LECTURE NOTES 14

EM RADIATION FROM AN ARBITRARY SOURCE:

We now apply the formalism/methodology that we have developed in the previous lectures on low-order multipole *EM* radiation {E(1), M(1), E(2), M(2)} to an ***arbitrary*** configuration of electric charges and currents, only restricting these to be ***localized*** charge and current distributions, contained within a finite volume v' near the origin:

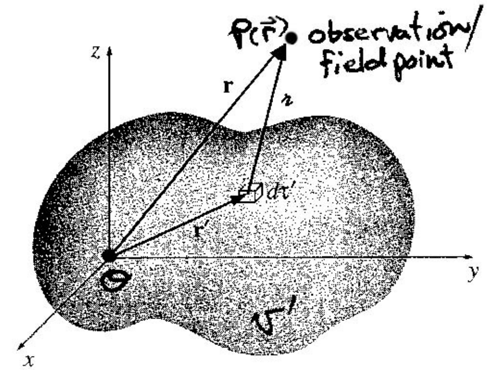
$$\vec{\imath} = \vec{r} - \vec{r}'(t_r)$$

$$\imath = \left|\vec{r} - \vec{r}'(t_r)\right|$$

$$\imath = \sqrt{r^2 + r'^2(t_r) - 2\vec{r}\cdot\vec{r}'(t_r)}$$

$$\Delta t = t - t_r = \imath/c$$

$$t_r = t - \imath/c$$



For arbitrary, localized {total} electric charge and current density distributions $\rho_{tot}(\vec{r}', t_r)$ and $\vec{J}_{tot}(\vec{r}', t_r)$, the ***retarded*** scalar and vector potentials, respectively are:

$$V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}',t_r)}{\imath}\,d\tau' = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}',t-\imath/c)}{\imath}\,d\tau' \quad \text{with} \quad t_r = t - \imath/c$$

$$\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{tot}(\vec{r}',t_r)}{\imath}\,d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{tot}(\vec{r}',t-\imath/c)}{\imath}\,d\tau' \quad \text{and} \quad \imath = \sqrt{r^2 + r'^2(t_r) - 2\vec{r}\cdot\vec{r}'(t_r)}$$

For *EM* ***radiation***, we assume that the observation / field point $\vec{r}$ is ***far away*** from the localized source charge / current distribution, such that: $r'_{max} \ll r$ or: $r'_{max}/r \ll 1$.

Then keeping only up to terms linear in $\left(\frac{r'}{r}\right)$: $\imath = r\sqrt{1 + \left(\frac{r'(t_r)}{r}\right)^2 - \frac{2\vec{r}\cdot\vec{r}'(t_r)}{r^2}} \simeq r\sqrt{1 - \frac{2\vec{r}\cdot\vec{r}'(t_r)}{r^2}}$

But: $\sqrt{1-\in} \simeq 1 - \tfrac{1}{2}\in$ for $\in \ll 1$ $\Rightarrow$ $\therefore$ $\imath \simeq r\left(1 - \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right) = r\left(1 - \frac{\hat{r}\cdot\vec{r}'(t_r)}{r}\right)$

And: $\frac{1}{\imath} \simeq \frac{1}{r}\frac{1}{\left(1 - \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right)} \approx \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right) = \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}'(t_r)}{r}\right)$ using: $\frac{1}{1-\in} \simeq 1 + \in$ for: $\in \ll 1$

Now: $\rho_{tot}(\vec{r}\,',t_r) = \rho_{tot}\left(\vec{r}\,',t-\frac{\mathcal{R}}{c}\right) \simeq \rho_{tot}\left(\vec{r}\,',t-\frac{r}{c}\left(1-\frac{\vec{r}\cdot\vec{r}\,'(t_r)}{r^2}\right)\right) \approx \rho_{tot}\left(\vec{r}\,',t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}\,'(t_r)}{c}\right)$

Expand $\rho_{tot}(\vec{r}\,',t_r)$ as a Taylor series in the ***present*** time t about the ***retarded*** time, at the ***origin*** $\{\vec{r}\,'=0\}$:

Defining the ***retarded*** time at the ***origin***: $t_o \equiv t - r/c$ {valid in the "far-zone limit}

Then: $\rho_{tot}(\vec{r}\,',t_r) \simeq \rho_{tot}(\vec{r}\,',t_o) + \dot{\rho}_{tot}(\vec{r}\,',t_o)\left(\frac{\hat{r}\cdot\vec{r}\,'}{c}\right) + \frac{1}{2!}\ddot{\rho}_{tot}(\vec{r}\,',t_o)\left(\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)^2 + \frac{1}{3!}\dddot{\rho}_{tot}(\vec{r}\,',t_o)\left(\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)^3 + \ldots$

Where: $\dot{\rho}_{tot}(\vec{r}\,',t_o) \equiv \frac{d\rho_{tot}(\vec{r}\,',t_o)}{dt_r}$ *etc.*

We can drop / neglect ***all*** higher-order terms ***beyond*** the $\dot{\rho}_{tot}$ term, ***provided that***:

$$r'_{\max} \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}}, \ldots$$ is satisfied…

For a ***harmonically*** oscillating system (*i.e.* one with angular frequency ω), each of these ratios *e.g.* $\frac{c}{|\ddot{\rho}/\dot{\rho}|}$, *etc*. is $= \frac{c}{\omega}$ and thus we have: $r'_{\max} \ll \frac{c}{\omega}$ if $r_{\max} = d \ll \frac{c}{\omega}$, then $\left(\frac{\omega d}{c}\right) \ll 1$, or equivalently{***here***}: $\left(\frac{\omega r'_{\max}}{c}\right) \ll 1$.

The two approximations $\frac{r'_{\max}}{r} \ll 1$ and $\frac{\omega r'_{\max}}{c} \ll 1$, or more generally: $\frac{r'_{\max}|\ddot{\rho}/\dot{\rho}|}{c} \ll 1$ *etc…* amount to keeping only the first-order {the lowest-order, non-negligible} terms in r'.

The ***retarded*** ***scalar*** potential $V_r(\vec{r},t)$ then becomes:

$$\begin{aligned} V_r(\vec{r},t) &= \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}\,',t_r)}{\mathcal{R}}\,d\tau' = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{1}{\mathcal{R}}\rho_{tot}(\vec{r}\,',t_r)\,d\tau' \\ &\simeq \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{1}{r}\left(1+\frac{\hat{r}\cdot\vec{r}\,'(t_o)}{r}\right)\left(\rho_{tot}(\vec{r}\,',t_o)+\dot{\rho}_{tot}(\vec{r}\,',t_o)\left(\frac{\hat{r}\cdot\vec{r}\,'(t_o)}{c}\right)+\ldots\right)d\tau' \\ &\approx \frac{1}{4\pi\varepsilon_o r}\left[\int_{v'}\rho_{tot}(\vec{r}\,',t_o)\,d\tau' + \left(\frac{\hat{r}}{r}\right)\cdot\int_{v'}\vec{r}\,'(t_o)\rho_{tot}(\vec{r}\,',t_0)\,d\tau' + \left(\frac{\hat{r}}{c}\right)\cdot\int_{v'}\vec{r}\,'(t_o)\dot{\rho}_{tot}(\vec{r}\,',t_o)\,d\tau' + \ldots\right] \end{aligned}$$

$$V_r(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o r}\left[\overbrace{\int_{v'}\rho_{tot}(\vec{r}\,',t_o)\,d\tau'}^{=Q_{tot}(t_o)} + \left(\frac{\hat{r}}{r}\right)\cdot\overbrace{\int_{v'}\vec{r}\,'(t_o)\rho_{tot}(\vec{r}\,',t_o)\,d\tau'}^{=\vec{p}_{tot}(t_o)} + \left(\frac{\hat{r}}{c}\right)\cdot\frac{d}{dt}\overbrace{\int_{v'}\vec{r}\,'(t_o)\rho_{tot}(\vec{r}\,',t_o)\,d\tau'}^{=\vec{p}_{tot}(t_o)} + \ldots\right]$$

Or: $$V_r(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{Q_{tot}(t_o)}{r} + \frac{\hat{r}\bullet\vec{p}_{tot}(t_o)}{r^2} + \frac{\hat{r}\bullet\dot{\vec{p}}_{tot}(t_o)}{cr} + \ \ldots\ \right]$$

In the ***static*** limit: monopole term — dipole term — vanishes in the ***static*** limit

The ***retarded*** ***vector*** potential, to first order in $r'\,(\kappa \simeq r)$ {with $t_o \equiv t - r/c$} then becomes:

$$\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{tot}(\vec{r}',t_r)}{\kappa}d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{tot}(\vec{r}',t-\kappa/c)}{\kappa}d\tau' \simeq \frac{\mu_o}{4\pi r}\int_{v'}\vec{J}_{tot}(\vec{r}',t_o)\,d\tau'$$

Griffiths Problem 5.7 (*p*. 214) showed that for ***localized*** electric charge / current distributions contained in the source volume v', that:

$$\int_{v'}\vec{J}_{tot}(\vec{r}',t_r)\,d\tau' = \frac{d\vec{p}_{tot}(\vec{r},t)}{dt} = \dot{\vec{p}}_{tot}(\vec{r},t)$$

Thus: $$\vec{A}_r(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\dot{\vec{p}}_{tot}(t_o)}{r}$$

Note that $\vec{p}_{tot}(t_o)$ is ***already*** **first** **order** in r' $\Rightarrow$ any ***additional*** refinements are therefore **second** **order** in r'; thus, the higher-order terms can be neglected/ignored (***here***).

Next, we calculate the ***retarded*** $\vec{E}$ and $\vec{B}$ fields. Since we are only interested in the *EM* ***radiation*** fields (in the "far-zone" limit), we drop / neglect $1/r^2$, $1/r^3$, $1/r^4$, *etc.* terms, and keep only the $1/r$ radiation-field terms.

Note that the radiation terms come entirely from those terms in the Taylor series expansions for $\rho_{tot}(\vec{r}',t_o)$ and $\vec{J}_{tot}(\vec{r}',t_o)$ in which we differentiate the argument t_o of $\rho_{tot}(\vec{r}',t_o)$, $\vec{J}_{tot}(\vec{r}',t_o)$.

Since ***retarded*** time: $t_o \equiv t - r/c$ then: $\vec{\nabla}t_o = -\frac{1}{c}\vec{\nabla}r$ but: $\vec{\nabla}r = \hat{r}$ $\therefore$ $\vec{\nabla}t_o = -\frac{1}{c}\hat{r}$

Thus: $$\vec{\nabla}V_r(\vec{r},t) \simeq \vec{\nabla}\left[\frac{1}{4\pi\varepsilon_o}\frac{\hat{r}\bullet\dot{\vec{p}}_{tot}(t_o)}{cr}\right] \approx \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\bullet\ddot{\vec{p}}_{tot}(t_o)}{cr}\right]\vec{\nabla}t_o = -\frac{1}{4\pi\varepsilon_o c^2}\frac{\left[\hat{r}\bullet\ddot{\vec{p}}_{tot}(t_o)\right]}{r}\hat{r}$$

And: $$\frac{\partial\vec{A}_r(\vec{r},t)}{\partial t} \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\ddot{\vec{p}}_{tot}(t_o)}{r}$$

The ***retarded*** electric field for *EM* radiation in the "far-zone" limit is:

$$\vec{E}_{\rm r}(\vec{r},t) = -\vec{\nabla}V_{\rm r}(\vec{r},t) - \frac{\partial \vec{A}_{\rm r}(\vec{r},t)}{\partial t} \simeq +\frac{1}{4\pi\varepsilon_o c^2}\frac{\left[\hat{r}\bullet\ddot{\vec{p}}_{tot}(t_o)\right]}{r}\hat{r} - \frac{\mu_o}{4\pi}\frac{\ddot{\vec{p}}_{tot}(t_o)}{r} \quad \text{but: } \boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$$

$$\therefore \boxed{\vec{E}_{\rm r}(\vec{r},t) \simeq \frac{\mu_o}{4\pi r}\left[\left(\hat{r}\bullet\ddot{\vec{p}}_{tot}(t_o)\right)\hat{r} - \ddot{\vec{p}}_{tot}(t_o)\right] = \frac{\mu_o}{4\pi r}\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}_{tot}(t_o)\right)\right]} \quad \{\text{using the BAC-CAB rule}\}$$

where the second time-derivative of the total electric dipole moment $\ddot{\vec{p}}_{tot}(t_o)$ is evaluated at the ***retarded*** time $t_o \equiv t - r/c$ and computed from the origin, $\vartheta\ \{\vec{r}\,' = 0\}$: $\ddot{\vec{p}}_{tot}(t_o) = \ddot{\vec{p}}_{tot}(0, t - r/c)$.

The ***retarded*** magnetic field for *EM* radiation in the "far-zone" limit is:

$$\vec{B}_{\rm r}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\rm r}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\vec{\nabla}\times\frac{\dot{\vec{p}}_{tot}(t_o)}{r} \approx \left(\frac{\mu_o}{4\pi r}\right)\left[\vec{\nabla}\times\dot{\vec{p}}(t_o)\right]$$

$$= \left(\frac{\mu_o}{4\pi}\right)\underbrace{\left[\vec{\nabla}t_o\times\ddot{\vec{p}}_{tot}(t_o)\right]} = -\frac{\mu_o}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}_{tot}(t_o)\right]$$

Where in first step we have used the relation $\vec{\nabla}\times\vec{v}(t_{\rm r}) = -\vec{a}(t_{\rm r})\times\vec{\nabla}t_{\rm r}$ {see "term (3)" P436 Lect. Notes 12 *p*. 11 and/or Griffiths Equation 10.55, *p*. 436} and in the last step on the RHS we have {again} used the relation $\vec{\nabla}t_o = -\frac{1}{c}\hat{r}$.

$$\therefore \boxed{\vec{B}_{\rm r}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}_{tot}(t_o)\right]}$$

where the second time-derivative of the total electric dipole moment $\ddot{\vec{p}}_{tot}(t_o)$ is evaluated at the ***retarded*** time $t_o \equiv t - r/c$ and computed from the origin, $\vartheta\ \{\vec{r}\,' = 0\}$: $\ddot{\vec{p}}_{tot}(t_o) = \ddot{\vec{p}}_{tot}(0, t - r/c)$.

If we use spherical-polar coordinates, with the $\hat{z}$-axis $\parallel \ddot{\vec{p}}_{Tot}(t_o)$, then noting that:

$$\boxed{\hat{r}\times\ddot{\vec{p}}_{tot}(t_o) = \ddot{p}_{tot}(t_o)\left[\hat{r}\times\hat{z}\right]} \quad \text{but: } \boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}} \qquad \boxed{\hat{r}\times\hat{r} = 0}$$

$$= \boxed{\ddot{p}_{tot}(t_o)\left[\hat{r}\times\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\right]} \qquad \boxed{\hat{r}\times\hat{\theta} = \hat{\varphi}}$$

$$= \boxed{-\ddot{p}_{tot}(t_o)\sin\theta\hat{\varphi}} \qquad \boxed{\hat{\theta}\times\hat{\varphi} = \hat{r}}$$

$$\boxed{\hat{\varphi}\times\hat{r} = \hat{\theta}}$$

Thus: $\boxed{\vec{E}_{\mathrm{r}}(r,\theta,t) \simeq \frac{\mu_o \ddot{p}_{tot}(t_o)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}}$ $\leftarrow$ $\boxed{\hat{r}\times(-\hat{\varphi}) = \hat{\theta}}$

And: $\boxed{\vec{B}_{\mathrm{r}}(r,\theta,t) \approx \frac{\mu_o \ddot{p}_{tot}(t_o)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\varphi}}$

and we also see that {again} $\boxed{\vec{B}_{\mathrm{r}}(r,\theta,t) = \frac{1}{c}\hat{r}\times\vec{E}_{\mathrm{r}}(r,\theta,t)}$, $\boxed{\vec{B}_{\mathrm{r}} \perp \vec{E}_{\mathrm{r}} \perp \hat{r}\ \{\parallel \hat{k}\}}$

The ***instantaneous*** ***retarded*** *EM* radiation energy density $u(r,\theta,t)$ in the "far-zone" limit is:

$$\begin{aligned}
u_{\mathrm{r}}^{rad}(r,\theta,t) &= \frac{1}{2}\left(\varepsilon_o E_{\mathrm{r}}^2(r,\theta,t) + \frac{1}{\mu_0}B_{\mathrm{r}}^2(r,\theta,t)\right)\\
&\simeq \frac{1}{2}\left[\frac{\varepsilon_o \mu_o^2 \ddot{p}^2(t_o)}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right) + \frac{1}{\mu_o}\frac{\mu_o^2 \ddot{p}^2(t_o)}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)\right] \quad \text{but: } \varepsilon_o = \frac{1}{\mu_o c^2}\\
&= \frac{1}{2}\left[\frac{\mu_o \ddot{p}^2(t_0)}{16\pi^2 c^2} + \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c^2}\right]\left(\frac{\sin^2\theta}{r^2}\right) = \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)
\end{aligned}$$

Thus: $\boxed{u_{\mathrm{r}}^{rad}(r,\theta,t) \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)}$ (Joules/m^3)

The ***instantaneous*** ***retarded*** Poynting's vector in the "far-zone" limit is:

$$\boxed{\vec{S}_{\mathrm{r}}^{rad}(r,\theta,t) = \frac{1}{\mu_o}\vec{E}_{\mathrm{r}}(r,\theta,t)\times B_{\mathrm{r}}(r,\theta,t)} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\boxed{\vec{S}_{\mathrm{r}}^{rad}(r,\theta,t) \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\underbrace{(\hat{\theta}\times\hat{\varphi})}_{=\hat{r}} = \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} = \vec{c}u_{\mathrm{r}}^{rad}(r,\theta,t)} \text{ with } \boxed{\begin{matrix}\vec{c}\equiv c\hat{r}\\ \hat{r}\parallel\hat{k}\end{matrix}}$$

The ***instantaneous*** ***retarded*** *EM* power radiated per unit solid angle in the "far-zone" limit is:

$$\boxed{\frac{dP_{\mathrm{r}}^{rad}(r,\theta,t)}{d\Omega} = \vec{S}^{rad}(r,\theta,t)\cdot r^2\hat{r} \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\sin^2\theta} \quad \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

The ***instantaneous*** ***retarded*** total *EM* power radiated into 4π steradians, with vector area element $d\vec{a}_\perp = r^2\sin\theta d\theta d\varphi\hat{r} = r^2 d\Omega\hat{r}$ in the "far-zone" limit is:

$$\boxed{P_{\mathrm{r}}^{rad}(t) = \int_S \vec{S}_{\mathrm{r}}^{rad}(r,\theta,t)\cdot d\vec{a}_\perp \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\sin\theta d\theta d\varphi = \frac{\mu_o \ddot{p}^2(t_o)}{6\pi^2 c}} \text{ (Watts)}$$

The ***instantaneous*** ***retarded*** *EM* radiation ***linear*** momentum density in the "far-zone" limit is:

$$\boxed{\vec{\wp}_r^{rad}(r,\theta,t) = \frac{1}{c^2}\vec{S}_r^{rad}(r,\theta,t) \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}} \quad \left(\frac{\text{kg}}{\text{m}^2\,\text{sec}}\right)$$

The ***instantaneous*** ***retarded*** *EM* radiation ***angular*** momentum density in the "far-zone" limit is:

$$\boxed{\vec{\ell}_r^{rad}(r,\theta,t) = \vec{r}\times\vec{\wp}_r^{rad}(r,\theta,t) = 0}$$

The scalar *EM* wave characteristic radiation impedance of the antenna associated with this lowest-order *EM* radiation is:

$$\boxed{Z_{rad} = \frac{|\vec{E}_r|}{|\vec{H}_r|} = \frac{|\vec{E}_r|}{\frac{1}{\mu_o}|\vec{B}_r|} = \mu_o c = \sqrt{\frac{\mu_o}{\varepsilon_o}} = Z_o = 120\pi\ \Omega \simeq 377\ \Omega}$$

The scalar *EM* wave radiation resistance of the antenna associated with this lowest-order *EM* radiation is:

$$\boxed{R_{rad} = \frac{P_{rad}}{I_o^2} = \frac{\mu_o \ddot{p}^2(t_o)}{6\pi c I_o^2}\ (Ohms)}$$

Note that in the above, we deliberately/consciously neglected the electric monopole {E(0)} term in the retarded scalar potential for "far-zone" limit, $r'_{\max} \ll r$:

$$\boxed{V_r^{E(0)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int_{v'}\rho(\vec{r}',t_o)\,d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{Q_{tot}(t_0)}{r}}$$

As mentioned previously (P436 Lect. Notes 13, *p*. 4), that because of electric charge conservation, a spherically-symmetric electric monopole moment cannot radiate transversely-polarized *EM* waves – spherical symmetry of the monopole moment restricts oscillations only to the radial direction – thus one could get radiation of one polarization from a certain *dΩ* solid angle element, but then radiation from other *dΩ*'s on the sphere also contribute, such that the ***net*** *EM* radiation from the entire sphere = 0 – total destructive interference. (Gauss' Law - $\int_{S'}\vec{E}\bullet d\vec{a} = Q_{tot}^{encl}/\varepsilon_o$ independent of the size of the spherically symmetric charge distribution enclosed by the surface S'.

Note also that for free-space *EM* radiation, $\vec{B}$ must be $\perp$ to $\vec{E}$, and with both $\vec{E}$ and $\vec{B} \perp$ to $\hat{k}$, the propagation direction. How do you do this for a ***spherically***-***symmetric*** source, where $\hat{k} = \hat{r}$?

Note also that if electric charge were ***not*** conserved, then we ***would*** get a retarded electric monopole field proportional to $1/r$: $\boxed{\vec{E}_r^{E(0)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o c}\frac{\dot{Q}(t_o)}{r}\hat{r}}$ $\Leftarrow$ *n.b.* this says nothing about the physical size of the spherically-symmetric charge distribution.

Contrast the behavior of ***transverse*** waves associated with *EM* radiation from a spherically-symmetric source (an oscillating electric monopole moment) (≡ no net *EM* radiation) to that of ***longitudinal*** sound waves / acoustic waves radiated from a spherically symmetric oscillating acoustic monopole sound source – *e.g.* a radially inward / outward oscillating sphere (a breathing bubble) – the latter of which very definitely can propagate / create sound precisely because sound waves are longitudinal, not transverse waves!!

Now think about the electron – for *EM* radiation fields, electric dipole / quadrupole / *etc.* higher *EM* moments ***break*** the rotational invariance / rotational symmetry associated with the spherical monopole electric charge distribution of the source – thus transverse *EM* waves (*EM* radiation) ***can*** couple to such electric monopole {E(0)} sources – and also ones that lack rotational invariance!!!

In the above Taylor series expansions for $\rho_{tot}(\vec{r}',t_r)$ and $\vec{J}_{tot}(\vec{r}',t_r)$, we only kept terms to first-order in r' in these expansions and then demonstrated that the first-order "far-zone" limit radiation terms were associated with the electric dipole moment {E(1)} .

For E(1) electric dipole *EM* radiation to first-order in r' for $r'_{\max} \ll r$ the ***instantaneous retarded*** scalar and vector potentials, electric and magnetic fields are:

$$V_r^{E(1)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\cdot\dot{\vec{p}}(t_o)}{cr}\right]$$

$$\vec{A}_r^{E(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\dot{\vec{p}}(t_o)}{r}\right]$$

n.b. proportional to $\dot{\vec{p}}(t_0)$ (first time derivative of $\vec{p}(t_o)$ - "velocity")

$$\vec{E}_r^{E(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}(t_o)\right)}{r}\right]$$

$$\vec{B}_r^{E(1)}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\ddot{\vec{p}}(t_o)}{r}\right]$$

n.b. proportional to $\ddot{\vec{p}}(t_0)$ (second time derivative of $\vec{p}(t_o)$ - "acceleration")

Suppose the (localized) charge / current distributions are such that there is no (time-varying) E(1) electric dipole moment, $\vec{p}(\vec{r}',t_r)=0$ and/or: $\dot{\vec{p}}(\vec{r}',t_r)=0$, $\ddot{\vec{p}}(\vec{r}',t_r)=0$.

Then the Taylor series expansion of $\rho_{tot}(\vec{r}',t_r)$ and $\vec{J}_{tot}(\vec{r}',t_r)$ to ***first order*** in r' would give ***nothing*** for potentials and fields associated with "far-zone" *EM* radiation. However, ***higher-order*** terms in these expansions ***might*** give rise to non-vanishing potentials and fields.

The ***second order*** terms in r' correspond to M(1) magnetic ***dipole*** and E(2) electric ***quadrupole*** *EM* radiation terms – in order to see/verify this, the second-order contribution needs to be / can be separated out into M(1) and E(2) terms.

Indeed, if we compare *e.g.* the ratio of *EM* power radiated for M(1) magnetic dipole *vs*. E(2) electric quadrupole radiation (in the "far-zone" limit):

$$\frac{\langle P_{M(1)}^{rad}\rangle}{\langle P_{E(2)}^{rad}\rangle} \simeq \frac{\left(\dfrac{\mu_o m_o^2 \omega^4}{12\pi c^3}\right)}{\left(\dfrac{\mu_o Q_{zz}^{e^2} \omega^6}{60\pi c^3}\right)} \quad \text{where:} \quad \begin{aligned} &m_o = \pi b^2 I_o = \pi b^2 q\omega \\ &I_o = q\omega \\ &Q_{zz}^e = qdd = \pi^2 b^2 q \\ &d = \pi b \end{aligned}$$

Thus:

$$\frac{\langle P_{M(1)}^{rad}\rangle}{\langle P_{E(2)}^{rad}\rangle} \simeq \frac{\left(\dfrac{\mu_o \pi^2 b^4 \omega^6 q^2}{12\pi c^3}\right)}{\left(\dfrac{\mu_o q^2 d^4 \omega^6}{60\pi c^3}\right)} = \frac{5\pi^2 b^4}{d^4} = \frac{5}{\pi^2} \simeq \frac{1}{2} \approx \mathcal{G}(1)$$

Similarly, the ***third*** ***order*** terms in r' in the Taylor series expansion of $\rho_{tot}(\vec{r}', t_r)$ and $\vec{J}_{tot}(\vec{r}', t_r)$ correspond to M(2) magnetic ***quadrupole*** and E(3) electric ***octupole*** radiation terms – *i.e.* the third-order contribution needs to be / can be separated out into M(2) and E(3) terms!

Similarly, the ***fourth*** ***order*** terms in r' in the Taylor series expansion of $\rho_{tot}(\vec{r}', t_r)$ and $\vec{J}_{tot}(\vec{r}', t_r)$ correspond to M(3) magnetic ***octupole*** and E(4) electric ***sextupole*** radiation terms – *i.e.* the fourth-order contribution can be separated out into M(3) and E(4) terms!

And so on, for each successive higher-order term r' in the Taylor series expansion of $\rho_{tot}(\vec{r}', t_r)$ and/or $\vec{J}_{tot}(\vec{r}', t_r)$!!!

Griffiths Example 11.2:

a.) An oscillating (*i.e.* harmonically varying) electric dipole has time-dependent dipole moment:

$$p(t_r) = p_o \cos(\omega t_r) \quad \text{where:} \quad \vec{p}(t_r) = p(t_r)\hat{z} = p_o \cos(\omega t_r)\hat{z}$$

$$\dot{p}(t_r) = \frac{dp(t_r)}{dt_r} = -\omega p_o \sin(\omega t_r)$$

$$\ddot{p}(t_r) = \frac{d\dot{p}(t_r)}{dt_r} = \frac{d^2 p(t_r)}{dt_r^2} = -\omega^2 p_o \cos(\omega t_r)$$

Then: $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$ with: $t_o \equiv t - r/c$

$$V_r^{E(1)}(\vec{r}, t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\cdot\dot{\vec{p}}(t_o)}{cr}\right] = \frac{-\omega p_o}{4\pi\varepsilon_o}\left[\frac{\hat{r}\cdot\hat{z}}{cr}\right]\sin(\omega t_o) = \frac{-\omega p_o}{4\pi\varepsilon_o cr}\cos\theta\sin(\omega t_o)$$

And:

$$\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\dot{\vec{p}}(t_o)}{r}\right] = -\frac{\mu_o \omega p_o}{4\pi r}\sin(\omega t_o)\hat{z}$$

$$\vec{\mathrm{E}}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times(\hat{r}\times\ddot{\vec{p}}(t_o))}{r}\right] = \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times(\hat{r}\times\hat{z})}{r}\right](-\omega^2 p_o)\cos(\omega t_o)$$

$$\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\ddot{\vec{p}}(t_o)}{r}\right] = -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\hat{z}}{r}\right](-\omega^2 p_o)\cos(\omega t_o)$$

$$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$$

But: $(\hat{r}\times\hat{z}) = \hat{r}\times(\cancel{\cos\theta\hat{r}} - \sin\theta\hat{\theta}) = -\sin\theta(\hat{r}\times\hat{\theta}) = -\sin\theta\hat{\varphi}$

And: $\hat{r}\times(\hat{r}\times\hat{z}) = (\hat{r}\times\hat{\varphi})*(-\sin\theta) = -\hat{\theta}(-\sin\theta) = +\sin\theta\hat{\theta}$

Thus:

$$V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq \frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$ with: $t_o \equiv t - r/c$

$$\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$$ where: $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$

$$\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

$$\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$$

Compare these results for the E(1) electric dipole *EM* radiation "far-zone" limit case with those we obtained P436 Lecture Notes 13 {see pages 8-11}, and/or P436 Lecture Notes 13.5 {the E(1)/M(1) summary / comparison page 11} – they are (of course) identical!

b.) A single, point electric charge q can have (by definition) an electric dipole moment $\vec{p}(t_{\mathrm{r}}) = q\vec{d}(t_{\mathrm{r}})$ where $\vec{d}(t_{\mathrm{r}})$ is the position vector of the point electric charge q at the ***retarded*** time t_{r} with respect to the (local) origin ϑ. (*n.b.* subject to all the caveats *r.e.* choice of origin for an EDM having a net charge – see P435 Lecture Notes. . .)

However: $$\dot{\vec{p}}(t_{\mathrm{r}}) = \frac{d\vec{p}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\frac{d\{\vec{d}(t_{\mathrm{r}})\}}{dt_{\mathrm{r}}} = q\vec{v}(t_{\mathrm{r}})$$

And: $$\ddot{\vec{p}}(t_r) = \frac{d\dot{\vec{p}}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\frac{d\vec{v}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\vec{a}(t_{\mathrm{r}})$$

n.b. these two quantities do ***not*** depend on the choice of origin !!!

$\vec{v}(t_{\mathrm{r}})$ = velocity vector of point electric charge q at the ***retarded*** time t_{r}

$\vec{a}(t_{\mathrm{r}})$ = acceleration vector of point electric charge q at the ***retarded*** time t_{r}

Everything goes through as before – get the same ***retarded*** scalar and vector potentials, same ***retarded*** $\vec{E}$ and $\vec{B}$ fields, same u, $\vec{S}$, P, *etc*.

In particular, the radiated *EM* E(1) power associated with a ***moving*** point charge q is:

$$\boxed{P_q \simeq \frac{\mu_o \ddot{p}^2(t_o)}{6\pi c}} \text{ (Watts)} \quad \text{But:} \quad \boxed{\ddot{p}(t_o) = q a(t_o)}$$

$$\therefore \boxed{P_q \simeq \frac{\mu_o q^2 a^2(t_o)}{6\pi c}}$$ ⇐ Famous Larmor formula (*EM* power radiated from a point charge q)

Note that the E(1) *EM* power radiated by a point charge q is proportional to the ***square*** of the acceleration a and also is proportional to the ***square*** of the electric charge q.

This is the origin of statement: "Whenever one ***accelerates*** an electric charge q, it ***radiates*** away *EM* energy in the form of (***real***) photons". It is the E(1) electric dipole term which dominates this radiation process.

n.b. This is ***also*** true for ***decelerating*** charged particles – the ***time-reversed*** situation!!!
$P_q \sim a^2$ ← doesn't care about sign of $\vec{a}$ {The *EM* interaction is ***time-reversal*** **invariant**}!!!

Radiation from accelerated / decelerated $+q$ vs. $-q$ charges is the ***same*** if $|+q| = |-q|$.
(P_q doesn't care about the ***sign*** of q!)

But: $P_q \sim q^2$ → so if ***double*** q → then P_q increases by factor of 4×!

⇒ For the ***same*** acceleration/deceleration, high-Z nuclei radiate *EM* energy {in the form of photons} ***much*** more than *e.g.* a proton (= hydrogen nucleus) – process is known as bremsstrahlung {= "braking radiation", auf Deutsch}.

e.g. A fully-stripped uranium nucleus ($Z_u = 92$) gives $92^2 = 8464\times$ more *EM* radiation than a proton for the ***same*** acceleration, a.

EM Power Radiated by a Moving Point Electric Charge:

The ***retarded*** electric field of an electric charge q in arbitrary motion is:

$$\boxed{\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\bullet\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]}$$

where: $\boxed{\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}(t_r)}$

$$\boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_r) = \vec{r} - \vec{w}(t_r)}$$

$$\boxed{\mathscr{r} = |\vec{r} - \vec{r}'(t_r)| = c\Delta t = c(t - t_r)}$$

or: $\boxed{t_r = t - \mathscr{r}/c}$.

The associated ***retarded*** magnetic field is:

$$\boxed{\vec{B}_r(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}_r(\vec{r},t)}$$

As mentioned before, the ***first*** term in $\vec{E}_r(\vec{r},t)$, $\dfrac{q}{4\pi\varepsilon_o}\dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\bullet\vec{u})^3}\left[(c^2 - v^2)\vec{u}\right]$

is known as the ***generalized*** ***Coulomb*** ***field***, or ***velocity*** ***field***.

The ***second*** term in $\vec{E}_{\rm r}(\vec{r},t)$, $\dfrac{q}{4\pi\varepsilon_o}\dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$ is known as the ***acceleration*** field (*a.k.a.* the ***radiation*** field).

The ***retarded*** Poynting's vector is: $\boxed{\vec{S}_{\rm r}(\vec{r},t)=\dfrac{1}{\mu_o}\left(\vec{E}_{\rm r}(\vec{r},t)\times\vec{B}_{\rm r}(\vec{r},t)\right)}$ where: $\boxed{\vec{B}_{\rm r}(\vec{r},t)=\dfrac{1}{c}\hat{\mathscr{r}}\times\vec{E}_{\rm r}(\vec{r},t)}$

Use the $\boxed{\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})}$ rule:

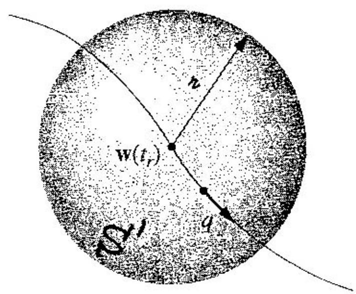
$$\boxed{\vec{S}_{\rm r}(\vec{r},t)=\frac{1}{\mu_o c}\left[\vec{E}_{\rm r}(\vec{r},t)\times\left(\hat{\mathscr{r}}\times\vec{E}_{\rm r}(\vec{r},t)\right)\right]=\frac{1}{\mu_o c}\left[E_{\rm r}^2(\vec{r},t)\hat{\mathscr{r}}-\left(\hat{\mathscr{r}}\cdot\vec{E}_{\rm r}\right)\vec{E}_{\rm r}(\vec{r},t)\right]}$$

However, note that not all of this *EM* energy flux constitutes *EM* ***radiation*** (***real*** photons) – some of it is still in the form of ***virtual*** photons, $\boxed{\vec{S}_{\rm r}(\vec{r},t)=\vec{S}_{\rm r}^{virt}(\vec{r},t)+\vec{S}_{\rm r}^{rad}(\vec{r},t)}$

The metaphor Griffiths uses, that of flies "attached" to a moving garbage truck, is a reasonable picture to imagine here….

- *n.b.* – In order to "detect" the total *EM* power ***radiated*** by a moving point charge *q*, we draw a ***huge*** sphere of radius $\mathscr{r}$ centered on the position of the charged particle $\vec{\rm w}(t_{\rm r})$ at the ***retarded*** time $\boxed{t_{\rm r}=t-\mathscr{r}/c}$ and wait the appropriate time interval $\boxed{\Delta t=t-t_{\rm r}=\mathscr{r}/c}$ for the *EM* radiation ***radiated*** at the ***retarded*** time $t_{\rm r}$ to arrive at the ***surface*** of the imaginary sphere.

Note that in the "far-zone" limit, the ***retarded*** time $t_{\rm r}$ is the correct retarded time for ***all*** points on the surface of the sphere S'.



- Again, since the area of the sphere $A_{sphere}=\pi\mathscr{r}^2$ ($\sim\mathscr{r}^2$), then any term in $\vec{S}_{\rm r}(\vec{r},t)$ that varies as $1/\mathscr{r}^2$ will yield a finite answer for radiated *EM* power, $P_{rad}=\oint_{S'}\vec{S}_{\rm r}(\vec{r},t)\cdot d\vec{a}_\perp$.
- However, note that terms in $\vec{S}_{\rm r}(\vec{r},t)$ that vary as $1/\mathscr{r}^3$, $1/\mathscr{r}^4$, $1/\mathscr{r}^5$ … *etc*. will contribute ***nothing*** to P_{rad} in the limit $\mathscr{r}\to\infty$.
- For this reason, ***only*** the acceleration fields represent ***true*** *EM* radiation (***real*** photons) – hence their other name, that of ***radiation*** fields:

$$\boxed{\vec{E}_{rad}(\vec{r},t)=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]}$$

The *EM* ***velocity*** fields ***do*** indeed carry *EM* energy – as the charged particle moves through space-time, this *EM* energy is ***dragged along with it*** – but it is ***not*** in the form of *EM* ***radiation***.

Note that $\vec{E}_{rad}(\vec{r},t)$ is $\perp\ \hat{\mathscr{r}}$ (due to the $\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$ term).

$\Rightarrow$ The second term in $\vec{S}_{rad}(\vec{r},t)$ ***vanishes***:

$$\vec{S}_{rad}(\vec{r},t)=\frac{1}{\mu_o c}\left[E_{rad}^2(\vec{r},t)\hat{\mathscr{r}}-\cancel{\left(\hat{\mathscr{r}}\bullet\vec{E}_{rad}(\vec{r},t)\right)}\vec{E}_{rad}(\vec{r},t)\right]=\frac{1}{\mu_o c}E_{rad}^2(\vec{r},t)\hat{\mathscr{r}}$$

Now if the point charge q ***happened*** to be {instantaneously} at ***rest*** ($\vec{v}(t_r)=0$) at the ***retarded*** time t_r, then: $\boxed{\vec{u}(t_r)=c\hat{\mathscr{r}}-\underbrace{\cancel{\vec{v}(t_r)}}_{=0}=c\hat{\mathscr{r}}}$ {***here***}. Then in ***this*** case:

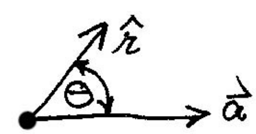
$$\begin{aligned}\vec{E}_{rad}(\vec{r},t)&=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}\right)^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet c\hat{\mathscr{r}}\right)^3}\left[\vec{\mathscr{r}}\times\left(c\hat{\mathscr{r}}\times\vec{a}\right)\right]\\&=\frac{q}{4\pi\varepsilon_o c^2}\frac{1}{\mathscr{r}^2}\left[\vec{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}\right)\right]=\frac{q}{4\pi\varepsilon_o c^2}\frac{1}{\mathscr{r}}\left[\hat{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}\right)\right]\\&=\frac{\mu_o q}{4\pi\mathscr{r}}\left[\hat{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}\right)\right]\qquad\left\{\text{since }\frac{1}{c^2}=\varepsilon_o\mu_o\right\}\\&=\frac{\mu_o q}{4\pi\mathscr{r}}\left[\left(\hat{\mathscr{r}}\bullet\vec{a}\right)\hat{\mathscr{r}}-\underbrace{\left(\hat{\mathscr{r}}\bullet\hat{\mathscr{r}}\right)}_{=1}\vec{a}\right]=\frac{\mu_o q}{4\pi\mathscr{r}}\left[\left(\hat{\mathscr{r}}\bullet\vec{a}\right)\hat{\mathscr{r}}-\vec{a}\right]\end{aligned}$$

Then {***here***} in ***this*** case {$\vec{v}(t_r)=0$}:

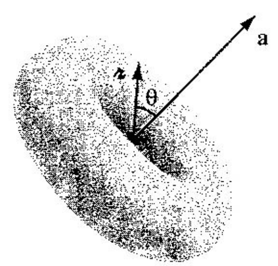
$$\vec{S}_{rad}(\vec{r},t)=\frac{1}{\mu_o c}E_{rad}^2(\vec{r},t)\hat{\mathscr{r}}=\frac{\mu_o q^2}{4\pi c\mathscr{r}^2}\left[a^2-\left(\hat{\mathscr{r}}\bullet\vec{a}\right)^2\right]\hat{\mathscr{r}}$$

But: $\hat{\mathscr{r}}\bullet\vec{a}=a\cos\theta$ where θ = opening angle between $\hat{\mathscr{r}}$ and acceleration $\vec{a}$.

$$\therefore\ \vec{S}_{rad}(\vec{r},t)=\frac{\mu_o q^2 a^2}{4\pi c\mathscr{r}^2}\left[1-\cos^2\theta\right]\hat{\mathscr{r}}=\frac{\mu_o q^2 a^2}{4\pi c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}}$$



Here again, we see that ***no*** power is radiated in the ***forward/backward*** directions ($\theta = 0$ and $\theta = \pi$) – radiated power is ***maximum*** when $\theta=\pi/2=90^o$, *i.e.* when $\hat{\mathscr{r}}\perp\vec{a}$ - get a donut-shaped intensity pattern about the instantaneous acceleration vector $\vec{a}(t_r)$:



The power radiated by ***this*** point charge (instantaneously at ***rest*** at time t_r) is:

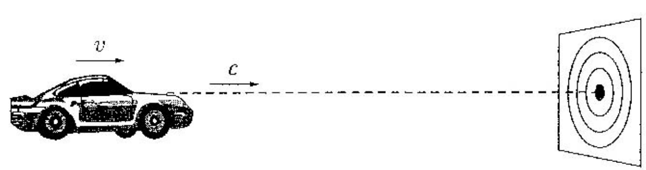
$$P_{rad}(t) = \oint_{S'} \vec{S}_{rad}(\vec{r},t)\bullet d\vec{a}_\perp = \frac{\mu_o q^2 a^2(t_r)}{16\pi^2 c}\int \frac{\sin^2\theta}{\cancel{r^2}}\cancel{r^2}\sin\theta d\theta d\varphi$$
$$= \frac{\mu_o q^2 a^2(t_r)}{\cancel{16}_8\,\pi^{\cancel{2}} c}\bullet \cancel{2\pi}\underbrace{\int_0^\pi \sin^3\theta d\theta}_{=4/3} = \frac{\mu_o q^2 a^2(t_r)}{\cancel{8}_2\,\pi}\bullet\frac{\cancel{4}}{3} = \frac{\mu_o q^2 a^2(t_r)}{6\pi c}$$

$$\boxed{P_{rad}(t) = \frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \Leftarrow \text{Larmor power formula \{again\} !!!}$$

This formula was derived assuming $\vec{v}(t_r)=0$, but in fact, we get the ***same*** formula as long as $v(t_r) \ll c$ (*i.e.* non-relativistic motion).

- An ***exact*** treatment of $\vec{v}(t_r)\neq 0$ is (much) more difficult / tedious.
- Note that in ***special relativity*** {inertial (non-accelerated) reference frames}, the choice $\vec{v}(t_r)=0$ merely represents a judicious choice of an (inertial) reference frame, with ***no*** loss of generality.
- If we can determine how $P_{rad}(t)$ transforms from one reference frame to another, then we can deduce the more general $\vec{v}(t_r)\neq 0$ result (Liénard) from the (Larmor) $\vec{v}(t_r)=0$ result. (See *e.g.* Griffiths problem 12.69, *p*. 545).
- For the $\vec{v}(t_r)\neq 0$ case, $\vec{E}_{rad}(\vec{r},t)$ is more complicated (than the $\vec{v}(t_r)=0$ case).
- For the $\vec{v}(t_r)\neq 0$ case, $\vec{S}_{rad}(\vec{r},t)$ = the rate of energy passing through the (imaginary) large-radius surface S' of the sphere, $\vec{S}_{rad}(\vec{r},t)$ is NOT the same as the rate of energy when it left the charged particle at the ***retarded*** time t_r.

Consider the example of a person firing a stream of bullets (photons) out the window of a ***moving*** car, ***parallel*** to the direction of motion of the car:



The rate at which the bullets strike a target, R_{tgt} (#/sec) is ***not*** the same as the rate of bullets leaving the gun, R_{gun} (#/sec) because of the relative motion of the car with respect to the target. This is again analogous to the Doppler effect. It is ***purely*** due to a motional geometrical factor (*i.e.* it is ***not*** due to special relativity). For bullets moving ***parallel*** to the car's velocity vector:

$$\boxed{R_{gun} = \left(1-\beta(t_r)\right)R_{tgt}} \quad \text{or:} \quad \boxed{R_{tgt} = \frac{1}{1-\beta(t_r)}R_{gun}} \quad \text{where:} \quad \boxed{\beta(t_r) \equiv \frac{v(t_r)}{c}}$$

Whereas for bullets moving ***anti-parallel*** to the car's velocity vector:

$$R_{gun} = \left(1+\beta(t_{\mathrm{r}})\right) R_{tgt} \quad \text{or:} \quad R_{tgt} = \frac{1}{1+\beta(t_{\mathrm{r}})} R_{gun} \quad \text{where:} \quad \vec{\beta}(t_{\mathrm{r}}) \equiv \frac{\vec{v}(t_{\mathrm{r}})}{c}$$

For ***arbitrary*** directions, with $\hat{\mathscr{r}} \equiv$ unit vector from car to target:

$$R_{gun} = \left(1-\hat{r}\bullet\vec{\beta}(t_{\mathrm{r}})\right) R_{tgt} \quad \text{or:} \quad R_{tgt} = \frac{1}{\left(1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_{\mathrm{r}})\right)} R_{gun} \quad \text{where:} \quad \vec{\beta}(t_{\mathrm{r}}) \equiv \frac{\vec{v}(t_{\mathrm{r}})}{c}$$

So if $\dfrac{dW}{dt}$ = rate of energy passing through sphere of radius $\mathscr{r}$ then the rate at which energy leaves the charge q is:

$$\frac{dW}{dt_{\mathrm{r}}} = \frac{dW}{dt}\cdot\frac{dt}{dt_{\mathrm{r}}} = \frac{dW}{dt}\Big/\frac{\partial t_{\mathrm{r}}}{\partial t} = \left(\frac{\vec{\mathscr{r}}\bullet\vec{u}}{c}\right)\frac{dW}{dt} \quad \text{since:} \quad \frac{\partial t_{\mathrm{r}}}{\partial t} = \frac{c}{\hat{\mathscr{r}}\bullet\vec{u}} = \frac{\mathscr{r}c}{\vec{\mathscr{r}}\bullet\vec{u}} \quad \text{with} \quad \vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}(t_{\mathrm{r}}).$$

(see P436 Lect. Notes 12, *p*. 14-15, and/or Griffiths problem 10.17, *p*. 441)

But:

$$\frac{\hat{\mathscr{r}}\bullet\vec{u}}{c} = \frac{\hat{\mathscr{r}}\bullet\left(c\hat{\mathscr{r}} - \vec{v}(t_{\mathrm{r}})\right)}{c} = 1-\hat{\mathscr{r}}\bullet\vec{v}(t_{\mathrm{r}})\big/c = 1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_{\mathrm{r}}) \equiv \kappa$$ = ***retardation factor***

Then:

$$\frac{dW}{dt_{\mathrm{r}}} = \left(\frac{\hat{\mathscr{r}}\bullet\vec{u}}{c}\right)\frac{dW}{dt} = \left(1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_{\mathrm{r}})\right)\frac{dW}{dt} = \kappa\frac{dW}{dt} \quad \text{where:} \quad \vec{\beta}(t_{\mathrm{r}}) \equiv \frac{\vec{v}(t_{\mathrm{r}})}{c}$$

Thus, the power radiated into a patch of area $da = \mathscr{r}^2 \sin\theta d\theta d\varphi = \mathscr{r}^2 d\Omega$ on the sphere S', where $d\Omega = \sin\theta d\theta d\varphi$ = solid angle into which the *EM* power is radiated into area element *da* on the surface of the sphere S', with $\vec{E}_{rad}(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_o}\dfrac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}\right)^3}\left[\vec{\mathscr{r}}\times\left(\vec{u}\times\vec{a}\right)\right]$ is given by:

$$\begin{aligned}
\frac{dP_{rad}(t_{\mathrm{r}})}{d\Omega} &= \left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})}{c}\right)S_{rad}(\vec{r},t)\mathscr{r}^2 = \left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})}{c}\right)\frac{1}{\mu_o c}E_{rad}^2\mathscr{r}^2 \\
&= \frac{q^2}{16\pi^2\varepsilon_o^2}\frac{1}{\mu_o c}\left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})}{c}\right)\frac{\mathscr{r}^2\cdot\mathscr{r}^2}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})\right)^6}\left|\vec{\mathscr{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)\right|^2 \\
&= \frac{q^2}{16\pi^2\varepsilon_o^2}\frac{1}{\mu_o c}\left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})}{c}\right)\frac{\mathscr{r}^4}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})\right)^6}\left|\vec{\mathscr{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)\right|^2 \\
&= \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)\right|^2}{\left(\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})\right)^5}
\end{aligned}$$

Thus:

$$\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)\right|^2}{\left(\hat{\mathscr{r}}\bullet\vec{u}(t_{\mathrm{r}})\right)^5}$$

Integrating $\int_{S'} \frac{dP_{rad}(t)}{d\Omega} d\Omega$ over the sphere S' (*i.e.* over θ and φ angles) is a ***pain***....
However, the result of this integration {again!} yields the famous Liénard formula:

$$\boxed{P_{rad}(t) = \frac{\mu_o q^2}{6\pi c}\gamma^6\left(a^2(t_r) - \left|\frac{\vec{v}(t_r)\times\vec{a}(t_r)}{c}\right|^2\right) = \frac{\mu_o q^2}{6\pi c}\gamma^6\left(a^2(t_r) - \left|\vec{\beta}(t_r)\times\vec{a}(t_r)\right|^2\right)}$$

Where: $\boxed{\vec{\beta}(t_r) \equiv \vec{v}(t_r)/c}$ and: $\boxed{\gamma(t_r) \equiv \frac{1}{\sqrt{1-\beta^2(t_r)}}}$ = ***Lorentz factor***.

$\boxed{0 \le \beta \le 1}$ $\boxed{1 \le \gamma \le \infty}$

Note that the Liénard formula reduces to the Larmor formula for $P_{rad}(t)$ when $v \ll c$.
Note also that when $v \to c$, the γ^6 factor in the Liénard formula goes "berserk" – as the charged particle travels closer and closer to the speed of light c, the more one tries to accelerate it (in order to make it travel even closer to the speed of light, c), it radiates away more and more of the (absorbed) energy as $v \to c$!!!

$\Rightarrow$ ***very*** high energy electron accelerators are problematic in this regard, because the electron is ***so*** light, mass-wise, *e.g.* relative to the proton: $m_e = 0.511\, MeV/c^2$ whereas $m_p = 938.28\, MeV/c^2$.

Griffiths Example 11.3:

Suppose $\vec{v}(t_r)$ and $\vec{a}(t_r)$ are ***instantaneously collinear*** (*i.e.* ***parallel*** to each other). Find the angular distribution of radiated power $\frac{dP_{rad}(t)}{d\Omega}$ when $\boxed{\vec{v}(t_r)\cdot\vec{a}(t_r) = v(t_r)a(t_r)}$ (*i.e.* when $\vec{v}(t_r) \parallel \vec{a}(t_r)$)

Then in this case: $\boxed{\vec{u}(t_r) = \left(c\hat{\mathscr{r}} - \vec{v}(t_r)\right)}$ because $\vec{v}(t_r) \parallel \vec{a}(t_r)$

Thus: $\boxed{\vec{u}(t_r)\times\vec{a}(t_r) = \left(c\hat{\mathscr{r}} - \vec{v}(t_r)\right)\times\vec{a}(t_r) = c\hat{\mathscr{r}}\times\vec{a}(t_r) - \overbrace{\vec{v}(t_r)\times\vec{a}(t_r)}^{=0} = c\hat{\mathscr{r}}\times\vec{a}(t_r)}$

Then: $\boxed{\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{\mathscr{r}}\cdot\vec{u}(t_r)\right)^5} = \frac{q^2c^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{\mathscr{r}}\cdot\vec{u}(t_r)\right)^5}}$ $\boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$

Work on ***denominator*** term: $\boxed{\hat{\mathscr{r}}\cdot\vec{u}(t_r) = \hat{\mathscr{r}}\cdot\left(c\hat{\mathscr{r}} - \vec{v}(t_r)\right) = c - \hat{\mathscr{r}}\cdot\vec{v}(t_r) = c\left(1 - \hat{\mathscr{r}}\cdot\vec{\beta}(t_r)\right) = \kappa c}$ $\boxed{\kappa \equiv 1 - \hat{\mathscr{r}}\cdot\vec{\beta}(t_r)}$

Work on ***numerator*** term: $\boxed{\hat{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}(t_r)\right) = \left(\hat{\mathscr{r}}\cdot\vec{a}(t_r)\right)\hat{\mathscr{r}} - \overbrace{\left(\hat{\mathscr{r}}\cdot\hat{\mathscr{r}}\right)}^{=1}\vec{a}(t_r) = \left(\hat{\mathscr{r}}\cdot\vec{a}(t_r)\right)\hat{\mathscr{r}} - \vec{a}(t_r)}$

Thus: $\boxed{\left|\hat{\mathscr{r}}\times\left(\hat{\mathscr{r}}\times\vec{a}(t_r)\right)\right|^2 = a^2(t_r) - \left(\hat{\mathscr{r}}\cdot\vec{a}(t_r)\right)^2}$

Then: $$\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2c^2}{16\pi^2\varepsilon_o}\frac{\left[a^2(t_r)-\left(\hat{\imath}\bullet\vec{a}(t_r)\right)^2\right]}{c^5\left(1-\hat{\imath}\bullet\vec{\beta}(t_r)\right)^5} = \frac{q^2}{16\pi^2\varepsilon_o c^3}\frac{\left[a^2(t_r)-\left(\hat{\imath}\bullet\vec{a}(t_r)\right)^2\right]}{\left(1-\hat{\imath}\bullet\vec{\beta}(t_r)\right)^5}$$

If we let the $\hat{z}$-axis point along $\vec{v}(t_r)$ – along $\vec{\beta}(t_r) = \vec{v}(t_r)/c$ {and hence also along $\vec{a}(t_r)$}:

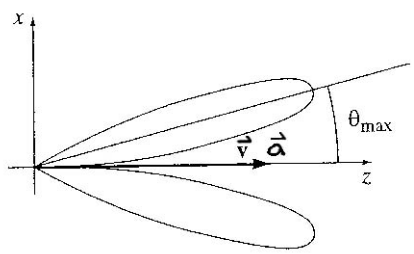
Then: $\hat{\imath}\bullet\vec{a} = a\cos\theta$ and: $\hat{\imath}\bullet\vec{v} = v\cos\theta$ or: $\hat{\imath}\bullet\vec{\beta} = \beta\cos\theta$ where θ = opening angle between $\hat{\imath}$ and acceleration $\vec{a}$, as shown on page 12 above.

Thus: $$\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2a^2(t_r)}{16\pi^2\varepsilon_o c^3}\frac{\left(1-\cos^2\theta\right)}{\left(1-\beta(t_r)\cos\theta\right)^5}$$ but: $$\frac{1}{c^2} = \varepsilon_o\mu_o$$

$$\therefore \quad \frac{dP_{rad}(t)}{d\Omega} = \frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}$$ with: $$\beta(t_r) \equiv \frac{v(t_r)}{c}$$

When $\beta \to 0$: $$\frac{dP_{rad}(t)}{d\Omega} = \frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\sin^2\theta = \vec{S}_{rad}^{\vec{v}=0}(\vec{r},t)\bullet\imath^2\hat{\imath}$$

When $\beta \to 1$: The donut of *EM* radiation intensity is ***folded forward*** by the factor $1/\left(1-\beta\cos\theta\right)^5$:



Note that there is still ***no*** radiation ***precisely*** in the forward direction, rather it is in a ***cone*** which becomes increasingly narrow as $\beta \to 1$, of half-angle:

$$\theta_{max} \approx \sqrt{(1-\beta)/2}$$ {see Griffiths problem 11.15, *p*. 465}

The total *EM* power radiated into 4π steradians by the point charge for $\vec{v} \parallel \vec{a}$ is:

$$P_{rad}(t) = \int\frac{dP_{rad}(t)}{d\Omega}d\Omega = \frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\int\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}\sin\theta d\theta d\varphi$$
$$= \frac{\mu_o q^2a^2(t_r)}{8\pi c}\int_{\theta=0}^{\theta=\pi}\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}\sin\theta d\theta$$

Let: $u = \cos\theta \quad \theta = 0 \to u = +1$
$du = -\sin\theta d\theta \quad \theta = \pi \to u = -1$

Then: $$P_{rad}(t)=\frac{\mu_o q^2 a^2(t_r)}{8\pi c}\int_{-1}^{1}\frac{(1-u^2)}{(1-\beta u)^5}du$$ Integrate by parts: $\int v du = uv - \int u dv$

$$P_{rad}(t)=\frac{\mu_o q^2 a^2(t_r)}{8\pi c}\left[\frac{4}{3}\left(1-\beta^2(t_r)\right)^{-3}\right]=\frac{\mu_o q^2 a^2(t_r)}{6\pi c}\left[\frac{1}{\left(1-\beta^2(t_r)\right)^3}\right]$$

But: $\gamma(t_r)\equiv\frac{1}{\sqrt{1-\beta^2(t_r)}}$ with: $\vec{\beta}(t_r)\equiv\vec{v}(t_r)/c$

$$\therefore \quad P_{rad}(t)=\frac{\mu_o q^2 a^2(t_r)}{6\pi c}\gamma^6(t_r)$$

This is the same/identical result as obtained ***directly*** from the Liénard formula when $\vec{v}(t_r)\parallel\vec{a}(t_r)$. It is also known as the ***classical*** formula for ***bremsstrahlung*** ("braking radiation" in german).

Again, note that because $P_{rad}(t)\sim a^2(t_r)$, the *EM* power radiated doesn't depend on the ***sign*** of $\vec{a}(t_r)$ – *i.e.* whether the charged particle is ***accelerating*** or ***decelerating***.

Now it can also be shown that the Lorentz factor $\gamma = E/mc^2$, where $E=\sqrt{(pc)^2+(mc^2)^2}$ = ***total*** relativistic energy associated with a charged particle moving with $\vec{\beta}(t_r)\equiv\vec{v}(t_r)/c$. Thus, when $v\to c$, for a given {high} total energy *E*, then $\gamma\sim 1/mc^2$ and thus: $P_{rad}(t)\sim 1/m^6$.

Comparing *EM* bremsstrahlung radiation from an accelerated electron {$m_e = 0.511\,MeV/c^2$} *vs*. that of *e.g*. an accelerated muon {$m_\mu = 105.66\,MeV/c^2$}, for the same total energy *E*, an electron will radiate $(m_\mu/m_e)^6 \simeq (206.8)^6 = 7.8\times10^{13}$ times more *EM* energy than a muon. This explains why muons have such high ***penetrating power*** in traversing matter – they lose relatively little energy via bremsstrahlung, whereas high-energy electrons radiate *EM* energy "like crazy" in matter.

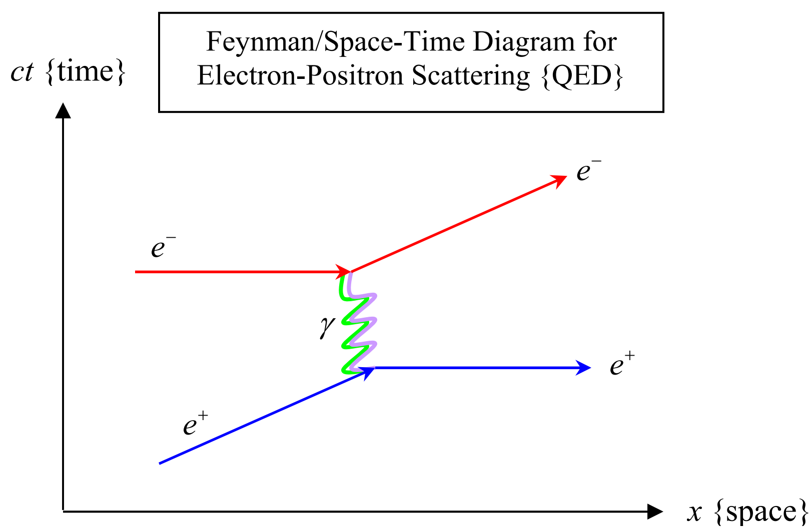
The Radiation Reaction on a Radiating Charged Particle

According to the laws of classical electrodynamics, an accelerating electric charge radiates electromagnetic energy in the form of real photons (= quanta of the *EM* radiation field).

By conservation of energy, the *EM* radiation carries off / carries away energy – which must come at the expense of the charged particle's ***kinetic*** energy {since its rest mass cannot change}.

In other words, one puts in energy to accelerate the charged particle, but the charged particle winds up being accelerated ***less*** than *e.g.* an electrically neutral particle {of the same rest mass of the charged particle}, for the same amount of input energy!

The devil is in the ***microscopic*** details of precisely how this is accomplished in both cases. At the microscopic level, an electrically charged particle of mass *m* is accelerated/increases its {**kinetic**} energy $\boxed{T=(\gamma-1)mc^2}$ by ***absorbing*** *EM* energy (either in the form of virtual or real photons) from a ***source*** of *EM* field(s). In order to accelerate/increase the {***kinetic***} energy $\boxed{T=(\gamma-1)mc^2}$ of an electrically ***neutral*** particle, it too must interact, at the microscopic level, via one of the four fundamental forces of nature, with a ***source*** {of fields} associated with that fundamental force.



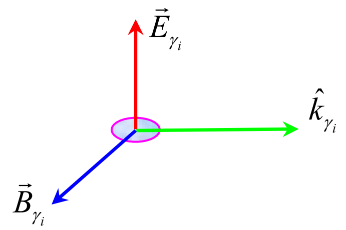
In the electromagnetic case, if an electrically charged particle is decelerated and radiates *EM* energy away in the form of {real} photons, by energy conservation, the change in the kinetic energy of the charged particle ***must*** equal the sum of the energies associated with each of the *n* individual {real} photons radiated by the charged particle:

$$\boxed{\Delta KE_q=\sum_{i=1}^{n}E_{\gamma_i}=\sum_{i=1}^{n}hf_{\gamma_i}}$$

This implies that the radiation must {somehow!} exert a force, $\vec{F}_{rad}$ ***back*** on the electrically charged particle – *i.e.* a ***recoil*** force, analogous to that associated with firing a bullet from a gun. Thus, ***linear*** momentum p must ***also*** conserved in this process. In the emission of *EM* radiation {real photons}, ***linear*** momentum $\boxed{p_{\gamma_i} = h/\lambda_{\gamma_i} = hf_{\gamma_i}/c}$ is also carried away by each of the {real} photons. This comes at the expense of the charged particle's momentum $\vec{p}_q$ and (non-relativistically, for $v_q \ll c$): $\boxed{KE_q = p_q^2/2m}$

$$\boxed{\Delta\vec{p}_q^{\,recoil} c = \sum_{i=1}^{n} \vec{p}_{\gamma_i} c = \sum_{i=1}^{n} \frac{h}{\lambda_{\gamma_i}} c\hat{k}_{\gamma_i} = \sum_{i=1}^{n} hf_{\gamma_i} \hat{k}_{\gamma_i}}$$

$\hat{k}_{\gamma_i}$ = wave vector for the i^{th} photon



Thus, if a similarly accelerated/decelerated neutral particle ***doesn't*** radiate force quanta {of some kind} because it is accelerated/decelerated, then because the electrically-charged particle ***does*** radiate *EM* quanta {real photons} in the acceleration/deceleration process, then we can see that the final-state $KE_q < KE_o$ for similarly accelerated/decelerated neutral particle of the same mass *m* and initial/original kinetic energy as that of the electrically-charged particle.

<u>The Radiation Reaction Force on a Charged Particle</u>

For a non-relativistic particle ($v_q \ll c$) the Larmor formula for the total instantaneous *EM* radiated power is:

$$\boxed{P_{rad}(t) \simeq \frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \quad \text{(Watts)}$$

Conservation of energy would then imply that this radiated *EM* power = the instantaneous rate at which the charged particle loses energy, due to the effect of the *EM* radiation back-reaction / recoil force $\vec{F}_{rad}(t_r)$:

$$\boxed{P_q(t_r) = \frac{dW(t_r)}{dt} = \vec{F}_{rad}(t_r) \bullet \vec{v}(t_r) = -\frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \quad \text{(Watts)}$$

This relation / equation is actually ***wrong***. Why???

The reason is, that we calculated the radiated *EM* power by integrating Poynting's vector $\vec{S}_{rad}(\vec{r},t)$ for the *EM* radiation associated with the accelerating point charged particle over an "infinite" sphere of radius $\mathscr{r}$; in this calculation the *EM* ***velocity*** fields played ***no*** role, since they fall off too rapidly as a function of $\mathscr{r}$ to make any contribution to $P_{rad}(t)$. However, the *EM* ***velocity*** fields ***do*** carry energy – because the ***total*** retarded electric field associated with the electrically charged particle is the sum of ***two*** terms – the *EM* ***velocity*** field ***and*** the *EM* ***acceleration*** field terms:

$$\boxed{\vec{E}_r^{tot}(\vec{r},t) = \vec{E}_r^{v}(\vec{r},t) + \vec{E}_r^{a}(\vec{r},t)}$$

The total ***retarded*** *EM* energy density associated with the total ***retarded*** electric field is:

$$u_{\mathrm{E}}^{tot}(\vec{r},t)=\tfrac{1}{2}\varepsilon_o E_{\mathrm{r}}^{tot^2}(\vec{r},t)=\tfrac{1}{2}\varepsilon_o\left(\vec{E}_{\mathrm{r}}^{v}(\vec{r},t)+\vec{E}_{\mathrm{r}}^{a}(\vec{r},t)\right)^2$$
$$=\tfrac{1}{2}\varepsilon_o\left[E_{\mathrm{r}}^{v^2}(\vec{r},t)+2\vec{E}_{\mathrm{r}}^{v}(\vec{r},t)\bullet\vec{E}_{\mathrm{r}}^{a}(\vec{r},t)+E_{\mathrm{r}}^{a^2}(\vec{r},t)\right]$$

- First term: Energy stored in ***velocity*** field only (***virtual*** photons) — Generalized Coulomb fields only
- Second term: Cross term!!! Energy stored in ***mixture*** of ***velocity*** and ***acceleration*** field (both ***virtual*** & ***real*** photons!!) — "Conversion" field ***virtual*** → ***real*** photons {and vice versa!}
- Third term: Energy stored in ***acceleration*** field only (***real*** photons) — Radiation fields only

Note that:

The Generalized Coulomb fields vary as $\sim 1/\mathscr{r}^4$
The "Conversion" fields vary as $\sim 1/\mathscr{r}^3$
The Radiation fields vary as $\sim 1/\mathscr{r}^2$

Neither the Generalized Coulomb field nor the "Conversion" field contribute to *EM* radiation in the "far-zone" limit $r' \ll r$

Clearly, the first two terms in the *EM* energy density formula associated with the electric field have energy associated with them. However, this energy ***stays with*** the charged particle – it is ***not*** radiated away.

As the charged particle accelerates / decelerates, energy is exchanged between the charged particle and the velocity and acceleration fields. For the latter term (the last/ 3rd term in $u_{\mathrm{E}}^{tot}(\vec{r},t)$ above), this energy is irretrievably carried away (by ***real*** photons) out to $r=\infty$.

Thus, $$P_q(t_{\mathrm{r}})=\frac{dW(t_{\mathrm{r}})}{dt}=\vec{F}_{rad}(t_{\mathrm{r}})\bullet\vec{v}(t_{\mathrm{r}})=-\frac{\mu_o q^2 a^2(t_{\mathrm{r}})}{6\pi c}$$ only accounts for the last / 3rd term ($E_{\mathrm{r}}^{a^2}(\vec{r},t)$) in $u_{\mathrm{E}}^{tot}(\vec{r},t)$ above.

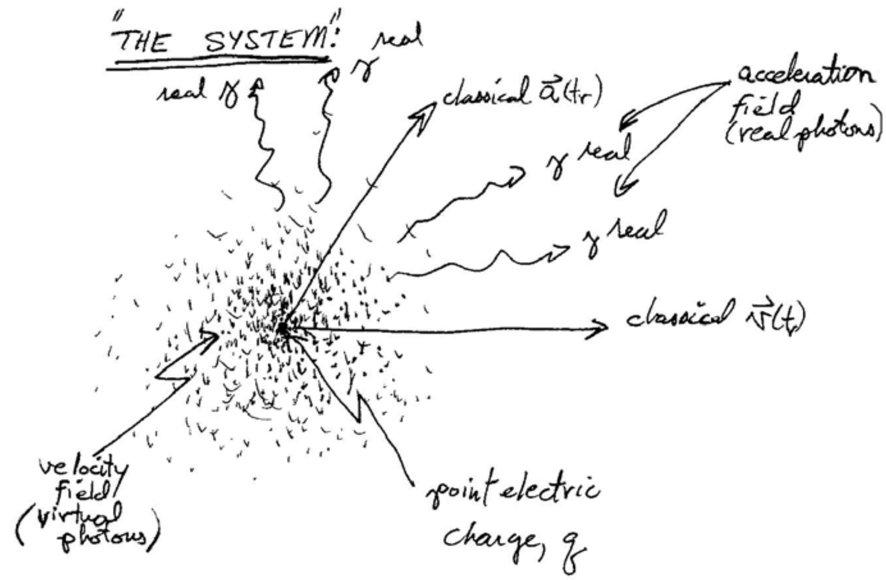
If we want to know the ***total*** recoil force exerted by the *EM* velocity ***and*** the *EM* acceleration fields on the point charge, then we need to know the ***total*** instantaneous power lost, not ***just*** the ***radiation-only*** contribution.

Thus, in this sense, the term "radiation" (back)-reaction is a misnomer because it should more appropriately be called an *EM* field (back)-reaction. Note further that this *EM* field (back)-reaction is also intimately connected with the issue of the so-called "hidden" *EM* momentum.

Shortly, we'll see that $\vec{F}_{rad}(t_{\mathrm{r}})$ is determined by the time derivative of the acceleration $\vec{a}(t_{\mathrm{r}})$, and can be non-zero even when the acceleration $\vec{a}(t_{\mathrm{r}})$ is instantaneously ***zero***!
(The charged particle is ***not*** radiating at that ***retarded*** instant in time!)

By energy conservation, the energy ***lost*** by the electrically charged particle in a given ***retarded*** time interval $\Delta t_r \equiv t_{r_2} - t_{r_1}$ $\left(t_{r_2} > t_{r_1}\right)$ ***must*** equal the energy ***carried*** ***away*** by the *EM* radiation, ***plus*** whatever ***extra*** energy has been pumped into the *EM* velocity/generalized Coulomb field.

If we consider time intervals $\Delta t_r = t_{r_2} - t_{r_1}$ such that "the system" (consisting of the point-charged particle *q* and the *EM* velocity field – see drawing on following page) *returns* to its initial state, then (assuming that the energy in the *EM* velocity fields is the same at time t_{r_2} as at time t_{r_1}, then the only net energy loss ***is*** in the form of *EM* radiation (due to the emission of *n* ***real*** photons).



Thus, while ***instantaneously*** $P_q(t_r) = \dfrac{dW(t_r)}{dt} = \vec{F}_{rad}(t_r)\bullet\vec{v}(t_r) = -\dfrac{\mu_o q^2 a^2(t_r)}{6\pi c}$ ***is*** incorrect, by suitably ***averaging*** this relation over a finite time interval, it ***is*** valid, with the restriction that state of "the system" is ***identical*** at the ***retarded*** times t_{r_1} and t_{r_2}:

$$\frac{1}{\Delta t_r}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \vec{F}_{rad}(t'_r)\bullet\vec{v}(t'_r)\,dt'_r = -\frac{1}{\Delta t_r}\frac{\mu_o q^2}{6\pi c}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} a^2(t'_r)\,dt'_r$$

For the case of ***periodic***/***harmonic*** motion, this means that the above integrals must be carried out over at least one (or more) complete / full cycles, $\Delta t_r \equiv t_{r_2} - t_{r_1} = n\,\tau$, $n = 1, 2, 3, \ldots$

For ***non***-***periodic*** motion, the condition that "the system" be identical at times t_{r_1} and t_{r_2} is more difficult to achieve – it is ***not*** enough that the instantaneous velocities and accelerations be equal at t_{r_1} and t_{r_2}, since the (retarded) fields farther out (at the ***present*** time $t = t_r + \mathscr{r}/c$) depend on $\vec{v}(t_r)$ and $\vec{a}(t_r)$ at the earlier ***retarded*** time t_r !!!

For ***non-periodic*** motion, the condition that "the system" be identical at times t_{r_1} and t_{r_2} technically requires that not only $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$, but ***all*** higher derivatives of $\vec{v}(t_r)$ must ***also*** likewise be equal at times t_{r_1} and t_{r_2} !!!

However, in practice, for ***non-periodic*** motion, since the *EM* velocity fields fall off rapidly with r, it is ***sufficient*** that $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$, for a ***brief*** time interval, $\Delta t_r = t_{r_2} - t_{r_1}$.

The RHS of the above equation can be integrated by parts:

$$\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} a^2(t'_r)\,dt'_r = \int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\frac{d\vec{v}(t'_r)}{dt_r}\right)\left(\frac{d\vec{v}(t'_r)}{dt_r}\right)dt'_r = \left(\vec{v}(t'_r)\bullet\frac{d\vec{v}(t'_r)}{dt_r}\right)\Bigg|_{t_{r_1}}^{t_{r_2}} - \int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \underbrace{\frac{d^2\vec{v}(t'_r)}{dt_r^2}}_{\equiv\dot{\vec{a}}(t'_r)}\bullet\vec{v}(t'_r)\,dt'_r$$

Because of the restriction on $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$ at the time endpoints t_{r_1} and t_{r_2},

The term: $$\left(\vec{v}(t'_r)\bullet\frac{d\vec{v}(t'_r)}{dt_r}\right)\Bigg|_{t_{r_1}}^{t_{r_2}} = \vec{v}(t'_r)\bullet\vec{a}(t'_r)\Big|_{t_{r_1}}^{t_{r_2}} = 0$$

Thus: $$\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\vec{F}_{rad}(t'_r)\bullet\vec{v}(t'_r)\right)dt'_r = +\frac{\mu_o q^2}{6\pi c}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\dot{\vec{a}}(t'_r)\bullet\vec{v}(t'_r)\right)dt'_r$$

Or: $$\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\vec{F}_{rad}(t'_r) - \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)\right)\bullet\vec{v}(t'_r)\,dt'_r = 0$$

Mathematically, there are lots of ways this integral equation can be satisfied, but it will certainly be satisfied if:

$$\vec{F}_{rad}(t'_r) = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r) \quad \Leftarrow \text{ Abraham-Lorentz formula}$$

This relation is known as the Abraham-Lorentz formula for the *EM* "radiation reaction" force.

$\vec{F}_{rad}(t'_r) = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)$ is the ***simplest*** possible form the *EM* radiation reaction force can take.

Physically, note that this formula tells us ***only*** about the ***time-averaged*** force {albeit} over a very brief time interval $\Delta t_r = t_{r_2} - t_{r_1}$, of the force component ***parallel*** to $\vec{v}(t_r)$ - because of the original term $\left(\vec{F}_{rad}(t'_r)\bullet\vec{v}(t'_r)\right)$. As such, it tells us ***nothing*** about $\vec{F}_{rad_\perp}(t_r)$ $\perp$ to $\vec{v}(t_r)$.

n.b. These averages are also restricted to time intervals such that $\Delta t_r = t_{r_2} - t_{r_1}$ is chosen to ***ensure*** that $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ ***and*** $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$.

The Abraham-Lorentz radiation reaction force $\vec{F}_{rad}\left(t_{\rm r}'\right)=\frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}\left(t_{\rm r}'\right)$ also has disturbing, seemingly unphysical implications that are still not fully understood today, despite the passage of nearly a century!

Suppose a charged particle is subject to ***NO*** external forces. Then Newton's 2nd law says that:

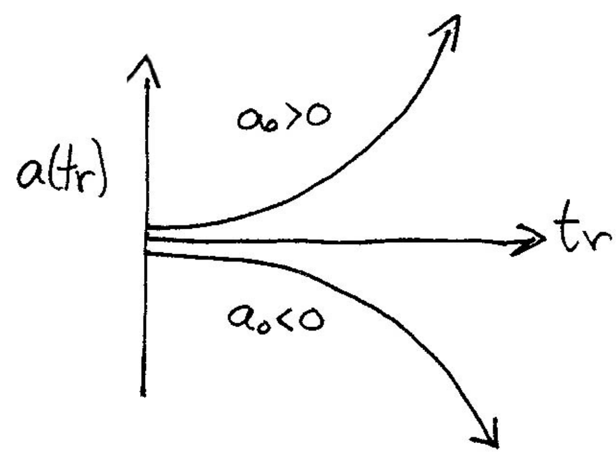
$$\boxed{\vec{F}_{rad}\left(t_{\rm r}'\right)=\frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}\left(t_{\rm r}'\right)=m\vec{a}\left(t_{\rm r}\right)}$$ where m = (real) rest mass of the charged particle.

Then: $$\boxed{\vec{F}_{rad}\left(t_{\rm r}'\right)=\frac{\mu_o q^2}{6\pi mc}m\dot{\vec{a}}\left(t_{\rm r}'\right)=m\tau\dot{\vec{a}}\left(t_{\rm r}'\right)=m\vec{a}\left(t_{\rm r}\right)}$$ or: $$\boxed{\frac{\mu_o q^2}{6\pi mc}\dot{\vec{a}}\left(t_{\rm r}\right)=\tau\dot{\vec{a}}\left(t_{\rm r}\right)=\vec{a}\left(t_{\rm r}\right)}$$

The solution to this linear, first-order homogeneous differential equation is: $\boxed{a\left(t_{\rm r}\right)=a_o e^{+t_{\rm r}/\tau}}$

where a_o = acceleration at the ***retarded*** zero of time, $t_{\rm r}=0$, and $\boxed{\tau\equiv\left(\frac{\mu_o q^2}{6\pi mc}\right)}$, which for the electron is a time constant of : $\boxed{\tau_e\simeq 6\times10^{-24}\,\text{sec}}$.

If $a_o \neq 0$, the acceleration exponentially increases ($+ve$, if $a_o > 0$, $-ve$, if $a_o < 0$) as time progresses! This is a runaway solution, which is *CRAZY* !!! This can ***only*** be avoided if $a_o \equiv 0$.



However, if the runaway solutions are ***excluded*** on physical grounds, then the charged particle develops an ***acausal*** behavior – *e.g.* if an external force is applied, the charged particle responds ***before*** the force acts!! This ***acausal*** "pre-acceleration" "jumps the gun" by only a short time $\tau_e\simeq 6\times10^{-24}$ sec, and since we know that quantum mechanics and uncertainty principle are operative on short distance/short timescales, perhaps this classical behavior shouldn't be ***too*** unsettling to us. Nevertheless, to many it is…. (see Griffiths Problem 11.19, *p*. 469 for more aspects/ramifications of the Abraham-Lorentz formula…)

Such difficulties also persist in the ***fully-relativistic*** version of the Abraham-Lorentz equation.

Griffiths Example 11.4 – *EM* Radiation Damping:

Calculate the *EM* radiation damping of an electrically charged particle attached to a spring of natural angular frequency ω_o with driving frequency = ω

The 1-dimensional equation of motion is:

$$\boxed{\begin{aligned} m\ddot{x}\left(t_{\rm r}\right) &= F_{spring}\left(t_{\rm r}\right)+F_{rad}\left(t_{\rm r}\right)+F_{driving}\left(t_{\rm r}\right)\\ &= -m\omega_o^2 x\left(t_{\rm r}\right)+m\tau\dddot{x}\left(t_{\rm r}\right)+F_{driving}\left(t_{\rm r}\right)\end{aligned}}$$

With the system oscillating at the driving frequency ω:

Instantaneous position: $x(t_r) = x_o \cos(\omega t_r + \delta)$

Instantaneous velocity: $\dot{x}(t_r) = -\omega x_o \sin(\omega t_r + \delta)$

Instantaneous acceleration: $\ddot{x}(t_r) = -\omega^2 x_o \cos(\omega t_r + \delta)$

Instantaneous jerk: $\dddot{x}(t_r) = +\omega^3 x_o \sin(\omega t_r + \delta) = -\omega^2 \underbrace{\left(-\omega x_o \sin(\omega t_r + \delta)\right)}_{=\dot{x}(t_r)}$

Thus: $\dddot{x}(t_r) = -\omega^2 \dot{x}(t_r)$

Thus: $m\ddot{x}(t_r) + \tau m\omega^2 \dot{x}(t_r) + m\omega_o^2 x(t_r) = F_{driving}(t_r)$

Define the ***damping constant***: $\gamma \equiv \omega^2 \tau$ (SI units: $1/\text{sec}$)

Then: $m\ddot{x}(t_r) + m\gamma\dot{x}(t_r) + m\omega_o^2 x(t_r) = F_{driving}(t_r)$ ← 2^{nd}-order linear inhomogeneous diff. eqn.

n.b. In this situation, the *EM* radiation damping is proportional to $\ddot{v}(t_r)$. Compare this to *e.g.* "normal" mechanical damping, which is proportional to $v(t_r)$ (*e.g.* friction / dissipation).

The Physical Basis of the Radiation Reaction

We derived the Abraham-Lorentz *EM* radiation reaction force $\vec{F}_{rad}(t'_r) = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)$ from consideration of conservation of energy in the *EM* radiation process, from what was observable in the far-field region, $\mathrm{r} \to \infty$.

Classically, if one tries to determine this radiation reaction force ***at*** the radiating point charge, we run into mathematical difficulties due to the mathematical point-behavior of the electric charge (*e.g.* at its origin) where the (static) electric field and corresponding scalar potential become singular, this problem correspondingly has infinite energy density at the point charge.

This singular nature is also the present for the retarded *EM* fields associated with a moving point charge:

$$\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

$$\vec{B}_r(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}_r(\vec{r},t) \quad \text{with:} \quad \vec{u} = c\hat{\mathscr{r}} - \vec{v}$$

Today, we know that quantum mechanics is operative, *e.g.* from the Heisenberg uncertainty principle on {*e.g.* 1-dimensional} distance scales of:

$\Delta x \Delta p_x \le \hbar$ where: $\hbar = h/2\pi$ = Planck's Constant / 2π, $h = 6.626\times10^{-34}$ J-sec

Then: $\Delta x \le \hbar/\Delta p_x$ but: $\Delta p_x c < m_e c^2$ {for electrons}

Note that: $hc = 1240$ eV-nm and: $1\text{ nm} = 10^{-9}$ m

$$\therefore \boxed{\Delta x \le \frac{\hbar c}{m_e c^2} = \frac{1240\text{ eV-nm}/2\pi}{0.511\text{ meV}} \simeq 0.386\times10^{-12}\,\text{m} = 386\text{ fm}} \qquad (1\text{ fm} = 10^{-15}\text{ m})$$

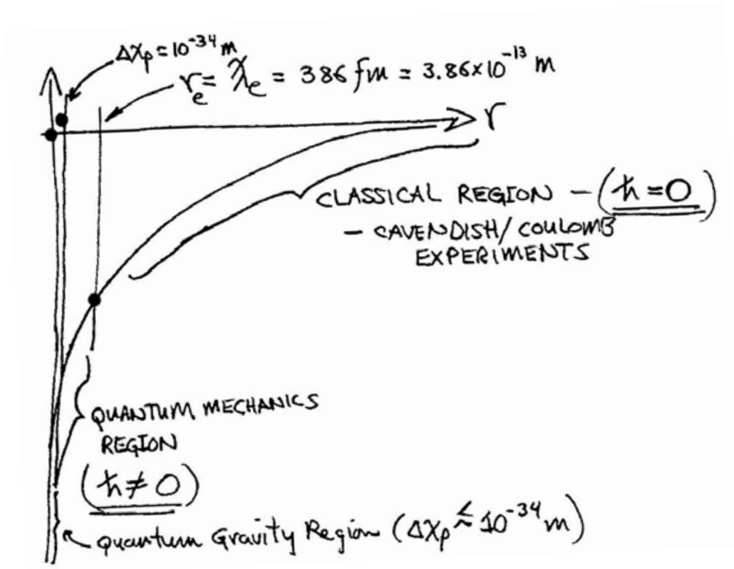
The quantity $\boxed{\lambda\!\!\!^{-}_e \equiv \frac{\lambda_e}{2\pi} \equiv \frac{\hbar c}{m_e c^2} = 386\text{ fm}}$ = reduced Compton wavelength of the electron

and: $\boxed{\lambda_e \equiv \frac{hc}{m_e c^2} = 2427\text{ fm}}$ = Compton wavelength of electron

Thus: $\boxed{\Delta x \le \lambda\!\!\!^{-}_e = \frac{\hbar c}{m_e c^2} = 386\text{ fm} = 386\times10^{-15}\,\text{m}}$

For short distance scales of order $\boxed{\Delta x \le \lambda\!\!\!^{-}_e = \hbar c/m_e c^2 = 386\text{ fm}}$ {and less} the behavior of an electron will be manifestly quantum mechanical in nature. Thus, we should ***not*** be surprised that when extrapolating ***classical*** *EM* theory into this short-distance regime, we obtain erroneous answers – we have ***no*** reason to expect ***classical*** theory to {continue to} hold in the ***quantum*** domain !!!

$$V_e(r) = -\frac{q}{4\pi\varepsilon_o}\frac{1}{r}$$



Similarly, we have ***no*** business extrapolating quantum mechanics to distance scales less than:

$$\boxed{r_e^{BH} = \frac{2G_N m_e}{c^2} = \frac{2\times6.673\times10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-1}\times9.109\times10^{-31}\,\text{kg}}{\left(3\times10^8\,\text{m/s}\right)^2} \simeq 1.35\times10^{-57}\,\text{m}} = \boxed{\text{Schwartzschild radius of electron (event horizon)}}$$

Where G_N = Newton's gravitational constant. The electron is a black hole at this distance scale – the Schwartzschild radius/event horizon of an electron is where space & time interchange roles!

However, ***long*** before this regime is reached, at distance scales corresponding to the Planck energy/Planck mass $\boxed{m_p c^2 = \sqrt{\hbar c^3/G_N} \simeq 2.2\times10^{-8}\,kg = 1.2\times10^{19}\,\text{GeV} = 1.2\times10^{28}\,\text{eV}}$ {1 GeV = 10^9 eV}, is the regime of ***quantum gravity***, where space-time itself becomes "***foam-like***" (*i.e.* not ***continuous***) – ***quantized/discretized*** {somehow…}. The distance scale where quantum gravity is operative is known as the Planck length: $\boxed{L_P = \sqrt{\hbar G_N / c^3} \simeq 1.6\times 10^{-35}\, m}$. The Planck length corresponds to a time-scale {known as the Planck time} of $\boxed{t_P = L_P/c = \sqrt{\hbar G_N / c^5} \simeq 5.4\times 10^{-44}\, sec}$.

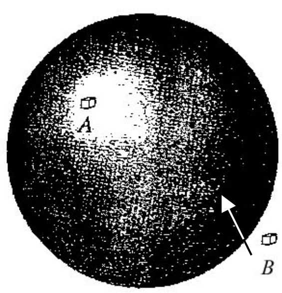
Nevertheless, back in the early 1900's, ignorance of quantum gravity and quantum mechanics did not stop Abraham, Lorentz, Poincaré {and many others} from applying classical *EM* theory - electrodynamics to calculate the self-force / radiation back-reaction on a point electric charge.

These efforts by-and-large modeled the point electron as {some kind of} spatially-extended electric charge distribution (of finite, but very small size), calculations could then be carried out and then (at the end of the calculation) the limit of the size of the charge distribution → 0.

In general (as we have already encountered this before in electrodynamics), the ***retarded*** classical/macroscopic *EM* force of one part (*A*) acting on another part (*B*) is ***not*** equal and opposite to the force of *B* acting on *A*, Newton's 3rd Law is seemingly violated:

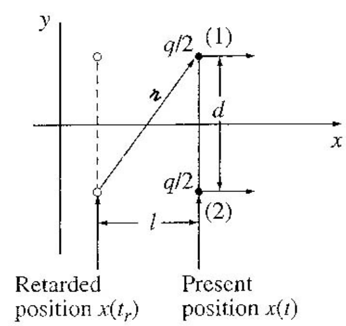
$$\boxed{\vec{F}_{\mathrm{r}}^{AB}\left(\vec{r}_B,t\right) \neq -\vec{F}_{\mathrm{r}}^{BA}\left(\vec{r}_A,t\right)}$$

Adding up the imbalances of such force pairs, we obtain the net force (imbalance) of a charge on itself – the "***self-force***" acting on the charge.



H.A. Lorentz originally calculated the classical self-force using a spherical charge distribution – tedious – see J.D. Jackson's Classical Electrodynamics, 3rd ed., sec. 16.3 and beyond if interested in these details….

A "less realistic" model of a charge is to use a ***rigid dumbbell*** in which the total charge q is divided into 2 halves separated by a fixed distance d (simplest possible charge arrangement to elucidate the self-force mechanism):



Assume that the dumbbell moves in $\hat{x}$-direction and (for simplicity) assume that the dumbbell is ***instantaneously*** at ***rest*** at the ***retarded*** time t_r. Then the ***retarded*** electric field at (1) due to (2) is:

$$\boxed{\vec{E}_{\mathrm{r}}^{21}\left(\vec{r}_1,t\right) = \frac{\frac{1}{2}q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}\right)^3}\left[\left(c^2+\vec{\mathscr{r}}\bullet\vec{a}\right)\vec{u}-\left(\vec{\mathscr{r}}\bullet\vec{u}\right)\vec{a}\right] = \frac{q}{8\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}\right)^3}\left[\left(c^2+\vec{\mathscr{r}}\bullet\vec{a}\right)\vec{u}-\left(\vec{\mathscr{r}}\bullet\vec{u}\right)\vec{a}\right]}$$

Here: $\boxed{\vec{u} = c\hat{\mathcal{R}}}$ {because $\vec{v}(t_r) = 0$ }

Note that: $\boxed{\vec{\mathcal{R}} = \ell\hat{x} + d\hat{y} = \mathcal{R}\hat{\mathcal{R}}}$ and thus: $\boxed{\mathcal{R} = \sqrt{\ell^2 + d^2}}$.

Note also that: $\boxed{\ell = fcn(\vec{a})}$ and: $\boxed{\vec{a}(t_r) = a(t_r)\hat{x}}$.

$\therefore$ $\boxed{\vec{\mathcal{R}}\bullet\vec{u}(t_r) = (\ell\hat{x} + d\hat{y})\bullet c\hat{\mathcal{R}} = \mathcal{R}\hat{\mathcal{R}}\bullet c\hat{\mathcal{R}} = c\mathcal{R}}$ and: $\boxed{\vec{\mathcal{R}}\bullet\vec{a}(t_r) = (\ell\hat{x} + d\hat{y})\bullet a(t_r)\hat{x} = a(t_r)\ell}$

We are in fact ***only*** interested in the $\hat{x}$-component of $\vec{E}_r^{21}(\vec{r},t)$, since the $\hat{y}$-components of $\vec{E}_r^{21}(\vec{r},t)$ and $\vec{E}_r^{12}(\vec{r},t)$ will ***cancel*** when we add forces on the two ends of the dumbbell.

Note further that since the two charges on the dumbbell are both moving in the ***same*** direction / ***parallel*** to each other, the ***magnetic*** forces associated one charge acting on the other will ***also*** cancel, thus Newton's 3rd Law is manifestly obeyed {***here***}, in ***this*** particular situation / configuration.

If $\boxed{\vec{u} = c\hat{\mathcal{R}}}$, then: $\boxed{\vec{u}_x = \vec{u}\bullet\hat{x} = \frac{c\vec{\mathcal{R}}}{\mathcal{R}}\bullet\hat{x}}$ and since: $\boxed{\vec{\mathcal{R}} = \ell\hat{x} + d\hat{y}}$ then: $\boxed{u_x = \frac{c}{\mathcal{R}}(l\hat{x} + d\hat{y})\bullet\hat{x} = \frac{c\ell}{\mathcal{R}}}$

Thus: $\boxed{E_{r_x}^{21}(\vec{r}_1,t) = \frac{q}{8\pi\varepsilon_o}\frac{\mathcal{R}}{\left(\vec{\mathcal{R}}\bullet\vec{u}(t_r)\right)^3}\left[\left(c^2 + \vec{\mathcal{R}}\bullet\vec{a}(t_r)\right)u_x - \left(\vec{\mathcal{R}}\bullet\vec{u}(t_r)\right)a_x(t_r)\right]}$

And: $\boxed{\left(\vec{\mathcal{R}}\bullet\vec{u}(t_r)\right) = c\mathcal{R}}$ and: $\boxed{\left(\vec{\mathcal{R}}\bullet\vec{a}(t_r)\right) = (\ell\hat{x} + d\hat{y})\bullet a(t_r)\hat{x} = a(t_r)\ell}$ since: $\boxed{\vec{a}(t_r) = a(t_r)\hat{x} = a_x(t_r)\hat{x}}$

Then:

$$\begin{aligned}
E_{r_x}^{21}(\vec{r}_1,t) &= \frac{q}{8\pi\varepsilon_o}\frac{\mathcal{R}}{c^3\mathcal{R}^3}\left[\left(c^2 + a(t_r)\ell\right)\frac{c\ell}{\mathcal{R}} - c\mathcal{R}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathcal{R}^2}\left[\left(c^2 + a(t_r)\ell\right)\frac{\ell}{\mathcal{R}} - \mathcal{R}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathcal{R}^2}\left[\frac{\ell c^2}{\mathcal{R}} + \frac{a(t_r)\ell^2}{\mathcal{R}} - \mathcal{R}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathcal{R}^2}\left[\frac{\ell c^2}{\mathcal{R}} + \frac{a(t_r)\ell^2 - \mathcal{R}^2 a(t_r)}{\mathcal{R}}\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathcal{R}^3}\left[\ell c^2 - \underbrace{\left(\mathcal{R}^2 - \ell^2\right)}_{=d^2}a(t_r)\right] \quad \text{but: } \mathcal{R}^2 = \ell^2 + d^2 \quad \text{or: } \mathcal{R}^2 - \ell^2 = d^2 \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathcal{R}^3}\left[\ell c^2 - d^2 a(t_r)\right]
\end{aligned}$$

Thus: $\boxed{E_{r_x}^{21}(\vec{r}_1,t) = \frac{q}{8\pi\varepsilon_o}\frac{\left(\ell c^2 - d^2 a(t_r)\right)}{c^2\left(\ell^2 + d^2\right)^{3/2}}}$ since: $\boxed{\mathcal{R} = \sqrt{\ell^2 + d^2}}$

By symmetry: $\boxed{E_{r_x}^{21}(\vec{r}_1,t) = E_{r_x}^{12}(\vec{r}_2,t)}$

$\therefore$ The net ***retarded*** force on the rigid dumbbell is:

$$\vec{F}_{\mathrm{r}}^{self}(\vec{r},t)=\vec{F}_{\mathrm{r}}^{\overline{21}}(\vec{r}_1,t)+\vec{F}_{\mathrm{r}}^{\overline{12}}(\vec{r}_2,t)=\tfrac{1}{2}q\vec{E}_{\mathrm{r}}^{\overline{21}}(\vec{r}_1,t)+\tfrac{1}{2}q\vec{E}_{\mathrm{r}}^{\overline{12}}(\vec{r}_2,t)=\frac{q^2}{8\pi\varepsilon_o c^2}\frac{\left(\ell c^2-d^2a(t_{\mathrm{r}})\right)}{\left(\ell^2+d^2\right)^{3/2}}\hat{x} \Leftarrow \textbf{Exact}$$

We now expand $\vec{F}_{\mathrm{r}}^{self}(\vec{r},t)$ in powers of *d*. Then when the size *d* of the electrically-charged dumbbell is taken to its limit of $d \to 0$, all positive powers will disappear.

Taylor's Theorem: $x(t)=x(t_{\mathrm{r}})+\dot{x}(t_{\mathrm{r}})(t-t_{\mathrm{r}})+\frac{1}{2!}\ddot{x}(t_{\mathrm{r}})(t-t_r)^2+\frac{1}{3!}\dddot{x}(t_{\mathrm{r}})(t-t_{\mathrm{r}})^3+\ldots$

Recall that: $\dot{x}(t_{\mathrm{r}})=v(t_{\mathrm{r}})=0$ and that: $\ell = fcn(\vec{a}(t_{\mathrm{r}}))$

Then: $\ell=\left[x(t_{\mathrm{r}})-x(t_{\mathrm{r}})\right]=\frac{1}{2}a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2+\frac{1}{6}\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^3+\ldots$ where: $\Delta t_{\mathrm{r}} \equiv (t-t_{\mathrm{r}})$

But: $c\Delta t_{\mathrm{r}}=\mathscr{r}=\sqrt{\ell^2+d^2} \Rightarrow c^2\Delta t_{\mathrm{r}}^2=\mathscr{r}^2=\ell^2+d^2$

Or:

$$\begin{aligned}
d&=\sqrt{\mathscr{r}^2-\ell^2}=\sqrt{c^2\Delta t_{\mathrm{r}}^2-\ell^2}=\sqrt{c^2\Delta t_{\mathrm{r}}^2-\left(\frac{1}{2}a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2+\frac{1}{6}\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^3+\ldots\right)^2}\\
&=\sqrt{c^2\Delta t_{\mathrm{r}}^2-c^2\Delta t_{\mathrm{r}}^2\left(\frac{a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}}{2c}+\frac{\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2}{6c}+\ldots\right)^2}=c\Delta t_{\mathrm{r}}\sqrt{1-\left(\frac{a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}}{2c}+\frac{\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2}{6c}+\ldots\right)^2}\\
&=c\Delta t_{\mathrm{r}}-\frac{a^2(t_{\mathrm{r}})}{8c}\Delta t_{\mathrm{r}}^3+\{\ \}\Delta t_{\mathrm{r}}^4+\ldots
\end{aligned}$$

We want Δt_{r} in terms of *d*. From above, it can be seen that we can solve for *d* in terms of Δt_{r}. But we can solve for Δt_{r} in terms of d using the ***reversion of series*** technique, which is a formal method that can be used to obtain an approximate value of Δt_{r} by ignoring all higher powers of Δt_{r}. To first order in *d*, we have:

$d \simeq c\Delta t_{\mathrm{r}} \Rightarrow \Delta t_{\mathrm{r}} \simeq \frac{d}{c}$ ← use this as an approximation for obtaining a cubic correction term:

$$d \simeq c\Delta t_{\mathrm{r}}-\frac{a^2(t_{\mathrm{r}})}{8c}\left(\frac{d}{c}\right)^3 \Rightarrow \Delta t_{\mathrm{r}} \simeq \frac{d}{c}+\frac{a^2(t_{\mathrm{r}})d^3}{8c^5}$$

Keep going. . . $\Delta t_{\mathrm{r}} \simeq \frac{1}{c}d+\frac{a^2(t_{\mathrm{r}})}{8c^5}d^3+\{\ \}d^4+\ldots$

Thus: $$\ell = \left[x(t) - x(t_r)\right] = \frac{1}{2}a(t_r)\Delta t_r^2 + \frac{1}{6}\dot{a}(t_r)\Delta t_r^3 + \ldots \simeq \frac{a(t_r)}{2c}d^2 + \frac{\dot{a}(t_r)}{6c^3}d^3 + \{\ \}d^4 + \ldots$$

Then: $$\vec{F}_r^{self}(\vec{r},t) = \frac{q^2}{8\pi\varepsilon_o c^2}\frac{\left(\ell c^2 - d^2 a(t_r)\right)}{\left(\ell^2 + d^2\right)^{3/2}}\hat{x} \simeq \frac{q^2}{4\pi\varepsilon_o}\left[-\frac{a(t_r)}{4c^2 d} + \frac{\dot{a}(t_r)}{12c^3} + \{\ \}d + \ldots\right]\hat{x}$$

{Note that $a(t_r)$ and $\dot{a}(t_r)$ are evaluated at the ***retarded*** time t_r.}

Using the Taylor series expansion of $a(t_r)$, we can rewrite this result in terms of the ***present*** time:

$$a(t_r) = a(t) + \dot{a}(t)(t_r - t) + \ldots = a(t) - \dot{a}(t)\Delta t_r + \ldots = a(t) - \dot{a}(t)\frac{d}{c} + \ldots$$

Then: $$\vec{F}_r^{self}(\vec{r},t) \simeq \frac{q^2}{4\pi\varepsilon_o}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + \{\ \}d + \ldots\right]\hat{x}$$

The first term inside the brackets on the RHS is proportional to acceleration of the charge q. If we put it on LHS, then by Newton's 2nd Law $\vec{F} = m\vec{a}$, we see that it ***adds*** to the mass m of the dumbbell – there is ***inertia*** associated with accelerating an electrically-charged particle.

The total inertial mass of the dumbbell is therefore:

$$m_{tot} = m_{dumbbell} + \frac{1}{4\pi\varepsilon_o}\left(\frac{q^2}{4dc^2}\right) = m_{dumbbell} + \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{dc^2}\right)$$

Or: $$m_{tot}c^2 = m_{dumbbell}c^2 + \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)$$ ⇐ rest mass ***energy***, $E = mc^2$

Note that the {repulsive} ***electrostatic*** potential energy associated with this dumbbell is:

$$U_E(r=d) = \left(\tfrac{1}{2}q\right)V(r=d) = \frac{\left(\frac{1}{2}q\right)^2}{4\pi\varepsilon_o d} = \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right) \text{ (Joules)}$$

The fact that this works out "perfectly" is ***simply*** due to the fact that the initial choice of the dumbbell's orientation was deliberately/consciously chosen to be ***transverse*** to the direction of motion. For a ***longitudinally*** oriented dumbbell, the *EM* mass correction is ***half*** this amount. For a ***spherical charge distribution***, the *EM* mass correction is a factor of ¾ !!

The second term inside the brackets on the RHS of the $\vec{F}_{\mathrm{r}}^{self}(\vec{r},t)$ relation is the *EM* ***radiation reaction*** term:

$$\vec{F}_{rad}^{\,\mathrm{int}}(\vec{r},t)=\frac{q^2\dot{a}(t)}{12\pi\varepsilon_o c^3}\hat{x}=\frac{\mu_o q^2\dot{a}(t)}{12\pi c}\hat{x}$$

Note that $\vec{F}_{rad}^{\,\mathrm{int}}(\vec{r},t)$ differs from Abraham-Lorentz result by a factor of 2×:

$$\vec{F}_{rad}^{\,A\text{-}L}(\vec{r},t)=\frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t)$$

The reason for the factor of 2 difference is that physically, $\vec{F}_{rad}^{\,\mathrm{int}}(\vec{r},t)$ is force of one end of the dumbbell acting on the on other – *i.e.* an *EM* interaction between the two ends of the dumbbell.

There is also a force of ***each end*** of the dumbbell acting on ***itself*** – an *EM* ***self-interaction*** $\vec{F}_{rad}^{\,self}(\vec{r},t)$ for ***each*** end. When the *EM* ***self-interactions*** for each end are included (see Griffiths Problem 11.20, *p*. 473), the ***total*** *EM* ***radiation-reaction*** is:

$$\vec{F}_{rad}^{\,tot}(\vec{r},t)=\vec{F}_{rad}^{\,\mathrm{int}}(\vec{r},t)+2\vec{F}_{rad}^{\,self}(\vec{r},t)=\frac{\mu_o q^2\dot{a}(t)}{6\pi c}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right]\hat{x}=\frac{\mu_o q^2\dot{a}(t)}{6\pi c}\hat{x}$$

which agrees perfectly with Abraham-Lorentz ***radiation-reaction*** force formula.

Thus, physically we see that the *EM* ***radiation reaction*** is due to the force of the charge acting on ***itself*** – an {apparent} ***self-force***!

Note also that $\vec{F}_{rad}(\vec{r},t)$ does ***NOT*** depend on d ($\vec{F}_{rad}(\vec{r},t)$ is valid/well-behaved in limit of the size of the dumbbell, $d \to 0$).

However, note that: $m_{tot}c^2 = m_{dumbbell}c^2+\frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)\to\infty$ when $d \to 0$!!!

The inertial mass of the classical electron becomes ***infinite*** when when $d \to 0$, because:

$$U_E(r=d)=\left(\tfrac{1}{2}q\right)V(r=d)=\frac{\left(\frac{1}{2}q\right)^2}{4\pi\varepsilon_o d}=\frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)\to\infty \text{ when } d\to 0 \text{ !!!}$$

{But we already knew this, as we learned long ago, in P435/last semester…}

Note that this unpleasant/awkward problem also persists in the fully-relativistic, quantum electrodynamical theory {QED}. Infinities/singularities there are dealt with/side-stepped by a process known as ***mass renormalization***, so as to avoid such infinities – look only at mass differences / energy differences…

