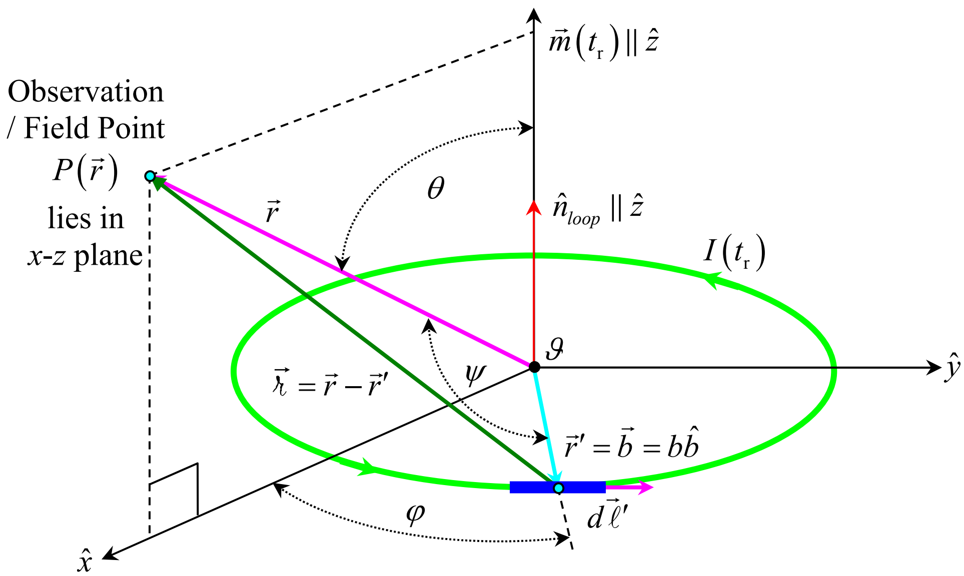
LECTURE NOTES 13.5

M(1) Magnetic Dipole Radiation:

A harmonically time-varying ***retarded*** current $I(t_r) = I\cos(\omega t_r)$ flows in a circular loop of radius b {chosen for convenience's sake in to lie in the *x-y* plane} as shown in the figure below, and has associated with it an oscillating magnetic dipole moment:

$$\boxed{\vec{m}(t_r) = I(t_r)\vec{A}_{loop} = \pi b^2 I\cos(\omega t_r)\hat{z}} \text{ where: } \boxed{\vec{A}_{loop} = A_{loop}\hat{n}_{loop} = \pi b^2\hat{z}} \text{ with: } \boxed{\hat{n}_{loop} = \hat{z}}.$$

Or: $$\boxed{\vec{m}(t_r) = m\cos(\omega t_r)\hat{z}} \text{ where: } \boxed{m \equiv \pi b^2 I}$$



Note that there is ***no*** volume electric charge density $\rho(\vec{r}, t_r)$ associated with the current flowing in the loop, thus the ***retarded*** scalar potential $V_r^{M(1)}(\vec{r},t) = 0$ and thus $\vec{\nabla} V_r^{M(1)}(\vec{r},t) = 0$ {***here***}.

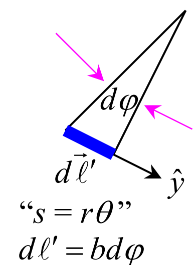
The ***retarded*** vector potential is:

$$\boxed{\vec{A}_r^{M(1)}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right)\int \frac{I\cos\left[\omega(t - \mathscr{r}/c)\right]}{\mathscr{r}}\, d\vec{\ell}'} \text{ with: } \boxed{t_r = t - \mathscr{r}/c} \text{ and: } \boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_r)}$$

n.b. $\vec{A}_r^{M(1)}(\vec{r},t)$ follows the direction of (conventional) current flow $\Rightarrow \vec{A}_r^{M(1)}(\vec{r},t) \parallel \hat{\varphi}$-direction.

For convenience's sake, the observation / field point $P(\vec{r})$ is chosen directly above the $\hat{x}$-axis in the *x-z* plane (see above figure). From (azimuthal / circular symmetry) associated with this problem, one can see that *e.g.* the $\hat{x}$-component contributions to $\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)$ from symmetrically-placed current segments $I(t)d\vec{\ell}'$ on either side of the $\hat{x}$-axis will cancel each other, $\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)$ at this observation point in the *x-z* plane will point in the $\hat{y}$-direction, but for an arbitrary field point, we see that $\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)$ points in the $\hat{\varphi}$-direction since:

$\boxed{\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}}$ (in spherical-polar and/or cylindrical coordinates)
(*n.b.* the angle $\varphi = 0$ for points in the *x-z* plane!)



Thus, for the choice of the observation / field point $P(\vec{r})$ in *x-z* plane (see figure above): $\boxed{d\vec{\ell}' = bd\varphi\,\hat{y}\cdot\cos\varphi = b\cos\varphi d\varphi\,\hat{y}}$ (*n.b.* the $\cos\varphi$ term simply picks off the $\hat{y}$-component of $d\vec{\ell}'$), but note that $d\vec{\ell}'$ is actually || to $\hat{\varphi}$.

Then: $$\boxed{\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right) Ib \int_{\varphi=0}^{\varphi=2\pi} \frac{\cos\left[\omega\left(t - \mathscr{r}/c\right)\right]}{\mathscr{r}} \cos\varphi d\varphi\,\hat{\varphi}}$$

From the Law of Cosines: $\boxed{\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi}}$ where: $\boxed{\psi = \cos^{-1}\left(\hat{r}\cdot\hat{b}\right)}$ {See above figure}

$\vec{r}$ lies in the *x-z* plane: $\boxed{\vec{r} = r\sin\theta\hat{x} + r\cos\theta\hat{z}}$ and $\vec{b}$ lies in *x-y* plane: $\boxed{\vec{b} = b\cos\varphi\hat{x} + b\sin\varphi\hat{y}}$

$$\therefore \boxed{\vec{r}\cdot\vec{b} = rb\cos\psi = \left(r\sin\theta\hat{x} + r\cos\theta\hat{z}\right)\cdot\left(b\cos\varphi\hat{x} + b\sin\varphi\hat{y}\right) = rb\sin\theta\cos\varphi}$$

$$\therefore \boxed{\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\varphi}}$$

Here again, we are interested in far-zone *EM* radiation solutions, *i.e.* the observer is far away from source, such that the characteristic dimension of the source is such that $\boxed{b \ll r}$.

Then keeping only the 1st non-trivial term in the Taylor series expansion of $\mathscr{r}$ with $b \ll r$:

$$\boxed{\mathscr{r} = r\sqrt{1 + \cancel{\left(\frac{b}{r}\right)^2} - 2\left(\frac{b}{r}\right)\sin\theta\cos\varphi} \simeq r\sqrt{1 - 2\left(\frac{b}{r}\right)\sin\theta\cos\varphi}} \text{ with: } \boxed{\left(\frac{b}{r}\right) \ll 1} \text{ and: } \boxed{\begin{array}{l}\sqrt{1-\epsilon} \simeq 1 - \tfrac{1}{2}\epsilon \\ \text{for } \epsilon \ll 1\end{array}}$$

$$\therefore \boxed{\mathscr{r} \approx r\left(1 - \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \quad \text{for: } \boxed{\left(\frac{b}{r}\right) \ll 1}$$

And: $$\boxed{\frac{1}{\mathscr{r}} \approx \frac{1}{r\left(1 - \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \approx \frac{1}{r}\left(1 + \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \quad \text{since: } \frac{1}{1-\epsilon} \simeq 1 + \epsilon \text{ for } \epsilon \ll 1.$$

Now:

$$\cos\left[\omega\left(t-\frac{\mathscr{r}}{c}\right)\right]\approx\cos\left[\omega\left(t-\frac{r}{c}\right)+\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right]$$
$$=\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\cos\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)-\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\sin\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)$$

Here (again) we assume for "far-zone" radiation that $\boxed{b\ll\lambda}$ and $\boxed{\lambda\ll r}$, *i.e.* $\boxed{b\ll\lambda\ll r}$ and since $\boxed{b\ll\lambda}$ and $\boxed{\lambda=c/f=2\pi c/\omega}$ $\Rightarrow$ $\boxed{b\ll\frac{c}{\omega}}$ or: $\boxed{\frac{\omega b}{c}\ll1}$.

Then:

$$\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\simeq\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\overbrace{\cos0}^{=1}-\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)$$
$$\approx\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Thus:

$$\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)=\left(\frac{\mu_o}{4\pi}\right)Ib\int_{\varphi=0}^{\varphi=2\pi}\frac{\cos\left[\omega\left(t-\mathscr{r}/c\right)\right]}{\mathscr{r}}\cos\varphi d\varphi\,\hat{\varphi}$$
$$\approx\left(\frac{\mu_o}{4\pi}\right)\frac{Ib}{r}\int_{\varphi=0}^{\varphi=2\pi}\left(1+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)$$
$$*\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos\varphi d\varphi\,\hat{\varphi}$$
$$=\left(\frac{\mu_o}{4\pi}\right)\frac{Ib}{r}\int_{\varphi=0}^{\varphi=2\pi}\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right.$$
$$\left.+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b^2}{rc}\right)\sin^2\theta\cos^2\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos\varphi d\varphi\,\hat{\varphi}$$

However, note that: $\boxed{\left(\frac{b}{r}\right)\ll1}$ and: $\boxed{\left(\frac{\omega b}{c}\right)\ll1}$, thus: $\boxed{\left(\frac{\omega b^2}{rc}\right)=\left(\frac{b}{r}\right)\left(\frac{\omega b}{c}\right)\lll1}$,

i.e. $\boxed{\left(\frac{\omega b^2}{rc}\right)=\left(\frac{b}{r}\right)\left(\frac{\omega b}{c}\right)}$ is a <u>2nd-order</u> term, so we drop/neglect it !!!

$\therefore$

$$\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)\approx\left(\frac{\mu_o}{4\pi}\right)\frac{Ib}{r}\int_{\varphi=0}^{\varphi=2\pi}\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right.$$
$$\left.+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos\varphi\,d\varphi\,\hat{\varphi}$$

Note that the first term in the above integral: $\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos\varphi\, d\varphi = 0$

The second term is: $-\left(\frac{\omega b}{c}\right)\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi$

The third term is: $+\left(\frac{b}{r}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi$ But: $\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi = \pi$

Thus: $\vec{A}_r^{M(1)}(\vec{r},t) \approx \left(\frac{\mu_o}{4\pi}\right)\frac{Ib^2}{r}(\pi\sin\theta)\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi}$

But: $m \equiv \pi b^2 I = A_{loop} I$

$$\therefore \quad \vec{A}_r^{M(1)}(\vec{r},t) \approx \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r}\sin\theta\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi} \quad **$$

Note that in the ***static*** limit ($\omega \to 0$), only the 1st term in the { … } brackets survives:

$\vec{A}_r^{M(1)}(\vec{r}) \approx \frac{\mu_o m}{4\pi r^2}\sin\theta\,\hat{\varphi}$ = vector potential for a (***static***) magnetic dipole.

Note also that in the "far-zone" for M(1) *EM* radiation, with $b \ll \lambda \ll r$ or: $\frac{c}{\omega} \ll r$

If $\frac{c}{\omega} \ll r$, then $\frac{\omega}{c} \gg \left(\frac{1}{r}\right)$ ∴ Drop the 1st term in { … } in the above expression for $\vec{A}_r^{M(1)}(\vec{r},t)$:

$$\therefore \quad \vec{A}_r^{M(1)}(\vec{r},t) \approx \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r}\sin\theta\left\{\cancelto{\approx 0}{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi}$$

Thus: $\vec{A}_r^{M(1)}(\vec{r},t) \approx -\left(\frac{\mu_o}{4\pi}\right)\frac{\omega m}{cr}\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$

Or: $\vec{A}_r^{M(1)}(\vec{r},t) \approx -\frac{\mu_o m\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$

Then: $\vec{E}_r^{M(1)}(\vec{r},t) = \cancelto{=0}{-\vec{\nabla}V_r^{M(1)}(\vec{r},t)} - \frac{\partial\vec{A}_r^{M(1)}(\vec{r},t)}{\partial t} = -\frac{\partial\vec{A}_r^{M(1)}(\vec{r},t)}{\partial t}$

Thus: $\vec{E}_r^{M(1)}(\vec{r},t) \approx +\frac{\mu_o m\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$

And:

$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$$
$$=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)-\cancelto{=0}{\frac{\partial A_\theta}{\partial\varphi}}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\cancelto{=0}{\frac{\partial A_r}{\partial\theta}}-\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta}+\frac{1}{r}\left[\frac{\partial}{\partial r}\cancelto{=0}{(rA_\theta)}-\cancelto{=0}{\frac{\partial A_r}{\partial\theta}}\right]\hat{\varphi}$$

Or:

$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx\frac{1}{r\sin\theta}\left(-\frac{\mu_o m\omega}{4\pi cr}\right)\frac{\partial}{\partial\theta}(\sin^2\theta)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$
$$-\frac{1}{r}\left(\frac{-\mu_o m\omega}{4\pi c}\right)\sin\theta\frac{\partial}{\partial r}\left(\left(\frac{\cancel{r}}{\cancel{r}}\right)\sin\left[\omega\left(t-{r}/{c}\right)\right]\right)\hat{\theta}$$

Or:

$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx-\frac{\mu_o m\omega}{4\pi cr^2}\frac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$
$$+\frac{\mu_o m\omega}{4\pi cr}\left(\frac{-\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

Or:

$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx-\left(\frac{\mu_o m\omega}{4\pi cr}\right)\left\{2\left(\frac{1}{r}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}+\left(\frac{\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}$$

Again, $\frac{\omega}{c}\gg\left(\frac{1}{r}\right)$ ∴ Drop the 1st term in { … } in the above expression for $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$

$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx-\left(\frac{\mu_o m\omega}{4\pi cr}\right)\left\{\cancel{2\left(\frac{1}{r}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}+\left(\frac{\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}$$

Thus:
$$\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx-\frac{\mu_o m\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

Note that in the ***static*** limit ($\omega\to 0$), ***first*** going back to using the expression for $\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ as given in ** on the previous page, and ***then*** using $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$, we do indeed obtain the familiar result: $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r})\approx-\left(\frac{\mu_o m}{4\pi r^3}\right)\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)$.

Therefore, for M(1) magnetic dipole radiation in the "far-zone" limit, with $b\ll\lambda\ll r$, we have:

$$V_{\text{r}}^{\text{M(1)}}(\vec{r},t)=0$$
$$\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)\approx-\frac{\mu_o m\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$$ where: $m=\pi b^2 I$

The ***retarded*** electric and magnetic fields associated with M(1) magnetic dipole radiation in the "far-zone" limit, with $\boxed{b \ll \lambda \ll r}$ are:

$$\boxed{\vec{E}_{\text{r}}^{\text{M(1)}}(\vec{r},t) = -\frac{\partial \vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)}{\partial t} \approx +\frac{\mu_o m\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$$

$$\boxed{\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t) \approx -\frac{\mu_o m\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

Here again, note that: $\boxed{\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}_{\text{r}}^{\text{M(1)}}(\vec{r},t)}$ with: $\left(\hat{r}\times\hat{\varphi} = -\hat{\theta}\right)$

{since: $\hat{r}\times\hat{\theta} = \hat{\varphi}$ $\quad \hat{\varphi}\times\hat{r} = \hat{\theta}$ $\quad \hat{\theta}\times\hat{\varphi} = \hat{r}$ }

Note that $\vec{E}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ and $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$:

a) both have same $\sim 1/r$ dependence (as does $\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$).

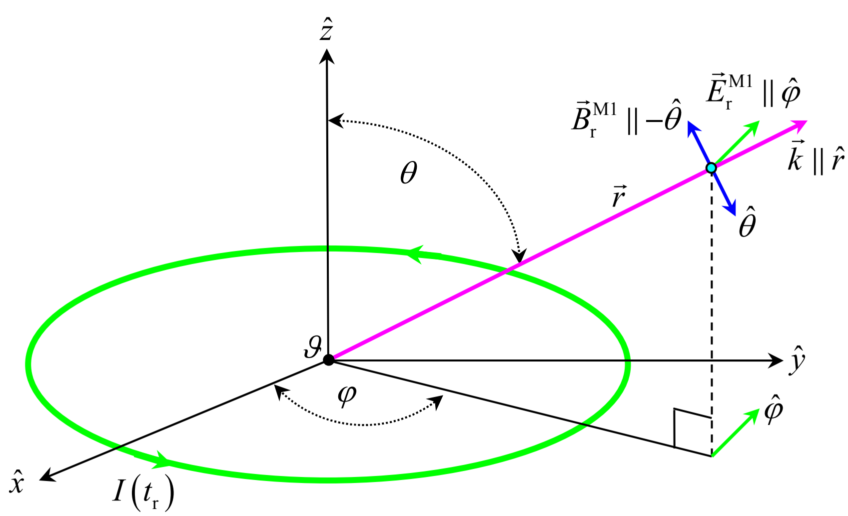
b) both have same $\sim \sin\theta$ dependence (as does $\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$).

c) both are ***in-phase*** with each other – both have same $\cos\left[\omega(t-r/c)\right]$ factors.

d) both are 90° ***out-of-phase*** with $\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$.

e) $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ is $\perp$ to $\vec{E}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ as it must be.

Note also that: $\vec{A}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$, $\vec{E}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ and $\vec{B}_{\text{r}}^{\text{M(1)}}(\vec{r},t)$ ***vanish*** (*i.e.* = 0) when $\theta = 0$ and $\theta = \pi$ *i.e.* at the poles, along the $\hat{z}$ -axis (as we also saw in the case of E(1) electric dipole radiation).



The *EM* ***radiation*** energy density $u_{\rm M(1)}^{rad}(\vec{r},t)$ associated with the oscillating M1 magnetic dipole for far-zone *EM* radiation $\{b \ll \lambda \ll r\}$ is:

$$u_{\rm M(1)}^{rad}(\vec{r},t) = u_{\rm M(1)}^{E}(\vec{r},t) + u_{\rm M(1)}^{M}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o \vec{E}_{\rm r}^{\rm M(1)}(\vec{r},t)\cdot\vec{E}_{\rm r}^{\rm M(1)}(\vec{r},t) + \frac{1}{\mu_0}\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\cdot\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\right)$$
$$\approx \frac{1}{2}\left\{\varepsilon_o \frac{\mu_o^2 m^2\omega^4}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{1}{\mu_o}\frac{\mu_o^{2} m^2\omega^4}{16\pi^2 c^4}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}$$

But: $c^2 = \frac{1}{\varepsilon_o\mu_o}$ or: $\varepsilon_o = \frac{1}{\mu_o c^2}$, so again we see that $u_{\rm M(1)}^{E}(\vec{r},t) = u_{\rm M(1)}^{M}(\vec{r},t)$ in the "far-zone" limit $b \ll \lambda \ll r$, and thus:

$$u_{\rm M(1)}^{rad}(\vec{r},t) \approx \left(\frac{\mu_o m^2\omega^4}{16\pi^2 c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] \quad \left(\frac{Joules}{m^3}\right)$$ for $b \ll \lambda \ll r$ with $m \equiv \pi b^2 I$

The *EM* energy radiated by oscillating magnetic dipole in the far-zone limit $\{b \ll \lambda \ll r\}$ is given by Poynting's vector:

$$\vec{S}_{\rm M(1)}^{rad}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}_{\rm r}^{\rm M(1)}(\vec{r},t)\times\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t) \approx -\frac{1}{\mu_o}\frac{\mu_o^{2} m^2\omega^4}{16\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\left[\hat{\varphi}\times\hat{\theta}\right]}_{=-\hat{r}}$$

$\hat{r}\times\hat{\theta} = \hat{\varphi}$

$\hat{\theta}\times\hat{\varphi} = \hat{r}$

$\hat{\varphi}\times\hat{r} = \hat{\theta}$

$$\vec{S}_{\rm M(1)}^{rad}(\vec{r},t) \approx +\frac{\mu_o m^2\omega^4}{16\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} \quad \left(\frac{Watts}{m^2}\right)$$

$\Rightarrow$ Radial outward flow of *EM* energy for: $b \ll \lambda \ll r$ "far zone" limit

The *EM* radiation ***linear*** momentum density associated with an oscillating magnetic dipole, in the far zone limit $\{b \ll \lambda \ll r\}$ is given by:

$$\vec{\wp}_{\rm M(1)}^{rad}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}_{\rm M(1)}^{rad}(\vec{r},t) = \frac{1}{c^2}\vec{S}_{\rm M(1)}^{rad}(\vec{r},t) \approx +\frac{\mu_o m^2\omega^4}{16\pi^2 c^5}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} \quad \left(\frac{kg}{m^2\text{-sec}}\right)$$

$\Rightarrow$ Radial outward *EM* linear momentum flow for: $b \ll \lambda \ll r$ "far zone" limit

The *EM* radiation ***angular*** momentum density associated with an oscillating magnetic dipole, in the far zone $\{b \ll \lambda \ll r\}$ is given by:

$$\vec{\ell}_{\rm M(1)}^{rad}(\vec{r},t) = \vec{r}\times\vec{\wp}_{M(1)}^{rad}(\vec{r},t) \approx \frac{\mu_o m^2\omega^4}{16\pi^2 c^5}\left(\frac{\sin^2\theta}{r}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\left[\hat{r}\times\hat{r}\right]}_{=0} = 0 \quad \left(\frac{kg}{m\text{-sec}}\right)$$

$\Rightarrow$ ***No*** angular momentum flow for: $b \ll \lambda \ll r$ "far zone" limit

n.b. Again, the ***exact*** $\vec{\ell}_{\rm E(1)}^{rad}(\vec{r},t) \not\equiv 0$ *i.e.* ignore ***restrictions*** on far-zone limit, keep ***all*** higher-order terms . . . we have neglected $\vec{E}_{\rm r}^{\rm M(1)} \sim \hat{r}$ term which is ***non-negligible*** in the ***near-zone*** $(d \sim r)$ and also in the so-called ***intermediate***, or ***inductive zone*** $(\lambda \sim r)$.

Time-Averaged Quantities for M(1) Radiation from an Oscillating Magnetic Dipole:

The time-averaged *EM* radiation energy density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle u^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Joules}{m^3}\right) \text{ for: } \boxed{b \ll \lambda \ll r} \text{ "far-zone" limit, with: } \boxed{m \equiv \pi b^2 I}.$$

The time-averaged |Poynting's vector|, which is also the ***intensity*** $I^{rad}_{\mathrm{M(1)}}$ of *EM* radiation associated with an oscillating magnetic dipole is:

$$\boxed{I^{rad}_{\mathrm{M(1)}}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{\mathrm{M(1)}}(\vec{r})\right|\right\rangle = \frac{1}{2}c\varepsilon_o\left\langle\left(E^{\mathrm{M(1)}}_{\mathrm{r}}(\vec{r},t)\right)^2\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{"far-zone" limit}\end{array}}$$

We also see that: $\boxed{I^{rad}_{\mathrm{M(1)}}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right|\right\rangle = c\left\langle u^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle} \left(\frac{Watts}{m^2}\right)$.

The time-averaged *EM* radiated power associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle P^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle = \int_S \left\langle\vec{S}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle\cdot d\vec{a}_\perp \approx \frac{\mu_o m^2\omega^4}{32\pi^2 c^3}\underbrace{\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\sin^2\theta\sin\theta d\theta d\varphi}_{=\frac{4}{3}2\pi=\frac{8\pi}{3}}}$$

∴ The time-averaged radiated power is:

$$\boxed{\left\langle P^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right)} (Watts) \text{ for: } \boxed{b \ll \lambda \ll r} \text{ "far-zone" limit}$$

n.b. $\left\langle P^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle$ has ***no*** *r*-dependence!

The time-averaged *EM* radiation ***linear*** momentum density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle \vec{\wp}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle = \frac{1}{c^2}\left\langle\vec{S}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle\hat{r} \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}} \left(\frac{kg}{m^2\text{-sec}}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{"far-zone" limit}\end{array}}$$

The time-averaged *EM* radiation ***angular*** momentum density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle \vec{\ell}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle = \vec{r}\times\left\langle\vec{\wp}^{rad}_{\mathrm{M(1)}}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^5}\right)\left(\frac{\sin^2\theta}{r}\right)(\hat{r}\times\hat{r}) \equiv 0} \left(\frac{kg}{m\text{-sec}}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{"far-zone" limit}\end{array}}$$

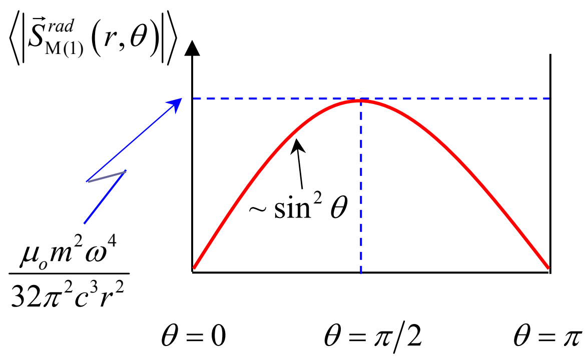
n.b. Again, the ***exact*** $\left\langle \vec{\ell}^{rad}_{\mathrm{M(1)}}(\vec{r})\right\rangle \not\equiv 0$ *i.e.* ignore ***restrictions*** on far-zone limit, keep ***all*** higher-order terms . . . we have neglected the $\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}} \sim \hat{r}$ term which is ***non-negligible*** in the ***near-zone*** $(d \sim r)$ and also in the so-called ***intermediate***, or ***inductive zone*** $(\lambda \sim r)$.

Again, note that because: $\boxed{I_{M(1)}^{rad}(\vec{r}) \equiv \left\langle \left| \vec{S}_{M(1)}^{rad}(\vec{r},t) \right| \right\rangle \approx \left(\frac{\mu_o m^2 \omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right)$

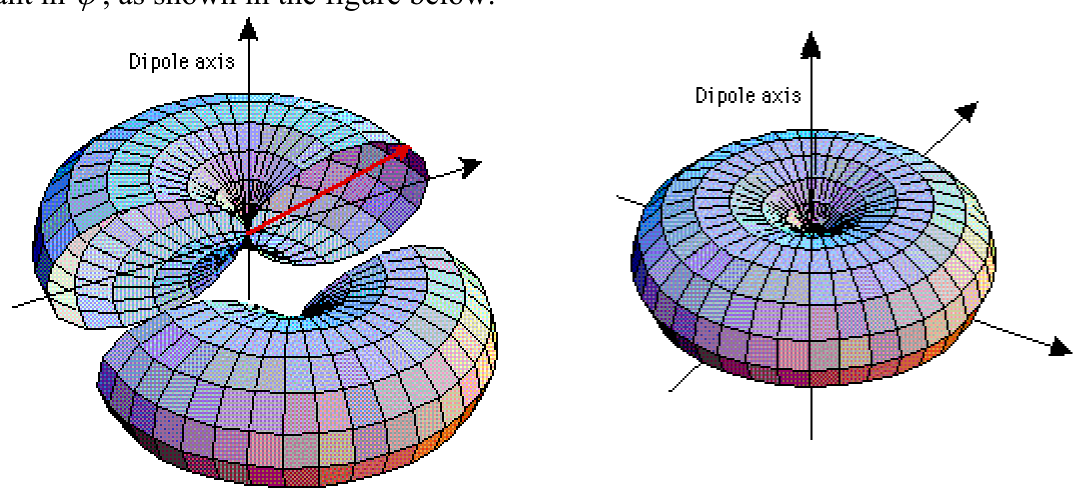
$\Rightarrow \boxed{\left\langle \vec{S}_{M(1)}^{rad}(r,\theta=0,\varphi) \right\rangle = \left\langle \vec{S}_{M(1)}^{rad}(r,\theta=\pi,\varphi) \right\rangle = 0}$ since: $\boxed{\sin^2 0 = \sin^2 \pi = 0}$

i.e. ***no*** *EM* radiation occurs along the ***axis*** of the magnetic dipole ($\hat{z}$ axis)

EM radiation for M(1) electric dipole is {also} peaked/maximum at $\theta = \pi/2$ (then $\sin^2\theta = 1$) *i.e.* maximum *EM* radiation occurs $\perp$ to the axis of the magnetic dipole:



Thus, the intensity profile $I_{M(1)}^{rad}(\vec{r})$ in 3-D {for fixed r} for M(1) magnetic dipole *EM* radiation is {again} ***donut-shaped*** {as in the case of E(1) electric dipole *EM* radiation} - it is rotationally invariant in φ, as shown in the figure below:



It is useful / illuminating to compare sources, retarded potentials, retarded fields, energy densities, Poynting's Vector, power radiated, linear and angular-momentum densities associated with E(1) electric dipole radiation *vs*. M(1) magnetic dipole radiation <u>in the far-zone limit</u>, with $\boxed{d = \pi b \ll \lambda \ll r}$:

	<u>E(1) Oscillating Electric Dipole</u>	**<u>M(1) Oscillating Magnetic Dipole</u>**
<u>Source Charge</u>:	$q(\vec{r},t_r) = q\delta(z \pm d/2)\cos(\omega t_r)$	$q(\vec{r},t_r) = 0$
<u>Source Currents</u>:	$I(\vec{r},t_r) = -q\omega\sin(\omega t_r)$ {on $\hat{z}$ -axis, $\lvert z\rvert < d/2$}	$I(\vec{r},t_r) = I\cos(\omega t_r)$ {in x-y plane, radius b}
<u>*EM* Moments</u>:	$\vec{p}(\vec{r},t_r) = q(\vec{r},t_r)\vec{d},\ \vec{d} = d\hat{z}$ $p = qd = \lvert\vec{p}\rvert = q\lvert\vec{d}\rvert$	$\vec{m}(\vec{r},t_r) = I(\vec{r},t_r)\vec{A}_{loop} = \pi b^2 I(\vec{r},t_r)$ $m = \pi b^2 I = \lvert\vec{m}\rvert$
Retarded Scalar Potential	$V_r^{E(1)}(\vec{r},t) \approx -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$	$V_r^{M(1)}(\vec{r},t) = 0$
Retarded Vector Potential	$\vec{A}_r^{E(1)}(r,t) \approx -\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$	$\vec{A}_r^{M(1)}(\vec{r},t) \approx -\frac{\mu_o m\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Electric Field	$\vec{E}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$	$\vec{E}_r^{M(1)}(\vec{r},t) \approx +\frac{\mu_o m\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Magnetic Field	$\vec{B}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$	$\vec{B}_r^{M(1)}(\vec{r},t) \approx -\frac{\mu_o m\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Time-Avg'd *EM* Energy Density	$\left\langle u_{E(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$\left\langle u_{M(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting Vector/Intensity	$I_{E(1)}^{rad}(\vec{r}) \equiv \left\langle\left\lvert\vec{S}_{E(1)}^{rad}(\vec{r},t)\right\rvert\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2c}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$I_{M(1)}^{rad}(\vec{r}) \equiv \left\langle\left\lvert\vec{S}_{M(1)}^{rad}(\vec{r})\right\rvert\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\left\langle P_{E(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right)$	$\left\langle P_{M(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right)$
Time-Avg'd *EM* Linear Momentum Density	$\left\langle\vec{\wp}_{E(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$	$\left\langle\vec{\wp}_{M(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2c^5}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\left\langle\vec{\ell}_{E(1)}^{rad}(\vec{r},t)\right\rangle = 0$	$\left\langle\vec{\ell}_{M(1)}^{rad}(\vec{r},t)\right\rangle = 0$

Note for "equal" strength *EM* moments, where $\boxed{p = qd}$, $\boxed{m = \pi b^2 I}$, $\boxed{I = q\omega}$ ***and*** we let $\boxed{d = \pi b}$:

Then: $$\boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}\left(\vec{r},t\right)\right\rangle} = \left(\frac{\mu_o m^2 \omega^4}{12\pi c^3}\right) \Bigg/ \left(\frac{\mu_o p^2 \omega^4}{12\pi c}\right) = \frac{m^2}{p^2 c^2} = \left(\frac{m}{pc}\right)^2 = \left(\frac{\pi b^2 I}{qdc}\right)^2}$$

$$\therefore \quad \boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}\left(\vec{r},t\right)\right\rangle} = \left(\frac{\omega b}{c}\right)^2}$$

But in the "far-zone" limit $\boxed{d = \pi b \ll \lambda \ll r}$ we have: $\boxed{\left(\frac{\omega b}{c}\right) \ll 1}$!!!

$$\therefore \quad \boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}\left(\vec{r},t\right)\right\rangle} = \left(\frac{\omega b}{c}\right)^2 \ll 1}$$

Thus, for "equal" strength *EM* dipole moments (as defined above), the oscillating E(1) electric dipole radiates ***vastly*** more power in the form of *EM* waves than does an oscillating M(1) magnetic dipole.

$\Rightarrow$ This is ***why*** *e.g*. all commercial radio & television stations use electric dipole antennae to broadcast their signals!

Note also that the structure of the $\vec{E}$ and $\vec{B}$ fields for E(1) electric dipole *vs*. M(1) magnetic dipole radiation, in the "far-zone" limit $\boxed{d = \pi b \ll \lambda \ll r}$ are very similar, except that the $\vec{E}$ and $\vec{B}$ field vectors for the M(1) case are ***rotated*** by 90° (*i.e.* $\hat{\theta}$ and $\hat{\varphi}$ are interchanged), compared to the E(1) case:

E(1) Electric dipole: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)} \sim \hat{\theta}}$ $\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)} \sim \hat{\varphi}}$

M(1) Magnetic dipole: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)} \sim -\hat{\varphi}}$ $\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)} \sim -\hat{\theta}}$

The {scalar} *EM* wave ***characteristic*** ***radiation*** ***impedance*** of an antenna is exactly as we defined the characteristic impedance of a waveguide; noting here that we are dealing with manifestly transverse waves for *EM* wave radiation from *e.g*. either an E(1) electric dipole or M(1) magnetic dipole antenna:

$$Z_{antenna}(\vec{r}) \equiv \frac{\left|\vec{E}_{\perp}^{rad}(\vec{r})\right|}{\left|\vec{H}_{\perp}^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}_{\perp}^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}_{\perp}^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}^{rad}(\vec{r})\right|}$$

For both E(1) electric dipole and M(1) magnetic dipole radiation, we see that the *EM* wave characteristic radiation impedances of these antennae in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$) with $c = 1/\sqrt{\varepsilon_o \mu_o}$ are:

$$Z_{antenna}^{\mathrm{E}(1)}(\vec{r}) = Z_{antenna}^{\mathrm{M}(1)}(\vec{r}) = \mu_o c = \sqrt{\frac{\mu_o}{\varepsilon_o}} \equiv Z_o = 120\pi\ \Omega = 377\ \Omega$$

Where:

$\mu_o = 4\pi \times 10^{-7}$ Henrys/m = magnetic permeability of free space / vacuum

$\varepsilon_o = 8.85 \times 10^{-12}$ Farads/m = electric permittivity of free space / vacuum

And: $$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}\ \mathrm{Henrys/m}}{8.85 \times 10^{-12}\ \mathrm{Farads/m}}} = 120\pi\ \Omega = 377\ \Omega$$

= {scalar} characteristic impedance of free space/the vacuum.

Thus we see that E(1) electric dipole and M(1) magnetic dipole antennae (in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$)) are ***perfectly*** **impedance-matched** for propagation of E(1) and/or M(1) *EM* waves into free space / vacuum!

Note also that in the "far-zone" ($d = \pi b \ll \lambda \ll r$), both the E(1) and M(1) *EM* wave characteristic radiation impedances $Z_{antenna}^{\mathrm{E}(1)}(\vec{r})$, $Z_{antenna}^{\mathrm{M}(1)}(\vec{r})$ have ***no*** spatial and/or frequency dependence.

The *EM* Wave Radiation Resistance of an Antenna:

The {scalar} *EM* wave ***radiation resistance*** of an antenna R_{rad} is defined in terms of the antenna power P_{rad} and the amplitude of the current I flowing in the antenna:

$$\boxed{P_{rad}^{antenna} \equiv I^2 R_{rad}^{antenna}} \quad \text{or:} \quad \boxed{R_{rad}^{antenna} \equiv P_{rad}^{antenna}/I^2} \quad \text{(Ohms)}$$

$p = qd$ For E(1) electric dipole antenna: $I = q\omega$ = amplitude of current flowing in ***dipole***

$m = \pi b^2 I$ For M(1) magnetic dipole antenna: $I =$ amplitude of current flowing in ***loop***

In the "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\text{E(1)}} \approx \frac{\mu_o p^2\omega^4}{12\pi c^3 I^2} = \frac{\mu_o q^2 d^2\omega^4}{12\pi c\, q^2\omega^2} = \frac{\mu_o\omega^2 d^2}{12\pi c}(\Omega)}$$

$$\boxed{R_{rad}^{\text{M(1)}} \approx \frac{\mu_o m^2\omega^4}{12\pi c^3 I^2} = \frac{\mu_o\pi^2 b^4 I^2\omega^4}{12\pi c^3 I^2} = \frac{\mu_o\pi\omega^4 b^4}{12c^3}(\Omega)}$$

n.b. $R_{rad}^{\text{E(1),M(1)}}$ are both frequency-dependent

In the "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\text{E(1)}} \approx \frac{\mu_o\omega^2 d^2}{12\pi c} = \frac{\omega^2 d^2}{12\pi c^2}(\mu_o c) = \frac{\omega^2 d^2}{12\pi c^2}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{\omega^2 d^2}{12\pi c^2} Z_{rad}^{\text{E(1)}}}$$

$$\boxed{R_{rad}^{\text{M(1)}} \approx \frac{\mu_o\pi\omega^4 b^4}{12c^3} = \frac{\pi\omega^4 b^4}{12c^4}(\mu_o c) = \frac{\pi\omega^4 b^4}{12c^4}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{\pi\omega^4 b^4}{12c^4} Z_{rad}^{\text{M(1)}}}$$

But: $\boxed{Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = Z_{rad}^{\text{E(1)}} = Z_{rad}^{\text{M(1)}}}$

$\therefore$ In the "far-zone" limit, $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\text{E(1)}} \approx \frac{\omega^2 d^2}{12\pi c^2} Z_o = \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o} \quad \text{and:} \quad \boxed{R_{rad}^{\text{M(1)}} \approx \frac{\pi\omega^4 b^4}{12c^4} Z_o = \frac{1}{12\pi^3}\left(\frac{\omega\pi b}{c}\right)^4 Z_o}$$

However, in the "far-zone" limit, $d = \pi b \ll \lambda \ll r$ we have: $\boxed{\left(\frac{\omega d}{c}\right) = \left(\frac{\omega\pi b}{c}\right) \ll 1}$

Thus, we see that ***both*** the ***radiation resistances*** $R_{rad}^{\text{E(1),M(1)}}$ associated with E(1) electric dipole and M(1) magnetic dipole antennae in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$) are ***much*** less than the ***characteristic radiation impedances*** $\boxed{Z_{rad}^{\text{E(1),M(1)}} = Z_o = 120\pi\ \Omega \simeq 377\ \Omega}$ of these antennae:

$$\boxed{R_{rad}^{\text{E(1)}} \approx \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o \ll Z_o = 377\,\Omega} \quad \text{and:} \quad \boxed{R_{rad}^{\text{M(1)}} \approx \frac{1}{12\pi^3}\left(\frac{\omega\pi b}{c}\right)^4 Z_o \ll Z_o = 377\,\Omega}$$

Taking the ***ratio*** of these *EM* wave radiation resistances {in the "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$} we ***also*** see that for $d = \pi b$:

$$\boxed{\left(\frac{R_{rad}^{\text{M(1)}}}{R_{rad}^{\text{E(1)}}}\right) = \frac{1}{\pi^2}\left(\frac{\omega d}{c}\right)^2 \ll 1} \quad \text{or:} \quad \boxed{R_{rad}^{\text{M(1)}} = \frac{1}{\pi^2}\left(\frac{\omega d}{c}\right)^2 R_{rad}^{\text{E(1)}} \ll R_{rad}^{\text{E(1)}}}$$

i.e. the M(1) magnetic dipole *EM* radiation resistance is ***much*** less than the E(1) electric dipole *EM* radiation resistance, for "equal" strength moments, as defined by $\boxed{I = q\omega}$ and $\boxed{d = \pi b}$.

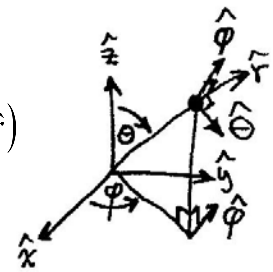
Polarization of E(1) Electric Dipole and M(1) Magnetic Dipole *EM* Radiation:

The *EM* radiation from E(1) electric dipole and/or M(1) magnetic dipole is ***linearly polarized*** in the "far-zone" limit $d = \pi b \ll \lambda \ll r$:

$$\boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}} \quad \boxed{\vec{E}_{\rm r}^{\rm M(1)}(\vec{r},t) \approx +\frac{\mu_o m\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$$

But: $\boxed{k = \omega/c}$ and note also that cos (x) = cos ($-x$) = ***even*** fcn (x).

$\therefore$ $\boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t) \sim -\cos(kr-\omega t)\hat{\theta}}$ and: $\boxed{\vec{E}_{\rm r}^{\rm M(1)}(\vec{r},t) \sim \cos(kr-\omega t)\hat{\varphi}}$



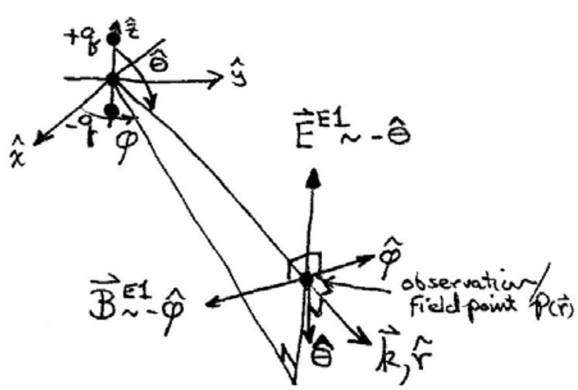
Note that: $\boxed{\cos(kr-\omega t)/r}$ is associated with ***spherical*** **outgoing** waves $\left(\vec{k} = k\hat{r}\right)$

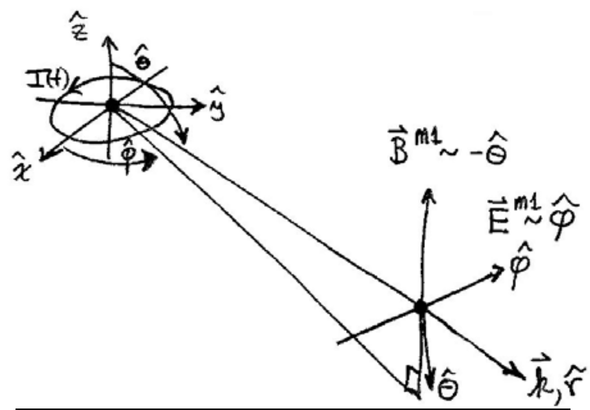
However for $r \to \infty$, ***spherical*** outgoing waves $\to$ ***plane*** outgoing waves.

If: $\hat{r}$ = propagation direction, *e.g.* $\hat{r} = \hat{z}$, then: cos ($kr - \omega t$) $\to$ cos ($kz - \omega t$).

Thus, in the "far-zone" limit, ($d = \pi b \ll \lambda \ll r$) for $r \to \infty$ (*i.e.* $r \gg \lambda$):

Polarization of E(1) Electric Dipole Radiation: $\perp$ **to** **Polarization of M(1) Magnetic Dipole Radiation:**





$$\boxed{\vec{E}_{\rm r}^{\rm E(1)} \parallel -\hat{\theta}\ \{\parallel \vec{p} = qd\hat{z}\ \text{ when }\ \theta = 90^o\}}$$
$$\boxed{\vec{S}_{\rm r}^{\rm E(1)} \sim -\hat{\theta}\times-\hat{\varphi} = +\hat{r}}$$

$$\boxed{\vec{E}_{\rm r}^{\rm M(1)} \parallel \hat{\varphi}\ \{\&\vec{B}_{\rm r}^{\rm M(1)} \parallel \vec{m} = m\hat{z}\ \text{ when }\ \theta = 90^o\}}$$
$$\boxed{\vec{S}_{\rm r}^{\rm M(1)} \sim \hat{\varphi}\times-\hat{\theta} = +\hat{r}}$$

Recall from P435 Lecture Notes 8 on the {generalized} multipole expansion of $V(\vec{r},t)$ {and $\vec{A}(\vec{r},t)$ } the order ℓ of the multipole was related to the spherical harmonic $Y_{\ell,m}(\theta,\varphi)$.

- E(1) ***linear*** electric ***dipole*** and M(1) ***linear*** magnetic ***dipole*** radiation corresponds to the $\ell = 1, m = 0$ terms in the multipole expansion.
- ***Rotating*** electric and magnetic ***dipoles*** (see *e.g.* Griffiths Problem 11.4, *p*. 450) correspond to $\ell = 1, m = \pm 1$ terms in the multipole expansion.
- Electric and magnetic ***quadrupoles*** {of various kinds} correspond to $\ell = 2, m = \pm 2, \pm 1, 0$ terms in the multipole expansion, and so on…
- The polarization of the *EM* radiation associated with each such multipole therefore depends on the ℓ & m values, and thus on the associated spherical harmonic $Y_{\ell,m}(\theta,\varphi)$, and thus can be linearly polarized (LP), or circularly polarized (RCP and/or LCP) !!!

The Time-Averaged Power Radiated by an *EM* Source:

The time-averaged *EM* radiated power associated with an oscillating electric and/or magnetic multipole of order ℓ {in the "far-zone" limit $d = \pi b \ll \lambda \ll r$ }is:

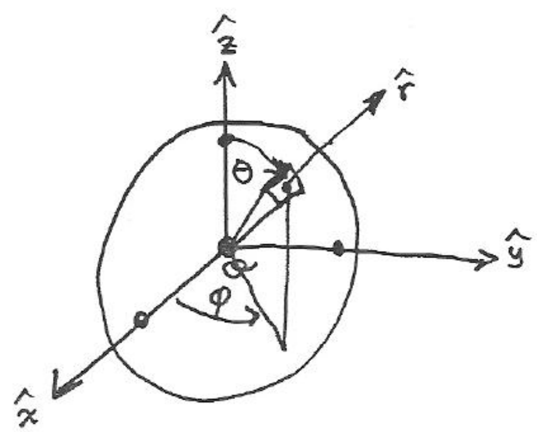
$$\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle = \int_S \left\langle \vec{S}^{rad}_{\ell-pole}(\vec{r},t)\right\rangle \cdot d\vec{a}_\perp$$

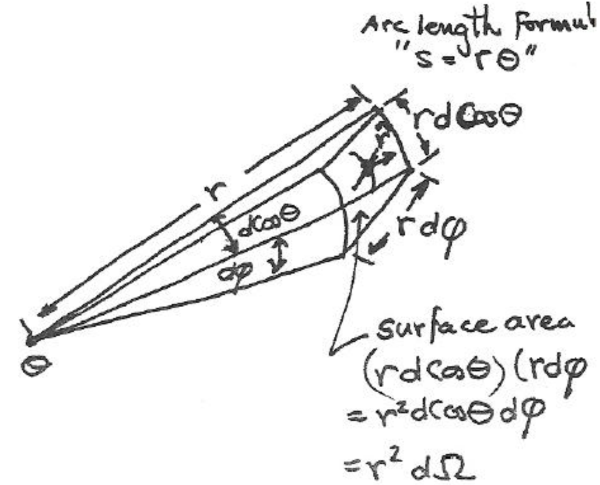
Where:

$$d\vec{a}_\perp = r^2 \sin\theta d\theta d\varphi \hat{r} = r^2 d\cos\theta d\varphi \hat{r} = r^2 d\Omega(\theta,\varphi)\hat{r}$$

$$d\Omega(\theta,\varphi) = d\cos\theta d\varphi = \sin\theta d\theta d\varphi$$ = solid angle (units = *steradians*)

$$\int d\Omega(\theta,\varphi) = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} d\cos\theta d\varphi = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \sin\theta d\theta d\varphi = 4\pi \quad (steradians)$$





Since: $$\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle = \int_\Omega \left(\frac{d\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle}{d\Omega}\right) d\Omega = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \left(\frac{d\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle}{d\Omega}\right) d\cos\theta d\varphi \quad (Watts)$$

We see that the ***angular*** ***power*** associated with an ℓ^{th} -order multipole is:

$$\frac{d\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle}{d\Omega} = I^{rad}_{\ell-pole}(\vec{r})r^2 = \left\langle\left|\vec{S}^{rad}_{\ell-pole}(\vec{r},t)\right|\right\rangle r^2 = \left\langle \vec{S}^{rad}_{\ell-pole}(\vec{r},t)\right\rangle \cdot r^2\hat{r} \quad \left(\frac{Watts}{steradian}\right)$$

Then, in the "far-zone" limit { $d = \pi b \ll \lambda \ll r$ }:

$$\frac{d\left\langle P^{rad}_{E(1)}(\vec{r},t)\right\rangle}{d\Omega} = I^{rad}_{E(1)}(\vec{r})r^2 = \left\langle\left|\vec{S}^{rad}_{E(1)}(\vec{r},t)\right|\right\rangle r^2 \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{\cancel{r^2}}\right)\cancel{r^2} = \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c}\right)\sin^2\theta \quad \left(\frac{Watts}{steradian}\right)$$

$$\frac{d\left\langle P^{rad}_{M(1)}(\vec{r},t)\right\rangle}{d\Omega} = I^{rad}_{M(1)}(\vec{r})r^2 = \left\langle\left|\vec{S}^{rad}_{M(1)}(\vec{r},t)\right|\right\rangle r^2 \approx \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{\cancel{r^2}}\right)\cancel{r^2} = \left(\frac{\mu_o m^2\omega^4}{32\pi^2 c^3}\right)\sin^2\theta \quad \left(\frac{Watts}{steradian}\right)$$

Again, in the "far-zone" limit $\{ d = \pi b \ll \lambda \ll r \}$ the ratio:

$$\boxed{\left(\frac{d\left\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\right)\Big/\left(\frac{d\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\right)=\left(\frac{m}{pc}\right)^2=\left(\frac{\pi b^2 I}{qdc}\right)^2=\left(\frac{\not{\pi} b^{\not{2}} \not{q}\omega}{\not{q}\not{\pi}\not{b}c}\right)=\left(\frac{\omega b}{c}\right)\ll 1} \text{ for } \boxed{\begin{matrix} I = q\omega \\ d = \pi b \end{matrix}}$$

Thus, we see that for the same θ and φ, the time-averaged angular power radiated by an M(1) magnetic dipole is $\ll$ than the angular power radiated by an E(1) electric dipole in the "far-zone" limit $\{ d = \pi b \ll \lambda \ll r \}$, for "equal" strength moments, as defined by $\boxed{I = q\omega}$ and $\boxed{d = \pi b}$:

$$i.e. \boxed{\left(\frac{d\left\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\right)=\left(\frac{\omega b}{c}\right)\left(\frac{d\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\right)\ll\left(\frac{d\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\right)}$$

for $\boxed{I = q\omega}$ and $\boxed{d = \pi b}$ in the "far-zone" limit $\{ d = \pi b \ll \lambda \ll r \}$.

