

LECTURE NOTES 13

ELECTROMAGNETIC RADIATION

In P436 Lect. Notes 4-10.5 (Griffiths *ch.* 9-10), we discussed the propagation of macroscopic *EM* waves, but we have not yet discussed how macroscopic *EM* waves are created. Using what we learned in P436 Lect. Notes 12, we can now discuss how macroscopic *EM* waves are created.

“Encrypted” into Maxwell’s equations:

$$\begin{array}{ll}
 1) \quad \boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = -\frac{\rho_{tot}(\vec{r}, t)}{\epsilon_0}} & 3) \quad \boxed{\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}} \\
 2) \quad \boxed{\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0} & 4) \quad \boxed{\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0 \vec{J}_{tot}(\vec{r}, t) + \mu_0 \epsilon_0 \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}}
 \end{array}$$

is the physics associated with radiation of electromagnetic waves/electromagnetic energy, arising from the acceleration {and/or deceleration} of electric charges (and/or electric currents).

In the P436 Lecture Notes #12, we derived the retarded electromagnetic fields associated with a moving point charge q from the retarded Liénard-Wiechert potentials:

$$\begin{array}{l}
 \boxed{V_r(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\kappa\lambda}} \quad \text{where:} \quad \boxed{\lambda \equiv c\Delta t_r = c(t - t_r)}, \quad \boxed{\vec{\lambda} \equiv \vec{r}(t) - \vec{r}'(t_r)} \\
 \boxed{\vec{A}_r(\vec{r}, t) = \frac{\mu_0 q}{4\pi} \frac{\vec{v}(t_r)}{\kappa\lambda}} \quad \text{and:} \quad \boxed{\kappa \equiv 1 - \hat{\lambda} \cdot \vec{v}(t_r)/c = 1 - \hat{\lambda} \cdot \vec{\beta}(t_r)} = \text{“retardation” factor} \\
 \text{With:} \quad \boxed{\vec{A}_r(\vec{r}, t) = \vec{\beta}(t_r)(V_r(\vec{r}, t)/c)} \quad \boxed{\vec{\beta}(t_r) \equiv \vec{v}(t_r)/c} \quad \text{and:} \quad \boxed{c^2 = 1/\epsilon_0\mu_0}
 \end{array}$$

We also derived the corresponding retarded electric and magnetic fields associated with a moving point electric charge q :

$$\begin{array}{l}
 \boxed{\vec{E}_r(\vec{r}, t) = -\vec{\nabla} V_r(\vec{r}, t) - \frac{\partial \vec{A}_r(\vec{r}, t)}{\partial t}} \\
 \boxed{\vec{E}_r(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\lambda}{(\vec{\lambda} \cdot \vec{u}(t_r))^3} \left[\overbrace{\left(c^2 - v^2(t_r) \right) \vec{u}(t_r)}^{\text{term for generalized Coulomb field/velocity field}} + \overbrace{\vec{\lambda} \times (\vec{u}(t_r) \times \vec{a}(t_r))}^{\text{term for radiation/acceleration field}} \right]} \\
 \boxed{\vec{B}_r(\vec{r}, t) = \vec{\nabla} \times \vec{A}_r(\vec{r}, t)} \quad \text{where:} \quad \boxed{\vec{u}(t_r) \equiv c\hat{\lambda} - \vec{v}(t_r)} \quad \text{and:} \quad \boxed{\vec{B}_r(\vec{r}, t) = \frac{1}{c} \hat{\lambda} \times \vec{E}_r(\vec{r}, t)} \\
 \boxed{\vec{B}_r(\vec{r}, t) = \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{\lambda}{(\vec{\lambda} \cdot \vec{u}(t_r))^3} \hat{\lambda} \times \left[\overbrace{\left(c^2 - v^2(t_r) \right) \vec{u}(t_r)}^{\text{term for generalized Coulomb field/velocity field}} + \overbrace{\vec{\lambda} \times (\vec{u}(t_r) \times \vec{a}(t_r))}^{\text{term for radiation/acceleration field}} \right]}
 \end{array}$$

Microscopically:

The **acceleration** {and/or **deceleration**} of electric charges q and/or time-varying electric current densities (e.g. $\vec{J} = nq\vec{v}$; $\partial\vec{J}/\partial t = nq\partial\vec{v}/\partial t \sim nq\vec{a}$) “converts” (a portion of the) **virtual** photons (associated with the “static” **Coulomb** field, which individually have zero total energy/zero-frequency) to **real** photons (which individually have finite total energy/finite frequency f), which then freely propagate outward/away from the source of time-varying electric charge and/or electric current at the speed of light, c {in vacuum / free space}.

Since real photons individually carry energy/linear momentum/angular momentum, macroscopic EM waves carry energy/linear momentum angular momentum away from the source, in an irreversible manner – these EM waves propagate away from the source \forall time. Energy/momentum **must** be input to the charged particle for this to happen – energy/momentum are {both} conserved in the radiation process.

{Note also that we can **reverse** the **arrow of time** $t \rightarrow -t$ in this process and thus learn about the **absorption** of energy/linear momentum/angular momentum by electric charges/currents from incoming/incident EM waves. . . }

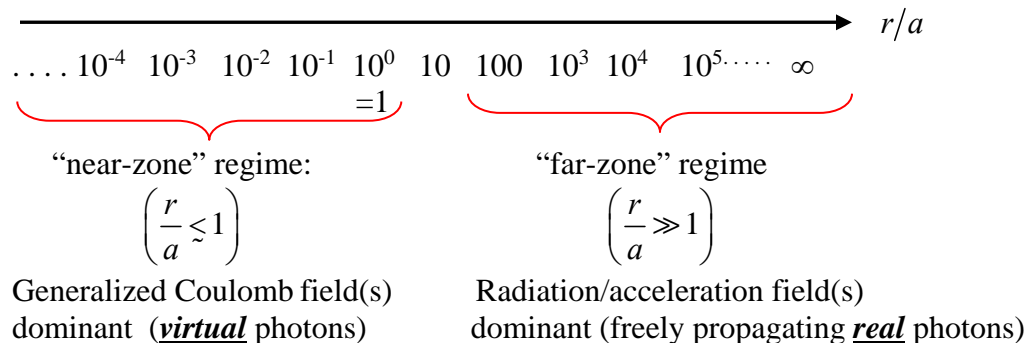
The total instantaneous power $P_r(\vec{r}, t)$ associated with **radiation** of EM waves from a source (assumed to be **localized**) is obtained by integrating the retarded Poynting’s vector $\vec{S}_r(\vec{r}, t)$ over a **large** spherical shell of radius $r \gg a =$ characteristic dimension of a localized source – this is known as the “far-field” limit, when $r \rightarrow \infty$:

$$P_r(\vec{r}, t) = \oint_{S'} \vec{S}_r(\vec{r}', t) \cdot d\vec{a}'_{\perp} = \frac{1}{\mu_0} \oint_{S'} (\vec{E}_r(\vec{r}', t) \times \vec{B}_r(\vec{r}', t)) \cdot d\vec{a}'_{\perp}$$

The instantaneous power **radiated** is the limit of $P_r(\vec{r}, t)$ as $r \rightarrow \infty$: $P_r^{rad}(t) \equiv \lim_{r \rightarrow \infty} P_r(\vec{r}, t)$

The physical reason for this definition is simple. In the so-called “near-zone”, when $r \lesssim a$, the (generalized) Coulomb field(s) (microscopically consisting of **virtual** photons) are dominant in this region – thus, time-varying but **non-radiating** \vec{E} and \vec{B} **fields** are present in proximity to the source. The near-zone EM fields fall off/decrease/diminish as $\sim 1/\lambda^2$ from the source.

In reality, for **finite** r , there is always a mixture of **radiating** and **non-radiating** EM fields present that is associated with any source. Expressed in a graphical manner in terms of r/a :



The instantaneous EM power associated with the Generalized Coulomb field is:

$$P_r^{GCF}(\vec{r}, t) = \oint_{S'} \vec{S}_r^{GCF}(\vec{r}', t) \cdot d\vec{a}'_{\perp} = \frac{1}{\mu_0} \oint_{S'} (\vec{E}_r^{GCF}(\vec{r}', t) \times \vec{B}_r^{GCF}(\vec{r}', t)) \cdot d\vec{a}'_{\perp}$$

But: $\vec{E}_r^{GCF}(\vec{r}, t) \sim 1/\lambda^2$ (even faster than this, if the **net** charge = 0, e.g. for higher order EM **moments** associated with **electric** dipoles, quadrupoles, octupoles, etc. ...)

And: $\vec{B}_r^{GCF}(\vec{r}, t) \sim 1/\lambda^2$ (even faster than this, if the **net** charge = 0, e.g. for higher order EM **moments** associated with **magnetic** dipoles, quadrupoles, octupoles, etc. ...)

$\Rightarrow \vec{S}_r^{GCF}(\vec{r}, t) \sim 1/\lambda^4$ (ever faster, for high-order EM moments than a point charge distribution)

But: $A_{\perp}^{\text{sphere}} = 4\pi\lambda^2$ = area of sphere of radius λ .

$$\therefore P_r^{GCF}(\vec{r}, t) \sim \frac{1}{\lambda^4} \cdot \lambda^2 \sim \frac{1}{\lambda^2} \Rightarrow \text{EM power associated with Generalized Coulomb fields is only appreciable **near** the source.}$$

Note that $\lim_{r \rightarrow \infty} P_r^{GCF}(\vec{r}, t) = 0$ i.e. no EM power is associated with G.C.F. at $r = \infty$

\Rightarrow “static” sources do not **radiate** EM energy.

On the other hand, the instantaneous EM power associated with the radiation/acceleration fields is:

$$P_r^{\text{rad}}(\vec{r}, t) = \oint_{S'} \vec{S}_r^{\text{rad}}(\vec{r}', t) \cdot d\vec{a}'_{\perp} = \frac{1}{\mu_0} \oint_{S'} (\vec{E}_r^{\text{rad}}(\vec{r}', t) \times \vec{B}_r^{\text{rad}}(\vec{r}', t)) \cdot d\vec{a}'_{\perp}$$

But: $\vec{E}_r^{\text{rad}} \sim 1/\lambda$ and $\vec{B}_r^{\text{rad}} \sim 1/\lambda \Rightarrow \vec{S}_r^{\text{rad}}(\vec{r}, t) \sim 1/\lambda^2, A_{\perp}^{\text{sphere}} \sim \lambda^2$

$\therefore P_r^{\text{rad}}(\vec{r}, t) \sim 1$ (i.e. $P_r^{\text{rad}}(\vec{r}, t)$ is **independent** of the radius of the enclosing surface S')

Thus, we can simply pick $r \rightarrow \infty$ to **eliminate** the $P_r^{GCF}(\vec{r}, t)$ **contribution**!!!

{n.b. for {unphysical} **non-localized** sources of time-varying EM radiation – e.g. **infinite** planes, **infinitely** long wires, **infinite** solenoids, etc. this requires a different approach altogether... }

In general, **arbitrary** configurations of **localized**, time-dependent electric charge and/or electric current density distributions, $\left[\partial\rho(t_r)/\partial t_r \equiv \dot{\rho}(t_r) \right]$ and $\left[\partial\vec{J}(t_r)/\partial t_r \equiv \dot{\vec{J}}(t_r) \right]$ can/do produce EM radiation/freely-propagating EM waves.

As we learned in P435 (last semester), from the principle of linear superposition, we can always decompose an **arbitrary** electric charge and/or current distribution into a linear combination of EM moments of the electric charge/current distribution, i.e. electric monopole (electric charge), electric and magnetic dipole, electric and magnetic quadrupole, etc. ... moments. This is true {separately} for both static and time-varying EM moments of the electric charge and/or current distribution(s).

For a **point** electric **monopole** field $\{E(0)\}$, *i.e.* $\rho(\vec{r}, t_r) = q(t_r - (t - \lambda/c)) \delta^3(\vec{r}')$

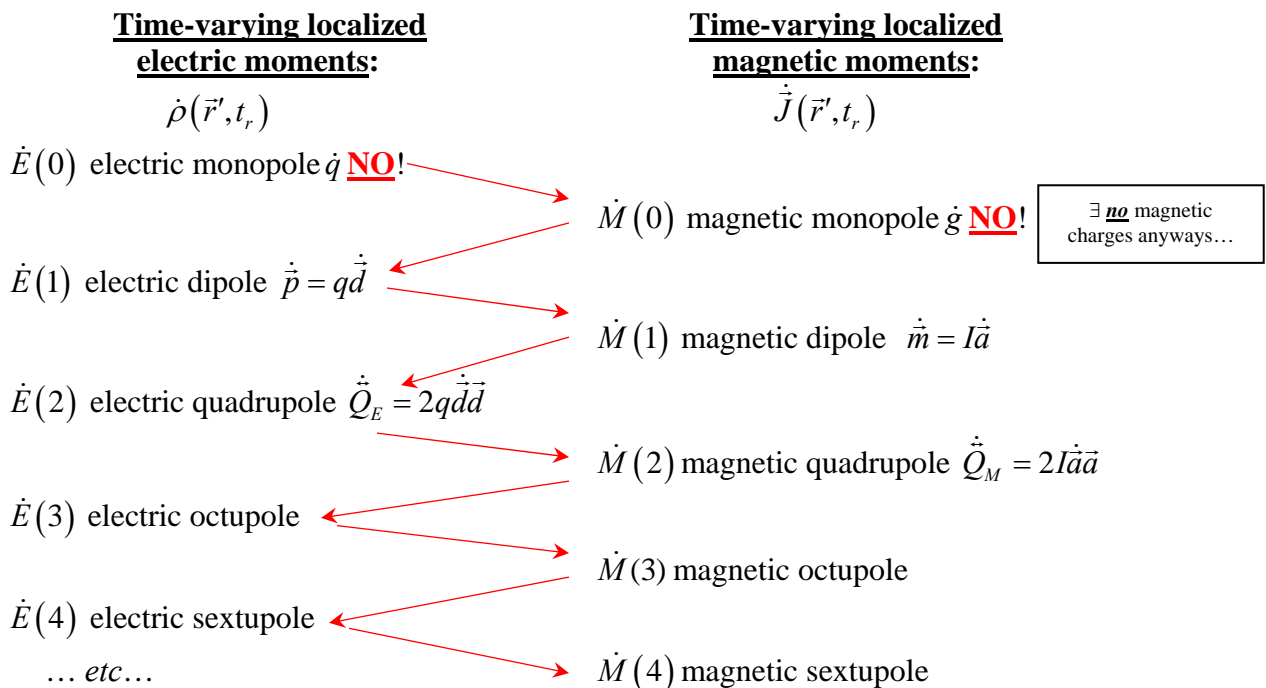
$$V_r^{(E0)}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho(\vec{r}', t_r)}{\lambda} d\tau' = \frac{q(t_r - (t - \lambda/c))}{4\pi\epsilon_0 \lambda}$$

Where $q(t_r) = \text{total}$ electric charge of the source at the **retarded** time t_r . But electric charge is (always) conserved, and furthermore, (by definition) a **localized** source is one that does **not** have electric charge q flowing into or away from it. Therefore, the electric monopole moment contribution/portion associated with the (retarded) potential(s) and *EM* fields is of necessity **static** – *i.e.* the electric monopole moment q has **no EM radiation** associated with it. In other words, there can be no **net transversely** polarized *EM* radiation emitted from a **spherically-symmetric** charge distribution! {See *e.g.* J. D. Jackson *Classical Electrodynamics* 3rd ed. p. 410 for additional/further details.}

The lowest-order **electric** multipole moment capable of producing *EM* radiation is that associated with a time-varying **electric dipole moment** $\vec{p}(\vec{r}', t_r) = q\vec{d}(\vec{r}', t_r)$ or: $= q(\vec{r}', t_r)\vec{d}$.
 ⇒ Electric dipole (E1) radiation originates from $\dot{\rho}(\vec{r}', t_r)$

The lowest-order **magnetic** multipole moment capable of producing *EM* radiation is that associated with a time-varying **magnetic dipole moment** $\vec{m}(\vec{r}', t_r) = I\vec{a}(\vec{r}', t_r)$ or: $= I(\vec{r}', t_r)\vec{a}$.
 ⇒ Magnetic dipole (M1) radiation originates from $\dot{\vec{J}}(\vec{r}', t_r)$

Each time-varying, localized, higher-order *EM* moment contributes in alternating **succession** between $\dot{\rho}(\vec{r}', t_r)$ and $\dot{\vec{J}}(\vec{r}', t_r)$ (*i.e.* electric vs. magnetic) multipole moment terms:



∴ We will consider/discuss the case of *EM* radiation from an oscillating E(1) electric dipole and then discuss case of radiation from an arbitrary localized source consisting of an arbitrary linear combination of time-varying *EM* moments, $\sum_{n=1}^{\infty} (a_n \dot{E}(n) + b_n \dot{M}(n))$, where $\dot{E}(n)$ and $\dot{M}(n)$ are n^{th} -order time-varying electric and magnetic multipole moments, respectively.

E(1) Electric Dipole Radiation:

Consider an oscillating (*i.e.* harmonic/sinusoidally time-varying) electric dipole: $\vec{p}(t) = q\vec{d}(t)$ where the charge separation distance varies in time: $\vec{d}(t) = \vec{d}(t) \hat{z} = d \cos(\omega t) \hat{z}$, $\omega = 2\pi f$

Then: $\vec{p}(t) = qd \cos(\omega t) \hat{z} = p \cos(\omega t) \hat{z}$, with: $p = qd$.

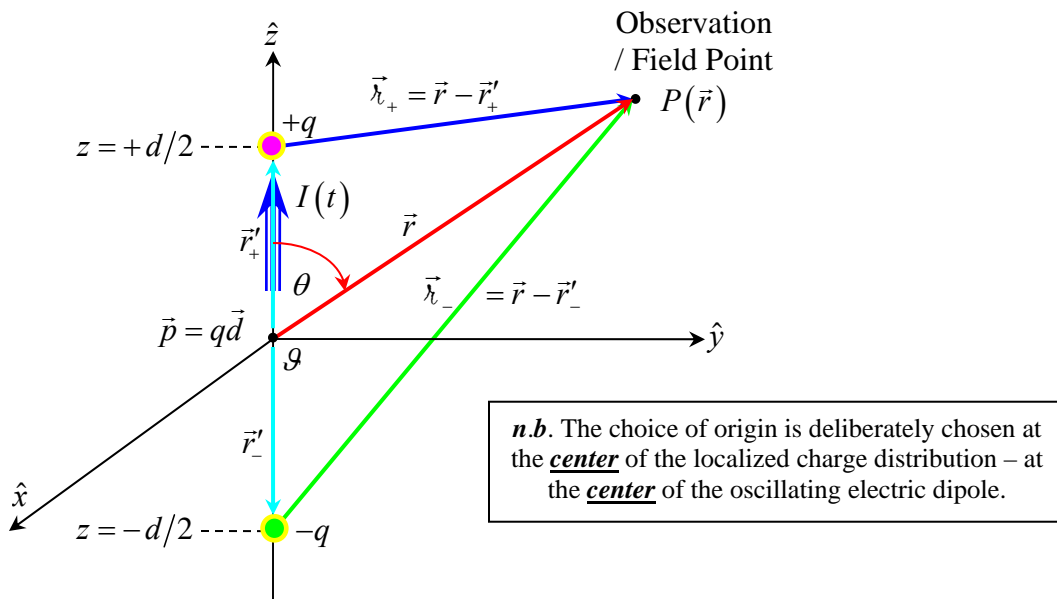
Equivalently, we can alternatively think of this as: $\vec{p}(t) = q(t) \vec{d}$, with: $\vec{d} = d\hat{z} = \text{constant}$, and with time-varying/oscillating electric charge: $q(t) = q \cos(\omega t)$.

Then: $\vec{p}(t) = qd \cos(\omega t) \hat{z} = p \cos(\omega t) \hat{z}$, with: $p = qd$. {*n.b.* same result!}

Either way one views/thinks about this, the physics associated with a harmonically time-varying/oscillating electric dipole moment $\vec{p}(t) = p \cos(\omega t) \hat{z} = qd \cos(\omega t) \hat{z}$ is unchanged.

n.b. ∃ an electric current associated with the oscillating electric dipole: $\vec{I}(t) = \frac{dq(t)}{dt} \hat{z}$, $\vec{I}(t=0) = 0$

A picture of this, for a given moment/instant/snapshot in time $t = 0$ is shown below:



n.b. ∃ exist (as always) some subtleties associated with the calculation of the retarded potentials associated with moving point charges – we will address these subsequently, but not right here / right now... so, we'll stick with the oscillating charge $q(t) = q \cos(\omega t)$ version for now...

Now $\vec{p}(t) = p \cos(\omega t) \hat{z}$ refers to the time-dependence associated with itself. An observer at field point $P(\vec{r})$ at \vec{r} “sees” the effects of the time-varying $\vec{p}(t)$ manifest themselves at a finite time later, $t = t_r + \lambda/c$ or: $t_r = t - \lambda/c$ due to the **retarded** nature of this problem.

Thus, $\vec{p}(t)$ used in the formulae for the **retarded** scalar and vector potentials **must** be evaluated at the **retarded time** t_r , i.e. $\vec{p}(t) \rightarrow \vec{p}(t_r) = q(t_r) \vec{d} = qd \cos(\omega t_r) \hat{z}$.

$$V_r^{E(1)}(\vec{r}, t) = \frac{q(t_r^+)}{4\pi\epsilon_0\lambda_+} - \frac{q(t_r^-)}{4\pi\epsilon_0\lambda_-} = \left(\frac{q}{4\pi\epsilon_0} \right) \frac{\cos(\omega t_r^+)}{\lambda_+} - \left(\frac{q}{4\pi\epsilon_0} \right) \frac{\cos(\omega t_r^-)}{\lambda_-} = \left(\frac{q}{4\pi\epsilon_0} \right) \left[\frac{\cos(\omega t_r^+)}{\lambda_+} - \frac{\cos(\omega t_r^-)}{\lambda_-} \right]$$

charge at $+d/2\hat{z}$
charge at $-d/2\hat{z}$
charge at $+d/2\hat{z}$
charge at $-d/2\hat{z}$

$$\vec{A}_r^{E(1)}(\vec{r}, t) = \left(\frac{\mu_0}{4\pi} \right) \int \frac{I(t_r)}{\lambda} d\vec{\ell}' \quad \text{where:} \quad I(t_r) = \frac{dq(t_r)}{dt} = -q\omega \sin(\omega t_r) \quad \text{and:} \quad d\vec{\ell}' = dz\hat{z}$$

Explicitly inserting the retarded time(s): $t_r^\pm = t - \lambda_\pm/c$:

$$V_r^{E(1)}(r, t) = \left(\frac{q}{4\pi\epsilon_0} \right) \left[\frac{\cos(\omega(t - \lambda_+/c))}{\lambda_+} - \frac{\cos(\omega(t - \lambda_-/c))}{\lambda_-} \right]$$

$$\vec{A}_r^{E(1)}(\vec{r}, t) = - \left(\frac{\mu_0 q \omega}{4\pi} \right) \int_{z=-d/2}^{z=+d/2} \frac{\sin[\omega(t - \lambda/c)]}{\lambda} dz \hat{z}$$

Let us first focus our attention on calculating $V_r^{E(1)}(r, t)$. From the law of cosines {see P435 Lecture Notes 8 *r.e.* the derivation of the **static** multipole moment expansion}:

$$\lambda_\pm = \sqrt{r^2 \mp rd \cos \theta + (d/2)^2}$$

However, we want to investigate *EM* radiation in the “far zone” where $r \gg d$. For **this** situation we can make the following approximation:

$$\lambda_\pm = r \sqrt{1 \mp \left(\frac{d}{r} \right) \cos \theta + \frac{1}{4} \left(\frac{d}{r} \right)^2} \approx r \sqrt{1 \mp \left(\frac{d}{r} \right) \cos \theta} \quad \text{But: } \sqrt{1 \mp \epsilon} \approx 1 \mp \frac{1}{2} \epsilon \text{ for } \epsilon \ll 1.$$

Thus: $\lambda_\pm \approx r \left(1 \mp \frac{1}{2} \left(\frac{d}{r} \right) \cos \theta \right) = r \left(1 \mp \left(\frac{d}{2r} \right) \cos \theta \right)$ for $r \gg d$.

Similarly/correspondingly:

$$\frac{1}{\lambda_\pm} \approx \frac{1}{r \left(1 \mp \frac{1}{2} \left(\frac{d}{r} \right) \cos \theta \right)} \approx \frac{1}{r} \left(1 \pm \left(\frac{d}{2r} \right) \cos \theta \right) \text{ for } r \gg d, \text{ since: } \frac{1}{1 \mp \epsilon} \approx 1 \pm \epsilon \text{ for } \epsilon \ll 1.$$

Likewise, for the $\cos(\omega(t - \lambda_{\pm}/c))$ term, for the “far zone”, when $r \gg d$ we have:

$$\begin{aligned} \cos(\omega(t - \lambda_{\pm}/c)) &\approx \cos \left[\omega \left(t - \frac{r}{c} \left(1 \mp \frac{d}{2r} \cos \theta \right) \right) \right] = \cos \left[\omega \left(t - \frac{r}{c} \right) \pm \left(\frac{\omega d}{2c} \right) \cos \theta \right] \\ &= \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \cdot \cos \left[\left(\frac{\omega d}{2c} \right) \cos \theta \right] \mp \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \cdot \sin \left[\left(\frac{\omega d}{2c} \right) \cos \theta \right] \end{aligned}$$

In order to proceed **further**, we need to make an **additional** simplifying assumption, namely that the **characteristic spatial dimension** a of the **source (here, $a = d$)** is \ll **wavelength** λ of the **emitted radiation**, i.e. $d \ll \lambda$ { $\lambda = c/f$ }. Thus we have: $d \ll c/f$ { $f = \omega/2\pi$ }, or: $d \ll 2\pi c/\omega$ or: $d \ll c/\omega$.

n.b. This assumption is tantamount/physically equivalent to saying that we will neglect any/all **time-retardation** effects associated with finite *EM* propagation delay times over the **dimensions characteristic of/associated** with the **source** – i.e. changes in charge/current are essentially **coherent/instantaneous** over the {small} spatial dimensions of the source, relative to the wavelength of the emitted radiation.

Suppose we have a source (e.g. an **atom**) with $a = d = 1 \text{ nm} = 10 \text{ \AA}$ emitting a $f = 1 \text{ Hz}$ sine-wave. Since *EM* radiation travels propagates at $1 \text{ ft} \approx 30 \text{ cm}$ per nanosecond, a 1 nm dimension source doesn't run into finite propagation decay time **problems** until:

$$c\Delta t \approx a = d \text{ (here) } \text{ i.e. } c\Delta t \approx 1 \text{ nm} \Rightarrow \Delta t \approx \frac{10^{-9}}{3 \times 10^8} \approx 0.3 \times 10^{-17} \text{ sec} \Rightarrow \underline{\underline{f \approx 3 \times 10^{17} \text{ Hz}}}$$

Thus, provided that we **additionally are** in the regime of $d \ll \lambda$, or $d \ll c/\omega$, i.e. $\left(\frac{\omega d}{c} \right) \ll 1$.

Noting that if: $\left(\frac{\omega d}{c} \right) \ll 1$, then: $\left| \left(\frac{\omega d}{2c} \right) \cos \theta \right| \ll 1 \quad \forall \theta \{ 0 \leq \theta \leq \pi \}$.

Then from the Taylor series expansions of $\cos(x) \approx 1$ and $\sin(x) \approx x$ for **very** small $x \ll 1$, we see that:

$$\cos \left[\left(\frac{\omega d}{2c} \right) \cos \theta \right] \approx \cos(0) \approx 1 \quad \text{and:} \quad \sin \left[\left(\frac{\omega d}{2c} \right) \cos \theta \right] \approx \left(\frac{\omega d}{2c} \right) \cos \theta$$

Thus:
$$\cos(\omega(t - \lambda_{\pm}/c)) \approx \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \mp \left(\frac{\omega d}{2c} \right) \cos \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right]$$

Thus:

$$\begin{aligned} V_r^{E(1)}(r, \theta, t) &\approx \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{r} \left(1 + \left(\frac{d}{2r} \right) \cos \theta \right) \left[\cos \left[\omega \left(t - \frac{r}{c} \right) \right] - \left(\frac{\omega d}{2c} \right) \cos \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right] \right. \\ &\quad \left. - \frac{1}{r} \left(1 - \left(\frac{d}{2r} \right) \cos \theta \right) \left[\cos \left[\omega \left(t - \frac{r}{c} \right) \right] + \left(\frac{\omega d}{2c} \right) \cos \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right] \right\} \end{aligned}$$

Expanding this out:

$$\begin{aligned}
 V_r^{E(1)}(r, \theta, t) \approx \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r}\right) & \left\{ \cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right) \cos\theta \cos\left[\omega\left(t-\frac{r}{c}\right)\right] \right. \\
 & - \left(\frac{\omega d}{2c}\right) \cos\theta \sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \cancel{\frac{\omega d^2}{4rc} \cos^2\theta \sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \\
 & \left. - \cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right) \cos\theta \cos\left[\omega\left(t-\frac{r}{c}\right)\right] \right. \\
 & \left. - \left(\frac{\omega d}{2c}\right) \cos\theta \sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \cancel{\frac{\omega d^2}{4rc} \cos^2\theta \sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \right\}
 \end{aligned}$$

Thus:

$$\begin{aligned}
 V_r^{E(1)}(r, \theta, t) & \approx \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r}\right) \left\{ \left(\frac{d}{r}\right) \cos\theta \cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \left(\frac{\omega d}{c}\right) \cos\theta \sin\left[\omega\left(t-\frac{r}{c}\right)\right] \right\} \\
 & = \frac{qd}{4\pi\epsilon_0} \left(\frac{\cos\theta}{r}\right) \left\{ \left(\frac{1}{r}\right) \cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \left(\frac{\omega}{c}\right) \sin\left[\omega\left(t-\frac{r}{c}\right)\right] \right\}
 \end{aligned}$$

But: $p = |\vec{p}| = qd$. Hence in the “far-zone” $d \ll r$ **and** $d \ll \lambda$:

$$V_r^{E(1)}(r, \theta, t) \approx \frac{p}{4\pi\epsilon_0} \left(\frac{\cos\theta}{r}\right) \left\{ -\left(\frac{\omega}{c}\right) \sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{1}{r}\right) \cos\left[\omega\left(t-\frac{r}{c}\right)\right] \right\}$$

In the “far-zone” $d \ll r$, with the **additional** restriction that we’ve also imposed on the source EM radiation: $d \ll \lambda$. We now **additionally** require/impose a **third** restriction that the “far-zone” **also** be such that $\lambda \ll r$, thus we have the hierarchical relation: $d \ll \lambda \ll r$ for “far-zone” EM radiation, namely that for $\lambda \ll r \Rightarrow \left(\frac{c}{\omega}\right) \ll r$, then $\Rightarrow \left(\frac{\omega}{c}\right) \gg \left(\frac{1}{r}\right)$ i.e. $\frac{1}{\lambda} \gg \frac{1}{r}$ for $\lambda \ll r$.

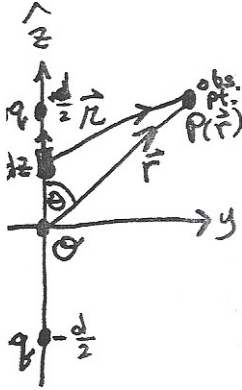
Thus for the far-zone, when $d \ll \lambda \ll r$ we can **neglect** the second term in the above expression for $V_r^{E(1)}(r, \theta, t)$.

Then: $V_r^{E(1)}(r, \theta, t) \approx -\frac{p\omega}{4\pi\epsilon_0 c} \left(\frac{\cos\theta}{r}\right) \sin\left[\omega\left(t-\frac{r}{c}\right)\right]$ in the far-zone, for $d \ll \lambda \ll r$.

Note that in the **static limit**, when $\omega \rightarrow 0$ it **is** necessary to **retain** the second term in the above expression; we obtain in this limit: $V_r^{E(1)}(r, \theta) \approx -\frac{p}{4\pi\epsilon_0} \left(\frac{\cos\theta}{r^2}\right)$ {cf w/ P435 Lect. Notes – same!}

Now let us focus our attention on calculating $\vec{A}_r^{E(1)}(\vec{r}, t)$:

$$\vec{A}_r^{E(1)}(\vec{r}, t) = -\left(\frac{\mu_0 q \omega}{4\pi}\right) \int_{z=-d/2}^{z=+d/2} \frac{\sin[\omega(t-\lambda/c)]}{\lambda} dz \hat{z}$$



Because the integration itself introduces a factor of d , then to **first order**

in $(d/r) \ll 1$: $\lambda = \sqrt{r^2 - 2rz \cos \theta + z^2} \approx r$ with: $|z| \leq \frac{d}{2}$

Thus: $\int_{z=-d/2}^{z=+d/2} \frac{\sin[\omega(t-\lambda/c)]}{\lambda} dz \approx \frac{\sin[\omega(t-r/c)]}{r} d$

Then: $\vec{A}_r^{E(1)}(r, t) \approx -\frac{\mu_0 (qd) \omega}{4\pi} \left(\frac{1}{r} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right) \hat{z}$ but: $p = qd$

Thus: $\vec{A}_r^{E(1)}(r, t) \approx -\frac{\mu_0 p \omega}{4\pi} \left(\frac{1}{r} \right) \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{z}$

Note that in the **static limit**, when $\omega \rightarrow 0$ then $\vec{A}_r^{E(1)}(r, t) \rightarrow 0$ as we expect.

Now that we have obtained the (retarded) scalar and vector potentials $V_r^{E(1)}(r, t)$ and $\vec{A}_r^{E(1)}(r, t)$ it is a “straight forward” exercise to compute the associated (retarded) EM fields, $\vec{E}_r^{E(1)}(r, t)$ and $\vec{B}_r^{E(1)}(r, t)$:

$$\vec{E}_r^{E(1)}(\vec{r}, t) = -\vec{\nabla} V_r^{E(1)}(\vec{r}, t) - \frac{\partial \vec{A}_r^{E(1)}(\vec{r}, t)}{\partial t} \quad \text{and:} \quad \vec{B}_r^{E(1)}(\vec{r}, t) = \vec{\nabla} \times \vec{A}_r^{E(1)}(\vec{r}, t)$$

In **spherical** coordinates:

$$\begin{aligned} \vec{\nabla} V_r^{E(1)}(\vec{r}, t) &\approx \left[\frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \hat{\phi} \right] \left\{ \frac{-p\omega \cos \theta}{4\pi \epsilon_0 c r} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right\} \\ &= -\frac{p\omega}{4\pi \epsilon_0 c} \left\{ \cos \theta \left[-\frac{1}{r^2} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] - \frac{\omega}{rc} \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \right] \hat{r} - \frac{1}{r^2} \sin \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\theta} \right\} \\ &= +\frac{p\omega}{4\pi \epsilon_0 cr} \left\{ \left(\frac{\omega}{c} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \cos \theta \hat{r} + \left(\frac{1}{r} \right) \left[\cos \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{r} + \sin \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\theta} \right] \right\} \end{aligned}$$

But for “far-zone” EM radiation, $d \ll \lambda \ll r$ we have: $\frac{\omega}{c} \gg \frac{1}{r}$

$$\therefore \underbrace{\left(\frac{\omega}{c} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \cos \theta}_{\sim \mathcal{O}(1)} \gg \underbrace{\left(\frac{1}{r} \right) \left\{ \cos \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] + \sin \theta \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \right\}}_{\sim \mathcal{O}(1)}$$

So we can neglect/drop the $\left(\frac{1}{r}\right)\left\{\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\hat{r}+\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\hat{\theta}\right]\right\}$ terms.

$$\therefore \vec{\nabla}V_r^{E(1)}(\vec{r},t) \approx +\frac{p\omega^2\cos\theta}{4\pi\epsilon_0c^2r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$

$$\text{And: } \frac{\partial\vec{A}_r^{E(1)}(\vec{r},t)}{\partial t} \approx -\frac{\mu_0p\omega}{4\pi}\left(\frac{1}{r}\right)\frac{\partial}{\partial t}\left(\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{z} = -\frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$$

But: $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$ in spherical coordinates.

$$\therefore \frac{\partial\vec{A}_r^{E(1)}(\vec{r},t)}{\partial t} \approx -\frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$

Then for far-zone EM radiation, with $d \ll \lambda \ll r$: $\vec{E}_r^{E(1)}(\vec{r},t) = -\vec{\nabla}V_r^{E(1)}(\vec{r},t) - \frac{\partial\vec{A}_r^{E(1)}(\vec{r},t)}{\partial t}$

$$\vec{E}_r^{E(1)}(\vec{r},t) \approx -\frac{p\omega^2}{4\pi\epsilon_0c^2r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r} + \frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$

$$\text{But: } c^2 = \frac{1}{\epsilon_0\mu_0} \text{ or: } \frac{1}{c^2} = \epsilon_0\mu_0$$

$$\therefore \vec{E}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r} + \frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r} - \frac{\mu_0p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta\hat{\theta}$$

$$\text{Or: } \vec{E}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_0p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

$$\text{Then: } \vec{B}_r^{E(1)}(\vec{r},t) = \vec{\nabla} \times \vec{A}_r^{E(1)}(\vec{r},t)$$

$$\text{with: } \vec{A}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_0p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z} = -\frac{\mu_0p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$

Thus:

$$\vec{B}_r^{E(1)}(\vec{r},t) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\overset{=0}{A_r}\right) - \frac{\overset{=0}{\partial A_r}}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\overset{=0}{\partial A_r}}{\partial\phi} - \frac{\partial}{\partial r}\left(r\overset{=0}{A_r}\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

Thus:

$$\begin{aligned}
 \vec{B}_r^{E(1)}(\vec{r}, t) &= \frac{1}{r} \left[\frac{\partial}{\partial r} (rA_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} \\
 &\approx \left(\frac{1}{r} \right) \left[-\frac{\mu_0 p \omega}{4\pi} \right] \left\{ \frac{\partial}{\partial r} \left(\frac{1}{f} \cdot f \sin \left[\omega \left(t - \frac{r}{c} \right) \right] (-\sin \theta) \right) - \frac{1}{r} \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \frac{\partial \cos \theta}{\partial \theta} \right\} \hat{\phi} \\
 &= -\frac{\mu_0 p \omega}{4\pi r} \left\{ + \left(\frac{\omega}{c} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \sin \theta + \left(\frac{1}{r} \right) \sin \left[\omega \left(t - \frac{r}{c} \right) \right] \sin \theta \right\} \hat{\phi} \\
 &= -\frac{\mu_0 p \omega}{4\pi r} \left\{ \left(\frac{\omega}{c} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] + \underbrace{\left(\frac{1}{r} \right) \sin \left[\omega \left(t - \frac{r}{c} \right) \right]}_{\therefore \text{neglect}} \right\} \sin \theta \hat{\phi}
 \end{aligned}$$

Again, $\left(\frac{\omega}{c} \right) \gg \left(\frac{1}{r} \right)$ here, because $\lambda \ll r$, thus

$$\vec{B}_r^{E(1)}(\vec{r}, t) \approx -\frac{\mu_0 p \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\phi} \quad \text{and:} \quad \vec{E}_r^{E(1)}(\vec{r}, t) \approx -\frac{\mu_0 p \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\theta}$$

Now since $\hat{r} \times \hat{\theta} = \hat{\phi}$, once again we see that: $\vec{B}_r^{E(1)}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}_r^{E(1)}(\vec{r}, t)$, i.e. $\vec{B} \perp \vec{E}$ and $\vec{B} \perp \hat{r}$

Note also that:

- $\vec{E}_r^{E(1)}$ and $\vec{B}_r^{E(1)}$ both vary as $\sim 1/r$.
- $\vec{E}_r^{E(1)}(\vec{r}, t)$ and $\vec{B}_r^{E(1)}(\vec{r}, t)$ are **in-phase** with each other.
- $\vec{E}_r^{E(1)}(\vec{r}, t)$ and $\vec{B}_r^{E(1)}(\vec{r}, t)$ have the **same** angular dependence ($\sim \sin \theta$).

The **EM radiation** energy density, $u_{E(1)}^{rad}(\vec{r}, t)$ associated with the oscillating E(1) electric dipole for far-zone EM radiation $\{ d \ll \lambda \ll r \}$ is:

$$\begin{aligned}
 u_{E(1)}^{rad}(\vec{r}, t) &= u_{E(1)}^{Erad}(\vec{r}, t) + u_{E(1)}^{Mrad}(\vec{r}, t) = \frac{1}{2} \left(\epsilon_0 \vec{E}_r^{E(1)}(\vec{r}, t) \cdot \vec{E}_r^{E(1)}(\vec{r}, t) + \frac{1}{\mu_0} \vec{B}_r^{E(1)}(\vec{r}, t) \cdot \vec{B}_r^{E(1)}(\vec{r}, t) \right) \\
 &\approx \frac{1}{2} \left(\frac{\epsilon_0 \mu_0^2 p^2 \omega^4}{16\pi^2} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] + \frac{\mu_0^2 p^2 \omega^4}{16\pi^2 c^2} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \right) \left(\frac{\text{Joules}}{m^3} \right) \\
 &= \frac{1}{2} \left(\frac{\mu_0 p^2 \omega^4}{16\pi^2 c^2} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] + \frac{\mu_0 p^2 \omega^4}{16\pi^2 c^2} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \right)
 \end{aligned}$$

n.b. $u_{E(1)}^{Erad}(\vec{r}, t) = u_{E(1)}^{Mrad}(\vec{r}, t)$ using: $c^2 = 1/\epsilon_0 \mu_0$ or: $\epsilon_0 = 1/\mu_0 c^2$.

$\therefore u_{E(1)}^{rad}(\vec{r}, t) \approx \frac{\mu_0 p^2 \omega^4}{16\pi^2 c^2} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \left(\frac{\text{Joules}}{m^3} \right)$ for: $d \ll \lambda \ll r$ "far zone" limit

The *EM* energy radiated by an oscillating electric dipole, in the “far zone” $\{d \ll \lambda \ll r\}$ limit is given by Poynting’s vector:

$$\vec{S}_{E(1)}^{rad}(\vec{r}, t) = \frac{1}{\mu_o} \left(\vec{E}_r^{E(1)}(\vec{r}, t) \times \vec{B}_r^{E(1)}(\vec{r}, t) \right)$$

$\hat{r} \times \hat{\phi} = \hat{\theta}$

$\hat{\theta} \times \hat{\phi} = \hat{r}$

$\hat{\phi} \times \hat{r} = \hat{\theta}$

$$\vec{S}_{E(1)}^{rad}(\vec{r}, t) \approx \left(\frac{1}{\mu_o} \right) \left(\frac{\cancel{\mu_o} p \omega^2}{4\pi r} \right) \left(\frac{\mu_o p \omega^2}{4\pi r c} \right) \sin^2 \theta \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \underbrace{\left[\hat{\theta} \times \hat{\phi} \right]}_{=\hat{r}}$$

Or:
$$\vec{S}_{E(1)}^{rad}(\vec{r}, t) \approx \frac{\mu_o p^2 \omega^4}{16\pi^2 c} \left(\frac{\sin^2 \theta}{r^2} \right) \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{r} \quad \left(\frac{Watts}{m^2} \right)$$

⇒ Radial outward flow of *EM* field energy for: $d \ll \lambda \ll r$ “far zone” limit

The *EM* radiation **linear** momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\vec{\phi}_{E(1)}^{rad}(\vec{r}, t) = \mu_o \epsilon_o \vec{S}_{E(1)}^{rad}(\vec{r}, t) = \frac{1}{c^2} \vec{S}_{E(1)}^{rad}(\vec{r}, t)$$

Or:
$$\vec{\phi}_{E(1)}^{rad}(\vec{r}, t) \approx \frac{\mu_o p^2 \omega^4}{16\pi^2 r^2 c^3} \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \sin^2 \theta \hat{r} \quad \left(\frac{kg}{m^2 \cdot sec} \right)$$

⇒ Radial outward *EM* field linear momentum for: $d \ll \lambda \ll r$ “far zone” limit

The *EM* radiation **angular** momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\vec{\ell}_{E(1)}^{rad}(\vec{r}, t) = \vec{r} \times \vec{\phi}_{E(1)}^{rad}(\vec{r}, t)$$

$$\vec{\ell}_{E(1)}^{rad}(\vec{r}, t) \approx \frac{\mu_o p^2 \omega^4}{16\pi^2 r^2 c^3} \cos^2 \left[\omega \left(t - \frac{r}{c} \right) \right] \sin^2 \theta (\hat{r} \times \hat{r}) \equiv 0 \quad \left(\frac{kg}{m \cdot sec} \right)$$

⇒ **No** *EM* field angular momentum for: $d \ll \lambda \ll r$ “far zone” limit

n.b. The **exact** $\vec{\ell}_{E(1)}^{rad}(\vec{r}, t) \neq 0$ *i.e.* ignore **restrictions** on far-zone limit, keep **all** higher-order terms . . . we have neglected $\vec{E}_r^{E(1)} \sim \hat{r}$ term which is non-negligible in the **near-zone** ($d \sim r$) and also in the so-called **intermediate**, or **inductive zone** ($\lambda \sim r$).

Time-Averaged Quantities for E(1) Radiation from an Oscillating Electric Dipole:

Recall the definition of time average: $\langle A(t) \rangle \equiv \frac{1}{\tau} \int_{t=0}^{t=\tau} A(t) dt = \frac{1}{\tau} \int_{t=0}^{t=\tau} A_o \cos^2 \omega t dt = \frac{1}{2} A_o$

The time-averaged EM radiation energy density associated with an oscillating electric dipole is:

$$\langle u_{E(1)}^{rad}(\vec{r}, t) \rangle \approx \left(\frac{\mu_o p^2 \omega^4}{32\pi^2 c^2} \right) \left(\frac{\sin^2 \theta}{r^2} \right) \left(\frac{\text{Joules}}{m^3} \right) \text{ for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit}$$

The time-averaged |Poynting's vector|, which is also the **intensity** $I_{E(1)}^{rad}$ of EM radiation associated with an oscillating electric dipole is:

$$I_{E(1)}^{rad}(\vec{r}) \equiv \langle |\vec{S}_{E(1)}^{rad}(\vec{r}, t)| \rangle = \frac{1}{2} c \epsilon_o \langle (E_r^{E(1)}(\vec{r}, t))^2 \rangle \approx \left(\frac{\mu_o p^2 \omega^4}{32\pi^2 c} \right) \left(\frac{\sin^2 \theta}{r^2} \right) \left(\frac{\text{Watts}}{m^2} \right) \text{ for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit}$$

We also see that: $I_{E(1)}^{rad}(\vec{r}) \equiv \langle |\vec{S}_{E(1)}^{rad}(\vec{r}, t)| \rangle = c \langle u_{E(1)}^{rad}(\vec{r}, t) \rangle \left(\frac{\text{Watts}}{m^2} \right)$.

The time-averaged EM radiated power associated with an oscillating electric dipole is:

$$\begin{aligned} \langle P_{E(1)}^{rad}(\vec{r}, t) \rangle &= \int_S \langle \vec{S}_{E(1)}^{rad}(\vec{r}, t) \rangle \cdot d\vec{a}_\perp \approx \frac{\mu_o p^2 \omega^4}{32\pi^2 c} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \cancel{r^2} \sin^2 \theta \underbrace{\sin \theta d\theta d\phi}_{=d \cos \theta} \\ &\approx \frac{\mu_o p^2 \omega^4}{16\cancel{2}\pi^2 c^2} 2\pi \int_{\theta=0}^{\theta=\pi} \sin^2 \theta d \cos \theta = \frac{\mu_o p^2 \omega^4}{16\pi c^2} \int_{\theta=0}^{\theta=\pi} \sin^2 \theta d \cos \theta \end{aligned}$$

Let: $\boxed{u = \cos \theta}$, $\boxed{du = -d(\cos \theta)}$, $\boxed{\theta = 0 \Rightarrow u = 1}$, $\boxed{\theta = \pi \Rightarrow u = -1}$, $\boxed{\sin^2 \theta = 1 - \cos^2 \theta = 1 - u^2}$

$$\therefore \int_{-1}^{+1} (1 - u^2) du = \left(u - \frac{1}{3} u^3 \right) \Big|_{-1}^{+1} = +1 - \frac{1}{3} + 1 - \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$$

\therefore The time-averaged radiated power associated with an oscillating electric dipole is:

$$\langle P_{E(1)}^{rad}(\vec{r}, t) \rangle \approx \left(\frac{\mu_o p^2 \omega^4}{12\pi c} \right) (\text{Watts}) \text{ for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit} \quad \boxed{\text{n.b. } \langle P_{E(1)}^{rad}(\vec{r}, t) \rangle \text{ has } \underline{\text{no}} \text{ } r\text{-dependence!}}$$

Note that time-averaged radiated power varies as the 4th power of frequency!

The time-averaged EM radiation **linear** momentum density associated with an oscillating electric dipole is:

$$\langle \vec{g}_{E(1)}^{rad}(\vec{r}, t) \rangle = \frac{1}{c^2} \langle \vec{S}_{E(1)}^{rad}(\vec{r}, t) \rangle = \frac{1}{c} \langle u_{E(1)}^{rad}(\vec{r}, t) \rangle \hat{r} \approx \left(\frac{\mu_o p^2 \omega^4}{32\pi^2 c^3} \right) \left(\frac{\sin^2 \theta}{r^2} \right) \hat{r} \left(\frac{\text{kg}}{m^2 \text{-sec}} \right) \text{ for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit}$$

The time-averaged *EM* radiation **angular** momentum density associated with an oscillating electric dipole is:

$$\left\langle \vec{\ell}_{E(1)}^{rad}(\vec{r}, t) \right\rangle = \vec{r} \times \left\langle \vec{\rho}_{E(1)}^{rad}(\vec{r}, t) \right\rangle \approx \left(\frac{\mu_0 p^2 \omega^4}{32\pi^2 c^3} \right) \left(\frac{\sin^2 \theta}{r} \right) (\hat{r} \times \hat{r}) \equiv 0 \quad \left(\frac{kg}{m\text{-sec}} \right) \quad \text{for:} \quad \begin{matrix} d \ll \lambda \ll r \\ \text{"far-zone" limit} \end{matrix}$$

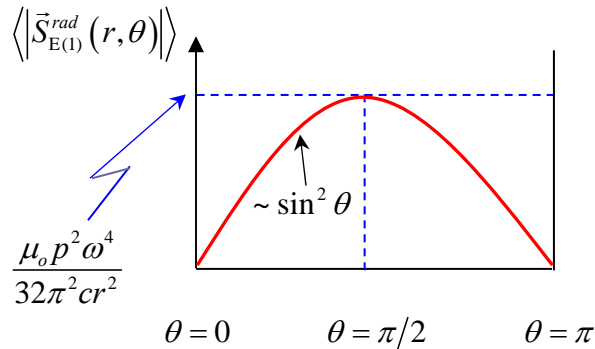
n.b. The **exact** $\left\langle \vec{\ell}_{E(1)}^{rad}(\vec{r}) \right\rangle \neq 0$ *i.e.* ignore **restrictions** on far-zone limit, keep **all** higher-order terms . . . we have neglected the $\vec{E}_r^{E(1)} \sim \hat{r}$ term which is non-negligible in the **near-zone** ($d \sim r$) and also in the so-called **intermediate**, or **inductive zone** ($\lambda \sim r$).

Note that because: $I_{E(1)}^{rad}(\vec{r}) \equiv \left\langle \vec{S}_{E(1)}^{rad}(\vec{r}, t) \right\rangle \approx \left(\frac{\mu_0 p^2 \omega^4}{32\pi^2 c} \right) \left(\frac{\sin^2 \theta}{r^2} \right) \left(\frac{Watts}{m^2} \right)$

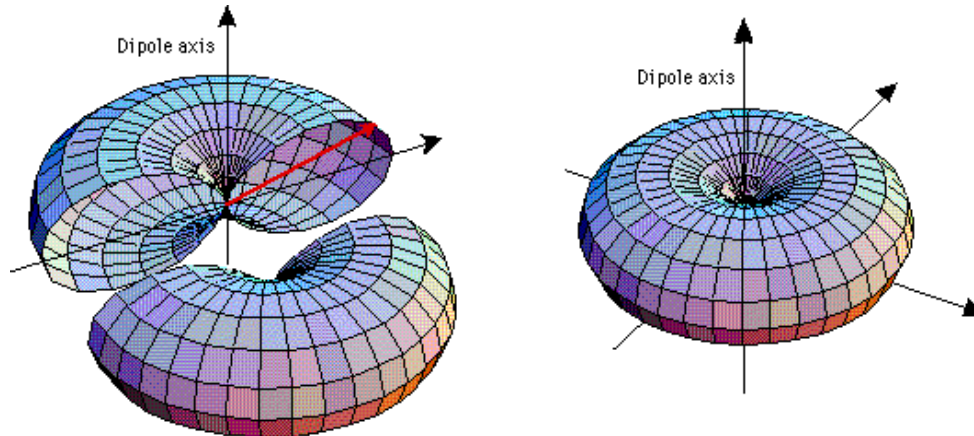
$\Rightarrow \left\langle \vec{S}_{E(1)}^{rad}(r, \theta = 0, \varphi) \right\rangle = \left\langle \vec{S}_{E(1)}^{rad}(r, \theta = \pi, \varphi) \right\rangle = 0$ since: $\sin^2 0 = \sin^2 \pi = 0$

i.e. **no** *EM* radiation occurs along the **axis** of the electric dipole (\hat{z} axis)

EM radiation for E(1) electric dipole is peaked/maximum at $\theta = \pi/2$ (then $\sin^2 \theta = 1$) *i.e.* maximum *EM* radiation occurs \perp to the axis of the electric dipole:



Thus, the intensity profile $I_{E(1)}^{rad}(\vec{r})$ in 3-D {for fixed r } for E(1) electric dipole radiation is **donut-shaped** - rotationally invariant in φ , as shown in the figure below:



Griffiths Example 11.1:

The time-averaged power for E(1) electric dipole radiation is $\langle P_{E(1)}^{rad} \rangle \approx \frac{\mu_0 p^2 \omega^4}{12\pi c}$.

Note that $\langle P_{E(1)}^{rad} \rangle \sim \omega^4$ (or $\sim f^4$, or $\sim \lambda^{-4}$)

For red light: $\lambda_{red} \approx 780 \text{ nm} \Rightarrow f_{red} = \frac{c}{\lambda_{red}} = \frac{3 \times 10^8}{780 \times 10^{-9}} \approx 3.85 \times 10^{14} \text{ Hz}$

For violet light: $\lambda_{violet} \approx 350 \text{ nm} \Rightarrow f_{violet} = \frac{c}{\lambda_{violet}} = \frac{3 \times 10^8}{350 \times 10^{-9}} \approx 8.57 \times 10^{14} \text{ Hz}$

Hence: $\left(\frac{\langle P_{E(1)}^{violet} \rangle}{\langle P_{E(1)}^{red} \rangle} \right) = \left(\frac{f_{violet}}{f_{red}} \right)^4 = \left(\frac{8.57 \times 10^{14}}{3.85 \times 10^{14}} \right)^4 = (2.23)^4 \approx 24.67$.

$\Rightarrow \langle P_{E(1)}^{rad} \rangle \sim \omega^4$ explains why the sky is blue! Sunlight {unpolarized light} incident on O₂ & N₂ molecules in the earth's atmosphere stimulates the O & N atoms – vibrates the {bound} atomic electrons at {angular} frequency ω , causing them to oscillate as electric dipoles! Solar EM radiation at a given angular frequency ω is thus absorbed and re-emitted in this EM radiation + atom **scattering** process.

The above formula for EM power radiated as E(1) electric dipole radiation by such atoms, by time-reversal invariance of the EM interaction, is also the EM power absorbed by atoms, thus we see that because of the ω^4 -dependence of $\langle P_{E(1)}^{rad} \rangle$, the higher frequency/shorter wavelength radiation (*i.e.* blue/violet light) is preferentially scattered **much more so** than the lower frequency/longer wavelength radiation (*i.e.* red light).

The Earth's sky appears blue {*e.g.* to an observer on the ground, or even *e.g.* a space shuttle astronaut in orbit around the earth} because the light from the sky is **scattered** (*i.e.* **re-radiated**) **light**, which is preferentially in the blue/violet portion of the visible light EM spectrum. The scattering of EM radiation off of atoms is known as **Rayleigh scattering**.

Note that precisely same physics also simultaneously explains why the Sun appears **red** *e.g.* to an observer on the ground at **sunrise** and **sunset** – because at these times of the day, path that the sunlight takes through the atmosphere is the longest, relative to that associated *e.g.* with its position at {local} noon. If the higher-frequency blue/violet light is preferentially scattered **out** of the beam of sunlight, what is left **in** the beam of sunlight **after** traversing the entire thickness of the Earth's atmosphere is the lower-frequency, orange-red light.

Note that the Sun is a **black-body radiator** – its *EM* spectrum peaks in the infra-red region – thus it is **NOT** flat by any means {also is affected by frequency-dependent absorption in the atmosphere}:

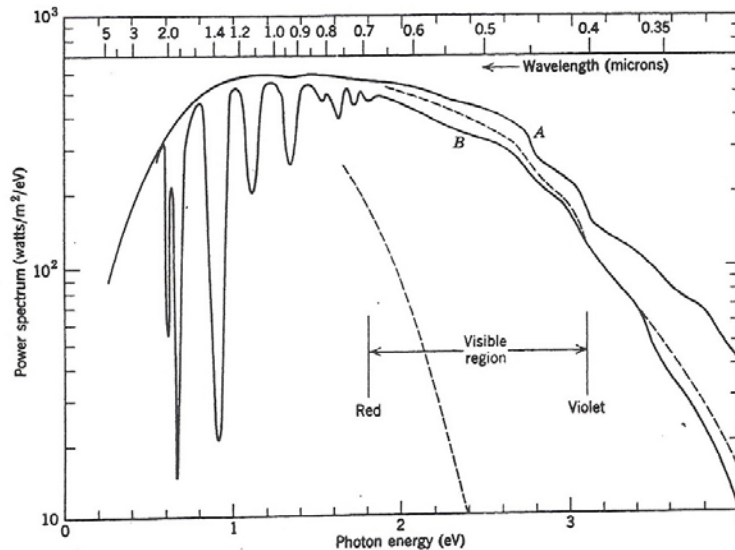
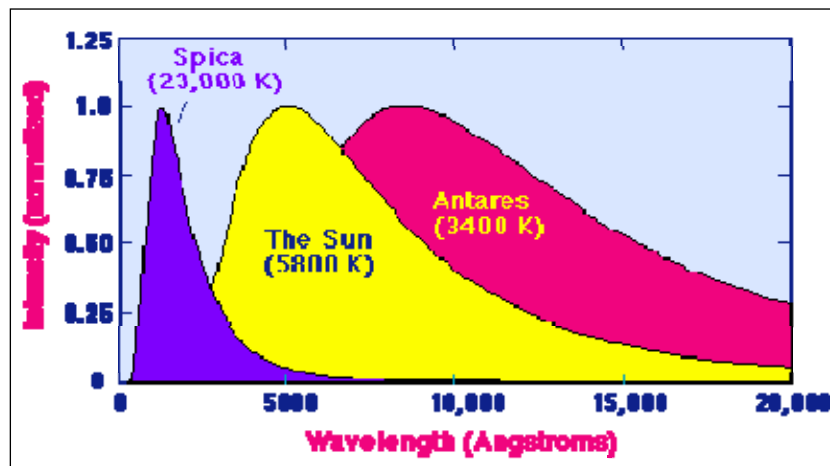


Figure 10.4 Power spectrum of solar radiation (in watts per square meter per electron volt) as a function of photon energy (in electron volts). Curve A is the incident spectrum above the atmosphere. Curve B is a typical sea-level spectrum with the sun at the zenith. The absorption bands below 2 eV are chiefly from water vapor and vary from site to site and day to day. The dashed curves give the expected sea-level spectrum at zenith and at sunrise-sunset if the only attenuation is from Rayleigh scattering by a dry, clean atmosphere.

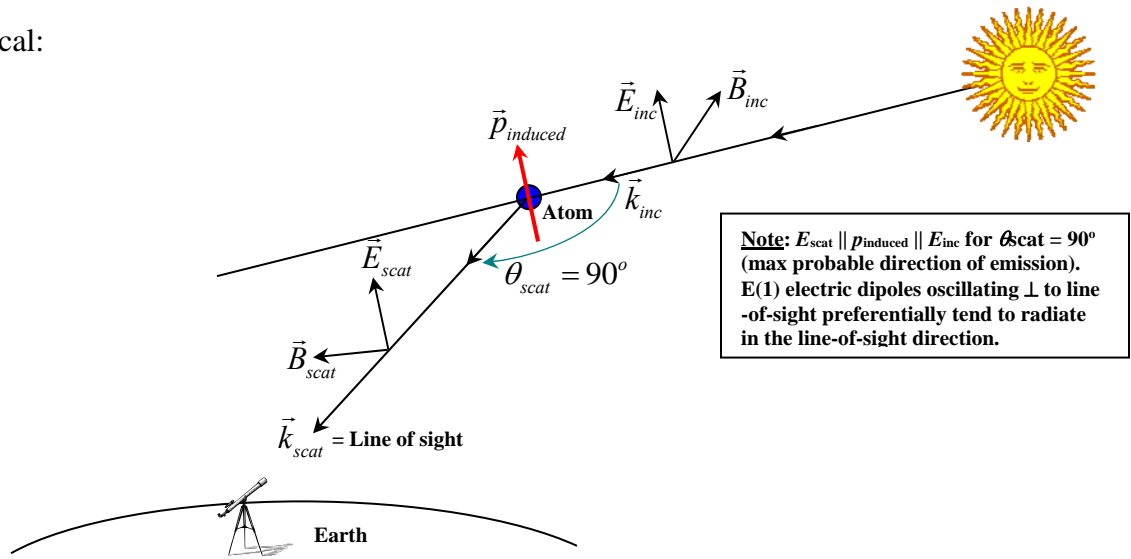
Note the log scale on the vertical axis! Thus, there is not much violet light in the Sun’s *EM* spectrum, and hence there is a delicate “balancing” act of flux of *EM* radiation from the Sun {convoluted} with its black-body spectrum and the scattering of this radiation by atoms in the Earth’s atmosphere – thus we see the sky as blue. Thus, if the black-body temperature of the sun was different, then the color of the Earth’s sky in the visible portion of the *EM* spectrum would also be different – compare the black-body spectra of our Sun *e.g.* with that of Spica (260 *ly* away in the Virgo constellation) and Antares (a red giant 600 *ly* away in the Scorpio constellation):



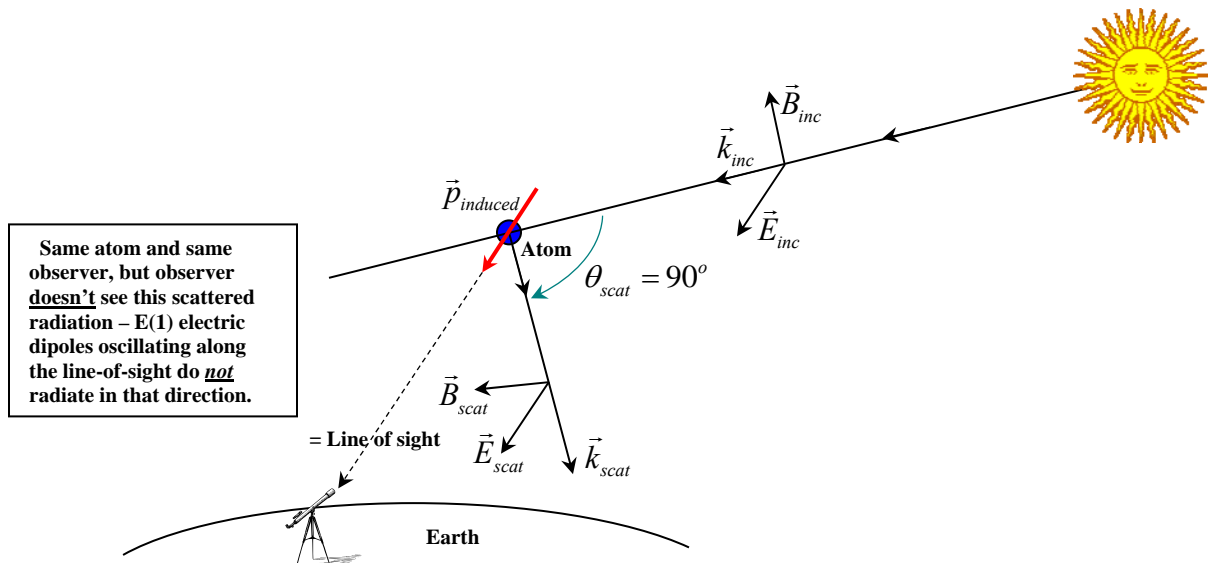
Light from the Sun is unpolarized (*i.e.* it consists of all polarizations, randomly oriented over time). However, because *EM* waves are **transversely** polarized (defined by the orientation of the \vec{E} -field vector) an incident *EM* plane wave from the Sun with polarization in a given direction (\perp to \vec{k} -propagation direction) will (transitorily) induce electric dipole moments in gas atoms in earth's atmosphere, via $\vec{p}_{mol}(\omega) = \alpha_{mol}(\omega)\vec{E}_{inc}$, where $\alpha_{mol}(\omega)$ is the **molecular polarizability** at {angular} frequency ω {see P435 Lect. Notes 12 and P436 Lect. Notes 7.5}.

The axis of induced electric dipole moments will be \parallel to the plane of polarization of incident wave at that instant, hence the scattered radiation emitted by the atom will be preferentially at $\theta = 90^\circ = \pi/2$ (*i.e.* \perp) to the axis of the (induced) electric dipole of gas atoms in earth's atmosphere. There are two specific/limiting cases to consider – (a) when the incident \vec{E} -field vector is vertical and (b) when the incident \vec{E} -field vector is horizontal. Random polarization is then an arbitrary linear combination of these two limiting cases:

(a.) \vec{E}_{inc} vertical:



(b.) \vec{E}_{inc} horizontal:



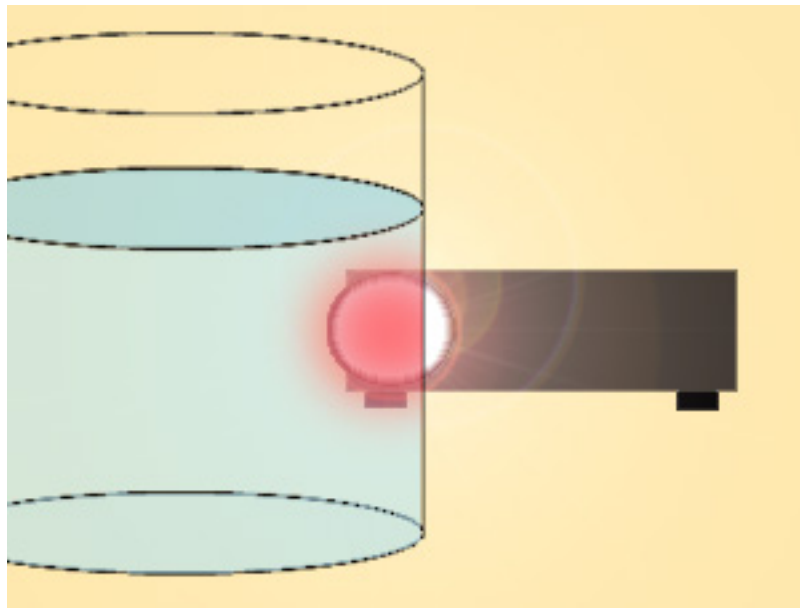
Because the blue light an observer sees from a given portion of the sky is due to the preferential scattering of E(1) electric dipole-type Rayleigh scattering of sunlight/solar EM radiation off of gas atoms in the Earth's atmosphere, with $\vec{E}_{scat} \perp$ to the line-of-sight, this radiation has a ***net*** polarization – *i.e.* the light from the sky is polarized, especially so away from the sun, *i.e.* in the northern portions of the sky {in the northern hemisphere} !!! You can very easily observe/explicitly verify this using a pair of polaroid sunglasses – try it some time!!!

It is beneficial to wear polaroid sunglasses *e.g.* when out boating on a lake – in order to reduce “glare” from {polarized} sunlight reflected off of the surface of the water!!!

As mentioned above, at sunrise or sunset, the sun appears red when an observer is looking directly at the sun, because the blue/violet light is $\sim 25\times$ more preferentially scattered out of the beam of light incident from the sun {per unit thickness of atmosphere} than red light. Thus sunlight when the sun is near the horizon consists predominantly of what remains – red light.

Note that this is also true for ***moonrise*** and ***moonset*** – the moon will {likewise} have a reddish hue at these times, and note that this is also true *e.g.* for the case of an eclipse of the moon by the Earth.

One can also observe this same phenomenon *e.g.* using a glass pitcher of milk diluted with water – because milk molecules are efficient Rayleigh scatterers of visible light! Here's a simple experiment that you can carry out at home, *e.g.* using a flashlight:



The {scalar} *EM* wave ***characteristic radiation impedance*** of an antenna is exactly as we defined the characteristic impedance of a waveguide; noting here that we are dealing with manifestly transverse waves for *EM* wave radiation from an E(1) electric dipole antenna:

$$Z_{\text{antenna}}(\vec{r}) \equiv \frac{|\vec{E}_{\perp}^{\text{rad}}(\vec{r})|}{|\vec{H}_{\perp}^{\text{rad}}(\vec{r})|} = \frac{|\vec{E}_{\perp}^{\text{rad}}(\vec{r})|}{\frac{1}{\mu_0} |\vec{B}_{\perp}^{\text{rad}}(\vec{r})|} = \frac{|\vec{E}^{\text{rad}}(\vec{r})|}{\frac{1}{\mu_0} |\vec{B}^{\text{rad}}(\vec{r})|}$$

Let's check the SI units of this definition:

$$\left(\frac{E}{B/\mu_0} \right) = \frac{(\text{Volts/m})}{\left(\frac{\text{Teslas}}{\frac{\text{Henrys}}{\text{m}}} \right)} = \frac{(\text{Volts/m})}{\left(\frac{\text{N/A-m}}{\left(\frac{\text{N}}{\text{A}^2} \right)} \right)} = \frac{\text{Volts}}{\text{Amps}} = \text{Ohms}$$

For E(1) electric dipole radiation, the *EM* wave characteristic radiation impedance in the “far-zone” limit ($d \ll \lambda \ll r$) with $c = 1/\sqrt{\epsilon_0 \mu_0}$ is:

$$\vec{E}_r^{\text{E}(1)}(\vec{r}, t) \approx -\frac{\mu_0 p \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\theta} \quad \text{and:} \quad \vec{B}_r^{\text{E}(1)}(\vec{r}, t) \approx -\frac{\mu_0 p \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \hat{\phi}$$

$$Z_{\text{antenna}}^{\text{E}(1)}(\vec{r}) = \frac{\mu_0 \cdot \frac{\mu_0 p \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right]}{\frac{\mu_0 p \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos \left[\omega \left(t - \frac{r}{c} \right) \right]} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv Z_0 = 120\pi \Omega = 377 \Omega$$

Where:

$$\mu_0 = 4\pi \times 10^{-7} \text{ Henrys/m} = \text{magnetic permeability of free space / vacuum}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ Farads/m} = \text{electric permittivity of free space / vacuum}$$

And: $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7} \text{ Henrys/m}}{8.85 \times 10^{-12} \text{ Farads/m}}} = 120\pi \Omega = 377 \Omega$ = {scalar} characteristic impedance of free space/the vacuum.

Thus we see that E(1) electric dipole antennae (in the “far-zone” limit ($d = \pi b \ll \lambda \ll r$)) are ***perfectly impedance-matched*** for propagation of E(1) *EM* waves into free space / vacuum!

Note also that the “far-zone” ($d \ll \lambda \ll r$) *EM* wave characteristic radiation impedance $Z_{\text{antenna}}^{\text{E}(1)}(\vec{r})$ has ***no*** spatial and/or frequency dependence.

The EM Wave Radiation Resistance of an Antenna:

The {scalar} EM wave **radiation resistance** of an antenna R_{rad} is defined in terms of the antenna power P_{rad} and the amplitude of the current I flowing in the antenna:

$$P_{rad}^{antenna} \equiv I^2 R_{rad}^{antenna} \quad \text{or:} \quad R_{rad}^{antenna} \equiv P_{rad}^{antenna} / I^2 \quad (\text{Ohms})$$

For an E(1) electric dipole antenna: $I = q\omega =$ amplitude of current flowing in the dipole.

In the “far-zone” limit, *i.e.* $d \ll \lambda \ll r$:

$$R_{rad}^{E(1)} \approx \frac{\mu_o p^2 \omega^4}{12\pi c^3 I^2} = \frac{\mu_o \cancel{q^2} d^2 \omega^4}{12\pi c \cancel{q^2} \omega^2} = \frac{\mu_o \omega^2 d^2}{12\pi c} (\Omega) \quad \leftarrow \text{n.b. } R_{rad}^{E(1)} \text{ is frequency-dependent!}$$

In the “far-zone” limit, *i.e.* $d \ll \lambda \ll r$:

$$R_{rad}^{E(1)} \approx \frac{\mu_o \omega^2 d^2}{12\pi c} = \frac{\omega^2 d^2}{12\pi c^2} (\mu_o c) = \frac{\omega^2 d^2}{12\pi c^2} \sqrt{\frac{\mu_o}{\epsilon_o}} = \frac{\omega^2 d^2}{12\pi c^2} Z_{rad}^{E(1)} \quad \text{But: } Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = Z_{rad}^{E(1)}$$

\therefore In the “far-zone” limit, $d \ll \lambda \ll r$:

$$R_{rad}^{E(1)} \approx \frac{\omega^2 d^2}{12\pi c^2} Z_o = \frac{1}{12\pi} \left(\frac{\omega d}{c} \right)^2 Z_o$$

However, in the “far-zone” limit, $d \ll \lambda \ll r$ we have: $\left(\frac{\omega d}{c} \right) \ll 1$

Thus, we see that the EM wave **radiation resistance** $R_{rad}^{E(1)}$ associated with E(1) electric dipole antenna in the “far-zone” limit ($d \ll \lambda \ll r$) is ***much*** less than the EM wave **characteristic radiation impedance** $Z_{rad}^{E(1),M(1)} = Z_o = 120\pi \Omega \approx 377 \Omega$ of an electric dipole antenna:

$$R_{rad}^{E(1)} \approx \frac{1}{12\pi} \left(\frac{\omega d}{c} \right)^2 Z_o \ll Z_o = 377 \Omega$$