} = E_{tot}^{final} - E_{tot}^{initial} = \Delta E_{kin} = E_{kin}^{final} - E_{kin}^{initial}$ ← (final-initial) difference in total energy = (final-initial) difference in kinetic energy = work done ***on*** particle.

As we have already encountered elsewhere in *E&M*, Newton's 3rd Law of Motion ("For every action (force) there is an equal and opposite reaction") does ***NOT*** (in general) extend to the relativistic domain, because *e.g.* if two objects are separated in 3-D space, the 3rd Law is incompatible with ***the relativity of simultaneity***.

Suppose the 3-D force of A acting on B at some instant t is: $\vec{F}_{\overline{AB}}(\vec{r}_B,t) = +\vec{F}(\vec{r}_B,t)$

and the 3-D force of B acting on A at the same instant t is: $\vec{F}_{\overline{BA}}(\vec{r}_A,t) = -\vec{F}(\vec{r}_A,t)$

} As observed *e.g.* in ***lab*** IRF(S)

Then Newton's 3rd Law ***does*** apply in ***this*** reference frame.

However, a moving observer {moving relative to the above IRF(S)} will report that these equal-but-opposite 3-D forces occurred at ***different times*** as seen from his/her IRF(S'), thus in his/her IRF(S'), Newton's 3rd Law is ***violated*** (the two 3-D forces $\vec{F}'_{\overline{AB}}(\vec{r}'_B,t')$ and $\vec{F}'_{\overline{BA}}(\vec{r}'_A,t')$ at the ***same*** time t' in IRF(S') are ***quite unlikely*** to be ***equal and opposite***, *e.g.* if they are changing in time in IRF(S)).

Only in the case of ***contact interactions*** (*i.e.* 2 point particles at ***same*** point in space-time = (x_A, t_A)) where the two 3-D forces $\vec{F}_{\overline{AB}}(\vec{r}_B,t)$ and $\vec{F}_{\overline{BA}}(\vec{r}_A,t)$ are applied at the ***same*** point (x_A) at the ***same*** time, and in the {trivial} case where forces are ***constant***, does Newton's 3rd Law hold!

$$\vec{F}(\vec{r},t) = \frac{d\vec{p}(\vec{r},t)}{dt}$$

The observant student may have noticed that because $\vec{F}(\vec{r},t)$ is the derivative of the (relativistic) momentum $\vec{p}(\vec{r},t)$ with respect to the ***ordinary*** (and ***not*** the ***proper***) time t, it "suffers" from the same "ugly" behavior that "***ordinary***" velocity does, in Lorentz-transforming it from one IRF to another: both ***numerator*** and ***denominator*** of $\dfrac{d\vec{p}(\vec{r},t)}{dt}$ must be transformed.

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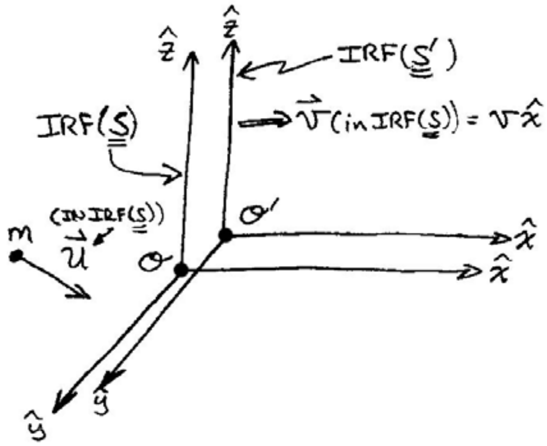
Thus, if we carry out a Lorentz transformation from IRF(*S*) to IRF(*S*′), along the $\hat{x}$ -axis where $\vec{v} = v\hat{x}$ is velocity vector of IRF(*S*′) as observed in IRF(*S*), and $\vec{u}$ is the velocity vector of a particle of mass *m* as observed in IRF(*S*):



Then: $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}}$ where: $\boxed{\beta \equiv \dfrac{v}{c}}$ with: $\boxed{\vec{v} = v\hat{x}}$

The γ and β factors are needed for the Lorentz transformation of kinematic quantities from IRF(*S*) → IRF(*S*′).

First, let us work out the $\hat{y}'$ and $\hat{z}'$ (*i.e.* the ***transverse***) components of the 3-D force $\vec{F}'(\vec{r}',t')$ as seen in IRF(*S*′) {they are simpler / easier to obtain. . . }:

Noting that: $\boxed{\vec{F} = \dfrac{d\vec{p}}{dt}}$, $\boxed{\vec{F}' = \dfrac{d\vec{p}'}{dt'}}$ and that: $\boxed{dt' = \gamma dt - \dfrac{\gamma\beta}{c}dx}$ and: $\boxed{u_x = \dfrac{dx}{dt}}$

In IRF(*S*′): $$\boxed{F'_y = \frac{dp'_y}{dt'} = \frac{dp_y}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_y}{dt}}{\gamma\left(1-\frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma\left(1-\left(\beta u_x/c\right)\right)}}$$

Similarly: $$\boxed{F'_z = \frac{dp'_z}{dt'} = \frac{dp_z}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_z}{dt}}{\gamma\left(1-\frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_z}{\gamma\left(1-\left(\beta u_x/c\right)\right)}}$$

Now calculate the $\hat{x}'$ -component of the force $\vec{F}'(\vec{r}',t')$ in IRF(*S*′):

In IRF(*S*′): $$\boxed{F'_x = \frac{dp'_x}{dt'} = \frac{\cancel{\gamma} dp_x - \cancel{\gamma}\beta dp^0}{\cancel{\gamma} dt - \frac{\cancel{\gamma}\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1-\frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \left(\beta/c\right)\frac{dE_{tot}}{dt}}{1-\left(\beta u_x/c\right)}}$$ where: $\boxed{p^0 = \dfrac{E_{tot}}{c}}$

But we have calculated $\dfrac{dE_{tot}}{dt}$ above / earlier: $\boxed{\dfrac{dE_{tot}}{dt} = \dfrac{d\vec{p}}{dt}\bullet\vec{u} = \vec{F}\bullet\vec{u} = \vec{u}\bullet\vec{F}}$ since: $\boxed{\vec{F} = \dfrac{d\vec{p}}{dt}}$

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$$\therefore \quad \boxed{F'_x = \frac{F_x - \beta(\vec{u}\cdot\vec{F})/c}{(1-(\beta u_x/c))}} \quad \boxed{F'_y = \frac{F_y}{\gamma(1-(\beta u_x/c))}} \quad \boxed{F'_z = \frac{F_z}{\gamma(1-(\beta u_x/c))}}$$

Relativistic "***ordinary***" *x*, *y*, *z* force components observed in IRF(*S′*) acting on particle of mass *m*, for a Lorentz transformation from lab IRF(*S*) to IRF(*S′*).

IRF(*S′*) moving with velocity $\vec{v} = v\hat{x}$ relative to IRF(*S*) (as seen in IRF(*S*)), $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta \equiv v/c$.

Particle of mass *m* is moving with "***ordinary***" velocity $\vec{u}$ as seen in IRF(*S*).

We see that ***only*** when the particle of mass *m* is ***instantaneously*** at rest in lab IRF(*S*) (*i.e.* $\vec{u}(t)=0$) will we then have a "simple" Lorentz transformation of the "***ordinary***" force $\vec{F} \rightarrow \vec{F}'$:

$$\boxed{\vec{u}=0}: \left\{ \begin{array}{l} \boxed{F'_x = F_x} \quad \Rightarrow \boxed{F'_{\|} = F_{\|}} \\ \boxed{F'_y = F_y/\gamma} \\ \boxed{F'_z = F_z/\gamma} \end{array} \right\} \Rightarrow \boxed{F'_{\perp} = F_{\perp}/\gamma}$$

← *n.b.* || force components are same/identical !!!

Where the subscripts || (⊥) refer to the parallel (perpendicular) components of the force with respect to the motion of IRF(*S′*) relative to IRF(*S*), respectively.

Note that for $\vec{u}=0$, the component of $\vec{F}$ || to the Lorentz boost direction is ***unchanged***.
For $\vec{u}=0$, the component of $\vec{F}$ ⊥ to the Lorentz boost direction is ***reduced*** by the factor $1/\gamma$.

Proper Force – The Minkowski Force:

In analogy to the definition of the ***proper*** time interval $d\tau$ and the ***proper*** velocity $\boxed{\vec{\eta} = d\vec{\ell}/d\tau}$ versus the "***ordinary***" time interval *dt* and the "***ordinary***" velocity $\boxed{\vec{u} = d\vec{\ell}/dt}$, we define a ***proper*** force $\vec{K}$ (also known as the Minkowski force), which is the derivative of the ***relativistic*** momentum $\vec{p}$ with respect to ***proper*** time $d\tau$:

$$\boxed{\vec{K} \equiv \frac{d\vec{p}}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{d\vec{p}}{dt}} \quad \text{but:} \quad \boxed{\frac{dt}{d\tau} = \frac{dt}{dt'} = \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-(u/c)^2}}}$$

$$\therefore \quad \boxed{\vec{K} \equiv \frac{d\vec{p}}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{d\vec{p}}{dt} = \gamma_u\vec{F}} \quad \text{where:} \quad \boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

$$\text{Thus:} \quad \boxed{\vec{K} = \gamma_u\vec{F} = \frac{1}{\sqrt{1-\beta_u^2}}\vec{F} = \frac{1}{\sqrt{1-(u/c)^2}}\vec{F}} \quad \text{where:} \quad \boxed{\gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-(u/c)^2}}} \quad \text{and:} \quad \boxed{\beta_u \equiv \frac{u}{c}}$$

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We can "4-vectorize" the Minkowski Force, because it's plainly / clearly a 4-vector:

ct: $$K^0 \equiv \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE_{tot}}{d\tau}$$ since: $p^0 = \frac{E_{tot}}{c} \Rightarrow K^0 =$ Proper rate at which energy of particle increases (or decreases) = (Proper **power** delivered to the particle)/c !

x: $$K^1 \equiv \frac{dp^1}{d\tau} = \gamma_u F^1 = \frac{1}{\sqrt{1-\beta_u^2}} F^1 = \frac{1}{\sqrt{1-(u/c)^2}} F^1$$

y: $$K^2 \equiv \frac{dp^2}{d\tau} = \gamma_u F^2 = \frac{1}{\sqrt{1-\beta_u^2}} F^2 = \frac{1}{\sqrt{1-(u/c)^2}} F^2$$

z: $$K^3 \equiv \frac{dp^3}{d\tau} = \gamma_u F^3 = \frac{1}{\sqrt{1-\beta_u^2}} F^3 = \frac{1}{\sqrt{1-(u/c)^2}} F^3$$

with: $$\gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-(u/c)^2}}$$

Thus: $$K^\mu \equiv \frac{dp^\mu}{d\tau}$$ ← Minkowski 4-vector force = **proper** 4-vector force.

Relativistic dynamics can be formulated in terms of either "**ordinary**" quantities or "**proper**" (particle rest frame) quantities. The latter is much neater / elegant, but it is (by its nature) restricted to the particle's rest frame IRF(S') {*n.b.* We can **always** Lorentz boost this "**proper**" result to **any** other inertial reference frame. . . }

There is a very simple reason for this! Since we humans live in the **lab** frame IRF(S) – we want to know everything about particle's trajectory, the forces acting on it, *etc*. in the **lab** because this is the **only** IRF that we can (easily) carry out **physical measurements** in – often, it is **not** possible to make physical measurements *e.g*. in a particle's rest frame / proper frame, especially if the particles are in relativistic motion (*e.g*. at Fermilab/LHC/… hadron colliders).

In the long run, we will (usually) be interested in the particle's trajectory as a function of "**ordinary**" time, so in fact the "**ordinary**" 4-force $F^\mu \equiv dp^\mu/dt$ is often more useful, even if it is more painful / cumbersome to calculate / compute…

We want to obtain the relativistic generalization of the classical Lorentz force law $\vec{F}_C = q\vec{E} + q\vec{u}\times\vec{B}$ {$\vec{u}$ = particle's "**ordinary**" velocity in IRF(S)}. Does the **classical** formula $\vec{F}_C$ correspond to the "**ordinary**" relativistic force $\vec{F}$, or to the **proper** / Minkowski force $\vec{K}$?

Thus, for the relativistic Lorentz force, should we write: $\vec{F} = q\vec{E} + q\vec{u}\times\vec{B} = q\left(\vec{E}+\vec{u}\times\vec{B}\right)$???

Or rather, should the relativistic Lorentz force relation be: $\vec{K} = q\vec{E} + q\vec{u}\times\vec{B} = q\left(\vec{E}+\vec{u}\times\vec{B}\right)$???

Since **proper** time and "**ordinary**" time are **identical** in **classical** physics / Euclidean / Galilean 3-space, **classical** physics **can't** tell us the answer.

It turns out that the Lorentz force law is an "**ordinary**" relativistic force law: $\vec{F} = q\left(\vec{E}+\vec{u}\times\vec{B}\right)$

We'll see **why** shortly… We'll also construct the **proper** / Minkowski *EM* force law, as well . . .

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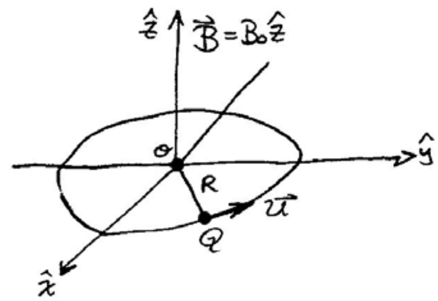
But first, some examples:

Griffiths Example 12.11: Relativistic Charged Particle Moving in a Uniform Magnetic Field

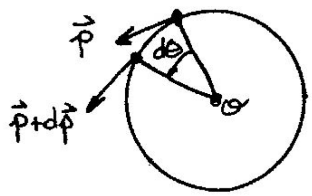
We've discussed this before, from a classical dynamics point of view:



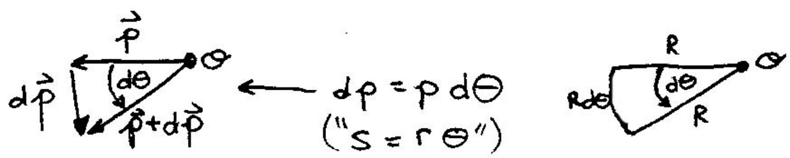
The typical trajectory of a charged particle (charge Q, mass m) moving in a uniform magnetic field is ***cyclotron*** motion. If the velocity of particle ($\vec{u}$) lies in the x-y plane and $\vec{B} = B_o\hat{z}$, then $\boxed{\vec{F} = Q\vec{u}\times\vec{B} = QuB_o\left(-\hat{r}\right) = -QuB_o\hat{r}}$ as shown on the right:

The magnetic force points radially inward – it provides the centripetal acceleration needed to sustain the circular motion. However, in ***special*** relativity the centripetal force is ***not*** mu^2/R (as it is in ***classical*** mechanics). Rather, it is: $\boxed{F = \dfrac{dp}{dt} = p\dfrac{d\theta}{dt} = p\dfrac{R}{R}\dfrac{d\theta}{dt} = p\dfrac{1}{R}\left(\dfrac{Rd\theta}{dt}\right) = p\dfrac{u}{R}}$.

Top View:



Vector Diagram:



$\boxed{\vec{F} = p\dfrac{u}{R}\left(-\hat{r}\right)}$ *n.b.* ***Classically***: $\boxed{\vec{p} = m\vec{u}}$ thus, ***classically***: $\boxed{\vec{F} = m\dfrac{u^2}{R}\left(-\hat{r}\right)}$

Thus, ***relativistically***: $\boxed{QuB_o\left(-\hat{r}\right) = p\dfrac{u}{R}\left(-\hat{r}\right)}$ or: $\boxed{QuB_o = p\dfrac{u}{R}}$ or: $\boxed{p = QB_oR}$

The ***relativistic*** cyclotron formula is ***identical*** to the ***classical*** / ***non-relativistic*** formula!

However ***here***, $\vec{p}$ is understood to be the ***relativistic*** 3-momentum: $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u}}$.

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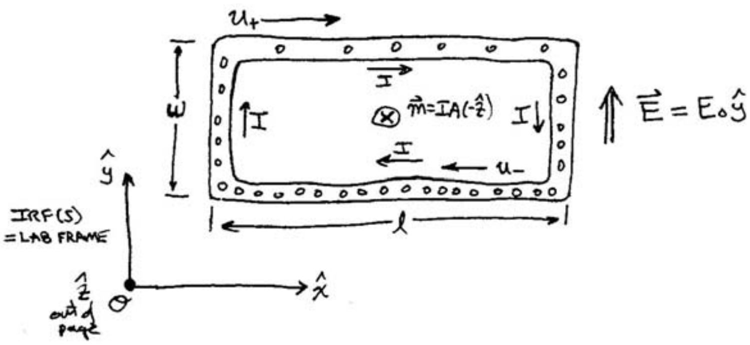
<u>Griffiths Example 12.12</u>: <u>Hidden Momentum</u>

Consider a magnetic dipole moment $\vec{m}$ modeled as a rectangular loop of wire (dimensions $\ell \times w$) carrying a steady current *I*. Imagine the current as a uniform stream of non-interacting positive charges flowing freely through the wire at constant speed *u*. (*i.e.* a fictitious kind of superconductor.) A uniform electric field $\vec{E}$ is applied as shown in the figure below:



The application of the external uniform electric field $\vec{E} = E_o \hat{y}$ changes the physics – the electric charges are ***<u>accelerated</u>*** in the ***<u>left</u>*** segment of the loop and ***<u>decelerated</u>*** in the ***<u>right</u>*** segment of the loop. [*n.b.* admittedly this is ***<u>not</u>*** a very realistic model, but other more realistic models ***<u>do</u>*** lead to the ***<u>same</u>*** result – see *e.g.* V. Hnizdo, Am. J. Phys. **65**, 92 (1997)].

Find the total momentum of all of the charges in the loop.

The momenta associated with the electric charges in the ***<u>left</u>*** and ***<u>right</u>*** segments of the loop ***<u>cancel</u>*** each other (*i.e.* $\vec{p}$ (in ***<u>left</u>*** segment) = $-\vec{p}$ (in ***<u>right</u>*** segment), so we ***<u>only</u>*** need to consider the momenta associated with the electric charges flowing in the ***<u>top</u>*** and ***<u>bottom</u>*** segments of the loop.

Suppose there are N_+ charges flowing in the ***<u>top</u>*** segment of the loop, moving in $+\hat{x}$ direction with speed $u_+ > u\left(\vec{E}=0\right)$ {because they underwent ***<u>acceleration</u>*** traveling on the LHS segment} and N_- charges flowing in the ***<u>bottom</u>*** segment of the loop, moving in the $-\hat{x}$ direction with speed $u_- < u\left(\vec{E}=0\right)$ {because they underwent ***<u>deceleration</u>*** traveling on the RHS segment}.

Note that the current $I = \lambda u$ ***<u>must</u>*** be the ***<u>same</u>*** in ***<u>all</u>*** four segments of the loop, otherwise charges would be piling up somewhere.

In particular: $I = I_+$ (top segment of loop) = I_- (bottom segment of loop), *i.e.* $\boxed{I = I_+ = I_-}$.

Since: $\boxed{\lambda \equiv \frac{Q_{tot}}{\ell} = \frac{NQ}{\ell}}$ then: $\boxed{I_+ = \lambda_+ u_+ = N_+\left(\frac{Q}{\ell}\right)u_+} = \boxed{I_- = \lambda_- u_- = N_-\left(\frac{Q}{\ell}\right)u_-}$

$$\therefore \quad \boxed{N_+\left(\frac{Q}{\ell}\right)u_+ = N_-\left(\frac{Q}{\ell}\right)u_- = I} \;\Rightarrow\; \boxed{N_+ u_+ = N_- u_- = \frac{I\ell}{Q}}$$

<u>Classically</u>, the linear momentum of each electric charge is $\vec{p}_{classical} = m_Q \vec{u}$ where m_Q = mass of the charged particle.

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The total ***classical*** linear momentum of the charged particles flowing to the ***right*** in the ***top*** segment of the loop is: $\boxed{\vec{p}_{+_{classical}}^{\text{top segment}} = \sum_{i=1}^{N_+} m_Q \vec{u}_+ = N_+ m_Q u_+ \left(+\hat{x}\right)}$

The total ***classical*** linear momentum of the charged particles flowing to the ***left*** in the ***bottom*** segment of the loop is: $\boxed{\vec{p}_{-_{classical}}^{\text{bottom segment}} = \sum_{i=1}^{N_-} m_Q \vec{u}_- = N_- m_Q u_- \left(-\hat{x}\right)}$

The net (or total) ***classical*** linear momentum of the charged particles flowing in the loop is:

$$\boxed{\begin{aligned}\vec{p}_{classical}^{\,tot} &= \cancel{\vec{p}_{classical}^{\text{left segment}}} + \vec{p}_{+_{classical}}^{\text{top segment}} + \cancel{\vec{p}_{classical}^{\text{right segment}}} + \vec{p}_{-_{classical}}^{\text{bottom segment}} = \vec{p}_{+_{classical}}^{\text{top segment}} + \vec{p}_{-_{classical}}^{\text{bottom segment}} \\ &= N_+ m_Q u_+ \hat{x} - N_- m_Q u_- \hat{x} = \left(N_+ u_+ - N_- u_-\right) m_Q \hat{x} = \left(I\ell/Q - I\ell/Q\right) m_Q \hat{x} = 0 \;\; !!!\end{aligned}}$$

Thus, $\boxed{\vec{p}_{classical}^{\,tot} = 0}$ as we expected, since we know the loop is not moving.

However, now let us consider the ***relativistic*** momentum:

$\boxed{\vec{p}_{rel} = \gamma_u m_Q \vec{u}}$ (even if $|\vec{u}| = u \ll c$) where: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-(u/c)^2}}}$

The total ***relativistic*** momentum of the charged particles flowing to the ***right*** in the ***top*** segment of the loop is: $\boxed{\vec{p}_{+_{rel}}^{\text{top segment}} = \gamma_{u^+} N_+ m_Q u_+ \left(+\hat{x}\right)}$ where: $\boxed{\gamma_{u^+} \equiv \frac{1}{\sqrt{1-\beta_+^2}} = \frac{1}{\sqrt{1-\left(u_+/c\right)^2}}}$.

The total ***relativistic*** momentum of the charged particles flowing to the ***left*** in the ***bottom*** segment of the loop is: $\boxed{\vec{p}_{-_{rel}}^{\text{bottom segment}} = \gamma_{u^-} N_- m_Q u_- \left(-\hat{x}\right)}$ where: $\boxed{\gamma_{u^-} \equiv \frac{1}{\sqrt{1-\beta_-^2}} = \frac{1}{\sqrt{1-\left(u_-/c\right)^2}}}$.

The net / total ***relativistic*** momentum is:

$$\boxed{\vec{p}_{rel}^{\,tot} = \vec{p}_{+_{rel}}^{\text{top segment}} + \vec{p}_{-_{rel}}^{\text{bottom segment}} = \left(\gamma_{u^+} N_+ m_Q u_+ - \gamma_{u^-} N_- m_Q u_-\right)\hat{x} = \left(\gamma_{u^+} N_+ u_+ - \gamma_{u^-} N_- u_-\right) m_Q \hat{x}}$$

But $\boxed{I = I_+ = I_-}$ gave us: $\boxed{N_+ u_+ = N_- u_- = \frac{I\ell}{Q}}$ $\therefore$ $\boxed{\vec{p}_{rel}^{\,tot} = \left(\gamma_{u^+} - \gamma_{u^-}\right) m_Q \left(\frac{I\ell}{Q}\right)\hat{x} \neq 0}$ because $\boxed{\gamma_{u^+} \neq \gamma_{u^-}}$!!!

Charged particles flowing in the ***top*** segment of the loop are moving ***faster*** than those flowing in the ***bottom*** segment of the loop.

The ***gain*** in energy $\left(\gamma_u mc^2\right)$ of the charged particles going up the ***left*** segment of the loop = the work done ***on*** the charges by the ***electric*** force ($W = QE_o w$) (w = height of the rectangle).

Thus, for a charged particle going up the ***left*** segment of the loop, the energy ***gain*** is:

$$\boxed{\Delta E = \gamma_{u^+} m_Q c^2 - \gamma_{u^-} m_Q c^2 = \left(\gamma_{u^+} - \gamma_{u^+}\right) m_Q c^2 = W = QE_o w} \;\Rightarrow\; \boxed{\left(\gamma_{u^+} - \gamma_{u^-}\right) = \frac{QE_o w}{m_Q c^2}}$$

Where E_o = the magnitude of the {uniform/constant} electric field.

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$$\therefore \boxed{\vec{p}_{rel}^{\,tot} = \left(\gamma_{u^+} - \gamma_{u^-}\right) m_Q \left(\frac{I\ell}{Q}\right)\hat{x} = \left(\frac{\cancel{Q}E_o w}{\cancel{m}_Q c^2}\right)\cancel{m}_Q\left(\frac{I\ell}{\cancel{Q}}\right)\hat{x} = \left(\frac{E_o I \ell w}{c^2}\right)\hat{x}}$$

But: $\boxed{\ell w = A}$ = area of the loop. $\therefore$ $\boxed{\vec{p}_{rel}^{\,TOT} = \frac{E_o IA}{c^2}\hat{x}}$ but: $\boxed{m = |\vec{m}| = IA}$ $\Rightarrow$ $\boxed{\vec{p}_{rel}^{\,TOT} = \frac{mE_o}{c^2}\hat{x}}$

But: $\boxed{\vec{m} = m(-\hat{z})}$ (see picture above) and: $\boxed{\vec{E} = E_o\hat{y}}$ *i.e.* $\boxed{\vec{m} \perp \vec{E}}$ {***here***}.

Thus, vectorially we {actually} have: $\boxed{\vec{p}_{rel}^{\,tot} = \frac{1}{c^2}\left(\vec{m}\times\vec{E}\right)}$ where: $\boxed{\left(\vec{m}\times E\right) = mE_o \overbrace{\left(-\hat{z}\times\hat{y}\right)}^{=+\hat{x}}}$

Thus a magnetic dipole moment $\vec{m}$ in the presence of an electric field $\vec{E}$ carries relativistic linear momentum $\vec{p}$, even though it is not moving !!!

n.b. it also (therefore) carries relativistic angular momentum $\boxed{\vec{L}_{rel} = \vec{r}\times\vec{p}_{rel}}$.

How big is this effect? Explicit numerical example - use “everyday” values:

$E_o = 1000\ V/m$
$I = 1\ Amp$
$A = (10\ cm)^2 = 0.01\ m^2$
$m = IA = 0.01\ A\text{-}m^2$

$\boxed{\left|p_{rel}^{tot}\right| = \frac{mE_o}{c^2} = \frac{10^{-2}\times 10^{+3}}{\left(3\times 10^8\right)^2} = 10^{-16}\ kg\text{-}m/s}$ Tiny !!! The $1/c^2$ factor ***kills*** this effect !!!

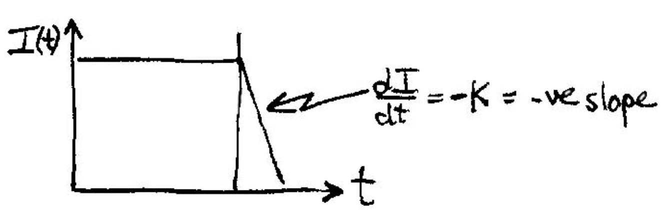
This so-called macroscopic hidden ***linear*** momentum is strictly relativistic, purely mechanical But note that it ***precisely*** cancels the ***electromagnetic*** linear momentum stored in the $\vec{E}$ and $\vec{B}$ fields!!! (Microscopically, the momentum imbalance arises from the imbalance of virtual photon emission on top segment of the loop *vs*. the bottom segment of the loop.)

Likewise, the corresponding hidden ***angular*** momentum precisely cancels the electromagnetic ***angular*** momentum stored in the $\vec{E}$ and $\vec{B}$ fields.

→ Now go back and take another look at Griffiths Example 8.3, pages 356-57. (The coax cable carrying uniform charge / unit length λ and steady current I flowing down / back cable.)

Let’s pursue this problem a little further…

Suppose there is a ***change*** in the current, *e.g.* suppose the current drops / decreases to zero. For simplicity’s sake, assume $\frac{dI}{dt} = -K$ (*i.e.* the current decreases linearly with time)



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Classically: $I(t) = I_+(t) = I_-(t)$ (as before)

$$I_+(t) = N_+(t)\left(\frac{Q}{\ell}\right)u_+$$
$$I_-(t) = N_-(t)\left(\frac{Q}{\ell}\right)u_-$$

$$N_+(t)\left(\frac{Q}{\ell}\right)u_+ = N_-(t)\left(\frac{Q}{\ell}\right)u_-$$

We assume that u, u_+ and u_- are unaffected by the change in the current with time.

Then: $\dfrac{dI}{dt} = \dfrac{dI_+(t)}{dt} = \dfrac{dI_-(t)}{dt} \Rightarrow \left(\dfrac{Q}{\ell}\right)u_+\dfrac{dN_+(t)}{dt} = \left(\dfrac{Q}{\ell}\right)u_-\dfrac{dN_-(t)}{dt} = -K$

$\therefore$ $\dfrac{dN_+(t)}{dt}u_+ = \dfrac{dN_-(t)}{dt}u_- = -\dfrac{K\ell}{Q}$ = constant (no time dependence on RHS of equation)

Then:

$$\frac{d\vec{p}_{+_{classical}}(t)}{dt} = \frac{dN_+(t)}{dt}m_Q u_+(+\hat{x}) = -\frac{K\ell m_Q}{Q}(+\hat{x}) = -\frac{K\ell m_Q}{Q}\hat{x}$$
$$\frac{d\vec{p}_{-_{classical}}(t)}{dt} = \frac{dN_-(t)}{dt}m_Q u_-(-\hat{x}) = -\frac{K\ell m_Q}{Q}(-\hat{x}) = +\frac{K\ell m_Q}{Q}\hat{x}$$

Constant

$\therefore$ The net / total classical time-rate of change of ***linear*** momentum is:

$$\vec{F}^{tot}_{classical}(t) = \frac{d\vec{p}^{\,tot}_{classical}(t)}{dt} = \frac{d\vec{p}_{+_{classical}}(t)}{dt} + \frac{d\vec{p}_{-_{classical}}(t)}{dt} = -\frac{K\ell m_Q}{Q}\hat{x} + \frac{K\ell m_Q}{Q}\hat{x} = 0$$

Thus: $\vec{F}^{tot}_{classical}(t) = \dfrac{d\vec{p}^{\,tot}_{classical}(t)}{dt} = 0$ as we expected, since the loop is not moving.

Now, let's investigate this situation ***relativistically***:

Since: $\vec{p}_{rel} = \gamma_u m_Q \vec{u} \Rightarrow \vec{p}_{+_{rel}} = \gamma_{u^+} m_Q \vec{u}_+$ and: $\vec{p}_{-_{rel}} = \gamma_{u^-} m_Q \vec{u}_-$

For individual charges with mass m_Q

Then: $\vec{p}^{\text{ top segment}}_{+_{rel}}(t) = \gamma_{u^+} N_+(t) m_Q u_+(+\hat{x})$ where: $\gamma_{u^+} \equiv \dfrac{1}{\sqrt{1-\beta_+^2}} = \dfrac{1}{\sqrt{1-(u_+/c)^2}}$.

And: $\vec{p}^{\text{ bottom segment}}_{-_{rel}}(t) = \gamma_{u^-} N_-(t) m_Q u_-(-\hat{x})$ where: $\gamma_{u^-} \equiv \dfrac{1}{\sqrt{1-\beta_-^2}} = \dfrac{1}{\sqrt{1-(u_-/c)^2}}$.

And: $\dfrac{d\vec{p}^{\text{ top segment}}_{+_{rel}}(t)}{dt} = \gamma_{u^+} m_Q u_+ \dfrac{dN_+(t)}{dt}(\hat{x}) = \text{ constant } = -\gamma_{u^+}\left(\dfrac{K\ell m_Q}{Q}\right)\hat{x}$

And: $\dfrac{d\vec{p}^{\text{ bottom segment}}_{-_{rel}}(t)}{dt} = \gamma_{u^-} m_Q u_- \dfrac{dN_-(t)}{dt}(-\hat{x}) = \text{ constant } = +\gamma_{u^-}\left(\dfrac{K\ell m_Q}{Q}\right)\hat{x}$

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The net / total time rate of change of relativistic linear momentum is:

$$\frac{d\vec{p}_{rel}^{\,tot}(t)}{dt} = \frac{d\vec{p}_{+_{rel}}^{\text{ top segment}}(t)}{dt} + \frac{d\vec{p}_{-_{rel}}^{\text{ bottom segment}}(t)}{dt} = -\gamma_{u^+}\left(\frac{K\ell m_Q}{Q}\right)\hat{x} + \gamma_{u^-}\left(\frac{K\ell m_Q}{Q}\right)\hat{x}$$
$$= \left(\gamma_{u^+} - \gamma_{u^-}\right)\left(\frac{K\ell m_Q}{Q}\right)\hat{x} \neq 0 \qquad \left(\gamma_{u^+} \neq \gamma_{u^-}\right)$$

From above (*p*. 20): $\left(\gamma_{u^+} - \gamma_{u^-}\right) = \frac{QE_o w}{m_Q c^2}$ where: E_o = electric field amplitude

$$\frac{d\vec{p}_{rel}^{\,tot}(t)}{dt} = -\left(\frac{\cancel{Q}E_o w}{\cancel{m}_Q c^2}\right)\left(\frac{K\ell \cancel{m}_Q}{\cancel{Q}}\right)\hat{x} = -\frac{E_o K\ell w}{c^2}\hat{x} = -\frac{E_o KA}{c^2}\hat{x}$$

$A = \ell \times w$ = cross-sectional area of the loop

Now: $\frac{dI}{dt} = -K$ and: $m = IA$ $\rightarrow$ $\frac{dm}{dt} = \frac{dI}{dt}A$ (Since A = constant).

$\therefore$ $\frac{dm}{dt} = -KA = \frac{dI}{dt}A$ = time rate of change of the magnetic dipole moment of the loop.

$\Rightarrow$ $\frac{d\vec{p}_{rel}^{\,tot}(t)}{dt} = \frac{1}{c^2}\frac{dm(t)}{dt}E_o\hat{x}$ but: $\left(\vec{m} = m(-\hat{z})\right) \perp \left(\vec{E} = E_o\hat{y}\right)$

$\therefore$ $\vec{F}_{rel}(t) = \frac{d\vec{p}_{rel}^{\,tot}(t)}{dt} = \frac{1}{c^2}\left(\frac{d\vec{m}(t)}{dt}\times\vec{E}\right) \neq 0$ (assuming external $\vec{E}$-field is constant in time)

Thus, $\exists$ a net "hidden" ***force*** acting on the magnetic dipole, when $dI/dt \neq 0$.

One ***might*** think that this net "hidden" force would be ***exactly*** cancelled / compensated for by a countering force due to the electromagnetic fields, as we saw in the static case ($dI/dt = 0$), with a steady current *I*. **But it isn't!!** Why??

As we saw for M(1) magnetic dipole radiation, a time-varying current in a loop produces *EM* radiation. Essentially there is a radiation reaction / back-force that acts on the "antenna" – a radiation pressure – much like the recoil / impulse from firing a bullet out of a gun – the short explosive "pulse" launches the bullet, but the gun is also kicked backwards, too.

The same thing happens here when $dI/dt \neq 0$ - the far zone *EM* radiation fields are produced (*i.e*. real photons) while $dI/dt \neq 0$ and carry away ***linear*** momentum, and since $dI/dt \neq 0$, $\exists$ a net force ***imbalance*** on the radiating object! (*n.b*. – *e.g*. by linear momentum conservation, a laser pen has a recoil force acting on it from emitting the laser radiation – a radiation back reaction)

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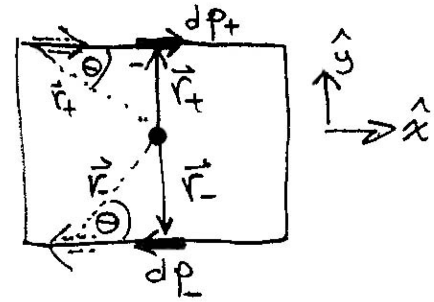
Likewise, the net "hidden" time rate of change of relativistic ***angular*** momentum is:

$$\boxed{\frac{d\vec{L}_{rel}^{tot}(t)}{dt} = \vec{r}\times\frac{d\vec{p}_{rel}^{tot}(t)}{dt} = \frac{1}{c^2}\frac{dm(t)}{dt}E_o(\vec{r}\times\hat{x})}$$

Which will also ***not*** be ***exactly*** cancelled either, for the ***same*** reason – the *EM* radiation field can / will carry away angular momentum…

In reality, in order to calculate $\frac{d\vec{L}_{rel}^{tot}(t)}{dt}$, we need to go back and integrate infinitesimal contributions along the (short) segments of upper and lower / top and bottom segments of the loop because $|\vec{r}\times\vec{p}| = rp\sin\theta$, $\theta = \measuredangle$ between $\vec{r}$ and $\vec{p}$.



Same for $\left|\vec{r}\times\frac{d\vec{p}}{dt}\right| = r\frac{dp}{dt}\sin\theta$.

Will get result that has geometrical factor of order ≤ 1.
→ Conclusions won't be changed by this, just actual #.

As we know, the time rate of change of angular momentum: $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$ = torque.

Thus, the time rate of change of the net / total "hidden" relativistic angular momentum $\frac{d\vec{L}_{rel}^{tot}(t)}{dt}$ = net "hidden" relativistic torque, $\vec{\tau}_{rel}^{tot}(t)$.

Thus: $$\boxed{\vec{\tau}_{rel}^{tot}(t) = \frac{d\vec{L}_{rel}^{tot}(t)}{dt} = \vec{r}\times\frac{d\vec{p}_{rel}^{tot}(t)}{dt} = \vec{r}\times\vec{F}_{rel}^{tot}(t) = \frac{1}{c^2}\vec{r}\times\left(\frac{d\vec{m}(t)}{dt}\times\vec{E}\right) \neq 0}$$

Which is ***not*** completely / ***exactly*** cancelled when $dI(t)/dt \neq 0$!!!

Linear momentum, angular momentum, energy, *etc*. are ***all*** conserved for this whole system, it's just that the *EM* radiation emitted from the antenna is free-streaming, carrying away all these quantities with it!

In the ***static*** situation I = constant, the "hidden" relativistic linear momentum and angular momentum is ***exactly*** cancelled by the linear momentum and angular momentum (respectively) carried by the (macroscopic) static electromagnetic fields $\vec{E}$ and $\vec{B}$. Microscopically, the field linear and angular momentum is carried by the static, virtual photons associated with the macroscopic $\vec{E}$ and $\vec{B}$ fields, cancelling the (macroscopic) "hidden" linear and angular relativistic momentum of the magnetic dipole in a uniform $\vec{E}$-field.

In the ***non-static*** situation $dI(t)/dt \neq 0$, ***virtual*** photons undergo space-time rotation, becoming ***real*** photons, which carry away {real} linear and angular momentum. "Hidden" relativistic linear and angular momentum is no longer ***exactly*** cancelled by the (now) real field linear and angular momentum associated with the *EM* radiation fields. It is only partially cancelled by remaining / extant virtual / near-zone / inductive zone *EM* fields.

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Griffiths Problem 12.36: Relativistic "Ordinary" Force

In ***classical*** mechanics Newton's 2nd Law is: $\boxed{\vec{F} = m\vec{a}}$.

The ***relativistic*** "***ordinary***" force relation; $\boxed{\vec{F}_{rel} = \dfrac{d\vec{p}}{dt}}$ cannot be so simply expressed.

$$\vec{F}_{rel} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma_u m\vec{u}\right) = \frac{d}{dt}\left(\frac{1}{\sqrt{1-(u/c)^2}} m\vec{u}\right) \quad \text{where: } \gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}$$

$$\vec{F}_{rel} = m\left\{\frac{\frac{d\vec{u}}{dt}}{\sqrt{1-(u/c)^2}} + \vec{u}\left(-\frac{1}{2}\right)\frac{\frac{1}{c^2}2\vec{u}\bullet\frac{d\vec{u}}{dt}}{\left(1-(u/c)^2\right)^{3/2}}\right\} \quad \text{where: } \boxed{\vec{a} \equiv \frac{d\vec{u}}{dt}} = \text{"ordinary" acceleration.}$$

$$\therefore \quad \boxed{\vec{F}_{rel} = \frac{m}{\sqrt{1-(u/c)^2}}\left\{\vec{a} + \frac{\vec{u}(\vec{u}\bullet\vec{a})}{c^2\left(1-(u/c)^2\right)}\right\} = \frac{m}{\sqrt{1-(u/c)^2}}\left\{\vec{a} + \frac{\vec{u}(\vec{u}\bullet\vec{a})}{\left(c^2-u^2\right)}\right\}} \quad Q.E.D.$$

Griffiths Problem 12.38: *Proper* Acceleration

We define the ***proper*** four-vector acceleration in the obvious way, as:

$$\boxed{\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \left(\alpha^0, \vec{\alpha}\right)} \quad \text{where: } \boxed{\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \left(\eta^0, \vec{\eta}\right) = \left(\gamma_u c, \gamma_u \vec{u}\right)} = \textit{proper} \text{ four-velocity}$$

a.) Find α^0 and $\vec{\alpha}$ in terms of $\vec{u}$ and $\vec{a}$ (= "***ordinary***" velocity, "***ordinary***" acceleration):

$$\alpha^0 = \frac{d\eta^0}{d\tau} = \frac{d\eta^0}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-(u/c)^2}}\frac{d}{dt}\left(\frac{c}{\sqrt{1-(u/c)^2}}\right) \quad \text{since: } \boxed{d\tau = \frac{1}{\gamma_u}dt} \Rightarrow \boxed{\frac{dt}{d\tau} = \gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}}$$

$$\alpha^0 = \frac{c}{\sqrt{1-(u/c)^2}}\frac{\left(-\frac{1}{\not{2}}\right)\left(-\frac{1}{c^2}\right)\not{2}\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^{3/2}} = \frac{1}{c}\frac{\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^2} \quad \text{where: } \boxed{\vec{a} \equiv \frac{d\vec{u}}{dt}}$$

Similarly:

$$\vec{\alpha} = \frac{d\vec{\eta}}{d\tau} = \frac{d\vec{\eta}}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-(u/c)^2}}\frac{d}{dt}\left(\frac{\vec{u}}{\sqrt{1-(u/c)^2}}\right) \quad \text{since: } \boxed{\vec{\eta} = \gamma_u\vec{u}} \quad \text{and: } \boxed{\gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}}$$

$$\vec{\alpha} = \frac{1}{\sqrt{1-(u/c)^2}}\left\{\frac{\vec{a}}{\sqrt{1-(u/c)^2}} + \vec{u}\frac{\left(-\frac{1}{\not{2}}\right)\left(-\frac{1}{c^2}\right)\not{2}\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^{3/2}}\right\}$$

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$$\vec{\alpha} = \frac{1}{\left(1-(u/c)^2\right)}\left\{\vec{a} + \frac{\vec{u}(\vec{u}\cdot\vec{a})}{\left(c^2-u^2\right)}\right\} = \gamma_u\left(\frac{\vec{F}_{rel}}{m}\right)$$ ← see Problem 12.36 above.

b.) Express $\alpha_\mu \alpha^\mu$ in terms of $\vec{u}$ and $\vec{a}$:

$$\alpha_\mu\alpha^\mu = -\left(\alpha^0\right)^2 + \vec{\alpha}\cdot\vec{\alpha} = -\frac{1}{c^2}\frac{(\vec{u}\cdot\vec{a})^2}{\left(1-(u/c)^2\right)^4} + \frac{1}{\left(1-(u/c)^2\right)^4}\left[\vec{a}\left(1-(u/c)^2\right) + \frac{1}{c^2}\vec{u}(\vec{u}\cdot\vec{a})\right]^2$$

$$= \frac{1}{\left(1-(u/c)^2\right)^4}\left\{-\frac{1}{c^2}(\vec{u}\cdot\vec{a})^2 + a^2\left(1-(u/c)^2\right)^2 + \frac{2}{c^2}\left(1-(u/c)^2\right)(\vec{u}\cdot\vec{a})^2 + \frac{1}{c^4}u^2(\vec{u}\cdot\vec{a})^2\right\}$$

$$= \frac{1}{\left(1-(u/c)^2\right)^4}\left\{a^2\left(1-(u/c)^2\right)^2 + \frac{(\vec{u}\cdot\vec{a})^2}{c^2}\left(-1+2-2\left(\frac{u}{c}\right)^2 + \left(\frac{u}{c}\right)^2\right)\right\}$$

Or: $$\alpha_\mu\alpha^\mu = \frac{1}{\left(1-(u/c)^2\right)^4}\left[a^2 + \frac{(\vec{u}\cdot\vec{a})^2}{\left(c^2-u^2\right)}\right]$$ ← *n.b.* ***Lorentz-invariant quantity*** – ***same*** in all IRFs.

c.) Show $\eta^\mu\alpha_\mu = 0$.

Recall that the "dot-product" of ***any*** two relativistic four-vectors is a ***Lorentz-invariant quantity***.

Thus, if we deliberately/consciously choose to evaluate $\eta_\mu\eta^\mu = \eta^\mu\eta_\mu = -\left(\eta^0\right)^2 + \vec{\eta}\cdot\vec{\eta}$ in the ***rest*** frame of an object, where $\vec{\eta}\cdot\vec{\eta} = 0$, $\beta = 0$, $\gamma = 1$ and $\eta^0 = c$, then:

$$\eta_\mu\eta^\mu = \eta^\mu\eta_\mu = -\left(\eta^0\right)^2 + \underbrace{\vec{\eta}\cdot\vec{\eta}}_{=0} = -\left(\eta^0\right)^2 = -c^2$$ = constant.

Note that $\eta^\mu\alpha_\mu = \eta^\mu\dfrac{d\eta_\mu}{d\tau}$ is ***also*** the "dot-product" of two relativistic four vectors $\{\eta^\mu$ and $\alpha_\mu\}$.

Note also that: $$\frac{d}{d\tau}\left(\eta^\mu\eta_\mu\right) = \frac{d\eta^\mu}{d\tau}\eta_\mu + \eta^\mu\frac{d\eta_\mu}{d\tau} = \alpha^\mu\eta_\mu + \eta^\mu\alpha_\mu = 2\alpha^\mu\eta_\mu$$

But: $\eta^\mu\eta_\mu = -c^2$ = constant (from above). Thus: $\dfrac{d}{d\tau}\left(\eta^\mu\eta_\mu\right) = \dfrac{d}{d\tau}\left(-c^2\right) = 0$ → $\therefore$ $\alpha^\mu\eta_\mu = 0$.

d.) Write the Minkowski / ***proper*** force version of Newton's 2nd law, $K^\mu = \dfrac{dp^\mu}{d\tau}$ in terms of the ***proper*** acceleration α^μ.

$$K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(m\eta^\mu\right) = m\frac{d\eta^\mu}{d\tau} = m\alpha^\mu$$

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e.) Evaluate the Lorentz-invariant 4-vector "dot product" $\boxed{K^{\mu}\eta_{\mu}}$:

$\boxed{K^{\mu}\eta_{\mu} = m\alpha^{\mu}\eta_{\mu}}$ but: $\boxed{\alpha^{\mu}\eta_{\mu} = 0}$ from part c.) above.

$\therefore \quad \boxed{K^{\mu}\eta_{\mu} = 0}$

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