# LECTURE NOTES 15

## Electrodynamics and Relativity

In the macroscopic "everyday" world $(v \ll c)$ that we are accustomed to living in, we know that the ***classical*** laws of mechanical physics obey Galileo's notion (or principle) of ***classical relativity***, as long as we are ***always*** in an ***inertial reference frame*** (*i.e.* a ***non-accelerating*** reference frame).

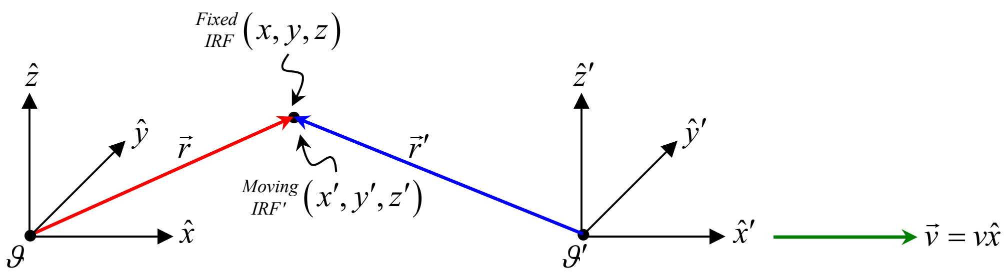
Newton's ***First Law of Motion*** holds in an ***inertial reference frame*** (IRF):

> "An object at rest remains at rest, and an object moving with (constant) speed $v$ remains moving at (constant) speed $v$ and in the same direction (*i.e.* $v =$constant in an IRF), unless acted upon by a net/non-zero/ unbalanced force" – but then, no longer in an IRF…

$$\Rightarrow \quad \vec{F}_{net} = 0 \text{ in an IRF } (\vec{a} = 0), \quad \underbrace{\vec{F}_{net} \neq 0 = m\vec{a}}_{\text{Newton's 2}^{\text{nd}}\text{ Law.}} \text{ in a non-IRF } (\vec{a} \neq 0)$$

In a ***Galilean transformation between*** two inertial references frames, *e.g.* one ***fixed*** $(v = 0)$ and one ***moving*** $(v' \neq 0)$ along the $\hat{x}$-axis. The two reference frames coincide at time $t = 0$:

**Fixed IRF:** **Moving IRF′:**



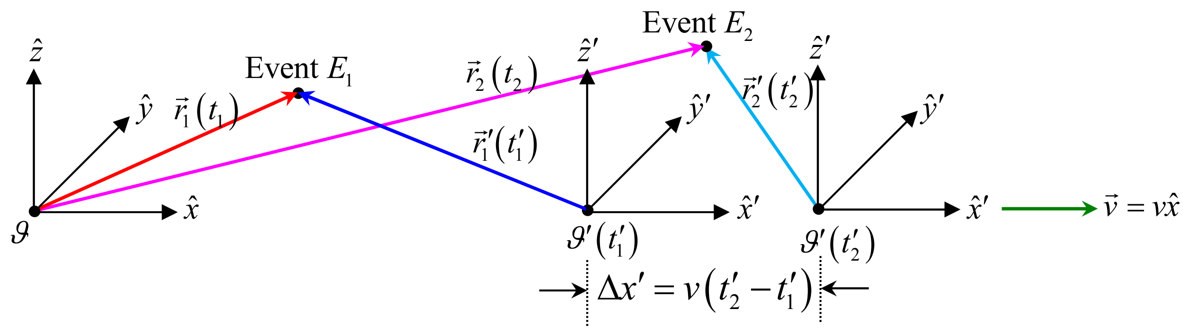

**Fixed IRF:** **Moving IRF′:**

$$\boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}} \qquad \boxed{\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}' = (x - vt)\hat{x} + y\hat{y} + z\hat{z}} \qquad \boxed{\hat{x}' \parallel \hat{x}}, \ \boxed{\hat{y}' \parallel \hat{y}}, \ \boxed{\hat{z}' \parallel \hat{z}}$$

$$\boxed{x' = x - vt}, \quad \boxed{y' = y}, \quad \boxed{z' = z} \quad \text{and:} \quad \boxed{t' \equiv t}$$

Two "events" $E_1$ and $E_2$ occurring in ***classical*** Galilean space-time:

**Fixed IRF:** **Moving IRF′:**




Event $E_1$ occurs at: $(\vec{r}_1,\ t_1)$ Event $E_1$ occurs at: $(\vec{r}_1',\ t_1')$

Event $E_2$ occurs at: $(\vec{r}_2, t_2)$ Event $E_2$ occurs at: $(\vec{r}_2', t_2')$

In a ***Galilean transformation***:

Separation distances (spatial intervals): $\boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}}$ and

Time differences (temporal intervals): $\boxed{\Delta t = t_2 - t_1}$

are the ***same*** / ***identical*** in ***all*** inertial reference frames, *i.e.* $\boxed{\Delta d' = \Delta d}$ and: $\boxed{\Delta t' = \Delta t}$.

Thus: $\boxed{\Delta d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}} = \boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}}$

And: $\boxed{\Delta t' = t_2' - t_1'} = \boxed{\Delta t = t_2 - t_1}$

No matter how fast an object is moving, in a ***Galilean transformation***, ***spatial*** separation ***distances*** are ***unchanged***/***constant*** and the ***rate of passage of time is also unchanged***/***constant***.

Note also that in ***classical*** Galilean physics ∃ there exists ***no*** notion of an ***absolute*** IRF.

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Does the ***principle of relativity*** also apply to ***electrodynamics***? *i.e.* are the physical ***laws*** of *E* & *M* also ***valid***/***same*** in all ***inertial reference frames***?

One might initially be tempted to say ***no***, because *e.g.* a stationary/fixed charge in one IRF₁ has only a ***static*** electric field $\vec{E}$ = constant associated with it, whereas an observer in another IRF₂ moving at constant velocity, $\vec{v}$ with respect to the first/fixed IRF₁ would see a magnetic field $\vec{B}$ associated with the (moving) charge. ⇒ *EM* theory pre-supposes ∃ a ***unique*** IRF (stationary) from which/with respect to which all velocities should be measured.

However, ***this*** notion ***is wrong***/***incorrect***!

**Another example**: Suppose we have a ***static magnetic field*** $\boxed{\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})}$ in (fixed) $IRF_1$ *e.g.* due to permanent bar magnet, $\boxed{\left|\vec{B}(\vec{r})\right| \sim 1/r^3}$ where $r$ = observer distance from the bar magnet. An observer in a moving $IRF_2$ passing by the permanent magnet will observe a (time varying) ***electric field*** due to $\boxed{\vec{E}(\vec{r}\,') = -\partial\vec{A}(\vec{r}\,')/\partial t}$ in her/his $IRF_2$.

**Another example**: In a ***fixed*** $IRF_1$, move a circular conducting loop (of radius $a$) *e.g.* at a constant velocity from a $\vec{B} = 0$ region into a $\vec{B} \neq 0$ region. Get an induced *EMF* $\boxed{\varepsilon_1^{IRF_1}(t) = -d\Phi_m(t)/dt}$. However, in $IRF_2$ ***of the loop*** (*i.e.*, imagine the observer is now sitting in the center of loop as the loop is moved from the $\vec{B} = 0$ region into the $\vec{B} \neq 0$ region), the observer in $IRF_2$ will "see" a time-varying magnetic field, which (by Faraday's Law) creates an electric field which induces {precisely} the ***same*** *EMF* (voltage/potential difference around the loop: $\boxed{\varepsilon_2^{IRF_2}(t') \equiv \varepsilon_1^{IRF_1}(t)}$.

Physicists in Maxwell's time (mid/late 1800's $\rightarrow$ early 1900's) grappled with the principle of relativity and electrodynamics – the consensus thinking at that time was that the $\vec{E}$ and $\vec{B}$-fields were "strains" in an invisible, all-pervasive/all-permeating medium known as the ***æther*** (a "jelly-like" substance, which ***also*** simultaneously had to be ~ infinitely rigid {because the speed of light, $c = \text{“}\sqrt{B/\rho}\text{”} = 1/\sqrt{\varepsilon_o\mu_o} = 3\times10^8\ m/s$ was already known to be very high at that time}. ***Transverse*** electromagnetic waves could ***not*** propagate without being ***immersed*** in such a "jelly-like" medium, or so they thought….

The "***absolute***" IRF, then, was the one in which the æther medium was at ***rest***, *i.e.* the ***rest*** frame of the æther.

Michelson and Morley's famous æther drift experiment carried out in the late 1880's – to accurately measure the earth's speed *w.r.t.* æther –was a ***null*** result!! They found that the speed of light $c$ was the ***same*** in ***all*** directions. This situation was not resolved for ~ 20 years, despite many theoretical and experimental efforts. All kinds of (***crazy***) things were proposed theoretically and investigated experimentally…. (*n.b.* ~ 100 years from now, perhaps some of ***today's*** current "theories" ***may also*** be viewed to be ***just*** as crazy …. $\Leftarrow$ ***think*** about this !!! )

It is certainly a credit to the genius and intellect of Albert Einstein, taking in all of what was then currently known theoretically and experimentally, to successfully develop his initial theory of ***special*** relativity (IRF's only) and then later, to ***general*** relativity (including ***non***-IRF's).

Einstein's two ***postulates*** of **special relativity**:

1) **Principle of Relativity**: Laws of physics apply/are the ***same*** in ***all*** IRF's

2) **Speed of light** $\boxed{c = 1/\sqrt{\varepsilon_o\mu_o} = 3\times10^8\ m/s}$ (in vacuum) ≡ ***same*** in ***all*** IRF's for ***all*** observers, ***regardless*** of the motion (*i.e.* the speed) of the source.

Einstein's 1st postulate elevates Galileo's Principle of Relativity (for classical mechanics) to encompass ***all*** physics. $\Rightarrow$ ∃ **NO** æther medium / ∃ **NO** ***absolute*** IRF for which *EM* waves "need" to propagate in.

Einstein's 2nd postulate has {even more} "brain-numbing" consequences for "mere mortals":

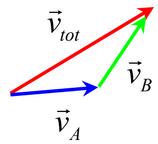
a.) ***Spatial*** intervals $\boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 z_1)^2}}$ *vs.* $\boxed{\Delta d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}}$
are ***NOT*** the same in ***different*** IRF's, *i.e.* $\boxed{\Delta d \neq \Delta d'}$ !!!

b.) ***Temporal*** intervals $\boxed{\Delta t = t_2 - t_1}$ *vs.* $\boxed{\Delta t' = t_2' - t_1'}$ are ***NOT*** the same in ***different*** IRF's, *i.e.* $\boxed{\Delta t \neq \Delta t'}$ !!!

In ***special relativity***, ***space and time*** are treated on an ***equal*** footing with each other, and so what ***is*** conserved/preserved is the so-called ***space-time interval***, as defined below:

$$\boxed{I \equiv (\Delta d)^2 - (c\Delta t)^2} = \boxed{I' \equiv (\Delta d')^2 - (c\Delta t')^2}$$

**In Galilean Relativity (Euclidean Space):**

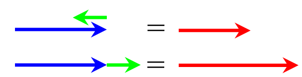



$\boxed{\vec{v}_{tot} = \vec{v}_A + \vec{v}_B}$ ⇐ The Galilean velocity addition rule is simple vector addition

*e.g.* A man walking down corridor of train at 5 mph ***relative*** to the train, but the train is moving at 50 mph relative to the ground:

$\boxed{v_{\text{ground}}^{\text{man}} = v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}$ parallel or anti-parallel velocity vector addition:

$$v_{\text{ground}}^{\text{man}} = 50 \pm 5 \text{ mph} = \begin{cases} 45 \text{ mph (man walking to the back of the train)} \\ 55 \text{ mph (man walking to the front of the train)} \end{cases}$$



In ***Galilean Relativity***, a beam of light emitted from a flashlight on a ***moving*** train will travel ***faster*** (or ***slower***) than a beam of light emitted from a flashlight on the ***ground***:

Flashlight on ***train***: $\boxed{v_{\text{ground}}^{\text{light}} = v_{\text{ground}}^{\text{train}} + c}$ $+\hat{x}$ direction; $\boxed{v_{\text{ground}}^{\text{light}} = v_{\text{ground}}^{\text{train}} - c}$ $-\hat{x}$ direction

*vs.*

Flashlight on ***ground***: $\boxed{v_{\text{ground}}^{\text{light}} = +c}$ $+\hat{x}$ direction; $\boxed{v_{\text{ground}}^{\text{light}} = -c}$ $-\hat{x}$ direction

Einstein's 2nd Postulate of Special Relativity says this ***doesn't*** happen!

⇒ **The Correct Einsteinian / Special Relativity Velocity Addition Formula (1-Dimension) is**:

Man on Train Problem:

$$\boxed{v_{\text{ground}}^{\text{man}} = \frac{v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \left(\dfrac{v_{\text{ground}}^{\text{train}} \cdot v_{\text{train}}^{\text{man}}}{c^2}\right)\right]}}$$

for ***parallel*** (+) and/or ***anti-parallel*** (−) velocity addition.

*n.b.* If $v' \ll c$, then in ***this*** limit, we obtain the **Galilean Velocity Addition Rule** – everyday world !

If the ***train*** is moving at the speed of light (with respect to the ***ground***) then: $v_{\text{ground}}^{\text{train}} = c$, and:

$$v_{\text{ground}}^{\text{man}} = \frac{c \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \dfrac{c \cdot v_{\text{train}}^{\text{man}}}{c^2}\right]} = \frac{\left(c \pm v_{\text{train}}^{\text{man}}\right)}{\left(1 \pm \dfrac{v_{\text{train}}^{\text{man}}}{c}\right)} = \frac{c\left(c \pm v_{\text{train}}^{\text{man}}\right)}{c\left(1 \pm \dfrac{v_{\text{train}}^{\text{man}}}{c}\right)} = \frac{c\cancel{\left(c \pm v_{\text{train}}^{\text{man}}\right)}}{\cancel{\left(c \pm v_{\text{train}}^{\text{man}}\right)}} = c \quad !!!$$

While Einstein was initially motivated by relativity issues concerning electrodynamics, relativity actually addresses the fundamental nature of ***space-time*** aspects of the universe in which we live – thereby encompassing ***all*** physics, ***all*** fundamental forces of nature!

As a consequence of this, the "speed of light", $c$ (in vacuum) is ***not*** just the maximum speed of *EM* waves /*EM* signals {***only***}, $c$ is the maximum speed of ***any/all*** waves/signals and/or particles, irrespective of their nature – because this maximum speed of propagation is associated with the fundamental nature of ***space-time*** itself!

Thus, the ***relativity*** of ***space-time*** tells us that: $c = c_{EM} = c_{\text{grav}} = c_{\text{weak}} = c_{\text{strong}}$
*i.e.* the speed of "light" $c$ is ***independent*** of/does ***not*** depend on the ***type*** of force!

However, in *E&M*, we ***do*** have the relation $c = 1\big/\sqrt{\varepsilon_o \mu_o} = c_{EM}$ where $\varepsilon_o$ and $\mu_o$ are ***macroscopic*** *EM* properties of the ***vacuum*** {"empty", matter-free space}:

$\mathscr{C} = C/\ell$: $\Rightarrow$ $\varepsilon_o = 8.85 \times 10^{-12}\left(Farads/m\right)$ = ***electric*** permittivity of free space/vacuum

$\mathscr{L} = L/\ell$: $\Rightarrow$ $\mu_o = 4\pi \times 10^{-7}\left(Henrys/m\right)$ = ***magnetic*** permeability of free space/vacuum

And: $Z_o = \sqrt{\mu_o/\varepsilon_o} = 120\pi\,\Omega \simeq 377\,\Omega$ = characterist impedance of free space/vacuum

The ***macroscopic*** *EM* parameters of free space, $\varepsilon_o$ and $\mu_o$ ***are*** intimately related/connected to the ***microscopic*** QED properties of the vacuum – *i.e.* the electrically-charged, virtual, particle-antiparticle pairs: $e^+e^-$, $\mu^+\mu^-$, $\tau^+\tau^-$, $d\bar{d}$, $u\bar{u}$, $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, $t\bar{t}$ and $W^+W^-$.

However, the virtual particle-antiparticle pairs of the vacuum ***also*** carry ***weak*** charges & ***weak*** moments (*all* spin-½ leptons & quarks, the spin-1 $W^{\pm}$, $Z^0$ bosons), ***strong*** charges & ***strong*** moments (*all* quark-antiquark pairs $\left(q\bar{q},\ q = d,u,s,c,b,t\right)$, and these particles also have ***mass***.

Thus: Q-Gravity: QWD: QCD: QED:

$$c_{\text{grav}} = 1\big/\sqrt{\varepsilon_{\text{g}}\mu_{\text{g}}}\,,\ = c_{\text{weak}} = 1\big/\sqrt{\varepsilon_{\text{w}}\mu_{\text{w}}}\,, = c_{\text{strong}} = 1\big/\sqrt{\varepsilon_{\text{s}}\mu_{\text{s}}}\,, = c_{EM} = 1\big/\sqrt{\varepsilon_o\mu_o} = c$$

and:

$$Z_{\text{grav}} = \sqrt{\frac{\mu_{\text{g}}}{\varepsilon_{\text{g}}}} = ??,\ Z_{\text{weak}} = \sqrt{\frac{\mu_{\text{w}}}{\varepsilon_{\text{w}}}} = ??,\ Z_{\text{strong}} = \sqrt{\frac{\mu_{\text{s}}}{\varepsilon_{\text{s}}}} = ??,\ Z_o^{EM} = \sqrt{\frac{\varepsilon_o}{\mu_o}} = 120\pi\,\Omega$$

$\Rightarrow$ Implies ***deep*** connections between the four fundamental forces of nature
(and the inter-relations between them – ***unification***?) – ***and*** – ***space-time***!!!

## The Geometry of Relativity:

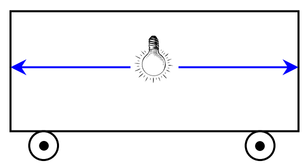
As mentioned earlier, Einstein's postulates (*esp*. #2) have several striking, "non-everyday" consequences:

a.) **Space** intervals $\Delta d$ are ***not*** the same in all IRF's $\Rightarrow$ Lorentz contraction in (boosted) IRF's
b.) **Time** intervals $\Delta t$ are ***not*** the same in all IRF's $\Rightarrow$ time dilation, and the relativity of simultaneity

In order to elucidate these phenomena, we consider a series of "gedanken" (thought) experiments:

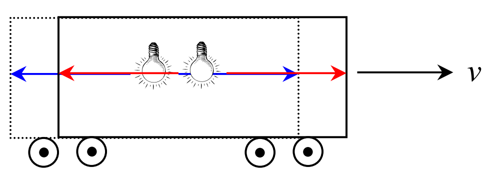
## The Relativity of Simultaneity:

**1st Gedanken Experiment**: Consider *e.g.* a freight train moving at ***constant*** speed, $v$ along a smooth, straight railroad track. In the center of one boxcar of the freight train is a light bulb, as shown in the figure below:

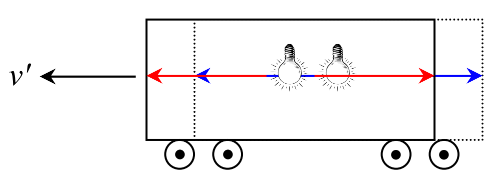


When the light bulb is switched on, light from the bulb spreads out in all directions / $4\pi$ steradians at the speed of light, $c$. Since the light bulb is ***equidistant*** from the two ends of the freight car, an ***observer on the train*** (*i.e.* in the IRF$'$ of the train) will find that the light reaches the front end of the freight car ***simultaneously*** with light reaching the back end of the freight car (consistent with our "everyday" world/Galilean experience) – these two "events" are ***simultaneous*** in ***this*** IRF$'$.

However, an ***observer on the ground***, watching the train go by with the light bulb turned on will ***not*** see these two "events" as being simultaneous. From his/her perspective, he/she sees the train moving forward, and therefore in this observer's IRF, since the beam of light heading towards the ***back*** of the freight car has a shorter distance to travel than the beam of light heading towards the ***front*** of the freight car, the observer on the ***ground*** will see the light hit the ***back*** of the freight car ***before*** the light hits the ***front*** of the freight car, as shown in the figure below:




***Another***, **3rd** ***observer on an express train*** (*e.g.* moving much faster than the ***freight*** train and moving in the ***same*** direction) "sees" the freight train from his/her IRF$''$ as "***going backwards***", hence would "see" the light from the light bulb hit the ***front*** of the freight car ***before*** the light hits the ***back*** of the freight car, as shown in the figure below:



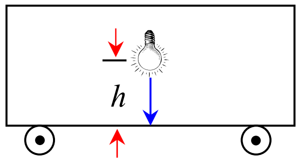

$\Rightarrow$ Two "events" that are simultaneous in one IRF are ***not*** in general simultaneous in ***other*** IRF's
*n.b.* If $c$ was *e.g.* 100 $m/s$ (and not $3\times10^8$ $m/s$), we ***all*** would have noticed/realized this, long ago...

**Time Dilation:**

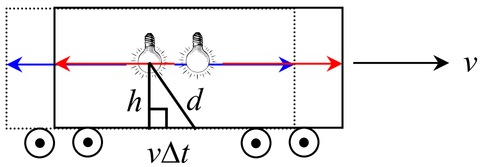
**2nd Gedanken Experiment**: How long of a time ***interval*** does it take for light to go from the light bulb *e.g.* to the ***floor*** of the freight car?

a.) To an observer ***on the train*** (*i.e.* in the ***rest*** frame/IRF′ of the train), he/she sees:




$\boxed{h = c\Delta t'}$ where $h$ = height of light bulb above the floor of the train, thus: $\boxed{\Delta t' = h/c}$

b.) To a ***stationary observer on the ground***/in the "***laboratory***" IRF, he/she sees:




$\boxed{v\Delta t}$ = ***horizontal*** distance that the train moves in time interval $\Delta t$ in the ***lab*** IRF.

Light from the ***ground observer's perspective*** (in ***lab*** IRF) travels a distance $d$ to reach the floor:

$\boxed{d = \sqrt{h^2 + (v\Delta t)^2}}$, which takes a time interval (in ***lab*** IRF): $\boxed{\Delta t = \frac{d}{c} = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c} > \Delta t' = \frac{h}{c}}$

for the light to hit the floor of freight car, from the ***ground observer's perspective*** (in ***lab*** IRF).

$\Rightarrow$ ***Moving*** clocks run ***slow*** $\Rightarrow$ ***time dilation*** !!!

Solve for the ***lab*** IRF time interval $\Delta t$: $\boxed{(c\Delta t)^2 = d^2 = h^2 + (v\Delta t)^2}$ $\Rightarrow$ $\boxed{(c^2 - v^2)\Delta t^2 = h^2}$

Thus: $\boxed{\Delta t = \frac{h}{\sqrt{c^2 - v^2}} = \frac{h/c}{\sqrt{1-(v/c)^2}} = \frac{h/c}{\sqrt{1-\beta^2}} = \gamma\, h/c}$ where: $\boxed{\beta \equiv v/c}$ and: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$

But: $\boxed{\Delta t' = h/c}$ $\therefore$ $\boxed{\Delta t = \frac{\Delta t}{\sqrt{1-\beta^2}} = \gamma\Delta t'}$ or: $\boxed{\Delta t' = \frac{1}{\gamma}\Delta t}$ where: $\boxed{1 \le \gamma \le \infty}$

$\boxed{\beta = 0}$ $\boxed{\beta = 1}$

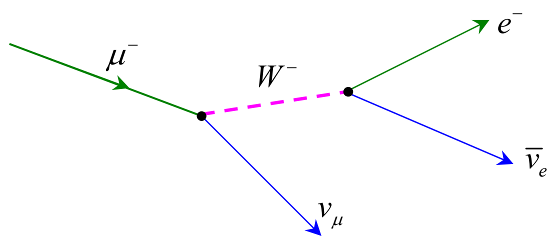
Thus: $\boxed{\Delta t = \gamma\Delta t' \ge \Delta t'}$

**Time Dilation (continued):**

**Griffiths Example 12.1:**

A muon $\left(\mu^{\pm}\right)$ is an unstable particle and has a mean lifetime of $\boxed{\tau'_{\mu} \simeq 2.2\mu s}$ in its ***rest frame*** IRF′.

Muons $\left.\begin{array}{l}\mu^{+} \rightarrow e^{+}\nu_{e}\bar{\nu}_{\mu} \\ \mu^{-} \rightarrow e^{-}\bar{\nu}_{e}\nu_{\mu}\end{array}\right\}$ decay via the charged weak interaction (mediated via the $W^{\pm}$ boson):




The muon is a heavier "cousin" of the electron – it has a rest mass of $m_{\mu} \simeq 105.66\, MeV/c^2$ $\left(m_e = 0.511 MeV/c^2\right)$ $\left(1 MeV/c^2 = 10^6\, eV/c^2\right)$

If a muon is traveling at 3/5 of the speed of light in the ***laboratory*** (*i.e*. the ground) ***reference frame***, then what is the mean lifetime of muon as observed in the ***lab reference frame*** {IRF}?

From above: $\boxed{\Delta t = \gamma \Delta t'}$ $\Rightarrow$ $\boxed{\tau_{\mu} = \gamma_{\mu}\tau'_{\mu}}$ where: $\boxed{\tau'_{\mu} \simeq 2.2\mu s}$. What is the Lorentz factor $\gamma_{\mu}$?

$\boxed{\beta_{\mu} = \frac{v}{c} = \frac{3}{5}}$ and: $\boxed{\gamma_{\mu} = \frac{1}{\sqrt{1-\beta_{\mu}^2}}}$

$$\therefore \quad \boxed{\gamma_{\mu} = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{1}{\sqrt{1-(9/25)}} = \frac{1}{\sqrt{16/25}} = \frac{1}{\sqrt{(4/5)^2}} = \frac{5}{4} \quad (\geq 1)}$$

$\therefore$ $\boxed{\tau_{\mu} = \gamma_{\mu}\tau'_{\mu} = \frac{5}{4}\tau'_{\mu}}$ *i.e.* $\boxed{\tau_{\mu} = \frac{5}{4} \times 2.2\mu s = 2.75\mu s}$ in the ***lab*** frame {IRF}.

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At Fermilab, beams of muons *e.g.* with momentum of $p_{\mu} = 211.32\ GeV/c$ $\left(1\,GeV = 10^9\,eV\right)$ can easily be produced. What is the mean lifetime of ***these*** muons, as observed in the ***lab frame***?

Again: $\boxed{\tau_{\mu} = \gamma_{\mu}\tau'_{\mu}}$, and the ***relativistic*** momentum of the muon in the ***lab*** IRF is:

$$\boxed{p_{\mu}c = \gamma_{\mu}\beta_{\mu}m_{\mu}c^2 = 211.32\, GeV}$$

$$\therefore \quad \boxed{\gamma_{\mu}\beta_{\mu} = \frac{p_{\mu}c}{m_{\mu}c^2} = \frac{211.32\, GeV}{105.66\, MeV} = 2.0 \times 1000 = 2000}$$

Thus: $\boxed{\gamma_{\mu}\beta_{\mu} = \frac{\beta_{\mu}}{\sqrt{1-\beta_{\mu}^2}} = 2000}$ or: $\boxed{\frac{\beta_{\mu}^2}{1-\beta_{\mu}^2} = (2000)^2}$

Solve for $\beta_{\mu}$: $\boxed{\beta_{\mu}^2 = (2000)^2\left(1-\beta_{\mu}^2\right) = (2000)^2 - (2000)^2\beta_{\mu}^2}$ $\Rightarrow$ $\boxed{\left\{(2000)^2 + 1\right\}\beta_{\mu}^2 = (2000)^2}$

Thus: $\boxed{\beta_\mu^2 = \frac{(2000)^2}{(2000)^2+1}}$ $\Rightarrow$ $\boxed{\beta_\mu = \sqrt{\frac{(2000)^2}{(2000)^2+1}}}$ or: $\boxed{\beta_\mu = 0.\underbrace{999999}_{6\text{ nines}} = \frac{v_\mu}{c}}$ $\Rightarrow$ $\boxed{v_\mu = 0.999999875c}$

$\therefore$ $\boxed{\gamma_\mu = \frac{1}{\sqrt{1-\beta_\mu^2}} \simeq \gamma_\mu\beta_\mu = 2000 \text{ !!!}}$

$\therefore$ $\boxed{\tau_\mu = \gamma_\mu \tau'_\mu = 2000 \times 2.2\mu s = 4.4\ ms = 0.0044\ sec}$ in the ***lab*** IRF !!!

The ratio $\tau_\mu / \tau'_\mu = \gamma_\mu = 2000 \Rightarrow$ a muon with ***lab*** momentum $p_\mu = 211.32\ GeV/c$ lives ***on average*** **2000×** ***longer*** in the ***lab frame*** {IRF} than in its ***own rest frame*** {IRF′} !!!

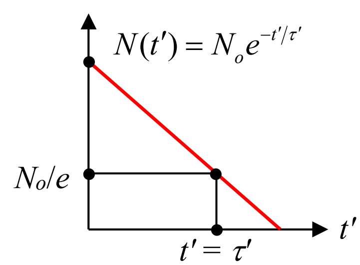
The "***proper***" ***decay length*** of an unstable particle (such as the muon) is defined as: $\boxed{\ell' \equiv c\tau'}$, where $\tau'$ = mean lifetime {*a.k.a.* "***proper***" lifetime}of the particle in its ***own rest frame*** {IRF′}.

*n.b.* In the particle's ***own rest frame*** {IRF′}, the particle is at ***rest***, so it has ***no*** decay length!

Thus, the ***proper*** decay length $\boxed{\ell' \equiv c\tau'}$ has ***no*** meaning ***for a particle at rest/in its own rest frame***.

The decay time ***distribution*** in the unstable particle's ***rest frame*** IRF′ is: $\boxed{N(t') = N_o e^{-t'/\tau'}}$ where $N_o$ = # of particles at $t$ = 0.

Semi-log plot of $N(t')$ *vs*. $t'$




For a muon with mean/proper lifetime $\boxed{\tau'_\mu \simeq 2.2\mu s}$ the "***proper***" ***decay length*** of the muon is:

$\boxed{\ell'_\mu \equiv c\tau'_\mu = 3\times10^8\ m/s \times 2.2\mu s = 6.6\times10^2\ m = 660 \text{ meters}}$.

Since $\boxed{\tau = \gamma\tau'}$, the decay time distribution of an unstable particle to an observer in the ***lab*** IRF is: $\boxed{N(t) = N_o e^{-t/\tau} = N_o e^{-t/\gamma\tau'}}$.

At the end of ***these*** lecture notes (see *p*.23), we show that the corresponding decay length for an unstable particle views by an observer in ***lab*** IRF is: $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau'}$. Note that when $\boxed{v \to 0}$, $\boxed{\beta = v/c \to 0}$, $\boxed{\gamma = 1/\sqrt{1-\beta^2} \to 1}$ and thus: $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau' \to 0}$ {as it should}.

Thus at Fermilab, for a beam of muons with $p_\mu = 211.32\ GeV/c$, they will travel a mean distance of: $\boxed{\ell_\mu = \gamma_\mu\beta_\mu\ell'_\mu = \gamma_\mu\beta_\mu c\tau'_\mu = 2000\ell'_\mu = 2000\times660\ m = 13.2\ km}$ before the beam of muons decay to $1/e = 0.368$ of their initial number, ***as seen in the lab frame***.

$\Rightarrow$ Explains why FNAL has ***lots*** of shielding to "range out" (*i.e.* absorb) the muons ***after*** passing through the HEP experiments that are using them for studies.

*n.b.* Since $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau'}$, then $\boxed{\ell = \ell' = c\tau'}$ occurs when $\boxed{\gamma\beta = 1}$ and thus $\boxed{\ell = \ell' = c\tau'}$ is the distance a beam of unstable particles travel in the lab frame before their # falls to $1/e = 0.368$ their initial #, traveling with $\boxed{\gamma\beta = 1}$.

Note further that since: $\boxed{pc = \gamma\beta mc^2}$ we also see that: $\boxed{\gamma\beta = 1}$ corresponds to: $\boxed{\gamma\beta = pc/mc^2 = 1}$.

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## Griffiths Example 12.2 The Twin Paradox: Time Travel Into The Future *Is* Possible!!!

A pair of identical twins celebrate their **21st** birthday by saying goodbye to each other. One is a medical doctor, the other, an astronaut. The astronaut blasts off in a rocket ship at a {constant} speed of $\boxed{v = \beta c = \tfrac{12}{13}c}$ and heads out towards $\alpha$ -centauri. After 5 years on ***her*** watch, she turns around and heads back to earth at the same {constant} speed to rejoin her ***twin sister*** doctor, who ***stayed at home on earth***.

**How old is each twin at their reunion**?

The ***traveling astronaut twin*** has aged 5 + 5 = 10 years in ***her own*** IRF' in making this round trip, and thus she arrives back home on her **31st** birthday.

However, ***viewed from her twin sister's earth-bound*** "IRF", the ***astronaut's*** clock has been running ***slower*** by a factor of:

$$\boxed{\gamma = \frac{\Delta t}{\Delta t'} = 1\Big/\sqrt{1-\beta^2} = 1\Big/\sqrt{1-\left(\tfrac{12}{13}\right)^2} = 1\Big/\sqrt{\left(\tfrac{169-144}{169}\right)^2} = \frac{13}{5} = 2.6}$$

Hence, the ***elapsed*** time for the ***astronaut twin*** to make the round trip, as ***viewed from the earth's*** "IRF" is therefore:

$$\boxed{\Delta t = \gamma\Delta t' = \frac{13}{5}\times 10 = \frac{130}{5} = 26 \text{ years}}$$

Thus, the astronaut's ***earth-bound twin sister*** (the doctor, who ***stayed*** on earth) is now herself 21 + 26 = **47** years old, *i.e.* the ***earth-bound doctor twin*** is celebrating her **47th** birthday.

$\Rightarrow$ Thus, the ***earth-bound doctor twin*** is 47 – 31 = **16** years ***older*** than her ***astronaut twin***!!!

The ***paradox*** associated with ***this*** situation arises when it is viewed from the ***astronaut twin's*** IRF – from ***her*** perspective, the ***astronaut twin*** sees the ***earth*** fly off away from her at $\boxed{v = \beta c = \tfrac{12}{13}c}$, then turn around after 5 years, and come back toward her at the same speed.

Thus, from the ***astronaut twin's*** perspective, ***she*** is at rest and her ***doctor twin*** is the one who is in ***motion*** – so shouldn't it be that the ***doctor twin*** is ***younger***? (*i.e.* < **31** years old)?

This paradox ***is*** resolved by consideration of ***which*** of the two twins ***actually experienced accelerations*** during the experiment. If both twins had stayed on earth, neglecting the earth's rotation and gravity, then they both would have experienced/undergone ***no*** accelerations.

 

However, the ***astronaut twin does*** experience ***accelerations/decelerations*** while on her rocket ship trip to $\alpha$-centauri and back – ***she*** has to ***go out***: accelerating from $0 \rightarrow \frac{12}{13}c$, then decelerating $\frac{12}{13}c \rightarrow 0$, then ***come back***: accelerating $0 \rightarrow \frac{12}{13}c$, then decelerating $\frac{12}{13}c \rightarrow 0$.

Thus, the ***traveling astronaut twin*** is ***not*** in an IRF at ***all*** times during her trip, while her earth-bound sister ***is*** in an "IRF" (neglecting earth's rotation and gravity) at ***all*** times.

The ***astronaut twin cannot*** claim to be an observer/in an IRF at ***all*** times, ***because*** of the ***accelerations/decelerations*** she experiences on her journey towards $\alpha$-centauri and back to earth.

See/work Griffiths problem 12.16 on how to analyze this problem correctly from astronaut twin's perspective.

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**An Actual Twin Paradox Experiment:**

A twin paradox experiment was carried out in the early 1970's using very high-precision cesium beam atomic clocks. Four commercial aircraft were flown around the world twice, two going ***east***, and two going ***west***. The atomic clocks were compared before and after each journey with identical {stationary} atomic clocks at the U.S. Naval Observatory.

Making allowances for the ***earth's rotation*** (*n.b.* we're actually living in a ***non***-IRF!!!) and the ***gravitational red shift*** (due to the earth's gravitational field – a general relativistic effect!), the ***average*** observed *vs*. calculated time differences of the aircraft-based clock *vs*. the ground-based clock, in nanoseconds was:

**Eastward Trip:** $\boxed{\Delta t_E^{obs} = -59 \pm 10 \text{ ns (observed)}}$ *vs.* $\boxed{\Delta t_E^{pred} = -40 \pm 23 \text{ ns (predicted)}}$

**Westward Trip:** $\boxed{\Delta t_W^{obs} = +273 \pm 7 \text{ ns (observed)}}$ *vs.* $\boxed{\Delta t_W^{pred} = +275 \pm 21 \text{ ns (predicted)}}$

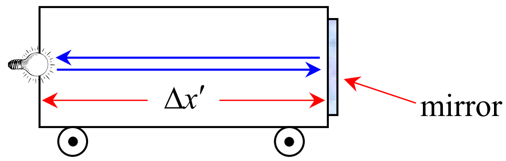
The kinematic effect of ***special*** relativity is $\approx$ comparable to that associated with the ***general*** relativistic effect.

$\Rightarrow$ **Moral of the story: Always** fly **eastward** – you will live {***hundreds*** of nanoseconds} ***longer*** !!!

See/read J.C. Hafele and R.E. Keating, Science 177, *p*.166-168 (**1972**) for further details.

## Lorentz Contraction:

**3rd Gedanken Experiment:** In the IRF′ of the freight train's boxcar, set up the boxcar such that the light bulb is at one end of the boxcar, a mirror at the other end, such that light signal can be sent down and back, as shown in the figure below:



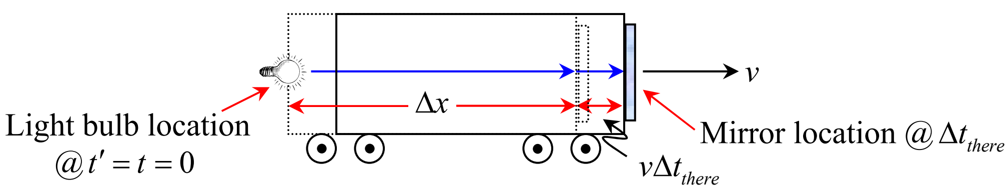

In the IRF′ of the freight train's boxcar, how long does it take the light signal to complete a round trip?

For an observer at ***rest*** on the train/in the train's IRF′, the round-trip time is $\boxed{\Delta t' = \Delta t'_{there} + \Delta t'_{back}}$:

$$\boxed{\Delta t' = \Delta t'_{there} + \Delta t'_{back} = \frac{\Delta x'}{c} + \frac{\Delta x'}{c} = 2\frac{\Delta x'}{c}}$$

For a ***stationary*** observer on the ***ground***/in the ground IRF, the round-trip time is $\boxed{\Delta t = \Delta t_{there} + \Delta t_{back}}$:

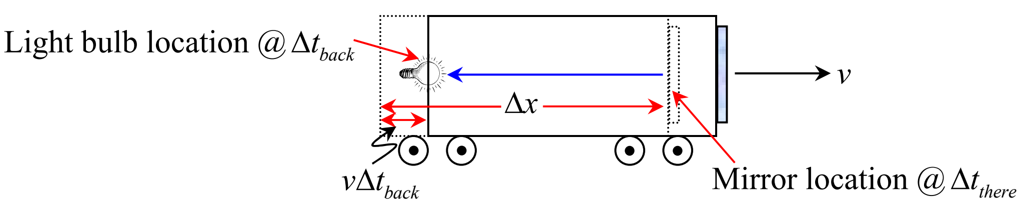
*a*.) The time interval required for the light signal to travel from the light bulb to the mirror = $\Delta t_{there}$ :




$$\boxed{\Delta t_{there} = \frac{\Delta x}{c} + \frac{v\Delta t_{there}}{c} = \frac{\Delta x + v\Delta t_{there}}{c}} \quad n.b. \quad \boxed{\Delta x \neq \Delta x'} \text{ and } \boxed{\Delta t \neq \Delta t'} \text{ !!!}$$

Solve for $\Delta t_{there}$ : $\boxed{\Delta t_{there} - \frac{v}{c}\Delta t_{there} = \left(1 - \frac{v}{c}\right)\Delta t_{there} = (1-\beta)\Delta t_{there} = \frac{\Delta x}{c}}$ $\therefore$ $\boxed{\Delta t_{there} = \frac{\Delta x}{c}\frac{1}{(1-\beta)}}$

*b*.) The time required for the light signal to travel from the mirror back to the light bulb = $\Delta t_{back}$ :




$$\boxed{\Delta t_{back} = \frac{\Delta x}{c} - \frac{v\Delta t_{back}}{c} = \frac{\Delta x - v\Delta t_{back}}{c}}$$

Solve for $\Delta t_{back}$ : $\boxed{\Delta t_{back} + \frac{v}{c}\Delta t_{back} = \left(1 + \frac{v}{c}\right)\Delta t_{back} = (1+\beta)\Delta t_{back} = \frac{\Delta x}{c}}$ $\therefore$ $\boxed{\Delta t_{back} = \frac{\Delta x}{c}\frac{1}{(1+\beta)}}$

∴ For a stationary observer on the ground/in the ground IRF, the round-trip time is:

$$\Delta t = \Delta t_{there} + \Delta t_{back} = \frac{\Delta x}{c}\frac{1}{(1-\beta)} + \frac{\Delta x}{c}\frac{1}{(1+\beta)} = \left[\frac{1}{(1-\beta)} + \frac{1}{(1+\beta)}\right]\frac{\Delta x}{c}$$
$$= \left[\frac{(1+\beta)+(1-\beta)}{(1-\beta)(1+\beta)}\right]\frac{\Delta x}{c} = \left[\frac{1+\cancel{\beta}+1-\cancel{\beta}}{(1-\beta^2)}\right]\frac{\Delta x}{c}$$
$$= 2\frac{1}{(1-\beta^2)}\frac{\Delta x}{c} = 2\gamma^2\frac{\Delta x}{c}$$

with: $\beta \equiv \frac{v}{c}$, $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

Round trip time as seen by an observer on ground/in the ***ground*** IRF: $\Delta t = 2\gamma^2\left(\frac{\Delta x}{c}\right)$

Round trip time as seen by an observer on the train/in the ***train*** IRF′: $\Delta t' = 2\left(\frac{\Delta x'}{c}\right)$

But the time dilation formula is: $\Delta t = \gamma \Delta t'$ or: $\Delta t' = \frac{1}{\gamma}\Delta t$

($\Delta t$: on ground, $\Delta t'$: on train; $\Delta t'$: on train, $\Delta t$: on ground)

Since: $\Delta t = 2\gamma^2\left(\frac{\Delta x}{c}\right)$ = Round trip time as seen by observer on the ***ground***.

And: $\Delta t' = 2\left(\frac{\Delta x'}{c}\right)$ = Round trip time as seen by observer on the ***train***, and $\Delta x \neq \Delta x'$

And: $\Delta t = \gamma \Delta t'$ = Time dilation formula. Thus: $\Delta t = \gamma \Delta t' = 2\gamma^2\left(\frac{\Delta x}{c}\right)$ or: $\Delta t' = 2\gamma\left(\frac{\Delta x}{c}\right)$

But we also have: $\Delta t' = 2\left(\frac{\Delta x'}{c}\right)$. Equating the two $\Delta t'$ expressions: $2\left(\frac{\Delta x'}{c}\right) = 2\gamma\left(\frac{\Delta x}{c}\right)$

$\Rightarrow$ $\Delta x' = \gamma \Delta x$ or: $\Delta x = \frac{1}{\gamma}\Delta x'$

($\Delta x'$: on train, $\Delta x$: on ground; $\Delta x$: on ground, $\Delta x'$: on train)

$\Delta t' = \frac{1}{\gamma}\Delta t$ or: $\Delta t = \gamma \Delta t'$

Therefore, for a stationary observer on the ground/in the ground IRF:

**Lorentz Contraction:** A ***moving*** meter stick is ***shortened***: $\Delta x = \frac{1}{\gamma}\Delta x'$ *n.b. **only*** along the direction of motion !!!

**Time Dilation:** A ***moving*** clock runs ***slow***: $\Delta t = \gamma \Delta t'$

Where $\Delta x$, $\Delta t$ = length and time interval in the rest frame of meter stick and clock. = ***ground*** IRF {***here***}

However, for an observer in the rest frame {IRF′} of the ***railroad car*** – his/her meter sticks are ***contracted*** by the same Lorentz factor $\gamma$, so all of his/her ***spatial*** measurements that he/she makes in the rest frame of the box car will come out the ***same*** as if the box car were at ***rest*** (with respect to the ***ground***) !!!

The spatial $H\times W\times L$ dimensions of the box car ***cannot*** change/be different moving *vs*. at rest!

From the observer's perspective/IRF′ on the ***train***, it's the objects on the ***ground*** that are ***shortened***!

How is it possible that ***both*** observers (A on the train, B on the ground) could be correct??

**They both are !!! Huh???**

We simply need to examine the ***details*** of the process whereby a length is ***actually measured***.

In order to measure the length of a board at ***rest*** (*w.r.t.* the measurer/observer), one simply lays a ruler down along it and measures it – *i.e.* record the readings of the ruler at each end of the board and subtract them: $\boxed{\Delta x = x_2 - x_1}$.

However, if the board is ***moving***, then one ***must*** read/measure the locations of the ends of the board at the ***same instant of time*** (in the ***measurer/observer's reference frame***).

⇒ Due to the ***simultaneity of relativity***, two observers (in two ***different*** IRF's) will ***disagree*** on what constitutes the "***same instant of time***" !!!

When the person on the ***ground*** measures the length of a ***moving*** box car, he/she reads the position of the two ends of the boxcar at the ***same instant*** of time in his/her IRF.

When the person on the ***train*** watches the person on the ***ground*** making this measurement, the train observer see the ***ground person*** reading the ***front-end*** of the box car ***first***, and the ***rear-end*** of the box car ***second*** – *i.e.* the observer on the ***train*** sees these ground-based measurements take place at ***different times/non-simultaneously*** !!!

***Both*** observers measure ***lengths*** in their ***own/respective*** IRF's ***correctly*** and ***each*** discover the ***other's*** meter stick to be ***shortened*** !!!

⇒ There is ***no*** inconsistency in this from the perspective of special relativity – it is {simply} a ***feature*** of special relativity !!!

**Griffiths Example 12.3: The Barn & Ladder Paradox – Another Gedanken Experiment:**

*n.b.* ∃ no ***direct*** experimental ***macroscopic*** confirmation of ***Lorentz contraction*** !!!
(unlike time dilation – which ***has*** been experimentally verified).

The technology doesn't yet exist to accelerate ***macroscopic*** objects to relativistic speeds, $v \approx c$ .

A farmer has a ladder that is ***too long*** to store in his barn.

The farmer figures that if he gets *e.g.* his daughter (who run much faster than the farmer) to move the ladder into barn at a high enough speed, the ladder would Lorentz-contract, and thus the fast-moving ladder would then fit into barn!

The farmer would then slam the barn door shut at instant the ***whole*** ladder fits into the barn.

However, the farmer's daughter pointed out to him that from her ***moving*** reference frame {IRF′}, the ***barn*** would ***Lorentz contract***, ***not*** the ladder!! ⇒ Thus, from ***her*** perspective (*i.e.* ***her*** IRF) the problem of fitting the ladder into barn would be ***aggravated*** by running relativistically fast into the barn with the ladder !!!

Who's right??? Will the ladder fit or won't it???

Again, they're ***both*** correct!!!

The statement: "the ladder is in the barn" means that ***all*** parts of the ladder (including both ends) ***are inside the barn at the same instant of time***. ⇐ This a condition that depends on the ***observer*** – *i.e.* ***his/her*** IRF.

There are ***two*** relevant "events" in ladder – barn problem:

*a*.) The ***back*** end of the ladder makes it in the door of barn.

*b*.) The ***front*** end of the ladder hits far wall of barn.

The farmer, at rest in the ***barn's*** IRF, sees *a*.) occur ***before*** *b*.) !!!

The daughter, at rest in the ***ladder's*** IRF′, sees *b*.) occur ***before*** *a*.) !!!

Contradiction??? ***No*** !!!

Just due to a difference in the ***simultaneity*** of "event" times due to ***two different*** IRF's!!!

**The Nature of Lorentz Contraction:**

A moving object is shortened ***only*** ***along*** (*i.e.* ***parallel*** to) its ***direction of motion/velocity vector***.

Spatial dimensions ***transverse/perpendicular*** to the direction of motion/velocity vector are ***not*** contracted/***not*** affected.

**A Gedanken Experiment for Lorentz-Noncontraction in the Transverse Direction(s):**

Is the ***height*** of a railroad boxcar {running along straight, horizontal railroad track} the ***same*** in all IRF's???

Build a tall wall along the side of the railroad tracks. Bring up the railroad car and stop it at the wall and mark its height $h$ on the wall, *e.g.* using ***blue*** paint. Now back off the train, get a running relativistic start and use *e.g.* ***red*** paint to mark the height of the train boxcar on the wall as the train flies past the wall.

***If*** the ***transverse*** directions ***are*** Lorentz-contracted, an observer on the ***ground*** would predict the ***red*** line to be ***lower*** than the ***blue*** one, whereas an observer on the ***train*** would predict the ***opposite*** – *i.e.* the ***red*** line would be ***higher*** than the ***blue*** line. Which line is ***lower***?

The principle of relativity says both observers/both IRF's are ***equally*** justified, but (here) both ***cannot*** be correct.

**The answer:** The red and blue lines ***exactly*** coincide $\Rightarrow$ there is **no** Lorentz contraction (or expansion) in the ***transverse*** direction.

Lorentz contraction ***only*** occurs in the ***longitudinal*** direction/along the direction of motion!

**Lorentz Transformations:**

Any physical process consists of one or more "events".

An "event" is something that occurs at a specific location ($x$, $y$, $z$) in space at a precise instant in time ($t$), *e.g.* the explosion of a firecracker.

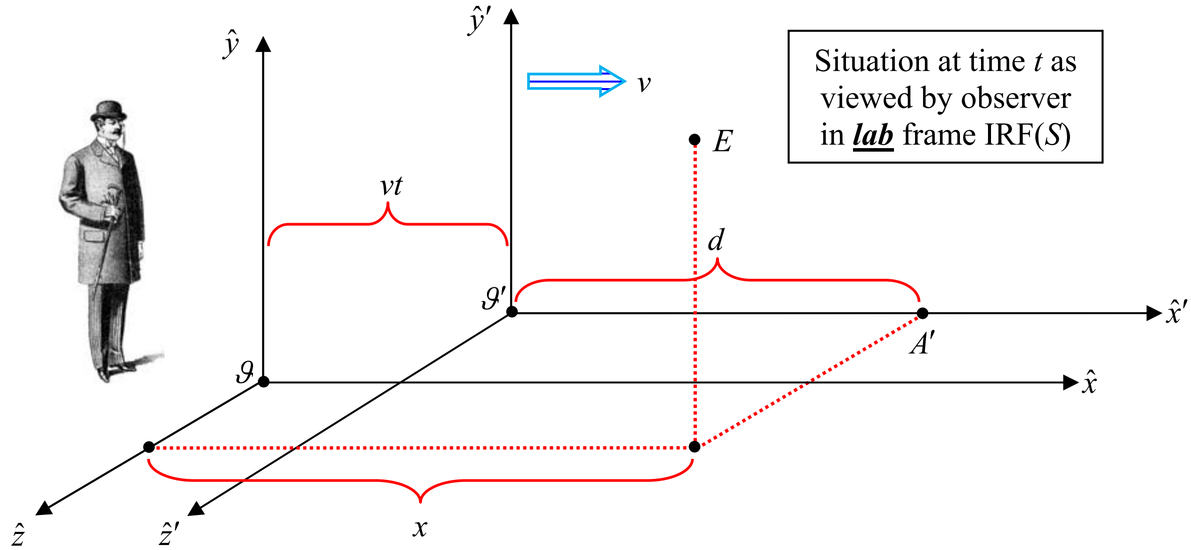
A ***Lorentz Transformation*** is the mathematical prescription for (properly) transforming the physics associated with an "event" as seen/observed in one IRF to another IRF.

*e.g.* Suppose we have {4-dimensional} ***space-time*** coordinates $(x, y, z, t)$ in IRF($S$) for an event, "$E$", and we want to know the space-time coordinates $(x', y', z', t')$ for the ***same*** event "$E$" in ***another*** IRF($S'$).

Suppose ***moving*** frame IRF($S'$) is moving at a velocity $\boxed{\vec{v} = v\hat{x}}$ ***relative*** to ***lab*** frame IRF($S$). The axes of the two coordinate systems coincide at time $t = 0$, *i.e.* the instant when the two origins $\vartheta$ and $\vartheta'$ coincide, as shown in the figure below:

**The Situation as Seen by an Observer in Lab Frame IRF(*S*) at Time *t*, for Event "*E*":**




At time $t$, the origin $\vartheta'(\text{in IRF}(S'))$ will be a distance $vt$ from the origin $\vartheta(\text{in IRF}(S))$.

$\therefore \boxed{x = d + vt}$ where $d$ = distance from $A'$ to $\vartheta'$ at time $t$, when event "$E$" occurs, as measured/observed in ***lab*** IRF($S$).

If we carry out a ***Galilean*** transformation from ***lab*** IRF($S$) to ***moving*** IRF($S'$):

$\boxed{d = x' = x - vt}$ $\leftarrow$ ***no*** spatial contraction along direction of motion.

$\boxed{y' = y}$

$\boxed{z' = z}$

$\boxed{t' = t}$ $\leftarrow$ ***no*** time dilation.

Where does the Galilean transformation ***go wrong***? $\boxed{d = x'}$ {and $\boxed{t' = t}$}.

$d$ = distance from $A'$ to $\vartheta'$ as measured in the ***lab*** frame IRF($S$)
$x'$ = distance from $A'$ to $\vartheta'$ as measured in ***moving*** frame IRF($S'$)

**Factually:** $\boxed{d \neq x'}$ (because of (special) relativity) !!!

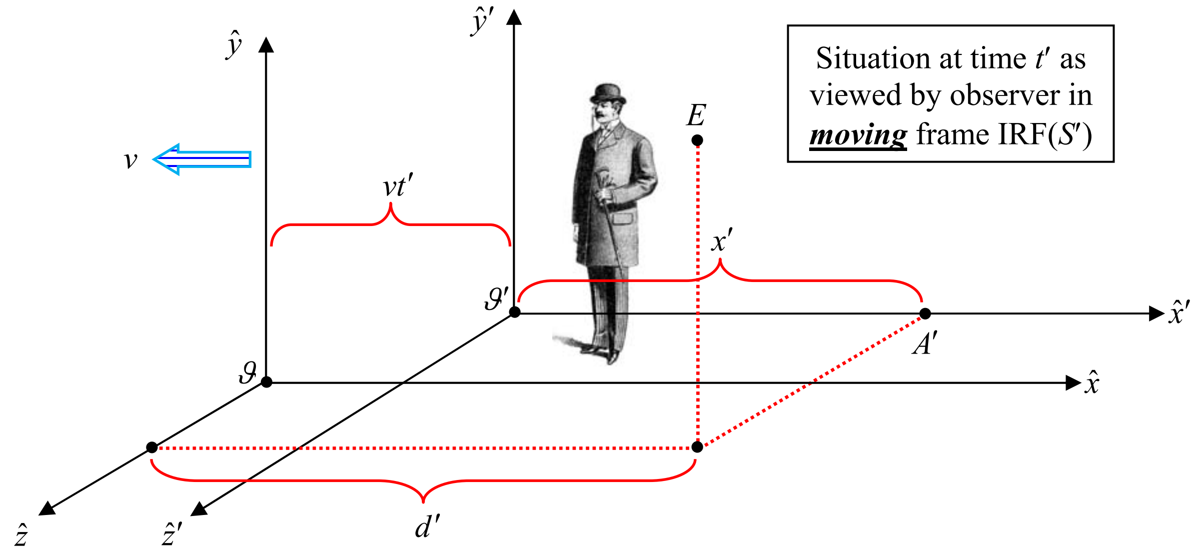
Because $A'$ and $\vartheta'$ are at ***rest*** in the ***moving*** IRF($S'$), from the perspective of an observer in the ***lab*** frame IRF($S$), $x'$ is the "moving meter stick", which appears ***Lorentz-contracted*** to an observer in the ***lab*** frame IRF(S).

Thus: $\boxed{d = \frac{1}{\gamma} x'}$ where: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$ and: $\boxed{\beta = \frac{v}{c}}$

Then: $\boxed{d = \frac{1}{\gamma} x' = x - vt}$ or: $\boxed{x' = \gamma(x - vt)}$

However, the ***same*** argument can be made from the perspective of an observer in the ***moving*** frame IRF($S'$), as shown in the figure below:

**The Situation as Seen by an Observer in Moving Frame IRF($S'$) at Time $t'$, for Event "$E$":**




If observers in IRF($S$) and IRF($S'$) both start their clocks $t = t' = 0$ when the origins $\vartheta$ and $\vartheta'$ coincide, then at time $t'$ in ***moving*** IRF($S'$), origin $\vartheta$ will be a distance $vt'$ from $\vartheta'$ and thus: $\boxed{x' = d' - vt'}$ where $d'$ = the distance from $A'$ to $\vartheta$ as measured in ***moving*** frame IRF($S'$) when event "$E$" occurs at time $t'$ as measured in ***moving*** frame IRF($S'$).

The times $t$ (in ***lab*** IRF($S$)) and $t'$ (in ***moving*** IRF($S'$) represent the same physical instant for event "$E$", viewed/observed from/in these two IRF's, respectively.

$x$ = distance from $A'$ to $\vartheta$ as measured in the ***lab*** frame IRF($S$) {drawing on previous page}.
$d'$ = distance from $A'$ to $\vartheta$ as measured in ***moving*** frame IRF($S'$).

Because $A'$ and $\vartheta$ are at rest in the ***lab*** frame IRF($S$), then {***here***} $x$ is the "moving meter stick" which appears ***Lorentz-contracted*** in the ***moving*** frame IRF($S'$).

Thus: $\boxed{d' = \frac{1}{\gamma} x}$ where: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$ and: $\boxed{\beta = \frac{v}{c}}$

Then: $\boxed{x' = d' - vt' = \frac{1}{\gamma} x - vt'}$ or: $\boxed{x = \gamma\left(x' + vt'\right)}$

**Time Dilation:**

We have: (1) $\boxed{x' = \gamma\left(x - vt\right) = \gamma x - \gamma vt}$

(2) $\boxed{x = \gamma\left(x' + vt'\right) = \gamma x' + \gamma vt'}$

Insert (1) into (2) and solve for $t'$ in terms of $t$ and $x$:

$$\boxed{x = \gamma x' + \gamma vt' = \gamma\left(\gamma x - \gamma vt\right) + \gamma vt' = \gamma^2 x - \gamma^2 vt + \gamma vt'}$$

Or: $\boxed{\gamma vt' = \gamma^2 vt - \gamma^2 x + x = \gamma^2 vt - \left(\gamma^2 - 1\right)x}$ $\Leftarrow$ divide both sides by $\gamma v$

$$t' = \gamma t - \frac{(\gamma^2 - 1)}{\gamma v} x$$ $\Leftarrow$ multiply both sides by $c$

$$ct' = \gamma ct - \frac{(\gamma^2 - 1)}{\gamma}\left(\frac{c}{v}\right) x = \gamma ct - \frac{(\gamma^2 - 1)}{\gamma\beta} x$$ where: $\beta = \frac{v}{c}$ and: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\gamma^2 = \frac{1}{1-\beta^2}$

$$ct' = \gamma ct - \frac{\left(\frac{1}{1-\beta^2} - 1\right)}{\gamma\beta} x = \gamma ct - \frac{\left(\frac{1}{1-\beta^2} - \frac{1-\beta^2}{1-\beta^2}\right)}{\gamma\beta} x$$

$$ct' = \gamma ct - \frac{\left(\frac{\not{1} - \not{1} + \beta^2}{1-\beta^2}\right)}{\gamma\beta} x = \gamma ct - \frac{\left(\frac{\beta^2}{1-\beta^2}\right)}{\gamma\beta} x = \gamma ct - \frac{\gamma^{\not{2}}\beta^{\not{2}}}{\not{\gamma}\,\not{\beta}} x = \gamma ct - \gamma\beta x = \gamma(ct - \beta x)$$

$\therefore$ $$ct' = \gamma(ct - \beta x)$$

Likewise, if we insert (2) into (1) and {instead} solve for $t$ in terms of $t'$ and $x'$, we obtain:

$$ct = \gamma(ct' + \beta x')$$

Thus, we now have all the ingredients needed for specifying our Lorentz transformation(s) to/from IRF($S$) $\rightleftharpoons$ IRF($S'$) along the $\hat{x} = \hat{x}'$ direction:

**Lorentz Transformation from IRF($S$) → IRF($S'$):**

$x' = \gamma(x - \beta ct)$ $\vec{v} = +v\hat{x}$ in IRF$(S)$

$y' = y$ $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$z' = z$

$ct' = \gamma(ct - \beta x)$

**Lorentz Transformation from IRF($S'$) → IRF($S$):**

$x = \gamma(x' + \beta ct')$ $\vec{v} = -v\hat{x}'$ in IRF$(S')$

$y = y'$ $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$z = z'$

$ct = \gamma(ct' + \beta x')$

**Griffiths Example 12.4: Simultaneity, Synchronization and Time Dilation**

In ***lab*** frame IRF(*S*), suppose event "*A*" occurs at $\boxed{x_A = 0,\ t_A = 0}$ and event "*B*" occurs at $\boxed{x_B = b,\ t_B = 0}$.
The two events "*A*" and "*B*" ***are*** simultaneous in ***lab*** IRF(*S*), because both occur at $\boxed{t_A = t_B = t = 0}$.
The two events "*A*" and "*B*" occur at ***different*** space-points in ***lab*** frame IRF(*S*): $\boxed{x_A = 0}$, $\boxed{x_B = b}$,

Events "*A*" and "*B*" do ***not*** occur simultaneously in ***moving*** frame IRF(*S′*) {moving with relative velocity $\beta\hat{x}$ w.r.t. ***lab*** frame}. The Lorentz transformation from ***lab*** IRF(*S*) to ***moving*** IRF(*S′*) gives:

$$\boxed{\begin{aligned} x'_A &= \gamma\left(x_A - \beta c t_A\right) = 0 \\ y'_A &= y_A \\ z'_A &= z_A \\ ct'_A &= \gamma\left(ct_A - \beta x_A\right) = 0 \end{aligned}} \quad \text{and:} \quad \boxed{\begin{aligned} x'_B &= \gamma\left(x_B - \beta c t_B\right) = \gamma b \\ y'_B &= y_B \\ z'_B &= z_B \\ ct'_B &= \gamma\left(ct_B - \beta x_B\right) = -\gamma\beta b \end{aligned}}$$

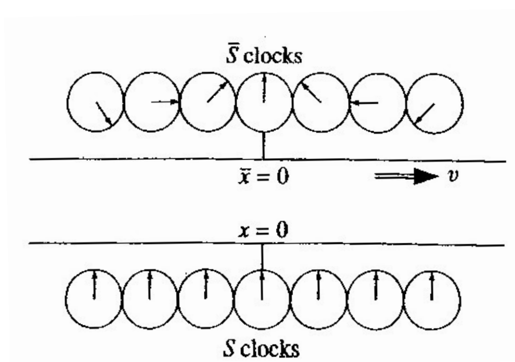
Thus, in IRF(*S*) {***lab*** frame}: $\boxed{x_A = 0,\ t_A = 0}$ and: $\boxed{x_B = b,\quad t_B = 0}$

Whereas in IRF(*S′*) {***moving*** frame}: $\boxed{x'_A = 0,\ t'_A = 0}$ but: $\boxed{x'_B = \gamma b,\ t'_B = -\gamma\beta b/c}$

Thus, we see that in the ***moving*** frame IRF(*S′*), event "*B*" occurs ***before*** event "*A*" !!!

*n.b.* Event "*A*" occurs at $t_A = 0$ and $t'_A = 0$ (***simultaneously***) in IRF(*S*) and IRF(*S′*), respectively, because the origins $\vartheta(S)$ and $\vartheta'(S')$ coincide (in ***space***) at $t = t_A = t_B = 0$.

$\Rightarrow$ Clocks that are synchronized in one IRF(*S*) are ***not*** synchronized in another IRF(*S′*), as can be seen in the following figure:




Suppose that at time $t = 0$ an observer in ***lab*** frame IRF(*S*), examines ***all*** clocks in ***moving*** frame IRF$(S')$. He/she discovers that the clocks in ***moving*** frame IRF(*S′*) all read ***different*** times, due to each of their seven different *x*-locations in ***lab*** frame IRF(*S*), and will vary/differ according to:

$$\boxed{t' = \gamma\left(t - \beta\frac{x}{c}\right) = -\gamma\beta\left(\frac{x}{c}\right)}$$ {since $t = 0$ in ***lab*** frame IRF(*S*) for all seven *x*-points}

$\Rightarrow$ For $x < 0$, clocks to the ***left*** of the origin $(x' < 0)$ in IRF(*S′*) are increasingly ***ahead*** $(t' > 0)$.
$\Rightarrow$ For $x > 0$, clocks to the ***right*** of the origin $(x' > 0)$ in IRF(*S′*) are increasingly ***behind*** $(t' < 0)$.

$\Rightarrow$ Non-synchronization of clocks in ***moving*** frame IRF($S'$) follows directly from the Lorentz transformation from (the synchronized) ***lab*** frame IRF($S$) $\rightarrow$ IRF($S'$).

*n.b.* From the viewpoint of an observer in ***moving*** frame IRF($S'$), if his/her clocks are synchronized in IRF($S'$) (*e.g.* all $t' = 0$) then it will be the clocks in ***lab*** frame IRF($S$) that are non-synchronized:

$$\boxed{t = \gamma\left(t' + \beta\frac{x'}{c}\right) = +\gamma\beta\left(\frac{x'}{c}\right)} \quad \{\text{since } t' = 0 \text{ in IRF}(S')\}$$

$\Rightarrow$ For $x' < 0$, clocks to the ***left*** of the origin $(x < 0)$ in IRF($S$) are increasingly ***behind*** $(t < 0)$.

$\Rightarrow$ For $x' > 0$, clocks to the ***right*** of the origin $(x > 0)$ in IRF($S$) are increasingly ***ahead*** $(t > 0)$.

If observer in ***lab*** frame IRF($S$) focuses his/her attention on a ***single*** clock in ***moving*** IRF($S'$), *e.g.* the clock located at $\boxed{x' = a}$ and watches it over a time interval $\Delta t$, how much time has elapsed on the ***moving*** clock? Here $x'$ is ***fixed*** in ***moving*** IRF($S'$) and: $\boxed{\Delta t \equiv t_2 - t_1}$, $\boxed{\Delta t' \equiv t_2' - t_1'}$.

Then: $\boxed{t = \gamma\left(t' + \beta\frac{x'}{c}\right)}$ $\quad \boxed{x' = a}$ is ***fixed*** in ***moving*** frame IRF($S'$)

Thus: $\boxed{t_2 = \gamma\left(t_2' + \beta\frac{a}{c}\right)}$

And: $\boxed{t_1 = \gamma\left(t_1' + \beta\frac{a}{c}\right)}$

$$\therefore \quad \begin{aligned}\Delta t = t_2 - t_1 &= \gamma\left(t_2' + \beta\left(\frac{a}{c}\right)\right) - \gamma\left(t_1' + \beta\left(\frac{a}{c}\right)\right) \\ &= \gamma t_2' + \cancel{\gamma\beta\left(\frac{a}{c}\right)} - \gamma t_1' - \cancel{\gamma\beta\left(\frac{a}{c}\right)} = \gamma t_2' - \gamma t_1' \\ &= \gamma\left(t_2' - t_1'\right) = \gamma\Delta t'\end{aligned}$$

$$\Rightarrow \quad \boxed{\Delta t = \gamma\Delta t'} \quad \text{or:} \quad \boxed{\Delta t' = \frac{1}{\gamma}\Delta t}$$

Time dilation formulae

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## Griffiths Example 12.5: Lorentz Contraction

Consider a stick moving with velocity $\vec{v} = +v\hat{x}$.

Its ***rest*** length as measured in the ***moving*** frame IRF($S'$) is: $\boxed{\Delta x' \equiv x_2' - x_1'}$

An observer in ***lab*** frame IRF($S$) wants to measure the length of this stick in his/her reference frame, *e.g.* at the same instant $t = 0$ in his/her IRF($S$). Thus in ***lab*** IRF(S): $\boxed{\Delta x \equiv x_2 - x_1}$ at $t = 0$.

Then in ***moving*** IRF($S'$):

$$\boxed{x_2' = \gamma\left(x_2 - vt\right) = \gamma x_2}$$

$$\boxed{x_1' = \gamma\left(x_1 - vt\right) = \gamma x_1}$$

$$\therefore \quad \boxed{\Delta x' = x_2' - x_1' = \gamma x_2 - \gamma x_1 = \gamma\left(x_2 - x_1\right) = \gamma \Delta x}$$

$$\Rightarrow \quad \boxed{\Delta x' = \gamma \Delta x} \quad \text{or:} \quad \boxed{\Delta x = \frac{1}{\gamma}\Delta x'}$$

Lorentz contraction formulae

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## Example: Show invariant interval(s) *are* invariant/independent of inertial reference frame:

For the most general case in space-time:

In ***lab*** frame IRF($S$): Event 1 is at $\left(x_1, t_1\right)$, Event 2 is at $\left(x_2, t_2\right)$, corresponding to:

In ***moving*** frame IRF($S'$): Event 1 is at $\left(x_1', t_1'\right)$, Event 2 is at $\left(x_2', t_2'\right)$.

Using the 1-D Lorentz transformations:

$$\boxed{x_1' = \gamma x_1 - \gamma\beta c t_1} \text{ and: } \boxed{x_2' = \gamma x_2 - \gamma\beta c t_2} \Rightarrow \boxed{\Delta x' = \left(x_2' - x_1'\right) = \gamma\left(x_2 - x_1\right) - \gamma\beta c\left(t_2 - t_1\right) = \gamma\Delta x - \gamma\beta c\Delta t}$$

$$\boxed{ct_1' = \gamma c t_1 - \gamma\beta x_1} \text{ and: } \boxed{ct_2' = \gamma c t_2 - \gamma\beta x_2} \Rightarrow \boxed{c\Delta t' = c\left(t_2' - t_1'\right) = \gamma c\left(t_2 - t_1\right) - \gamma\beta\left(x_2 - x_1\right) = \gamma c\Delta t - \gamma\beta\Delta x}$$

Does the invariant interval $I' = I$ ???

$$\begin{aligned}
I' &\equiv \left(\Delta x'\right)^2 - \left(c\Delta t'\right)^2 = \gamma^2\left(\Delta x - \beta c\Delta t\right)^2 - \gamma^2\left(c\Delta t - \beta\Delta x\right)^2 \\
&= \gamma^2\left(\Delta x\right)^2 - \cancel{2\gamma^2\beta\left(\Delta x c\Delta t\right)} + \gamma^2\beta^2\left(c\Delta t\right)^2 - \gamma^2\left(c\Delta t\right)^2 + \cancel{2\gamma^2\beta\left(\Delta x c\Delta t\right)} - \gamma^2\beta^2\left(\Delta x\right)^2 \\
&= \underbrace{\gamma^2\left(1 - \beta^2\right)}_{=1}\left(\Delta x\right)^2 - \underbrace{\gamma^2\left(1 - \beta^2\right)}_{=1}\left(c\Delta t\right)^2 \quad \text{but: } \gamma^2 = 1/\left(1 - \beta^2\right) \\
&= \left(\Delta x\right)^2 - \left(c\Delta t\right)^2 \equiv I \quad \text{Yes!!!}
\end{aligned}$$

 

**Griffiths Example 12.6: Einstein's 1-D Velocity Addition Rule**

*n.b.* IRF($S'$) is moving with velocity $\vec{v} = v\hat{x}$ relative to ***lab*** IRF($S$)

Suppose a particle moves a distance $dx$ in a time $dt$ in ***lab*** IRF($S$).

The speed of the particle as observed in ***lab*** frame IRF($S$) is thus: $u = \dfrac{dx}{dt}$

However, viewed from the ***moving*** frame IRF($S'$), the particle has moved a distance:

$dx' = \gamma\left(dx - \beta c dt\right)$ in a time: $dt' = \gamma\left(dt - \beta\dfrac{dx}{c}\right)$

Thus, the speed of the particle as observed in the ***moving*** frame IRF($S'$) is:

$$u' = \frac{dx'}{dt'} = \frac{\gamma\left(dx - \beta c dt\right)}{\gamma\left(dt - \beta\left(\frac{dx}{c}\right)\right)} = \frac{dx - \beta c dt}{dt - \beta\left(\frac{dx}{c}\right)} = \frac{\frac{dx}{dt} - \beta c}{1 - \left(\frac{\beta}{c}\right)\frac{dx}{dt}} = \frac{u - \beta c}{1 - u\frac{\beta}{c}} = \frac{u - v}{1 - \frac{uv}{c^2}}$$

**Einstein's 1-D Velocity Addition Rule for IRF($S'$) moving with velocity $\vec{v} = v\hat{x}$ relative to IRF($S$):**

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \quad \text{or:} \quad \beta'_{u'} = \frac{\beta_u - \beta}{1 - \beta_u\beta} \quad \text{where:} \quad \beta = \frac{v}{c},\ \beta_u = \frac{u}{c},\ \beta'_{u'} = \frac{u'}{c}$$

If the situation is ***reversed*** for the two IRF's, then the speed of the particle as observed in IRF($S$) in terms of its speed as observed in IRF($S'$) is given by:

**Einstein's 1-D Velocity Addition Rule for IRF($S$) moving with velocity $\vec{v}' = -v\hat{x}$ relative to IRF($S'$):**

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{or:} \quad \beta_u = \frac{\beta'_{u'} + \beta}{1 - \beta'_{u'}\beta} \quad \text{where:} \quad \beta = \frac{v}{c},\ \beta_u = \frac{u}{c},\ \beta'_{u'} = \frac{u'}{c}$$

*n.b.* Compare *e.g.* this last 1-D velocity addition result to that originally given at top of *p.* 4 above – they are the same/identical for $\vec{v} = \pm v\hat{x}$:

$$v_{\text{ground}}^{\text{man}} = \frac{v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \left(\frac{v_{\text{ground}}^{\text{train}} \cdot v_{\text{train}}^{\text{man}}}{c^2}\right)\right]}$$

## Derivation of Lab Frame Decay Length and Decay Time Relations $\boxed{\ell = \gamma\beta c\tau'}$ and $\boxed{\tau = \gamma\tau'}$ for Unstable Relativistic Particles

An unstable particle (*e.g.* a muon) is at ***rest*** in the ***moving*** frame IRF($S'$). An observer in IRF($S'$) sees this particle:

Created at the IRF($S'$) space-time coordinate: $\boxed{(x_1', y_1', z_1', t_1') = (0,0,0,0)}$

Decay at the IRF($S'$) space-time coordinate: $\boxed{(x_2', y_2', z_2', t_2') = (0,0,0,\tau')}$

$\tau'$ = the decay time of the particle as measured in IRF($S'$)

And: $\boxed{\Delta x' \equiv x_2' - x_1' = 0}$ and: $\boxed{\Delta t' \equiv t_2' - t_1' = \tau'}$. {*n.b.* the unstable particle is ***not*** moving in IRF($S'$).}

In the ***lab*** frame IRF($S$), if the unstable particle has velocity ***only*** in the $\hat{x}$-direction, *i.e.* $\boxed{\vec{v} = v\hat{x}}$, then if the two reference frames IRF($S$) and IRF($S'$) coincide at $\boxed{t_1 = t_1' = 0}$, then viewed by an observer in the ***lab*** frame IRF($S$), this particle is:

Created at the IRF($S$) space-time coordinate: $\boxed{(x_1, y_1, z_1, t_1) = (0,0,0,0)}$

Decays at the IRF($S$) space-time coordinate: $\boxed{(x_2, y_2, z_2, t_2) = (\ell,0,0,\tau)}$

$\tau$ = the decay time and $\ell$ = decay length of the particle as measured in IRF($S$)

And: $\boxed{\Delta x \equiv x_2 - x_1 = \ell}$ and: $\boxed{\Delta t \equiv t_2 - t_1 = \tau}$.

Next, we need to use the Lorentz Transformation from IRF($S'$) $\rightarrow$ IRF($S$):

$$x = \gamma(x' + \beta ct') \qquad \vec{v} = -v\hat{x} \text{ in IRF}(S')$$
$$y = y' \qquad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$
$$z = z'$$
$$ct = \gamma(ct' + \beta x')$$

Then:

$\boxed{x_2 = \gamma(x_2' + \beta ct_2') = \gamma(0 + \beta c\tau') = \gamma\beta c\tau'}$ and: $\boxed{ct_2 = \gamma(ct_2' + \beta x_2') = \gamma(c\tau' + \beta 0) = \gamma c\tau'}$ or: $\boxed{t_2 = \gamma\tau'}$

$\boxed{x_1 = \gamma(x_1' + \beta ct_1') = \gamma(0 + \beta c0) = 0}$ and: $\boxed{ct_1 = \gamma(ct_1' + \beta x_1') = \gamma(c0 + \beta 0) = 0}$ or: $\boxed{t_1 = t_1' = 0}$

Thus, for an observer in the ***lab*** frame IRF($S$): $\boxed{\ell = \Delta x \equiv x_2 - x_1 = \gamma\beta c\tau'}$ and: $\boxed{\tau = \Delta t \equiv t_2 - t_1 = \gamma\tau'}$.

$\therefore$ The decay length and decay time of an unstable particle in the ***lab*** frame IRF($S$) are respectively: $\boxed{\ell = \gamma\beta c\tau'}$ and: $\boxed{\tau = \gamma\tau'}$, where $\tau'$ = the decay time of the particle in its own ***rest*** frame IRF($S'$).

 