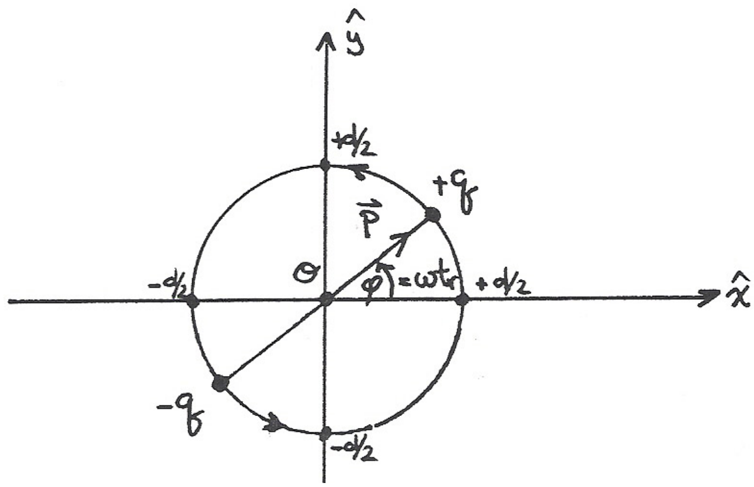
LECTURE NOTES 13.75

EM Radiation Fields Associated with a *Rotating* E(1) Electric Dipole

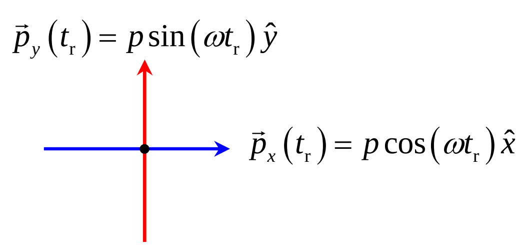
Griffiths Problem 11.4:

A ***static*** electric dipole $(p = qd)$ ***rotates*** CCW {as viewed from ***above***} in the *x*-*y* plane with ***constant*** angular frequency $\omega = 2\pi f$, as shown in the figure below:

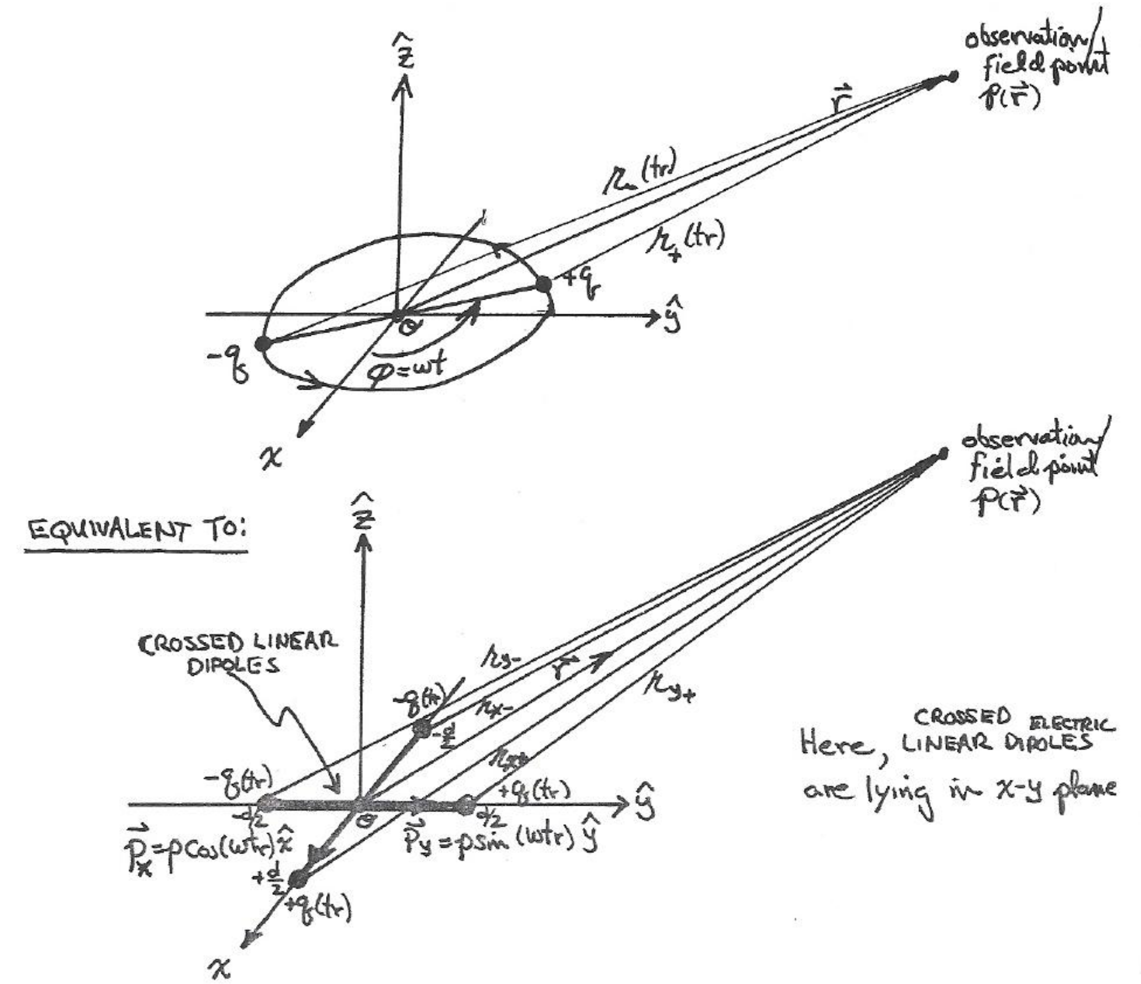


We show that a ***rotating static*** electric dipole is equivalent to ***two crossed quadrature-oscillating non-rotating*** electric dipoles, one || to the $\hat{x}$ axis, one || to the $\hat{y}$ axis, the latter of which is $90^\circ = \pi/2$ radians ***out-of-phase*** with the former. The rotating static electric dipole moment $\boxed{\vec{p}_{rot}(t_r) = p\,\hat{\rho}_{rot}(t_r)}$, where: $\boxed{\hat{\rho}_{rot}(t_r) = \cos\varphi(t_r)\,\hat{x} + \sin\varphi(t_r)\,\hat{y}}$ and {***here***}: $\boxed{\varphi(t_r) = \omega t_r}$ where: $\boxed{t_r = t + \mathscr{r}/c}$ is the ***retarded*** time, the time-dependence of the ***rotating static*** electric dipole can be mathematically described by:

$$\boxed{\begin{aligned}
\vec{p}_{rot}(t_r) &= p\,\hat{\rho}_{rot}(t_r) \\
&= qd\left[\cos\varphi(t_r)\,\hat{x} + \sin\varphi(t_r)\,\hat{y}\right] \\
&= qd\left[\cos(\omega t_r)\,\hat{x} + \sin(\omega t_r)\,\hat{y}\right] \\
&= \underbrace{q\cos(\omega t_r)}\,d\,\hat{x} + \underbrace{q\sin(\omega t_r)}\,d\,\hat{y} \\
&= \underbrace{q_x(t_r)\,d}\,\hat{x} + \underbrace{q_y(t_r)\,d}\,\hat{y} \\
&= \underbrace{p_x(t_r)\,\hat{x}} + \underbrace{p_y(t_r)\,\hat{y}} \\
&= \vec{p}_x(t_r) + \vec{p}_y(t_r)
\end{aligned}}$$



3-D geometry for ***rotating static*** electric dipole *vs*. ***crossed quadrature-oscillating*** dipole ***pair***:



By the ***principle of linear superposition***, we can add the separate contributions associated with $\vec{p}_x$ and $\vec{p}_y$ to obtain the {total(ly)} ***retarded*** scalar and vector potentials. We ***could*** simply brute-force/explicitly work this out from {either of} the above equivalent geometries, but we ***instead*** show {***here***} a different approach to solving this problem:

Recall for the linear oscillating electric dipole aligned along the $\hat{z}$ -axis $\boxed{\vec{p}_z(\vec{r},t_r) = p\cos(\omega t_r)\hat{z}}$ {in the "far-zone" limit, $d \ll \lambda \ll r$ } we obtained:

$$\boxed{V_{r_z}^{E(1)}(\vec{r},t) \approx -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] = -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{r\cos\theta}{r^2}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

But: $z = r\cos\theta$

$$\therefore \text{ For: } \vec{p}_z(\vec{r},t_r) = p\cos(\omega t_r)\hat{z} \quad \begin{cases} V_{r_z}^{E(1)}(\vec{r},t) \approx -\dfrac{p\omega}{4\pi\varepsilon_o c}\left(\dfrac{z}{r^2}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{r_z}^{E(1)}(\vec{r},t) \approx -\dfrac{\mu_o p\omega}{4\pi}\left(\dfrac{1}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{z} \end{cases}$$

Since: $x = r\sin\theta\cos\varphi$ and: $y = r\sin\theta\sin\varphi$, then:

$$\vec{p}_x(r,t_r) = p\sin(\omega t_r)\hat{x} \quad \begin{cases} V_{r_x}^{E(1)}(\vec{r},t) \approx -\dfrac{p\omega}{4\pi\varepsilon_o c}\left(\dfrac{x}{r^2}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] = -\dfrac{p\omega}{4\pi\varepsilon_0 c}\left(\dfrac{\sin\theta\cos\varphi}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{r_x}^{E(1)}(\vec{r},t) \approx -\dfrac{\mu_o p\omega}{4\pi}\left(\dfrac{1}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{x} \end{cases}$$

And:

$$\vec{p}_y(r,t_r) = p\sin(\omega t_r)\hat{y} \quad \begin{cases} V_{r_y}^{E(1)}(\vec{r},t) \approx -\dfrac{p\omega}{4\pi\varepsilon_o c}\left(\dfrac{y}{r^2}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right] = -\dfrac{p\omega}{4\pi\varepsilon_o c}\left(\dfrac{\sin\theta\sin\varphi}{r}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{r_y}^{E(1)}(\vec{r},t) \approx -\dfrac{\mu_o p\omega}{4\pi}\left(\dfrac{1}{r}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{y} \end{cases}$$

Then the totally ***retarded*** potentials {in the “far-zone” limit, $d \ll \lambda \ll r$ } are:

$$\begin{aligned} V_{r_{tot}}^{E(1)}(\vec{r},t) &= V_{r_x}^{E(1)}(\vec{r},t) + V_{r_y}^{E(1)}(\vec{r},t) \\ &\approx -\frac{p\omega}{4\pi\varepsilon_o c}\left\{\left(\frac{x}{r^2}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{y}{r^2}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &= -\frac{p\omega}{4\pi\varepsilon_o c}\left\{\left(\frac{\sin\theta\cos\varphi}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{\sin\theta\sin\varphi}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &= -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\sin\theta}{r}\right)\left\{\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \sin\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \end{aligned}$$

And:

$$\begin{aligned} \vec{A}_{r_{tot}}^{E(1)}(\vec{r},t) &= \vec{A}_{r_x}^{E(1)}(\vec{r},t) + \vec{A}_{r_y}^{E(1)}(\vec{r},t) \\ &\approx -\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\left\{\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{x} + \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{y}\right\} \end{aligned}$$

But:
$$\begin{aligned} \hat{x} &= \sin\theta\cos\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} - \sin\varphi\hat{\varphi} \\ \hat{y} &= \sin\theta\sin\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} + \cos\varphi\hat{\varphi} \end{aligned}$$

Thus: $\vec{A}_{r_{tot}}^{E(1)}(\vec{r},t) =$ ***big*** ***mess*** in spherical coordinates!!!

Instead of {mindlessly} bulldozing/grinding our way thru this, we can obtain

$$\boxed{\vec{E}^{E(1)}_{r_{tot}}(\vec{r},t) = -\vec{\nabla}V^{E(1)}_{r_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{E(1)}_{r_{tot}}(\vec{r},t)}{\partial t}} \quad \text{and} \quad \boxed{\vec{B}^{E(1)}_{r_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{E(1)}_{r_{tot}}(\vec{r},t)} \quad \text{by:}$$

a.) Using the already known form of $\vec{E}^{E(1)}_{r_z}(\vec{r},t)$ that we have previously obtained from the ***single*** oscillating dipole aligned along the $\hat{z}$-axis, $\vec{p}_z = p\cos(\omega t_r)\hat{z}$ – *i.e.* we simply ***rotate*** the $\vec{E}^{E(1)}_{r_z}(\vec{r},t)$ solution by 90° {and change the phase relation in the $\hat{y}$-direction} to obtain $\vec{E}^{E(1)}_{r_x}(\vec{r},t)$ and $\vec{E}^{E(1)}_{r_y}(\vec{r},t)$ associated with the $\vec{p}_x = p\cos(\omega t_r)\hat{x}$ and $\vec{p}_y = p\sin(\omega t_r)\hat{y}$ electric dipole moments respectively, and then:

b.) obtain the corresponding/associated *B*-fields using the relation $\boxed{B^{E(1)}_{r}(\vec{r},t) = \frac{1}{c}\left(\hat{r}\times\vec{E}^{E(1)}_{r}(\vec{r},t)\right)}$

Thus, recall for $\vec{p}_z = p\cos(\omega t_r)\hat{z}$ in the "far-zone" limit { $d \ll \lambda \ll r$ } that we obtained:

$$\boxed{\vec{E}^{E(1)}_{r_z}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}} \quad \text{however, note that:} \quad \boxed{\sin\theta\ \hat{\theta} = \cos\theta\hat{r} - \hat{z}}$$

$$\therefore \quad \boxed{\vec{E}^{E(1)}_{r_z}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi r}\left[\cos\theta\hat{r} - \hat{z}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} \quad \text{but:} \quad \boxed{\cos\theta = \frac{z}{r}}$$

$$\therefore \quad \boxed{\vec{E}^{E(1)}_{r_z}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi r}\left[\left(\frac{z}{r}\right)\hat{r} - \hat{z}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

$$\therefore \quad \boxed{\vec{E}^{E(1)}_{r_x}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi r}\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} \Leftarrow \text{for} \ \boxed{\vec{p}_x = p\cos(\omega t_r)\hat{x}}$$

$$\text{And:} \quad \boxed{\vec{E}^{E(1)}_{r_y}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi r}\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \Leftarrow \text{for} \ \boxed{\vec{p}_y = p\sin(\omega t_r)\hat{y}}$$

Thus, the totally ***retarded*** electric field {in the "far-zone" limit, $d \ll \lambda \ll r$ } is:

$$\boxed{\begin{aligned}\vec{E}^{E(1)}_{r_{tot}}(\vec{r},t) &= \vec{E}^{E(1)}_{r_x}(\vec{r},t) + \vec{E}^{E(1)}_{r_y}(\vec{r},t) \\ &\approx -\frac{\mu_o p\omega^2}{4\pi r}\left\{\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + \left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\end{aligned}}$$

$$\text{And:} \quad \boxed{B^{E(1)}_{r_{tot}}(\vec{r},t) = \frac{1}{c}\left(\hat{r}\times\vec{E}^{E(1)}_{r_{tot}}(\vec{r},t)\right)}$$

Thus, the totally ***retarded*** Poynting's vector for the rotating E(1) electric dipole is:

$$\vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) = \frac{1}{\mu_o}\left(\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\times\vec{B}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right) = \frac{1}{\mu_o c}\left[\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\left(\hat{r}\times\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)\right]$$

But: $\vec{A}\times\vec{B}\times\vec{C} = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$

$$\therefore\quad \vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) = \frac{1}{\mu_o c}\left\{\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2\hat{r} - \left(\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\bullet\hat{r}\right)\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right\}$$

But: $\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{Tot}}(\vec{r},t)\bullet\hat{r} = 0$ because:

$$\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\bullet\hat{r} = \left(\frac{x}{r}\right)\hat{r}\bullet\hat{r} - \hat{x}\bullet\hat{r} = \frac{x}{r} - \frac{x}{r} = 0$$

$$\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\bullet\hat{r} = \left(\frac{y}{r}\right)\hat{r}\bullet\hat{r} - \hat{y}\bullet\hat{r} = \frac{y}{r} - \frac{y}{r} = 0$$

$$\therefore\quad \vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) = \frac{1}{\mu_o c}\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2\hat{r}$$

In the "far-zone" limit $\{\, d \ll \lambda \ll r \,\}$:

$$\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2 \approx \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2 \left\{\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]^2 \cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] + \left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]^2 \sin^2\left[\omega\left(t-\frac{r}{c}\right)\right] \right.$$
$$\left. + \underbrace{2\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\bullet\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\left(t-\frac{r}{c}\right)\right]}_{=\vec{p}_x\leftrightarrow\vec{p}_y\ \textit{interference}\ \text{term !!!}}\right\}$$

Noting that: $\hat{x}\bullet\vec{r} = x$ and: $\hat{y}\bullet\vec{r} = y$:

Then: $\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]^2 = \left[\left(\frac{x}{r^2}\right)\vec{r} - \hat{x}\right]\bullet\left[\left(\frac{x}{r^2}\right)\vec{r} - \hat{x}\right] = \left(\frac{x^2}{r^2}\right) - 2\left(\frac{x}{r}\right)^2 + 1 = 1 - \left(\frac{x}{r}\right)^2$ $\Leftarrow$ p_x^2 term

And: $\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]^2 = \left[\left(\frac{y}{r^2}\right)\vec{r} - \hat{y}\right]\bullet\left[\left(\frac{y}{r^2}\right)\vec{r} - \hat{y}\right] = \left(\frac{y^2}{r^2}\right) - 2\left(\frac{y}{r}\right)^2 + 1 = 1 - \left(\frac{y}{r}\right)^2$ $\Leftarrow$ p_y^2 term

And: $\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\bullet\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right] = \frac{xy}{r^2} - \frac{xy}{r^2} - \frac{xy}{r^2} = -\frac{xy}{r^2}$ $\Leftarrow$ $\vec{p}_x \leftrightarrow \vec{p}_y$ interference term

Thus:

$$\left(E_{r_{tot}}^{E(1)}(\vec{r},t)\right)^2 \approx \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2 \left\{\left(1-\left(\frac{x}{r}\right)^2\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(1-\left(\frac{y}{r}\right)^2\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right] -2\left(\frac{xy}{r^2}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}$$

Or:

$$\begin{aligned}\left(E_{r_{tot}}^{E(1)}(\vec{r},t)\right)^2 &\approx \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2 \left\{\overbrace{\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]}^{=1}\right. \\ &\left. -\frac{1}{r^2}\left[x^2\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+2xy\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+y^2\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right]\right\} \\ &= \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2\left\{1-\frac{1}{r^2}\left[x\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+y\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]^2\right\}\end{aligned}$$

But: $x = r\sin\theta\cos\varphi$ and: $y = r\sin\theta\sin\varphi$

$$\therefore \begin{aligned}\left(\vec{E}_{r_{tot}}^{E(1)}(\vec{r},t)\right)^2 &\approx \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2\left\{1-\sin^2\theta\underbrace{\left[\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\sin\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]^2}_{=\cos\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]}\right\} \\ &= \left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2\left\{1-\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]\right\}\end{aligned}$$

Thus, in the "far-zone" limit, where $d \ll \lambda \ll r$, the totally ***retarded*** Poynting's vector for the rotating E(1) electric dipole is:

$$\vec{S}_{E(1)_{tot}}^{rad}(\vec{r},t) \approx \frac{1}{\mu_o c}\left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2\left\{1-\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]\right\}\hat{r} \quad \left(\frac{Watts}{m^2}\right)$$

Then the ***time-averaged*** totally ***retarded*** Poynting's vector in the "far-zone" limit { $d \ll \lambda \ll r$ } for the rotating E(1) electric dipole is:

$$\left\langle \vec{S}_{E(1)_{tot}}^{rad}(\vec{r})\right\rangle \approx \frac{1}{\mu_o c}\left(\frac{\mu_o p\omega^2}{4\pi r}\right)^2\left[1-\frac{1}{2}\sin^2\theta\right]\hat{r} \quad \left(\frac{Watts}{m^2}\right)$$

The ***time-averaged*** total power radiated ***per unit solid angle*** in the "far-zone" limit $\{ d \ll \lambda \ll r \}$ for the rotating dipole is:

$$\frac{d\left\langle P^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle}{d\Omega} = r^2\left\langle \vec{S}^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle \bullet \hat{r} \approx \frac{1}{\mu_o c}\left(\frac{\mu_o p\omega^2}{4\pi}\right)^2\left[1-\frac{1}{2}\sin^2\theta\right] \quad \left(\frac{Watts}{steradian}\right)$$

Note that the power angular distribution varies as: $\left(1-\frac{1}{2}\sin^2\theta\right)$

i.e. is associated with the $\ell=1,\ m=\pm 1$ spherical harmonic $Y^{m=\pm1}_{\ell=1}(\theta,\varphi)$

$\Rightarrow$ *z*-component of *EM* angular momentum $L_z \neq 0$ here !

The ***total time-averaged*** power radiated into 4π steradians in the "far-zone" limit $\{ d \ll \lambda \ll r \}$ is:

$$\begin{aligned}
\left\langle P^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle &= \int \frac{d\left\langle P^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle}{d\Omega} d\Omega = \int_S \left\langle \vec{S}^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle \bullet d\vec{a}_\perp \\
&\approx \frac{\mu_o}{c}\left(\frac{p\omega^2}{4\pi}\right)^2 \int \frac{1}{\cancel{r^2}}\left(1-\frac{1}{2}\sin^2\theta\right)\cancel{r^2}\sin\theta d\theta d\varphi \\
&= \frac{\mu_o p^2\omega^4}{16\pi^2 c}2\pi\left[\int_0^\pi \sin\theta d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta d\theta\right] \\
&= \frac{\mu_o p^2\omega^4}{8\pi c}\left(2-\frac{1}{2}\cdot\frac{4}{3}\right) = \frac{\mu_o p^2\omega^4}{8\pi c}\left(2-\frac{2}{3}\right) = \frac{\mu_o p^2\omega^4}{8\pi c}\left(\frac{4}{3}\right) = \frac{\mu_o p^2\omega^4}{6\pi c}
\end{aligned}$$

Thus, we see that $\left\langle P^{rad}_{E(1)_{tot}}(\vec{r},t)\right\rangle \approx \frac{\mu_o p^2\omega^4}{6\pi c} = 2\times\left\langle P^{rad}_{E(1)_z}(\vec{r},t)\right\rangle = 2\times$ times the time-averaged radiated power (in the "far-zone" limit) for a ***single*** E(1) oscillating electric dipole.

Note that ***in general***, using the ***principle of linear superposition***: $\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2$ and $\vec{B}_{tot} = \vec{B}_1 + \vec{B}_2$ thus, the ***total*** Poynting's vector is:

$$\begin{aligned}
\vec{S}_{tot} &= \frac{1}{\mu_o}\vec{E}_{tot}\times\vec{B}_{tot} = \frac{1}{\mu_o}\left[\left(\vec{E}_1+\vec{E}_2\right)\times\left(\vec{B}_1\times\vec{B}_2\right)\right] \\
&= \frac{1}{\mu_o}\left[\vec{E}_1\times\vec{B}_1 + \vec{E}_2\times\vec{B}_2 + \vec{E}_1\times\vec{B}_2 + \vec{E}_2\times\vec{B}_1\right]
\end{aligned}$$

Or: $\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2 + \frac{1}{\mu_o}\left(\vec{E}_1\times\vec{B}_2\right) + \frac{1}{\mu_o}\left(\vec{E}_2\times\vec{B}_1\right)$

In general, the ***cross terms*** {*i.e.* ***interference terms***} in Poynting's vector will ***not*** always cancel!!!

In the case (***here***) with the ***rotating*** physical electric dipole, they ***do*** vanish, because the fields of 1) and 2) are 90^o out of phase with each other, the cross-term(s) vanish in the time-averaging procedure. $\Rightarrow$ Total power: $P^{E(1)}_{ToT} = P^{E(1)}_1 + P^{E(2)}_2$ {***here***}.

***EM* Radiation – Low-Order Angular Distributions:**

Table 9.1 Some Angular Distributions: $|\mathbf{X}_{lm}(\theta,\phi)|^2$

l	m = 0	±1	±2
1 Dipole	$\frac{3}{8\pi}\sin^2\theta$	$\frac{3}{16\pi}(1+\cos^2\theta)$	
2 Quadrupole	$\frac{15}{8\pi}\sin^2\theta\cos^2\theta$	$\frac{5}{16\pi}(1-3\cos^2\theta+4\cos^4\theta)$	$\frac{5}{16\pi}(1-\cos^4\theta)$

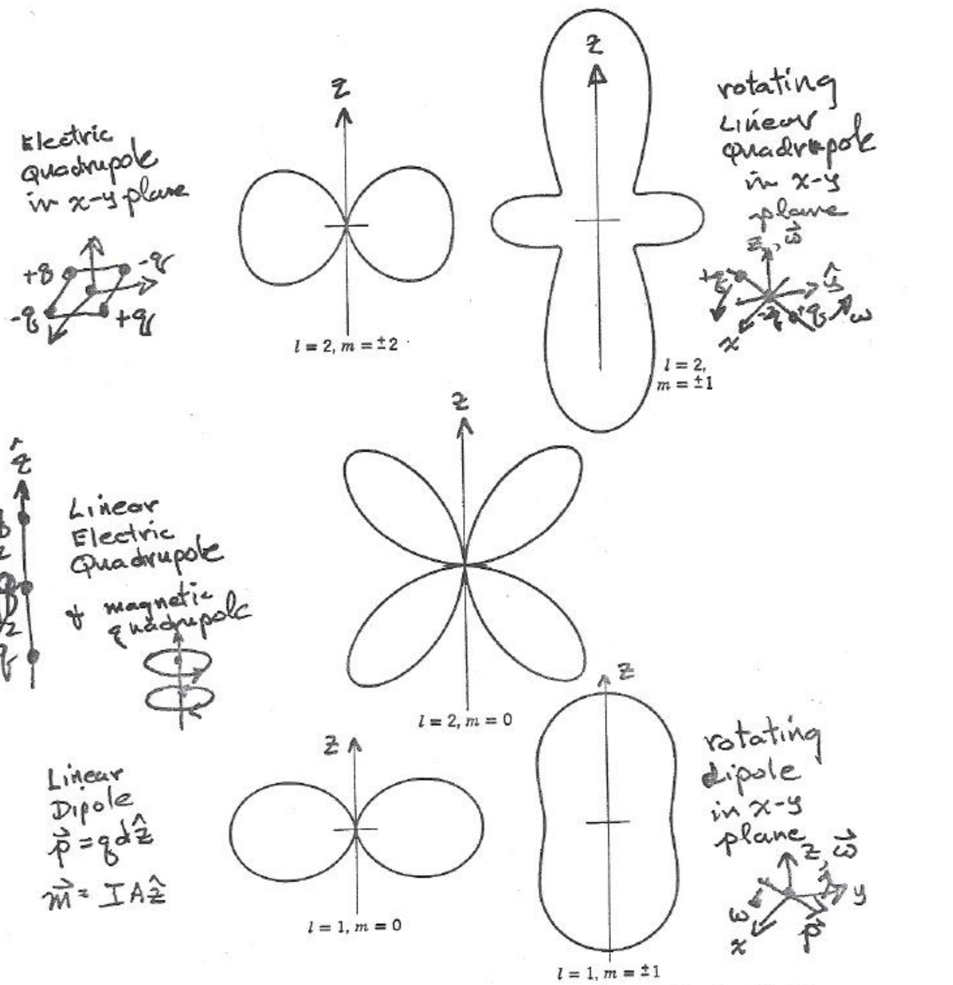
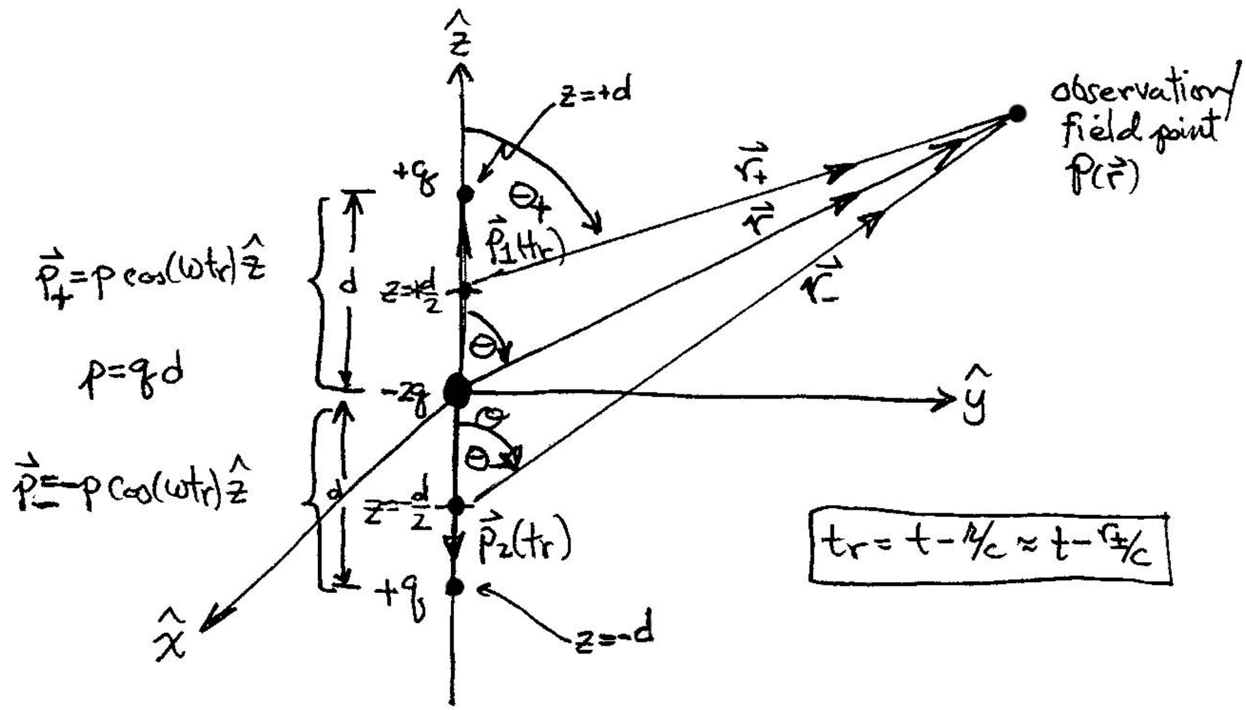


Figure 9.5 Dipole and quadrupole radiation patterns for pure (l, m) multipoles.

"Far-Zone" *EM* Radiation Fields Associated with a Oscillating Linear Electric Quadrupole, E(2)

Griffith's Problem 11.11:

Construct a linear electric quadrupole from two opposing linear electric dipoles.
Take two oppositely-oriented oscillating dipoles, one with $\vec{p}_+(t_r) = p\cos(\omega t_r)\hat{z}$ {where $p = qd$} with its center located at $z_1 = +d/2$ and another with $\vec{p}_-(t_r) = -p\cos(\omega t_r)\hat{z}$ with its center located at $z_2 = -d/2$ as shown in the figure below:



For *EM* radiation associated with this linear oscillating E(2) electric quadrupole in the "far-zone" limit { $d \ll \lambda \ll r$ }, keeping terms only to first order in d/r, *i.e.* $\boxed{\frac{\omega}{c} \gg \frac{1}{r}}$, using the ***principle of linear superposition***, the total(ly) ***retarded*** scalar potential is: $\boxed{V_{r_{tot}}^{E(2)}(\vec{r},t) = V_{r_+}^{E(1)}(\vec{r},t) + V_{r_-}^{E(1)}(\vec{r},t)}$

where: $\boxed{V_{r_\pm}^{E(1)}(\vec{r},t) \approx \mp \frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta_\pm}{r_\pm}\right)\sin\left[\omega\left(t - \frac{r_\pm}{c}\right)\right]}$ (see P436 Lecture Notes 13.5, *p*. 10)

1) Now: $$\boxed{r_\pm = \sqrt{r^2 + \left(d/2\right)^2 \mp 2r\left(d/2\right)\cos\theta} = r\sqrt{1 + \left(d/2r\right)^2 \mp \left(d/r\right)\cos\theta}}$$

$\boxed{r_\pm \approx r\sqrt{1 \mp \underbrace{\left(d/r\right)}_{\ll 1}\cos\theta}}$ but: $\boxed{d/r \ll 1}$, we will keep only ***linear*** terms in $\left(d/r\right)$

And: $$\boxed{\frac{1}{r_\pm} \approx \frac{1}{r\sqrt{1 \mp \underbrace{\left(d/r\right)}_{\ll 1}\cos\theta}} \approx \frac{1}{r}\left(1 \pm \left(d/2r\right)\cos\theta\right)}$$

2) And:

$$\cos\theta_\pm = \frac{r\cos\theta \mp \left(d/2\right)}{r_\pm} \approx \cancel{r}\left(\cos\theta \mp \frac{d}{2r}\right)\frac{1}{\cancel{r}}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$
$$= \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r} - \underbrace{\left(\frac{d}{2r}\right)^2}_{\ll 1}\cos\theta \approx \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r}$$
$$\approx \cos\theta \mp \left(\frac{d}{2r}\right)\left(1-\cos^2\theta\right) = \cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta$$

3) Then:

$$\sin\left[\omega\left(t - r_\pm/c\right)\right] \approx \sin\left\{\omega\left[t - \frac{r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right]\right\}$$
$$= \sin\left[\omega\left(t - r/c\right) \pm \left(\frac{\omega d}{2c}\right)\cos\theta\right] \quad \text{but:} \quad \sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$$
$$= \sin\left[\omega\left(t - r/c\right)\right]\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \pm \cos\left[\omega\left(t - r/c\right)\right]\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right]$$

But: $\left(\frac{\omega d}{c}\right) \ll 1$ in the "far-zone" limit $\{ d \ll \lambda \ll r \}$ $\therefore$ $\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx 1$ since: $\cos(\approx 0) \approx 1$

And: $\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \left(\frac{\omega d}{2c}\right)\cos\theta$ since $\sin\alpha \approx \alpha$ for $\alpha \ll 1$.

$$\therefore \quad \sin\left[\omega\left(t - r_\pm/c\right)\right] \approx \sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]$$

Then, keeping only terms linear in (d/r):

$$V_{r_\pm}^{E(1)}(\vec{r},t) \approx \mp \frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta_\pm}{r_\pm}\right)\sin\left[\omega\left(t - \frac{r_\pm}{c}\right)\right]$$
$$\approx \mp \frac{p\omega}{4\pi\varepsilon_o cr}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)\left(\cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta\right)$$
$$\times\left[\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left(\omega\left[t - r/c\right]\right)\right]$$
$$\approx \mp \frac{p\omega}{4\pi\varepsilon_o cr}\left\{\left(\cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta \pm \left(\frac{d}{2r}\right)\cos^2\theta - \left(\frac{d}{2r}\right)^2\sin^2\theta\cos\theta\right)\right.$$
$$\left.\times\left[\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right]\right\}$$

Thus:

$$
\begin{aligned}
V_{r_\pm}^{E(1)}(\vec{r},t) &\approx \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left(\cos\theta \pm \left(\frac{d}{2r}\right)\left(\cos^2\theta - \sin^2\theta\right)\right) \\
&\qquad \times\left[\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right] \\
&\approx \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cos\theta\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\
&\qquad \pm\left(\frac{d}{2r}\right)\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right] \\
&\qquad \left. + \underbrace{\cancel{\left(\frac{d}{2r}\right)\left(\frac{\omega d}{2c}\right)}}_{\ll 1}\left(\cos^2\theta - \sin^2\theta\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}
$$

Finally:

$$
\begin{aligned}
V_{r_\pm}^{E(1)}(\vec{r},t) &\approx \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cos\theta\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\
&\qquad \left. \pm\left(\frac{d}{2r}\right)\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}
$$

Then: $\boxed{V_{r_{Tot}}^{E(2)}(\vec{r},t) = V_{r_+}^{E(1)}(\vec{r},t) + V_{r_-}^{E(1)}(\vec{r},t)}$

$$
\begin{aligned}
V_{r_{tot}}^{E(2)}(\vec{r},t) &\approx -\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cancel{\cos\theta\sin\left[\omega\left(t - r/c\right)\right]} + \left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\
&\qquad \left. + \left(\frac{d}{2r}\right)\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\quad + \frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cancel{\cos\theta\sin\left[\omega\left(t - r/c\right)\right]} - \left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\
&\qquad \left. - \left(\frac{d}{2r}\right)\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\approx -\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\left(\frac{\omega d}{c}\right)\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right] - \left(\frac{d}{r}\right)\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\approx -\frac{p\omega^2 d}{4\pi\varepsilon_o c^2 r}\left\{\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right] - \underbrace{\cancel{\left(\frac{c}{\omega r}\right)}}_{\ll 1}\left(\cos^2\theta - \sin^2\theta\right)\sin\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}
$$

In the "far-zone" limit $\{\, d \ll \lambda \ll r \,\}$ $\boxed{\left(\frac{c}{\omega}\right) \ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right) \ll 1}$, keep only linear terms!

$\therefore$ In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{V_{r_{tot}}^{E(2)}(\vec{r},t) \approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]}$$

Now let's work on obtaining: $\boxed{\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) = \vec{A}_{r_+}^{E(1)}(\vec{r},t) + \vec{A}_{r_-}^{E(1)}(\vec{r},t)}$

Where: $\boxed{\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) \approx \mp\frac{\mu_o p\omega}{4\pi r_\pm c}\sin\left[\omega\left(t - r_\pm/c\right)\right]\hat{z}}$ (see P436 Lecture Notes 13.5, p.10)

Carrying out the same methodology as above, keeping only linear terms in $\left(\frac{d}{r}\right)$ and $\left(\frac{\omega d}{c}\right)$:

$$\begin{aligned}\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) &\approx \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)\left\{\sin\left[\omega(t - r/c\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right\} \\ &\approx \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left\{\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\ &\qquad \left. \pm \left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right] + \underbrace{\cancel{\left(\frac{d}{2r}\right)\left(\frac{\omega d}{2c}\right)}}_{\ll 1}\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right\}\end{aligned}$$

$$\boxed{\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) \approx \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left\{\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{d}{2r}\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right)\right\}}$$

Then in the "far-zone" limit $\{ d \ll \lambda \ll r \}$, with $\boxed{\left(\frac{c}{\omega}\right) \ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right) \ll 1}$:

$$\begin{aligned}\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) &\approx -\left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\cancel{\sin\left[\omega\left(t - r/c\right)\right]} + \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] + \left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\ &\quad + \left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\cancel{\sin\left[\omega\left(t - r/c\right)\right]} - \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] - \left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\ &\approx -\left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\left(\frac{\omega d}{c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] + \left(\frac{d}{r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\ &\approx -\left(\frac{\mu_o p\omega^2 d}{4\pi c r}\right)\cos\theta\left\{\cos\left[\omega\left(t - r/c\right)\right] + \underbrace{\cancel{\left(\frac{c}{\omega r}\right)}}_{\ll 1}\sin\left[\omega\left(t\; r/c\right)\right]\right\}\hat{z}\end{aligned}$$

$\therefore$ In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) \approx -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z}}$$

Then the totally ***retarded*** electric and magnetic fields associated with E(2) "far-zone" *EM* radiation from an oscillating linear electric quadrupole are:

$$\boxed{\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = -\vec{\nabla}V^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)}{\partial t}} \quad \text{and} \quad \boxed{\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)}$$

Note that: $\boxed{V^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]}$ has no explicit φ-dependence

Note that: $\boxed{\vec{A}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z}}$ also has no explicit φ-dependence,

and also note that $A^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = \{\ \}\hat{z}$, *i.e.* $A^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \parallel \hat{z}$.

Now:

$$\begin{aligned}
\vec{\nabla}V^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) &= \frac{\partial V^{\mathrm{E}(2)}_{\mathrm{r}_{Tot}}(\vec{r},t)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V^{\mathrm{E}(2)}_{\mathrm{r}_{Tot}}(\vec{r},t)}{\partial \theta}\hat{\theta} \\
&\approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\cos^2\theta\left\{-\frac{1}{r^2}\cos\left[\omega\left(t - r/c\right)\right] + \frac{1}{r}\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t - r/c\right)\right]\right\}\hat{r} \\
&\quad -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\frac{-2\cos\theta\sin\theta}{r^2}\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta} \\
&\approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left\{\frac{1}{r}\left(\frac{\omega}{c}\right)\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} - \frac{1}{r^2}\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\hat{r}\right. \\
&\qquad\qquad \left. -\frac{1}{r^2}\left(2\cos\theta\sin\theta\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\} \\
&\approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{1}{r}\right)\left(\frac{\omega}{c}\right)\left\{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r}\right. \\
&\qquad\qquad \left. -\underbrace{\left(\frac{c}{\omega r}\right)}_{\ll 1}\cos\theta\left[\cos\theta\hat{r} + 2\sin\theta\hat{\theta}\right]\cos\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}$$

In the "far-zone" limit { $d \ll \lambda \ll r$ }: $\boxed{\left(\frac{c}{\omega}\right) \ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right) \ll 1}$.

$\therefore$ In the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\vec{\nabla}V^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^3}\right)\left(\frac{\cos^2\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{r}}$$

Next: $$\frac{\partial \vec{A}^{E(2)}_{r_{tot}}}{\partial t}(\vec{r},t) \approx +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{z}$$ But: $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$

$$\therefore \quad \frac{\partial \vec{A}^{E(2)}_{r_{tot}}}{\partial t}(\vec{r},t) \approx +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$ in spherical coordinates

Then: $$\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) = -\vec{\nabla}V^{E(2)}_{r_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)}{\partial t}$$

Thus:

$$\begin{aligned}\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) &\approx +\left(\frac{p\omega^3 d}{4\pi\varepsilon_o c^3 r}\right)\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} \\ &\quad -\left(\frac{\mu_o p\omega^3 d}{4\pi c r}\right)\left\{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} - \cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\}\end{aligned}$$

But: $c^2 = \dfrac{1}{\varepsilon_o\mu_o}$ or: $\dfrac{1}{c^2} = \varepsilon_o\mu_o$

$$\begin{aligned}\therefore \quad \vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) &\approx \left(\frac{\mu_o p\omega^3 d}{4 p c r}\right)\cancel{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]}\hat{r} \\ &\quad -\left(\frac{\mu_o p\omega^3 d}{4\pi c r}\right)\left\{\cancel{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]}\hat{r} - \cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\}\end{aligned}$$

n.b. $\vec{\nabla}V^{E(2)}_{r}$ term cancels with 1st $\partial\vec{A}^{E(2)}_{r}/\partial t$ term!!!

Thus, in the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\dfrac{d}{r}\right)$, $\left(\dfrac{\omega d}{c}\right)$, $\left(\dfrac{c}{\omega r}\right)$:

$$\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) \approx +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}$$

Now let's work on obtaining: $\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)$

Note that $\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)$ has only a $\hat{z}$ component and has no explicit φ-dependence.

However, in spherical coordinates: $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$.

Hence, in the "far-zone" limit { $d \ll \lambda \ll r$ } $\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)$:

$$\begin{aligned}\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t) &\approx -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z} \\ &= -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]\end{aligned}$$

Thus in spherical coordinates:

$$\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}} = \vec{\nabla}\times\vec{A}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, \cancel{A_\varphi}\right) - \frac{\cancel{\partial A_\theta}}{\cancel{\partial\varphi}}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\cancel{\partial A_r}}{\cancel{\partial\varphi}} - \frac{\partial}{\partial r}\left(r\,\cancel{A_\varphi}\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}$$

$\therefore$ $$\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA^{\mathrm{E}(2)}_\theta\right) - \frac{\partial A^{\mathrm{E}(2)}_r}{\partial\theta}\right]\hat{\varphi}$$

Thus, in the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$:

$$\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx -\left(\frac{\mu_o p\omega^2 d}{4\pi c r}\right)\left\{-\left(\frac{\omega}{c}\right)\cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right] + \frac{1}{r}\left(2\cos\theta\sin\theta\right)\cos\left[\omega\left(t - r/c\right)\right]\right\}\hat{\varphi}$$
$$\approx +\left(\frac{\mu_o p\omega^3 d}{4\pi c^2 r}\right)\cos\theta\sin\theta\left\{\sin\left[\omega\left(t - r/c\right)\right] - \underbrace{\left(\frac{c}{\omega r}\right)}_{\ll 1} 2\cos\left[\omega\left(t - r/c\right)\right]\right\}\hat{\varphi}$$

$\therefore$ $$\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx \left(\frac{\mu_o p\omega^3 d}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\varphi}$$

And: $$\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) \approx \left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}$$

Note again that: $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)$ $\Leftarrow$ $\hat{r}\times\hat{\theta} = \hat{\varphi}$

Note also that: $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)\bullet\hat{r} = 0$, $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)\bullet\hat{r} = 0$ and $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t)\bullet\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}(\vec{r},t) = 0$

Note also that: $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ and $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ have the ***same*** angular dependence: $\sim\cos\theta\sin\theta$

$\Rightarrow$ E(2) electric quadrupole $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ and $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ fields ***both*** vanish for $\theta = 0, \tfrac{1}{2}\pi, \pi$!!!

Note also that: $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ and $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ both vary as $\sim 1/r$

Note also that: $\vec{E}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ and $\vec{B}^{\mathrm{E}(2)}_{\mathrm{r}_{tot}}$ are ***in-phase*** with each other $\sim\sin\left[\omega\left(t - r/c\right)\right]$

$\Rightarrow$ linearly polarized *EM* radiation from linear/axial electric quadrupole.
{*n.b.* ***NOT*** true for ***all*** types of electric quadrupoles!}

Now let's calculate: $u^{rad}_{\mathrm{E}(2)}(\vec{r},t)$, $\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t)$, $\vec{\wp}^{rad}_{\mathrm{E}(2)}(\vec{r},t)$, $\vec{\ell}^{rad}_{\mathrm{E}(2)}(\vec{r},t)$, $P^{rad}_{\mathrm{E}(2)}(\vec{r},t)$ *etc*. for "far-field"

EM radiation associated with the linear oscillating E(2) electric quadrupole:

<u>*EM* Energy Density for E(2) Linear Oscillating Electric Quadupole</u>:

$$u_{E(2)}^{rad}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o \vec{E}_r^{E(2)}(\vec{r},t)\bullet\vec{E}_r^{E(2)}(\vec{r},t)\right) + \frac{1}{\mu_o}\left(\vec{B}_r^{E(2)}(\vec{r},t)\bullet\vec{B}_r^{E(2)}(\vec{r},t)\right) \quad \left(\frac{Joules}{m^3}\right)$$

$$u_{E(2)}^{rad}(\vec{r},t) \approx \frac{1}{2}\left\{\varepsilon_o\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)^2\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right.$$

$$\left. + \frac{1}{\mu_o}\left(\frac{\mu_o p\omega^2 d}{4\pi c^2}\right)^2\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \quad \underline{\text{but}}: \varepsilon_o = \frac{1}{\mu_o c^2}$$

$$= \frac{1}{2}\left\{\underbrace{\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^4}} + \underbrace{\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^4}}\right\}\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t - r/c\right)\right]$$

n.b. EM radiation energy is {again} carried equally by $\vec{E}_r^{E(2)}$ and $\vec{B}_r^{E(2)}$

$$\therefore \quad u_{E(2)}^{rad}(\vec{r},t) \approx \left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t - r/c\right)\right] \quad \left(\frac{Joules}{m^3}\right)$$

In the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

<u>Poynting's Vector for E(2) Linear Oscillating Electric Quadrupole</u>:

$$\vec{S}_{E(2)}^{rad}(\vec{r},t) = \frac{1}{\mu_o}\left(\vec{E}_r^{E(2)} \times \vec{B}_r^{E(2)}\right) \approx \frac{1}{\mu_o}\left(\frac{\mu_o^2 p^2\omega^6 d^2}{16\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t - r/c\right)\right]\overbrace{\left(\hat{\theta}\times\hat{\varphi}\right)}^{=\hat{r}}$$

$$\vec{S}_{E(2)}^{rad}(\vec{r},t) \approx \left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t - r/c\right)\right]\hat{r} \quad \left(\frac{Watts}{m^2}\right)$$

In the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

Note again that: $\vec{S}_{E(2)}^{rad}(\vec{r},t) = \vec{c}\,u_{E(2)}^{rad}(\vec{r},t)$ where: $\vec{c} \equiv c\hat{r}$, $\hat{r} \parallel \hat{k}$

EM Linear Momentum Density for E(2) Linear Oscillating Electric Quadrupole:

$$\boxed{\vec{\wp}^{\,rad}_{\mathrm{E}(2)}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t) = \frac{1}{c^2}\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t) \approx \left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}} \quad \left(\frac{kg}{m^2\text{-}sec}\right)$$

In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

EM Angular Momentum Density for E(2) Linear Oscillating Electric Quadrupole:

$$\boxed{\vec{\ell}^{\,rad}_{\mathrm{E}(2)}(\vec{r},t) = \vec{r}\times\vec{\wp}^{\,rad}_{\mathrm{E}(2)}(\vec{r},t) = 0} \quad \left(\frac{kg}{m\text{-}sec}\right)$$

n.b. ***exact*** $\vec{\ell}^{\,rad}_{\mathrm{E}(2)}(\vec{r},t) \not\equiv 0$.

In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

Time-Averaged Quantities for E(2) Linear Oscillating Electric Quadrupole Radiation

Recall: $\boxed{\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{\tau}\int_0^\tau \sin^2(\omega t)\,dt = \frac{1}{2}}$

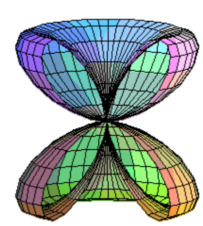
Define electric quadrupole moment: $\boxed{Q^e_{zz} \equiv qdd = pd}$ $(Coulomb - m^2)$

$$\boxed{\left\langle u^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2 d^2\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right) = \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)} \quad \left(\frac{Joules}{m^3}\right)$$

$$\boxed{I^{rad}_{\mathrm{E}(2)}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right|\right\rangle \approx \left(\frac{\mu_o p^2 d^2\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r} = \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)} \quad \left(\frac{Watts}{m^2}\right)$$

$$\boxed{I^{rad}_{\mathrm{E}(2)}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right|\right\rangle = c\left\langle u^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle}$$

Quadrupole radiation pattern, l=2, m=0



Time-averaged radiated power: $\boxed{\left\langle P^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle = \int_{S'}\left\langle\vec{S}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle\cdot d\vec{a}_\perp}$

where: $\boxed{d\vec{a}_\perp = r^2 d\Omega\,\hat{r}}$ and: $\boxed{d\Omega = \sin\theta\, d\theta\, d\varphi}$

$$\boxed{\left\langle P^{rad}_{\mathrm{E}(2)}(\vec{r})\right\rangle \approx \frac{\mu_o p^2 d^2\omega^6}{\cancel{32}\,\cancel{\pi^2}\,c^3\,\cancel{r^2}}\cdot\cancel{r^2}\cdot\cancel{2\pi}\int_{\theta=0}^{\theta=\pi}\left(\cos^2\theta\sin^2\theta\right)\sin\theta\, d\theta}$$

$$\boxed{\left\langle P^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle \approx \frac{\mu_o p^2 d^2\omega^6}{16\pi c^3}\int_{\theta=0}^{\theta=\pi}\cos^2\theta\left(1-\cos^2\theta\right)\sin\theta\, d\theta = \frac{\mu_o p^2 d^2\omega^6}{16\pi c^3}\int_{\theta=0}^{\theta=\pi}\int_{\theta=0}^{\theta=\pi}\left[\cos^2\theta-\cos^4\theta\right]\sin\theta\, d\theta}$$

Let: $\boxed{u = \cos\theta,\ \ du = -\sin\theta\, d\theta}$, then: $\boxed{\theta = 0:\ u = +1}$ and: $\boxed{\theta = \pi:\ u = -1}$

Then: $\int_{\theta=0}^{\theta=\pi}\left[\cos^2\theta-\cos^4\theta\right]\sin\theta d\theta=\int_{u=-1}^{u=+1}\left(u^2-u^4\right)du=\left(\frac{1}{3}u^3-\frac{1}{5}u^5\right)\Big|_{-1}^{+1}=\frac{2}{3}-\frac{2}{5}=\frac{10-6}{15}=\frac{4}{15}$

$$\therefore\quad \left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle \approx\left(\frac{\not{4}}{15}\right)\frac{\mu_o p^2d^2\omega^6}{\not{16}_4\,\pi c^3}=\frac{\mu_o p^2d^2\omega^6}{60\pi c^3}=\frac{\mu_o Q_{zz}^{e^2}\omega^6}{60\pi c^3}\quad (Watts)$$

Note that time-averaged E(2) *EM* power radiated in the "far-zone" limit is proportional to the ***square*** of the electric quadrupole moment $Q_{zz}^e=pd=qdd=qd^2$ $(Coulomb\text{-}m^2)$

Note also that time averaged E(2) *EM* radiated power $\sim\omega^6$ (*cf.* with $\sim\omega^4$ for E(1) *EM* radiation.

The time-averaged *EM* angular power radiated by E(2) linear electric quadrupole $\left(\ell=2,\ m=0\ Y_\ell^m\right)$:

$$\frac{d\left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\equiv\left\langle\vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle\bullet r^2\hat{r}\approx\left(\frac{\mu_o p^2d^2\omega^6}{32\pi^2c^4}\right)\cos^2\theta\sin^2\theta=\left(\frac{\mu_o Q_{zz}^{e^2}\omega^6}{32\pi^2c^3}\right)\cos^2\theta\sin^2\theta\quad\left(\frac{Watts}{steradian}\right)$$

Note that $d\left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle/d\Omega$ has ***zeros*** when $\theta=0,\tfrac{1}{2}\pi,\pi$!!!

The time-averaged E(2) *EM* linear momentum density in the "far-zone" limit is:

$$\left\langle\vec{\wp}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle\approx\left(\frac{\mu_o p^2\omega^6d^2}{32\pi^2c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}=\left(\frac{\mu_o Q_{zz}^{e^2}\omega^6}{32\pi^2c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}\quad\left(\frac{kg}{m^2\text{-}sec}\right)$$

The time-averaged E(2) *EM* angular momentum density in the "far-zone" limit is:

$$\left\langle\vec{\ell}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle=0\quad\left(\frac{kg}{m\text{-}sec}\right)$$

The *EM* wave characteristic impedance of an E(2) oscillating linear electric quadrupole antenna:

$$Z_{antenna}^{\mathrm{E}(2)}(\vec{r})\equiv\frac{\left|\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}{\left|\vec{H}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}=\frac{\left|\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}{\frac{1}{\mu_o}\left|\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}=\mu_o c=\sqrt{\frac{\mu_o}{\varepsilon_o}}=Z_o=120\pi\,\Omega\simeq377\,\Omega$$

The *EM* wave radiation resistance of an E(2) oscillating linear electric quadrupole antenna:
Recall that $I=q\omega$ for a linear oscillating linear electric ***dipole*** (also true ***here***).

$$\left\langle P_{\mathrm{E}(2)}^{rad}\right\rangle\approx\frac{\mu_o p^2d^2\omega^6}{60\pi c^3}\equiv I^2R_{\mathrm{E}(2)}^{rad}\quad\Rightarrow\quad R_{\mathrm{E}(2)}^{rad}=\frac{\left\langle P_{\mathrm{E}(2)}^{rad}\right\rangle}{I^2}\approx\frac{\mu_o p^2d^2\omega^6}{60\pi c^3I^2}=\frac{\mu_o p^2d^2\omega^6}{60\pi c^3(q\omega)^2}\quad\text{but: } p=qd$$

$$\therefore\quad R_{\mathrm{E}(2)}^{rad}\approx\frac{\mu_o \not{q}^2d^4\not{\omega}^{6\,4}}{60\pi c^3\not{q}^2\not{\omega}^2}=\frac{\mu_o\omega^4d^4}{60\pi c^3}=\frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4\mu_o c=\frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4Z_o$$

Where: $\boxed{Z_o = \mu_o c = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega}$ = impedance of free space / vacuum

But: $\boxed{\left(\dfrac{\omega d}{c}\right) \ll 1}$ in the "far-zone" limit { $d \ll \lambda \ll r$ }, thus we see that:

$$\boxed{R_{\text{E}(2)}^{rad} \approx \frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o \ll Z_o \simeq 377\,\Omega}$$

Comparison of *EM* Quantities for E(1) Oscillating Linear Electric Dipole *vs.* E(2) Oscillating Linear Electric Quadrupole in the "far-zone" limit $d \ll \lambda \ll r$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$

	$\ell=1,\ m=0$ **Oscillating E(1) Linear Electric Dipole**	$\ell=2,\ m=0$ **Oscillating E(2) Linear Electric Quadrupole**
Moments	$\vec{p}(t)=q(t)\vec{d},\ \vec{d}=dz,\ p=qd$	$\vec{\vec{Q}}^{e}_{zz}=q(t)\vec{d}\vec{d},\ Q^{e}_{zz}=qdd$
Retarded Scalar Potential	$V_r^{E(1)}(\vec{r},t)\approx-\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$	$V_r^{E(2)}(\vec{r},t)\approx-\left(\frac{Q^{e}_{zz}\omega^2}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$
Retarded Vector Potential	$\vec{A}_r^{E(1)}(r,t)\approx-\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$	$\vec{A}_r^{E(2)}(\vec{r},t)\approx-\left(\frac{\mu_o Q^{e}_{zz}\omega^2}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$
	$\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}$	
Retarded Electric Field	$\vec{E}_r^{E(1)}(\vec{r},t)\approx-\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$	$\vec{E}_r^{E(2)}(\vec{r},t)\approx+\left(\frac{\mu_o Q^{e}_{zz}\omega^3}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Retarded Magnetic Field	$\vec{B}_r^{E(1)}(\vec{r},t)\approx-\frac{\mu_o p\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$	$\vec{B}_r^{E(2)}(\vec{r},t)\approx+\left(\frac{\mu_o Q^{e}_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Time-Avg'd *EM* Energy Density	$\langle u^{rad}_{E(1)}(\vec{r},t)\rangle\approx\left(\frac{\mu_o p^2\omega^4}{32\pi^2c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$\langle u^{rad}_{E(2)}(\vec{r},t)\rangle\approx\left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting's Vect/Intensity	$I^{rad}_{E(1)}(\vec{r})\equiv\langle\lvert\vec{S}^{rad}_{E(1)}(\vec{r},t)\rvert\rangle\approx\left(\frac{\mu_o p^2\omega^4}{32\pi^2c}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$I^{rad}_{E(2)}(\vec{r})\equiv\langle\lvert\vec{S}^{rad}_{E(2)}(\vec{r},t)\rvert\rangle\approx\left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\langle P^{rad}_{E(1)}(\vec{r},t)\rangle\approx\left(\frac{\mu_o p^2\omega^4}{12\pi c}\right)$	$\langle P^{rad}_{E(2)}(\vec{r},t)\rangle\approx\frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}$
Time-Avg'd *EM* Linear Momentum Density	$\langle\vec{\wp}^{rad}_{E(1)}(\vec{r},t)\rangle\approx\left(\frac{\mu_o p^2\omega^4}{32\pi^2c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$	$\langle\vec{\wp}^{rad}_{E(2)}(\vec{r},t)\rangle\approx\left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\langle\vec{\ell}^{rad}_{E(1)}(\vec{r},t)\rangle=0$	$\langle\vec{\ell}^{rad}_{E(2)}(\vec{r},t)\rangle=0$
Characteristic Antenna Impedance	$Z^{E(1)}_{rad}=Z_o=\sqrt{\frac{\mu_o}{\varepsilon_o}}=120\pi\,\Omega\simeq377\,\Omega$ =	$Z^{E(2)}_{rad}=Z_o=\sqrt{\frac{\mu_o}{\varepsilon_o}}=120\pi\,\Omega\simeq377\,\Omega$
Antenna Radiation Resistance	$R^{E(1)}_{rad}\approx\frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2Z_o$	$R^{E(2)}_{rad}\approx\frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4Z_o$

Note the ratios of *EM* power radiated:

$$\left\langle P^{rad}_{\mathrm{E}(2)}\right\rangle \Big/ \left\langle P^{rad}_{\mathrm{E}(1)}\right\rangle \approx \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}\right) \Bigg/ \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right) = \frac{1}{5}\left(\frac{Q^e_{zz}}{p}\right)^2 \frac{\omega^2}{c^2} = \frac{1}{5}\left(\frac{qdd}{qd}\right)^2 \frac{\omega^2}{c^2} = \frac{1}{5}\left(\frac{\omega d}{c}\right)^2 \ll 1$$

Recall/compare to:

$$\left\langle \mathrm{P}^{rad}_{\mathrm{M}(1)}\right\rangle \Big/ \left\langle \mathrm{P}^{rad}_{\mathrm{E}(1)}\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right) \Bigg/ \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right) = \left(\frac{\omega b}{c}\right)^2 \ll 1$$ where $p = qd$ and $m = \pi b^2 I$,

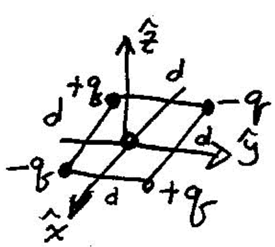
and $I = q\omega$ and $d = \pi b$ or: $b = d/\pi$.

Then: $$\left\langle P^{rad}_{\mathrm{M}(1)}\right\rangle \Big/ \left\langle P^{rad}_{\mathrm{E}(2)}\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right) \Bigg/ \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}\right) \simeq 5\left(\frac{b}{d}\right)^2 \approx 5\left(\frac{1}{\pi}\right)^2 \approx \frac{5}{\pi^2} \sim \frac{1}{2} \sim \vartheta(1)$$!!!

General comments for the ℓ^{th}-order, $m = 0$ electric multipole in "far-zone" limit, $d \ll \lambda \ll r$:

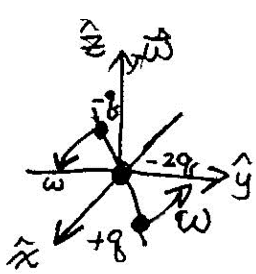
Each successive/higher power of ℓ brings in a multiplicative factor of $(\omega d/c) \ll 1$ to the retarded *EM* potentials and retarded *EM* fields, and thus brings in a multiplicative factor of $(\omega d/c)^2 \ll 1$ to the retarded *EM* energy densities, Poynting's vector, *EM* power radiated/ *EM* intensity, *EM* linear momentum density, *etc*.

- By similar methodology of above {plus suitable space-rotations}, we can obtain all of the above results for the oscillating E(2) quadrupole *e.g.* lying in the *x*-*y* plane as shown in the figure below:

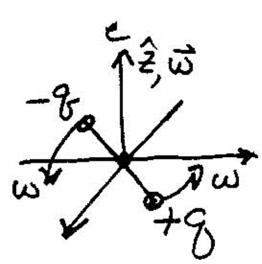


$\ell = 2,\ m = \pm 1$

- Similarly, we can also *e.g.* take the ***linear*** E(2) electric quadrupole (along $\hat{z}$ axis) $\Rightarrow$ place it in the *x*-*y* plane and have it ***rotate*** at angular frequency ω:



$\Rightarrow$ Get E(2) $\ell = 2, m = \pm 1$ results, analogous to linear E(1) electric dipole rotating in *x*-*y* plane $(\ell = 1, m = \pm 1)$:

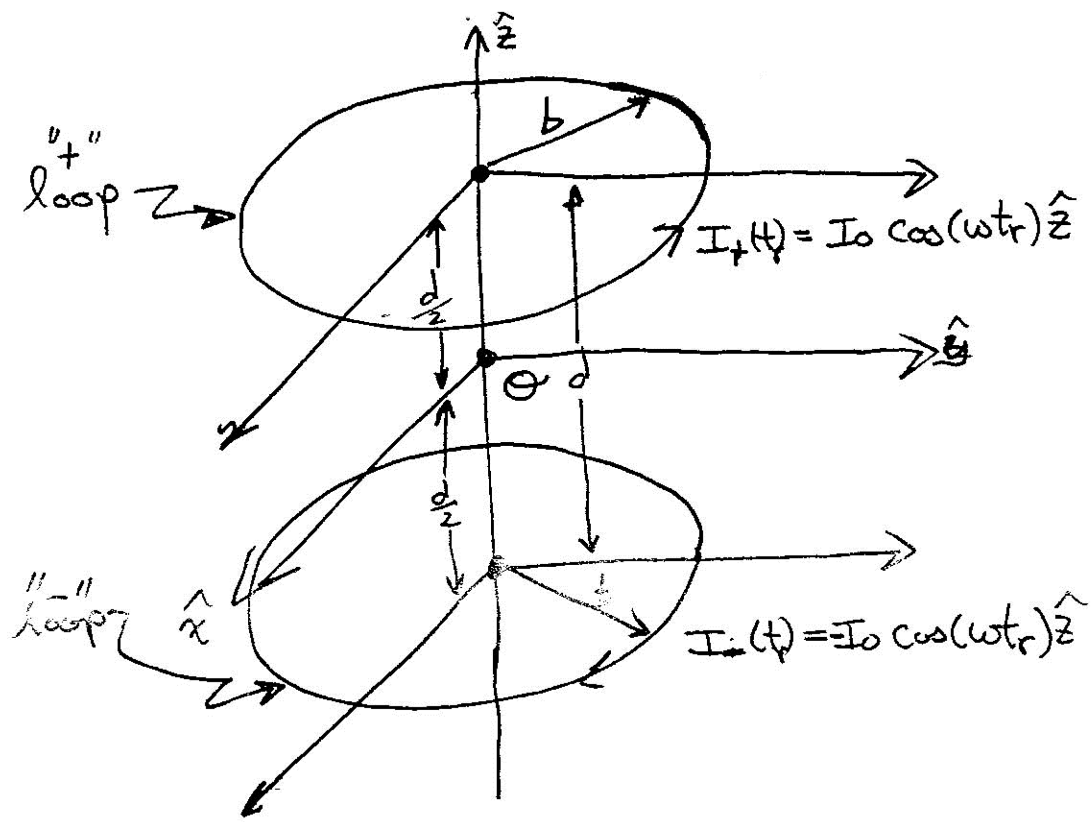


$\Rightarrow$ See angular distribution radiation patterns on page 8 of ***these*** P436 Lecture Notes.

<u>"Far-Zone" *EM* Radiation Fields Associated with a Oscillating Linear Magnetic Quadrupole, M(2)</u>

Instead of blindly/mindlessly grinding out the "far-zone" *EM* radiation field results for the oscillating linear magnetic quadrupole, we can, via use of the ***duality transform***, we can use the results from the oscillating E(2) linear electric quadrupole to obtain results for oscillating M(2) linear magnetic quadrupole, *i.e*. we will use the ***duality transform*** on the E(2) electric charge/ current density distributions/*EM* moments and the "far-zone" E(2) electric and magnetic fields:

$$\boxed{\begin{gathered} Q_{zz}^{e} \;\Rightarrow\; Q_{zz}^{m}/c \\ \left\{\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t),c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right\} \Rightarrow \left\{\vec{E}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t),c\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)\right\} \end{gathered}}$$



<u>Duality Transform {From P435 Lecture Notes #18 page 7-9}</u>:

$$\boxed{\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{M}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)} \end{pmatrix} = R_{DT}(\varphi)\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{E}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \end{pmatrix} = \begin{pmatrix} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{E}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \end{pmatrix}} \quad \text{where: } \boxed{\varphi = 90^\circ = \pi/2}.$$

Thus: $$\boxed{\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{M}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{E}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \end{pmatrix}} \Rightarrow \boxed{\begin{pmatrix} \vec{E}_{\mathrm{r}}^{\mathrm{M}(2)} = c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \\ c\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)} = -\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)} \end{pmatrix}}$$

n.b. φ is ***not*** a physical/space angle here!

The corresponding duality transform associated with the electric and magnetic charges is:

$$\boxed{\begin{pmatrix} e' \\ cg'_m \end{pmatrix} = R_{DT}(\varphi)\begin{pmatrix} e \\ cg_m \end{pmatrix} = \begin{pmatrix} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} e \\ cg_m \end{pmatrix}} \quad \text{where: } \boxed{\varphi = 90^\circ = \pi/2}.$$

Thus: $\boxed{\begin{pmatrix} e' \\ cg'_m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e \\ cg_m \end{pmatrix}} \Rightarrow \boxed{\begin{pmatrix} e' = cg_m \\ cg'_m = -e \end{pmatrix}}$ *i.e.* $\boxed{\{p = ed\} \Rightarrow \{-g_m d/c = -m/c\}}$

Hence for $\varphi = 90^\circ = \pi/2$:

$e \Rightarrow -g'_m/c$ for electric *vs*. magnetic ***monopole*** moments {E(0) & M(0)}.

$p = qd \Rightarrow -g'_m d/c = -m'/c$ for electric *vs*. magnetic ***dipole*** moments {E(1) & M(1)}.

$Q^e_{zz} = edd \Rightarrow -g'_m dd/c = -Q'^m_{zz}/c$ for electric *vs*. magnetic ***quadrupole*** moments {E(2) & M(2)}.

Note also that, as we saw for the case of the M(1) magnetic dipole, where the scalar potential was $V_{\rm r}^{\rm M(1)}(\vec{r},t) = 0$, likewise, for the case of the M(2) magnetic quadrupole, the scalar potential is also zero, *i.e.* $V_{\rm r}^{\rm M(2)}(\vec{r},t) = 0$.

We can then {easily} obtain $\vec{A}_{\rm r}^{\rm M(2)}(\vec{r},t)$ from $\vec{E}_{\rm r}^{\rm M(2)}(\vec{r},t)$, since:

$$\boxed{\vec{E}_{\rm r}^{\rm M(2)}(\vec{r},t) = -\vec{\nabla}\underbrace{\cancel{V_{\rm r}^{\rm M(2)}(\vec{r},t)}}_{=0} - \frac{\partial \vec{A}_{\rm r}^{\rm M(2)}(\vec{r},t)}{\partial t} = -\frac{\partial \vec{A}_{\rm r}^{\rm M(2)}(\vec{r},t)}{\partial t}}$$

Comparison of *EM* Quantities for E(2) Oscillating Linear Electric Quadrupole *vs.* M(2) Oscillating Linear Magnetic Quadrupole in the "far-zone" limit $d = \pi b \ll \lambda \ll r$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$

	$\ell = 2,\ m = 0$ **Oscillating E(2) Linear Electric Quadrupole**		$\ell = 2,\ m = 0$ **Oscillating M(2) Linear Magnetic Quadrupole**
Moments	$\vec{Q}^e_{zz} = q(t)\vec{d}\vec{d},\ Q^e_{zz} = qdd$	$Q^e_{zz} = Q^m_{zz}/c$	$\vec{Q}^m_{zz} = I\vec{A}_{\ell oop},\ \vec{A}_{\ell oop} = \pi b^2 \hat{z},\ Q^m_{zz} = I\pi^2 b$
Retarded Scalar Potential	$V_r^{E(2)}(\vec{r},t) \approx -\left(\frac{Q^e_{zz}\omega^2}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$		$V_r^{M(2)}(\vec{r},t) = 0$
Retarded Vector Potential	$\vec{A}_r^{E(2)} \approx -\left(\frac{\mu_o Q^e_{zz}\omega^2}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$		$\vec{A}_r^{M(2)} \approx -\left(\frac{\mu_o Q^m_{zz}\omega^2}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
			$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$
Retarded Electric Field	$\vec{E}_r^{E(2)} \approx +\left(\frac{\mu_o Q^e_{zz}\omega^3}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$		$\vec{E}_r^{M(2)} \approx +\left(\frac{\mu_o Q^m_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Magnetic Field	$\vec{B}_r^{E(2)} \approx +\left(\frac{\mu_o Q^e_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$		$\vec{B}_r^{E(2)} \approx -\left(\frac{\mu_o Q^m_{zz}\omega^3}{4\pi c^3}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Time-Avg'd *EM* Energy Density	$\langle u^{rad}_{E(2)}(\vec{r},t)\rangle \approx \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$		$\langle u^{rad}_{M(2)}(\vec{r},t)\rangle \approx \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^6}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting's Vect/Intensity	$I^{rad}_{E(2)}(\vec{r}) \equiv \langle\lvert\vec{S}^{rad}_{E(2)}\rvert\rangle \approx \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$		$I^{rad}_{M(2)}(\vec{r}) \equiv \langle\lvert\vec{S}^{rad}_{M(2)}\rvert\rangle \approx \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\langle P^{rad}_{E(2)}(\vec{r},t)\rangle \approx \frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}$		$\langle P^{rad}_{M(2)}(\vec{r},t)\rangle \approx \frac{\mu_o Q^{m^2}_{zz}\omega^6}{60\pi c^5}$
Time-Avg'd *EM* Linear Momentum Density	$\langle \vec{\wp}^{rad}_{E(2)}(\vec{r},t)\rangle \approx \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$		$\langle \vec{\wp}^{rad}_{M(2)}(\vec{r},t)\rangle \approx \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^7}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\langle \vec{\ell}^{rad}_{E(2)}(\vec{r},t)\rangle = 0$		$\langle \vec{\ell}^{rad}_{E(2)}(\vec{r},t)\rangle = 0$
Characteristic Antenna Impedance	$Z^{E(2)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$	=	$Z^{M(2)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$
Antenna Radiation Resistance	$R^{E(2)}_{rad} \approx \frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o$		$R^{M(2)}_{rad} \approx \frac{1}{60\pi}\left(\frac{\omega\pi b}{c}\right)^6 Z_o$

In the "far-zone" limit $\{ d = \pi b \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$ we (again) see that:

$$\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t) = \frac{1}{c}\hat{r} \times \vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)} \Leftarrow \boxed{\hat{r} \times \hat{\theta} = \hat{\varphi}} \quad \text{and:} \quad \boxed{\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t) = \frac{1}{c}\hat{r} \times \vec{E}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)} \Leftarrow \boxed{\hat{r} \times \hat{\varphi} = -\hat{\theta}}$$

Ratio of *EM* wave radiation resistances: $\boxed{R_{\mathrm{M}(2)}^{rad} \Big/ R_{\mathrm{E}(2)}^{rad} = \left(\frac{\omega \pi b}{c}\right)^2 = \left(\frac{\omega d}{c}\right)^2 \ll 1}$

Ratio of time-averaged *EM* power radiated: $\boxed{\left\langle P_{\mathrm{M}(2)}^{rad} \right\rangle \Big/ \left\langle P_{\mathrm{E}(2)}^{rad} \right\rangle = \left(\frac{\omega \pi b}{c}\right)^2 = \left(\frac{\omega d}{c}\right)^2 \ll 1}$