LECTURE NOTES 12

THE LIÉNARD-WIECHERT RETARDED POTENTIALS $V_r(\vec{r},t)$ AND $\vec{A}_r(\vec{r},t)$ FOR A MOVING POINT CHARGE

Suppose a point electric charge q is ***moving*** along a specified ***trajectory*** = locus of points of $\vec{w}(t_r)$ = ***retarded*** position vector (SI units = meters) of the point charge q at ***retarded*** time t_r.

The ***retarded*** time (in free space/vacuum) from point charge q to observer is determined by:

$$\boxed{c\Delta t = \mathscr{r} = |\vec{\mathscr{r}}|} \quad \text{where the time interval} \quad \boxed{\Delta t = t - t_r}$$

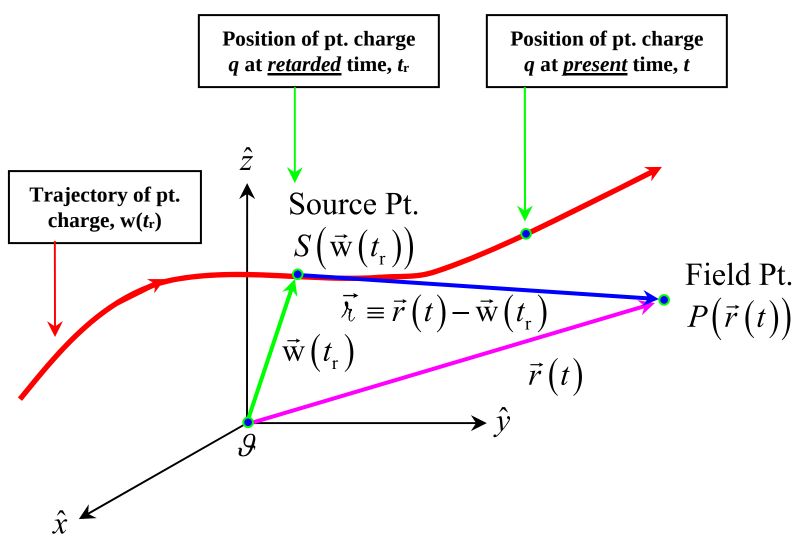
and: $$\boxed{\vec{\mathscr{r}} = \vec{r}(t) - \vec{r}'(t_r) = \vec{r}(t) - \vec{w}(t_r)}$$

Observation / field point at time t (doesn't move)

position of the point charge q at the ***retarded*** time t_r
$\vec{w}(t_r)$ = ***retarded*** ***position*** of charge at ***retarded*** time t_r

$$\boxed{\mathscr{r} = |\vec{\mathscr{r}}| = |\vec{r}(t) - \vec{w}(t_r)| = c\Delta t = c(t - t_r)}$$

$\mathscr{r}$ = separation distance of point charge q at the ***retarded*** position $\vec{w}(t_r)$ at the ***retarded*** time t_r to the observer's position at the field point $P(\vec{r}(t))$, which is at the ***present*** time, $t = t_r + \mathscr{r}/c$.
$\vec{\mathscr{r}}$ = vector separation distance between the two points, as shown in the figure below:



n.b. At most ***one*** **point** (***one*** and ***only*** one point) on the ***trajectory*** $\vec{w}(t_r)$ of the charged particle can be "in communication" with the (stationary) observer at field point $P(\vec{r}(t))$ at the present time t, because it takes a finite/causal amount of time $\Delta t = t - t_r$ for *EM* "news" to ***propagate*** from $\vec{w}(t_r)$ at the ***retarded*** time t_r to the observation/field point $P(\vec{r}(t))$ at position $\vec{r}$ at the ***present*** time $t = t_r + \mathscr{r}/c$.

In order to make this point conceptually clear, imagine replacing the point charge q moving along the ***retarded*** trajectory $\vec{w}(t_r)$ with a moving point light source. The stationary observer at the field point $P(\vec{r},t)$ at the ***present*** time t will see the point light source move along the ***retarded*** trajectory $\vec{w}(t_r)$; but it takes a finite time interval for the light (*EM* "news") to propagate from where the light source was at the ***retarded*** source point location $S(\vec{r}'(t_r)=\vec{w}(t_r))$ at the ***retarded*** time t_r to the observer's location at the field point $P(\vec{r},t)$ at the present time t.

This situation is ***precisely*** what an observer sees when looking at stars, planets, *etc*. in the night sky!

Suppose that {somehow} there ***were*** *e.g.* ***two*** such source points along the trajectory $\vec{w}(t_r)$ "in communication" with the observer at the field point $P(\vec{r}(t))$ at the present time t with retarded times t_{r_1} and t_{r_2} respectively.

Then: $\boxed{\mathscr{r}_1 = c(t-t_{r_1})}$ and: $\boxed{\mathscr{r}_2 = c(t-t_{r_2})}$ thus: $\boxed{\mathscr{r}_1 - \mathscr{r}_2 = c(t-t_{r_1}) - c(t-t_{r_2}) = c(t_{r_2}-t_{r_1})}$

$\Rightarrow$ The average velocity of ***this*** charged particle in the direction of the observer at $\vec{r}$ is $\vec{c}$!!!
{*n.b.* the velocity component(s) of this particle in ***other*** directions are ***not*** counted here}.
However, we know that ***nothing*** can move faster than the speed of light c !!!

$\therefore$ Only ***one*** retarded point $\vec{w}(t_r)$ can contribute to the potentials $V_r(\vec{r},t)$ and $\vec{A}_r(\vec{r},t)$ at the field point $P(\vec{r}(t))$ at any given moment in the ***present*** time, t for $v < c$!

$\Rightarrow$ For $v < c$, an observer at the field point $P(\vec{r}(t))$ at a given present time t "sees" the moving charged particle q in **only** ***one*** place.

{Note that a ***massless*** particle, such as a photon (which in free space/vacuum ***does*** move at the speed of light, c) ***could/can*** be "seen" by a stationary observer as being at more than one place at a given {present} time, t !!!
Note further that it is also possible that ***no*** points along the trajectory of the photon are accessible to an observer….}

A "naïve" / cursory reading of the formula for the retarded scalar potential

$$\boxed{V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{tot}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'}$$

might suggest that the ***retarded*** scalar potential for a moving point charge is {also} $\boxed{\frac{1}{4\pi\varepsilon_o}\frac{q}{\mathscr{r}}}$ (as in the ***static*** case), except that $\mathscr{r}$ = the separation distance is from observer position to the ***retarded*** position of the charge q.

However, this would be ***wrong*** – for a ***subtle*** conceptual reason!

It ***is*** true that for a **moving *point*** charge q, the denominator factor $1/\mathscr{r}$ ***can*** be taken outside of the integral, but note that {even} for a **moving *point*** charge, the integral: $\boxed{\int_{v'}\rho_{tot}(\vec{r}',t_r)\,d\tau' \neq q}$!!!

In order to calculate the ***total*** charge of a configuration, one must integrate $\rho_{tot}(\vec{r}', t_r)$ over the ***entire*** charge distribution at ***one instant of time***, but {***here***} the ***retardation*** $t_r = t - \mathscr{r}/c$ ***forces*** us to evaluate $\rho_{tot}(\vec{r}', t_r)$ at ***different times*** for ***different*** parts of the charge configuration!!!

Thus, if the source is ***moving***, we ***will*** obtain a ***distorted*** picture of the total charge!

Before integration: $\boxed{\mathscr{r} = |\vec{r}(t) - \vec{r}'(t_r)|}$ is a function of $\vec{r}(t)$ and $\vec{r}'(t_r)$

After integration: $\boxed{\vec{r}' = \vec{w}(t_r)}$ is ***fixed*** after integration: $\boxed{\rho_{tot}(\vec{r}', t_r) = q\delta^3(\vec{w}(t_r))}$

$\boxed{\mathscr{r} = |\vec{r}(t) - \vec{w}(t_r)|}$ is a function of $\vec{r}$ and t because $\boxed{t_r = t - \mathscr{r}/c}$.

One ***might*** think that this problem would be understandable *e.g.* for a moving ***extended*** charge distribution, but that it would disappear/go away/vanish for ***point*** charges. However it ***doesn't*** !!!

In Maxwell's equations of electrodynamics, formulated in terms of electric charge and current densities $(\rho_{tot}$ and $\vec{J}_{tot})$, a point charge = limit of extended charge when the size → zero.

For an ***extended*** charge distribution, the ***retardation*** effect in $\int_{v'} \rho_{tot}(\vec{r}', t_r) d\tau'$ throws in a factor of:

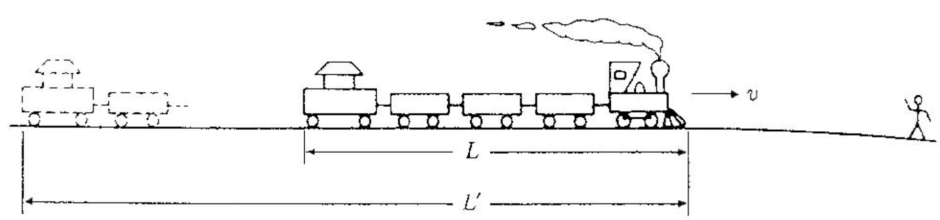
$$\boxed{\frac{1}{\kappa} \equiv \frac{1}{\left(1 - \hat{\mathscr{r}} \cdot \vec{v}(t_r)/c\right)} = \frac{1}{\left(1 - \hat{\mathscr{r}} \cdot \vec{\beta}(t_r)\right)}} \quad \text{where: } \boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$$

We define the ***retardation factor*** $\boxed{\kappa \equiv 1 - \hat{\mathscr{r}} \cdot \vec{v}(t_r)/c}$, where $\vec{v}(t_r)$ {more precisely $\vec{v}(\vec{r}'(t_r))$} is the velocity of the moving charged particle at the source position $\vec{r}'(t_r)$ at the ***retarded*** time t_r.

This is a ***purely geometrical*** effect, one which is analogous/similar to the Doppler effect. {However, it is ***not*** due to special / general relativity (yet)!!}

Consider a long train moving towards a stationary observer. Due to the finite propagation time of *EM* signals, the train actually appears (a little) longer than it really is! (If $c \approx 10$ *m/s* rather than 3×10^8 *m/s*, this motional effect ***would*** be readily apparent in the everyday world!!)

As shown in the figure below, light emitted from the caboose (end of the train) arriving at the observer at time *t* ***must*** leave the caboose ***earlier*** $(t_{r_{end}})$ than light emitted from the front of the train $(t_{r_{front}})$, both arrive simultaneously at the observer at the same ***present*** time *t*. The train is ***further away*** from the observer when light from the ***end*** of the train is emitted at the ***earlier*** time $(t_{r_{end}})$, compared to the train's location for the light emitted from the ***front*** of the train at the ***later*** time $(t_{r_{front}})$. The observer thus sees a ***distorted*** picture of the moving train at the ***present*** time *t*.



In the time interval Δt_c that the light from the caboose takes to travel the distance L' (see figure above) the train moves a distance $\Delta L \equiv (L' - L)$. Then since $c\Delta t_c = L'$, then: $\Delta t_c = L'/c$.

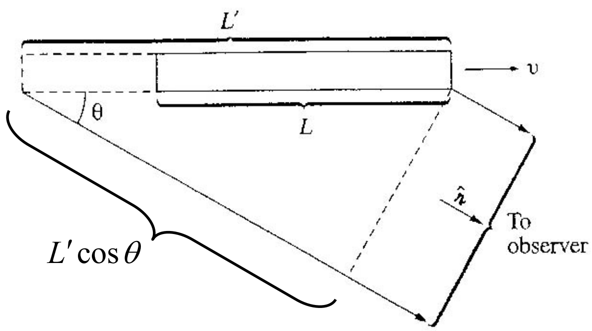
But during the same time interval Δt_c, the train moves a distance $\Delta L = v\Delta t_c = (L' - L)$, or:

$$\boxed{\Delta t_c = \frac{\Delta L}{v} = \frac{(L'-L)}{v}} \text{ but: } \boxed{\Delta t_c = \frac{L'}{c}} \text{ thus: } \boxed{\Delta t_c = \frac{L'}{c} = \frac{\Delta L}{v} = \frac{(L'-L)}{v}} \Rightarrow \boxed{L' = \frac{1}{1 - v/c} L}$$

⇒ Trains moving ***towards*** / ***approaching*** an observer ***appear longer***, by a factor of $1/(1 - v/c)$.

⇒ Conversely, it can similarly be shown that trains moving ***away*** / ***receding from*** an observer ***appear shorter*** by a factor of $1/(1 + v/c)$.

⇒ ***In general***, if the train's velocity vector $\vec{v}$ makes an angle θ with the observer's line of sight $\hat{\imath}$ (*n.b.* assuming that the train is far enough away from the observer that the solid angle subtended by the train is such that rays of light emitted from ***both*** ends of train are ***parallel***) the extra distance that light from the caboose must cover is $L'\cos\theta$ (see figure below). The corresponding time interval is $\Delta t_c = L'\cos\theta / c$. Note that the train also moves a distance $\Delta L = L' - L$ in this same time interval.



$$\boxed{\Delta t_c = L'\cos\theta / c} \text{ but: } \boxed{\Delta t_c = \frac{\Delta L}{v} = \frac{L'-L}{v}} \;\therefore\; \boxed{\Delta t_c = \frac{L'\cos\theta}{c} = \frac{\Delta L}{v} = \frac{L'-L}{v}} \text{ i.e. } \boxed{\frac{L'\cos\theta}{c} = \frac{L'-L}{v}}$$

$$\text{Or: } \boxed{L'\left(\frac{1}{v} - \frac{\cos\theta}{c}\right) = \frac{L}{v}} \text{ or: } \boxed{\frac{1}{v}L'\left(1 - \frac{v\cos\theta}{c}\right) = \frac{L}{v}} \text{ or: } \boxed{L' = \frac{1}{\left(1 - \frac{v\cos\theta}{c}\right)}L = \frac{1}{(1-\beta\cos\theta)}L} \text{ with } \boxed{\beta \equiv \frac{v}{c}}$$

From the above figure, the angle $\boxed{\theta = \cos^{-1}\left(\hat{\imath}\cdot\hat{v}\right)}$ = opening angle between $\hat{\imath}$ and $\hat{v}$.

Thus: $\boxed{\beta\cos\theta = \hat{\imath}\cdot\vec{\beta}}$ where: $\boxed{\vec{\beta} \equiv \frac{\vec{v}}{c}}$. Hence: $\boxed{L' = \frac{1}{(1-\beta\cos\theta)}L = \frac{1}{\left(1 - \hat{\imath}\cdot\vec{\beta}\right)}L}$.

Again, ***this*** retardation effect ***is*** due solely to the ***finite propagation time*** of the speed of light – it has ***nothing*** to do with special / general relativity – *e.g.* Lorentz contraction and/or time dilation and simultaneity.

The ***apparent*** volume τ' of the train is related to the ***actual*** volume of train $\tau'_\circ$ by:

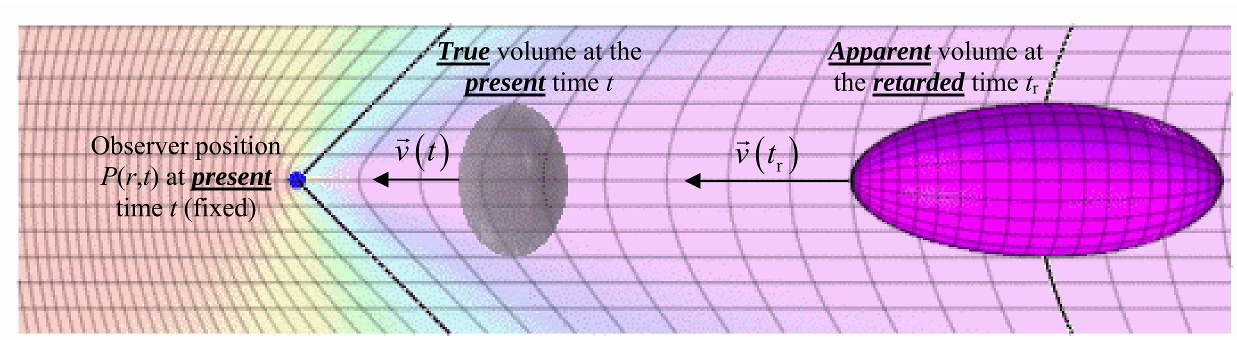
$$\boxed{\tau' = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c}\tau'_\circ = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)}\tau'_\circ = \frac{1}{\kappa}\tau'_\circ}$$ where $\boxed{\beta(t_r) \equiv \frac{v(t_r)}{c}}$ and $\boxed{\kappa \equiv 1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c = 1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)}$

and where $\boxed{\hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r} = \vec{\mathscr{r}}/|\vec{\mathscr{r}}|}$ = unit vector associated with the separation distance between the position of a {stationary} observer $\vec{r}(t)$ at the ***present*** time t to a position somewhere on the train $\vec{r}'(t_r)$ at the ***retarded*** time t_r. Explicitly: $\boxed{\hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r} = \vec{\mathscr{r}}/|\vec{\mathscr{r}}| = (\vec{r}(t)-\vec{r}'(t_r))/|\vec{r}(t)-\vec{r}'(t_r)|}$

The ***stationary*** observer's position vector $\vec{r}(t)$ is ***constant*** in time, whereas the ***retarded*** position vector of the ***moving*** train $\vec{r}'(t_r)$ ***changes*** in time.

Hence, whenever we carry out integrals of the type $\int_{v'} \rho_{tot}(\vec{r}',t_r)\,d\tau'$ {or $\int_{v'} \vec{J}_{tot}(\vec{r}',t_r)\,d\tau'$ } where the integrand(s) $\rho_{tot}(\vec{r}',t_r)$ {or $\vec{J}_{tot}(\vec{r}',t_r)$} are associated with {some kind of} moving charge {current} distribution(s), evaluated at the ***retarded*** time t_r, the ***apparent*** volume τ' of these integrals is modified by the factor $\boxed{\frac{1}{\kappa} = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c} = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)}}$ where: $\boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$, and the ***retardation factor*** $\boxed{\kappa \equiv 1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c = 1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)}$, *i.e.* $\boxed{d\tau' = \frac{1}{\kappa}d\tau'_\circ = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c}d\tau'_\circ = \frac{1}{1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)}d\tau'_\circ}$

The figure shown below graphically depicts this effect, for a snapshot-in-time $t = t_r + \mathscr{r}/c$:



⇒ See animated demo of this effect: https://en.wikipedia.org/wiki/Relativistic_Doppler_effect

Note that because the motional correction factor makes ***no*** reference to the actual physical size of the "particle", it ***is*** also relevant/important for ***point*** charged particles.

The ***retarded*** ***scalar*** potential associated with a ***point*** electric charge q ***moving*** along a ***retarded*** trajectory $\vec{\mathrm{w}}(t_r)$, with: $\boxed{\rho_q(\vec{r}',t_r) = q\delta^3(\vec{\mathrm{w}}(t_r))}$ is:

$$\boxed{\begin{aligned} V_r(\vec{r},t) &= \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_q(\vec{r}',t_r)}{\mathscr{r}}d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{1}{\mathscr{r}}\int_{v'}\frac{q\delta^3(\vec{\mathrm{w}}(t_r))}{\mathscr{r}}d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{1}{\mathscr{r}}\int_{v'}\frac{q\delta^3(\vec{\mathrm{w}}(t_r))}{\mathscr{r}(1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c)}d\tau'_{\circ} \\ &= \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{(1-\hat{\mathscr{r}}\bullet\vec{v}(\vec{\mathrm{w}}(t_r))/c)\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{(1-\hat{\mathscr{r}}\bullet\vec{\beta}(\vec{\mathrm{w}}(t_r)))\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa\mathscr{r}} \end{aligned}}$$

Where: $\boxed{\hat{\mathscr{r}} = \vec{\mathscr{r}}/\mathscr{r} = \vec{\mathscr{r}}/|\vec{\mathscr{r}}|}$, $\boxed{\vec{\mathscr{r}} = \vec{r}(t) - \vec{r}'(t_r) = \vec{r}(t) - \vec{\mathrm{w}}(t_r)}$

And: $\boxed{\kappa \equiv 1-\hat{\mathscr{r}}\bullet\vec{v}(\vec{\mathrm{w}}(t_r))/c = 1-\hat{\mathscr{r}}\bullet\vec{\beta}(\vec{\mathrm{w}}(t_r))}$ with: $\boxed{\vec{\beta}(\vec{\mathrm{w}}(t_r)) \equiv \frac{\vec{v}(\vec{\mathrm{w}}(t_r))}{c}}$,

Where: $\vec{v}(\vec{\mathrm{w}}(t_r))$ = velocity vector of charged particle evaluated at the ***retarded*** time $\boxed{t_r = t - \mathscr{r}/c}$.

The ***retarded*** current density $\vec{J}_{tot}(\vec{r}',t_r)$ for a ***rigid*** object is related to its ***retarded*** charge density $\rho_{tot}(\vec{r}',t_r)$ and its ***retarded*** velocity $\vec{v}(\vec{r}',t_r)$ by the relation:

$$\boxed{\vec{J}_{tot}(\vec{r}',t_r) = \rho_{tot}(\vec{r}',t_r)\vec{v}(\vec{r}',t_r)}.$$

The ***retarded*** ***vector*** potential associated with a ***point*** electric charge q ***moving*** with ***retarded*** velocity $\vec{v}(\vec{\mathrm{w}}(t_r))$ along a ***retarded*** trajectory $\vec{\mathrm{w}}(t_r)$, with: $\boxed{\vec{J}_q(\vec{r}',t_r) = \rho_q(\vec{r}',t_r)\vec{v}(\vec{r}',t_r) = q\vec{v}(\vec{r}',t_r)\delta^3(\vec{\mathrm{w}}(t_r))}$ is:

$$\boxed{\begin{aligned} \vec{A}_r(\vec{r},t) &= \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_q(\vec{r}',t_r)}{\mathscr{r}}d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{\rho_q(\vec{r}',t_r)\vec{v}(\vec{r}',t_r)}{\mathscr{r}}d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{q\vec{v}(\vec{r}',t_r)\delta^3(\vec{\mathrm{w}}(t_r))}{\mathscr{r}}d\tau' \\ &= \frac{\mu_o}{4\pi}\int_{v'}\frac{q\vec{v}(\vec{r}',t_r)\delta^3(\vec{\mathrm{w}}(t_r))}{\mathscr{r}(1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c)}d\tau'_{\circ} = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}(\vec{\mathrm{w}}(t_r))}{(1-\hat{\mathscr{r}}\bullet\vec{v}(\vec{\mathrm{w}}(t_r))/c)\mathscr{r}}\right) \\ &= \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}(\vec{\mathrm{w}}(t_r))}{(1-\hat{\mathscr{r}}\bullet\vec{\beta}(\vec{\mathrm{w}}(t_r)))\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\frac{q\vec{v}(\vec{\mathrm{w}}(t_r))}{\kappa\mathscr{r}} = \frac{\mu_o}{4\pi}\frac{q}{\kappa\mathscr{r}}\vec{v}(\vec{\mathrm{w}}(t_r)) \end{aligned}}$$

Thus, we have obtained the so-called Liénard-Wiechert ***retarded*** potentials for a ***point*** electric charge q ***moving*** with ***retarded*** velocity $\vec{v}(\vec{\mathrm{w}}(t_r))$ along a ***retarded*** trajectory $\vec{\mathrm{w}}(t_r)$:

$$\boxed{V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{(1-\hat{\mathscr{r}}\bullet\vec{\beta}(\vec{\mathrm{w}}(t_r)))\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa\mathscr{r}}}$$

$$\boxed{\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}(\vec{\mathrm{w}}(t_r))}{(1-\hat{\mathscr{r}}\bullet\vec{\beta}(\vec{\mathrm{w}}(t_r)))\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\frac{q}{\kappa\mathscr{r}}\vec{v}(\vec{\mathrm{w}}(t_r))}$$

Where: $\kappa \equiv 1 - \hat{\mathscr{r}} \cdot \vec{v}\left(\vec{w}\left(t_r\right)\right)/c = 1 - \hat{\mathscr{r}} \cdot \vec{\beta}\left(\vec{w}\left(t_r\right)\right)$ with: $\vec{\beta}\left(w\left(t_r\right)\right) \equiv \dfrac{\vec{v}\left(\vec{w}\left(t_r\right)\right)}{c}$.

Note that: $\vec{A}_r\left(\vec{r},t\right) = \dfrac{\vec{v}\left(\vec{w}\left(t_r\right)\right)}{c^2} V_r\left(\vec{r},t\right) = \dfrac{1}{c}\left[\dfrac{\vec{v}\left(\vec{w}\left(t_r\right)\right)}{c}\right] V_r\left(\vec{r},t\right) = \vec{\beta}\left(\vec{w}\left(t_r\right)\right)\dfrac{V_r\left(\vec{r},t\right)}{c}$ using: $c^2 = \dfrac{1}{\varepsilon_o \mu_o}$

Or: $\vec{A}_r\left(\vec{r},t\right) = \vec{\beta}\left(\vec{w}\left(t_r\right)\right)\dfrac{V_r\left(\vec{r},t\right)}{c}$ with: $\vec{\beta}\left(\vec{w}\left(t_r\right)\right) \equiv \dfrac{\vec{v}\left(\vec{w}\left(t_r\right)\right)}{c}$.

Where V_r is in *Volts*, $\vec{A}_r$ is in *Newtons/Ampere* (= ***momentum*** *per Coulomb*); they are related to each other by a factor of $1/c$ and $\vec{\beta}$ for the case of a ***moving point*** electric charge q.

Recall that the relativistic four-potential is: $A^\mu \equiv \left(V/c, \vec{A}\right)$, hence {***here***} the ***retarded*** relativistic four-potential for a ***moving point*** charge is: $A_r^\mu \equiv \left(V_r/c, \vec{A}_r\right) = \left(V_r/c, \vec{\beta}\left(\vec{w}\left(t_r\right)\right)V_r/c\right)$.

Griffith's Example 10.3:

Find/determine the Liénard-Wiechert ***retarded*** potentials associated with ***point*** charge q ***moving*** with ***constant*** velocity $\vec{v}$.

For convenience sake, define $t_r = 0 =$ ***retarded*** time the charged particle passes through the origin.

Then: $\vec{w}\left(t_r\right) = \vec{v}\left(t_r\right)t_r = \vec{v}t_r$ because $\vec{v}\left(t_r\right) = \vec{v}$ is a ***constant*** vector.

The ***retarded*** time is: $t_r = t - \mathscr{r}/c$ or: $\mathscr{r} = c\left(t - t_r\right) = c\Delta t_r$ with: $\Delta t_r \equiv t - t_r$

But: $\mathscr{r} = \left|\vec{r} - \vec{r}'\left(t_r\right)\right| = \left|\vec{r} - \vec{w}\left(t_r\right)\right| = \left|\vec{r} - \vec{v}t_r\right|$ but we ***also*** have: $\mathscr{r} = c\left(t - t_r\right) = c\Delta t_r$

Solve for the ***retarded*** time t_r by relating: $\mathscr{r} = \left|\vec{r} - \vec{v}t_r\right| = c\left(t - t_r\right)$

Square both sides: $\left|\vec{r} - \vec{v}t_r\right|^2 = c^2\left(t - t_r\right)^2 = c^2\left(t^2 - 2tt_r + t_r^2\right)$

But: $\left|\vec{r} - \vec{v}t_r\right|^2 = \left(\vec{r} - \vec{v}t_r\right)\cdot\left(\vec{r} - \vec{v}t_r\right) = r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2 t_r^2$

Thus: $r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2\left(t^2 - 2tt_r + t_r^2\right)$

Solve this quadratic equation for t_r: $\underbrace{\left(c^2 - v^2\right)}_{\equiv a} t_r^2 - 2\underbrace{\left(c^2 t - \vec{r}\cdot\vec{v}\right)}_{\equiv b} t_r + \underbrace{\left(c^2t^2 - r^2\right)}_{\equiv c} = 0$

$$t_r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow t_r = \frac{\left(c^2 t - \vec{r}\cdot\vec{v}\right) \pm \sqrt{\left(c^2 t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2t^2\right)}}{c^2 - v^2} \quad **$$

Which sign do we choose? + or – ?? Must make ***physical*** sense!

Consider the limit as $v \to 0$: $\boxed{t_r = t - \mathscr{r}/c}$ ← what we want!

And, if $v = 0$, the point charge q is at rest at the ***origin*** ($\vec{r}' = 0$), because it is there at time $t_r = 0$.

Then: $\boxed{\mathscr{r} = |\vec{r} - \vec{r}'| = |\vec{r}| = r}$

Thus, its ***retarded*** time should be: $\boxed{t_r = t - \mathscr{r}/c}$ → $\boxed{t_r = t - r/c}$ when $v \to 0$.

∴ We ***must*** choose the – sign on ***physical*** grounds, *i.e.* we ***must*** choose:

$$\boxed{t_r = \frac{\left(c^2 t - \vec{r}\bullet\vec{v}\right) - \sqrt{\left(c^2 t - \vec{r}\bullet\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}{c^2 - v^2}}$$

Now: $\boxed{\mathscr{r} = c\Delta t_r = c\left(t - t_r\right)}$ and: $\boxed{\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{\vec{r} - \vec{v}t_r}{c\Delta t_r} = \frac{\left(\vec{r} - \vec{v}t_r\right)}{c\left(t - t_r\right)}}$.

Therefore, the quantity:

$$\boxed{\begin{aligned}\mathscr{r}\left(1 - \hat{\mathscr{r}}\bullet\vec{v}/c\right) &= c\Delta t_r\left[1 - \frac{\vec{r} - \vec{v}t_r}{c\left(t - t_r\right)}\bullet\frac{\vec{v}}{c}\right] = c\left(t - t_r\right)\left[1 - \frac{\vec{r} - \vec{v}t_r}{c\left(t - t_r\right)}\bullet\frac{\vec{v}}{c}\right] = c\left(t - t_r\right) - \frac{\vec{r}\bullet\vec{v}}{c} + \frac{\vec{v}\bullet\vec{v}}{c}t_r \\ &= c\left(t - t_r\right) - \frac{\vec{r}\bullet\vec{v}}{c} + \frac{v^2}{c}t_r = \frac{1}{c}\left[\left(c^2 t - \vec{r}\bullet\vec{v}\right) - \left(c^2 - v^2\right)t_r\right]\end{aligned}}$$

Then, insert the ***retarded*** time t_r from the expression (**) {above} with the ***minus*** sign, *i.e.*:

$$\boxed{t_r = \frac{\left(c^2 t - \vec{r}\bullet\vec{v}\right) - \sqrt{\left(c^2 t - \vec{r}\bullet\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}{c^2 - v^2}}$$

into the above formula & carry out the algebra:

Thus: $\boxed{\mathscr{r}\underbrace{\left(1 - \hat{\mathscr{r}}\bullet\vec{v}/c\right)}_{\equiv\kappa} = \kappa\mathscr{r} = \frac{1}{c}\sqrt{\left(c^2 t - \vec{r}\bullet\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}$ with: $\boxed{\kappa \equiv \left(1 - \hat{\mathscr{r}}\bullet\vec{v}/c\right)}$ {***here***}

The general form of the Liénard-Wiechert ***retarded*** scalar and vector potentials associated with a ***point*** electric charge q ***moving*** with a ***time-dependent*** velocity $\vec{v}\left(\vec{w}\left(t_r\right)\right)$ are:

$$\boxed{V_r\left(\vec{r}, t\right) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\left(1 - \hat{\mathscr{r}}\bullet\vec{v}\left(\vec{w}\left(t_r\right)\right)/c\right)\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\left(1 - \hat{\mathscr{r}}\bullet\vec{\beta}\left(\vec{w}\left(t_r\right)\right)\right)\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa\mathscr{r}}}$$

$$\boxed{\vec{A}_r\left(\vec{r}, t\right) = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}\left(\vec{w}\left(t_r\right)\right)}{\left(1 - \hat{\mathscr{r}}\bullet\vec{v}\left(\vec{w}\left(t_r\right)\right)/c\right)\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}\left(\vec{w}\left(t_r\right)\right)}{\left(1 - \hat{\mathscr{r}}\bullet\vec{\beta}\left(\vec{w}\left(t_r\right)\right)\right)\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\frac{q}{\kappa\mathscr{r}}\vec{v}\left(\vec{w}\left(t_r\right)\right)}$$

Where: $\boxed{\kappa \equiv \left(1 - \hat{\mathscr{r}}\bullet\vec{v}\left(\vec{w}\left(t_r\right)\right)/c\right) = \left(1 - \hat{\mathscr{r}}\bullet\vec{\beta}\left(\vec{w}\left(t_r\right)\right)\right)}$ with: $\boxed{\vec{\beta}\left(w\left(t_r\right)\right) \equiv \frac{\vec{v}\left(\vec{w}\left(t_r\right)\right)}{c}}$.

The Liénard-Wiechert ***retarded*** scalar and vector potentials associated with a ***point*** charge q ***moving*** with ***constant*** velocity $\vec{v}$ are:

$$V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\left(1-\hat{\mathscr{r}}\cdot\vec{v}/c\right)\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa\mathscr{r}} = \frac{1}{4\pi\varepsilon_o}\frac{qc}{\sqrt{\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}}$$

$$\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}}{\left(1-\hat{\mathscr{r}}\cdot\vec{v}/c\right)\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\frac{q\vec{v}}{\kappa\mathscr{r}} = \frac{\mu_o}{4\pi}\frac{qc\vec{v}}{\sqrt{\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}}$$

where: $\kappa \equiv \left(1-\hat{\mathscr{r}}\cdot\vec{v}/c\right)$ = ***retardation factor*** {***here***} and: $\vec{\mathscr{r}} = \mathscr{r}\hat{\mathscr{r}}$.

Note again that {***here***}: $\vec{A}_r(\vec{r},t) = \vec{\beta}\left(V_r(\vec{r},t)/c\right)$ where: $\vec{\beta} \equiv \vec{v}/c$ = ***constant*** vector.

The Electromagnetic Fields Associated with a Moving Point Charge

We are now in a position to derive the ***retarded*** electric and magnetic ***fields*** associated with a ***moving point*** charge using the Liénard-Wiechert ***retarded potentials*** associated with a ***moving point*** charge:

$$V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\left(1-\hat{\mathscr{r}}\cdot\vec{\beta}\left(\vec{w}(t_r)\right)\right)\mathscr{r}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa\mathscr{r}}$$

$$\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\left(\frac{q\vec{v}\left(\vec{w}(t_r)\right)}{\left(1-\hat{\mathscr{r}}\cdot\vec{\beta}\left(\vec{w}(t_r)\right)\right)\mathscr{r}}\right) = \frac{\mu_o}{4\pi}\frac{q}{\kappa\mathscr{r}}\vec{v}\left(\vec{w}(t_r)\right)$$

with: $\vec{A}_r(\vec{r},t) = \vec{\beta}\left(\vec{w}(t_r)\right)V_r(\vec{r},t)/c$ with: $\vec{\beta}\left(\vec{w}(t_r)\right) \equiv \vec{v}\left(\vec{w}(t_r)\right)/c$ and: $c = 1/\sqrt{\varepsilon_o\mu_o}$ {in free space}

where: $\mathscr{r} = \left|\vec{r}-\vec{w}(t_r)\right|$

and: $t_r = t-\mathscr{r}/c$ = ***retarded*** time and: $\kappa \equiv \left(1-\hat{\mathscr{r}}\cdot\vec{v}\left(\vec{w}(t_r)\right)/c\right)$ = ***retardation factor***.

The equations for the ***retarded*** $\vec{E}$ and $\vec{B}$-fields in terms of their ***retarded*** potentials are:

$$\vec{E}_r(\vec{r},t) = -\vec{\nabla}V_r(\vec{r},t) - \frac{\partial\vec{A}_r(\vec{r},t)}{\partial t} \quad \text{and:} \quad \vec{B}_r(\vec{r},t) = \vec{\nabla}\times\vec{A}_r(\vec{r},t)$$

Again, the differentiation has various ***subtleties*** associated with it ***because***:

$$\vec{\mathscr{r}} = \vec{r}(t)-\vec{r}'(t_r) = \vec{r}(t)-\vec{w}(t_r)$$

and: $$\vec{v}(t_r) = \frac{\partial\vec{w}(t_r)}{\partial t} \equiv \dot{\vec{w}}(t_r)$$

} ***both*** quantities are evaluated at the ***retarded*** time $t_r = t-\mathscr{r}/c$

Note that: $\boxed{\mathscr{r} = |\vec{\mathscr{r}}| = |\vec{r} - \vec{r}'(t_r)| = |\vec{r} - \vec{w}(t_r)| = c\Delta t_r = c(t - t_r)}$ $\Rightarrow$ $\mathscr{r}$ is a function of ***both*** $\vec{r}$ and t.

Now: $\vec{\nabla} V_r(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_o}\vec{\nabla}\left\{\dfrac{1}{\mathscr{r}}\left[\dfrac{1}{(1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c)}\right]\right\}$ with: $\mathscr{r}\hat{\mathscr{r}} = \vec{\mathscr{r}}$

Thus: $\vec{\nabla} V_r(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_o}\vec{\nabla}\left\{\dfrac{1}{\mathscr{r} - \vec{\mathscr{r}}\bullet\vec{v}(t_r)/c}\right\} = \dfrac{q}{4\pi\varepsilon_o}\dfrac{-1}{\left[\mathscr{r} - \vec{\mathscr{r}}\bullet\vec{v}(t_r)/c\right]^2}\vec{\nabla}\left(\mathscr{r} - \vec{\mathscr{r}}\bullet\vec{v}(t_r)/c\right)$

But: $\mathscr{r} = c(t - t_r)$ $\therefore$ $\vec{\nabla}\mathscr{r} = \vec{\nabla}c(t - t_r) = -c\vec{\nabla}t_r$

Then, using Griffiths "Product Rule # 4" $\vec{\nabla}(\vec{A}\bullet\vec{B}) = (\vec{A}\bullet\vec{\nabla})\vec{B} + (\vec{B}\bullet\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A})$ on the second term $\vec{\nabla}(\vec{\mathscr{r}}\bullet\vec{v}(t_r)/c)$ we obtain:

$$\vec{\nabla}(\vec{\mathscr{r}}\bullet\vec{v}(t_r)/c) = \frac{1}{c}\overbrace{(\vec{\mathscr{r}}\bullet\vec{\nabla})\vec{v}(t_r)}^{(1)} + \frac{1}{c}\overbrace{(\vec{v}(t_r)\bullet\vec{\nabla})\vec{\mathscr{r}}}^{(2)} + \frac{1}{c}\overbrace{\vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v}(t_r))}^{(3)} + \frac{1}{c}\overbrace{\vec{v}(t_r)\times(\vec{\nabla}\times\vec{\mathscr{r}})}^{(4)}$$

For term (1):

$$(\vec{\mathscr{r}}\bullet\vec{\nabla})\vec{v}(t_r) = \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\vec{v}(t_r) = \mathscr{r}_x\frac{d\vec{v}(t_r)}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{d\vec{v}(t_r)}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{d\vec{v}(t_r)}{dt_r}\frac{\partial t_r}{\partial z}$$

Again: $\dfrac{d}{dt_r} = \dfrac{d}{dt}$ since: $t_r = t - \mathscr{r}/c$

$$\therefore \quad (\vec{\mathscr{r}}\bullet\vec{\nabla})v(t_r) = \frac{d\vec{v}(t_r)}{dt}\left[\mathscr{r}_x\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\right] = \vec{a}(t_r)(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)$$

where: $\vec{a}(t_r) \equiv \dfrac{d\vec{v}(t_r)}{dt_r} = \dfrac{d\vec{v}(t_r)}{dt}$ = acceleration of charged particle at the ***retarded*** time $t_r = t - \mathscr{r}/c$.

Second term (2): $\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_r) = \vec{r} - \vec{\mathrm{w}}(t_r)$

Thus: $$(\vec{v}(t_r)\bullet\vec{\nabla})\vec{\mathscr{r}} = (\vec{v}(t_r)\bullet\vec{\nabla})(\vec{r} - \vec{\mathrm{w}}(t_r)) = \underbrace{(\vec{v}(t_r)\bullet\vec{\nabla})\vec{r}}_{=(2a)} - \underbrace{(\vec{v}(t_r)\bullet\vec{\nabla})\vec{\mathrm{w}}(t_r)}_{=(2b)}$$

Term (2a):
$$(\vec{v}(t_r)\bullet\vec{\nabla})\vec{r} = \left(v_x(t_r)\frac{\partial}{\partial x} + v_y(t_r)\frac{\partial}{\partial y} + v_z(t_r)\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$
$$= v_x(t_r)\hat{x} + v_y(t_r)\hat{y} + v_z(t_r)\hat{z} = \vec{v}(t_r) \quad \{!!!\}$$

<u>Term (2b)</u>:

$$-\left(\vec{v}(t_r)\cdot\vec{\nabla}\right)\vec{w}(t_r) = -\left(v_x(t_r)\frac{\partial}{\partial x} + v_y(t_r)\frac{\partial}{\partial y} + v_z(t_r)\frac{\partial}{\partial z}\right)\vec{w}(t_r)$$

$$= -v_x(t_r)\frac{d\vec{w}(t_r)}{dt_r}\frac{\partial t_r}{\partial x} + v_y(t_r)\frac{d\vec{w}(t_r)}{dt_r}\frac{\partial t_r}{\partial y} + v_z(t_r)\frac{d\vec{w}(t_r)}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= -\left[v_x(t_r)\frac{\partial t_r}{\partial x} + v_y(t_r)\frac{\partial t_r}{\partial y} + v_z(t_r)\frac{\partial t_r}{\partial z}\right]\frac{d\vec{w}(t_r)}{dt_r}$$

But: $\vec{v}(t_r) = \dfrac{d\vec{w}(t_r)}{dt_r}$

Thus: $-\left(\vec{v}(t_r)\cdot\vec{\nabla}\right)\vec{w}(t_r) = -\vec{v}(t_r)\left[v_x(t_r)\dfrac{\partial t_r}{\partial x} + v_y(t_r)\dfrac{\partial t_r}{\partial y} + v_z(t_r)\dfrac{\partial t_r}{\partial z}\right] = -\vec{v}(t_r)\left(\vec{v}(t_r)\cdot\vec{\nabla}t_r\right)$

n.b. $-\left(\vec{v}(t_r)\cdot\vec{\nabla}\right)\vec{w}(t_r) = -\vec{v}(t_r)\left(\vec{v}(t_r)\cdot\vec{\nabla}t_r\right)$ is analogous / similar to: $\left(\vec{\mathscr{r}}\cdot\vec{\nabla}\right)\vec{v}(t_r) = \vec{a}\left(\vec{\mathscr{r}}\cdot\vec{\nabla}t_r\right)$

<u>Third term (3)</u>: $\vec{\mathscr{r}}\times\left(\vec{\nabla}\times\vec{v}(t_r)\right)$

First, work out: $\vec{\nabla}\times\vec{v}(t_r)$ Since: $\vec{A}\times\vec{B} \equiv \left(A_yB_z - A_zB_y\right)\hat{x} + \left(A_zB_x - A_xB_z\right)\hat{y} + \left(A_xB_y - A_yB_x\right)\hat{z}$

Then:

$$\vec{\nabla}\times\vec{v}(t_r) = \left(\frac{\partial v_z(t_r)}{\partial y} - \frac{\partial v_y(t_r)}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x(t_r)}{\partial z} - \frac{\partial v_z(t_r)}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y(t_r)}{\partial x} - \frac{\partial v_x(t_r)}{\partial y}\right)\hat{z}$$

$$= \left(\frac{dv_z(t_r)}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y(t_r)}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{dv_x(t_r)}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z(t_r)}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{dv_y(t_r)}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x(t_r)}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{z}$$

$$= \left(a_z(t_r)\frac{\partial t_r}{\partial y} - a_y(t_r)\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(a_x(t_r)\frac{\partial t_r}{\partial z} - a_z(t_r)\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(a_y(t_r)\frac{\partial t_r}{\partial x} - a_x(t_r)\frac{\partial t_r}{\partial y}\right)\hat{z}$$

$$= -\left\{\left(a_y(t_r)\frac{\partial t_r}{\partial z} - a_z(t_r)\frac{\partial t_r}{\partial y}\right)\hat{x} + \left(a_z(t_r)\frac{\partial t_r}{\partial x} - a_x(t_r)\frac{\partial t_r}{\partial z}\right)\hat{y} + \left(a_x(t_r)\frac{\partial t_r}{\partial y} - a_y(t_r)\frac{\partial t_r}{\partial x}\right)\hat{z}\right\}$$

$$= -\vec{a}(t_r)\times\vec{\nabla}t_r$$

$\therefore$ $\vec{\mathscr{r}}\times\left(\vec{\nabla}\times\vec{v}(t_r)\right) = \vec{\mathscr{r}}\times\left(-\vec{a}(t_r)\cdot\vec{\nabla}t_r\right) = -\vec{\mathscr{r}}\times\left(\vec{a}(t_r)\cdot\vec{\nabla}t_r\right)$

Fourth Term (4): $\vec{v}(t_r)\times(\vec{\nabla}\times\vec{\mathscr{r}})$ with: $\vec{\mathscr{r}}=\vec{r}-\vec{r}'(t_r)=\vec{r}-\vec{w}(t_r)$

First, work out: $(\vec{\nabla}\times\vec{\mathscr{r}})=\vec{\nabla}\times(\vec{r}-\vec{w}(t_r))=\vec{\nabla}\times\vec{r}-\vec{\nabla}\times\vec{w}(t_r)$

But: $\vec{\nabla}\times\vec{r}=\left(\overset{=0}{\frac{\partial z}{\partial y}}-\overset{=0}{\frac{\partial y}{\partial z}}\right)\hat{x}+\left(\overset{=0}{\frac{\partial x}{\partial z}}-\overset{=0}{\frac{\partial z}{\partial x}}\right)\hat{y}+\left(\overset{=0}{\frac{\partial y}{\partial x}}-\overset{=0}{\frac{\partial x}{\partial y}}\right)\hat{z}=0$ ⇐ Of course!!! Because $\vec{r}$ is position vector of field point/observer $P(\vec{r})$, $\vec{r}$ = ***constant*** vector !!!

$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$

From the result of term (3) above: $\vec{\nabla}\times\vec{v}(t_r)=-\vec{a}(t_r)\times\vec{\nabla}t_r$ we see that: $\vec{\nabla}\times\vec{w}(t_r)=-\vec{v}(t_r)\times\vec{\nabla}(t_r)$

$$\therefore\quad \vec{v}(t_r)\times(\vec{\nabla}\times\vec{\mathscr{r}})=\vec{v}(t_r)\times\left\{\overset{=0}{\cancel{\vec{\nabla}\times\vec{r}}}-\vec{\nabla}\times\vec{w}(t_r)\right\}=-\vec{v}(t_r)\times(\vec{\nabla}\times\vec{w}(t_r))=+\vec{v}(t_r)\times(\vec{v}(t_r)\times\vec{\nabla}t_r)$$

Collecting all of the above individual results (1) – (4) for the term:

$$\vec{\nabla}(\mathscr{r}-\vec{\mathscr{r}}\bullet\vec{v}(t_r)/c)=\vec{\nabla}\mathscr{r}-\frac{1}{c}\vec{\nabla}(\vec{\mathscr{r}}\bullet\vec{v}(t_r))$$
$$=\vec{\nabla}\mathscr{r}-\frac{1}{c}\left\{\underbrace{(\vec{\mathscr{r}}\bullet\vec{\nabla})\vec{v}(t_r)}_{(1)}+\underbrace{(\vec{v}(t_r)\bullet\vec{\nabla})\vec{\mathscr{r}}}_{(2)}-\underbrace{\vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v}(t_r))}_{(3)}+\underbrace{\vec{v}(t_r)\times(\vec{\nabla}\times\vec{\mathscr{r}})}_{(4)}\right\}$$

We see that:

$$\vec{\nabla}(\mathscr{r}-\vec{\mathscr{r}}\bullet\vec{v}(t_r)/c)=-c\vec{\nabla}t_r-\frac{1}{c}\left\{\vec{a}(t_r)(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)+\vec{v}(t_r)-\vec{v}(t_r)(\vec{v}(t_r)\bullet\vec{\nabla}t_r)\right.$$
$$\left.-\vec{r}\times(\vec{a}(t_r)\times\vec{\nabla}t_r)+\vec{v}(t_r)\times(\vec{v}(t_r)\times\vec{\nabla}t_r)\right\}$$

But: $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\bullet\vec{C})-\vec{C}(\vec{A}\bullet\vec{B})$

$$\therefore\quad -\vec{\mathscr{r}}\times(\vec{a}\times\vec{\nabla}t_r)=-\left\{\vec{a}(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)-\vec{\nabla}t_r(\vec{\mathscr{r}}\bullet\vec{a})\right\}=-\vec{a}(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)+\vec{\nabla}t_r(\vec{\mathscr{r}}\bullet\vec{a})$$

And: $\vec{v}\times(\vec{v}\times\vec{\nabla}t_r)=\vec{v}(\vec{v}\bullet\vec{\nabla}t_r)-\vec{\nabla}t_r(\vec{v}\bullet\vec{v})$

Then, some truly amazing/fortuitous cancellations occur:

$$\vec{\nabla}(\mathscr{r}-\vec{\mathscr{r}}\bullet\vec{v}(t_r)/c)=-c\vec{\nabla}t_r-\frac{1}{c}\left\{\cancel{\vec{a}(t_r)(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)}+\vec{v}(t_r)-\cancel{\vec{v}(t_r)(\vec{v}(t_r)\bullet\vec{\nabla}t_r)}\right.$$
$$\left.-\cancel{\vec{a}(t_r)(\vec{\mathscr{r}}\bullet\vec{\nabla}t_r)}+\vec{\nabla}t_r(\vec{\mathscr{r}}\bullet\vec{a}(t_r))+\cancel{\vec{v}(t_r)(\vec{v}(t_r)\bullet\vec{\nabla}t_r)}-\vec{\nabla}t_r(\vec{v}(t_r)\bullet\vec{v}(t_r))\right\}$$
$$=-c\vec{\nabla}t_r-\frac{1}{c}\left\{\vec{v}(t_r)+\left[(\vec{\mathscr{r}}\bullet\vec{a}(t_r))-v^2(t_r)\right]\vec{\nabla}t_r\right\}$$

Thus:

$$\begin{aligned}\vec\nabla V_r(\vec r,t) &= \frac{q}{4\pi\varepsilon_o}\frac{-1}{\left[\mathscr{r}-\vec{\mathscr{r}}\cdot\vec v(t_r)/c\right]^2}\vec\nabla\left(\mathscr{r}-\vec{\mathscr{r}}\cdot\vec v(t_r)/c\right)\\ &= \frac{q}{4\pi\varepsilon_o}\frac{+1}{\left[\mathscr{r}-\vec{\mathscr{r}}\cdot\vec v(t_r)/c\right]^2}\left\{c\vec\nabla t_r+\frac{1}{c}\left[\vec v(t_r)+\left(\vec{\mathscr{r}}\cdot\vec a(t_r)\right)-v^2(t_r)\vec\nabla t_r\right]\right\}\\ &= \frac{q}{4\pi\varepsilon_o}\frac{1}{\left[\mathscr{r}-\vec{\mathscr{r}}\cdot\vec v(t_r)/c\right]^2}\left\{\frac{\vec v(t_r)}{c}+\frac{1}{c}\left[c^2-v^2(t_r)+\vec{\mathscr{r}}\cdot\vec a(t_r)\right]\vec\nabla t_r\right\}\end{aligned}$$

Now, what is $\vec\nabla t_r$ {***here***}?? $\quad t_r = t-\frac{\mathscr{r}}{c} = t-\frac{\left|\vec r-\vec r'(t_r)\right|}{c} = t-\frac{\left|\vec r-\vec{\mathrm{w}}(t_r)\right|}{c}$

But we already found that: $\vec\nabla\mathscr{r} = -c\vec\nabla t_r$, and noting that: $\mathscr{r}=\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}$

Then: $\vec\nabla\mathscr{r} = \vec\nabla\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \frac{1}{2}\frac{1}{\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}}\vec\nabla\left(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}\right) = \frac{1}{2\mathscr{r}}\vec\nabla\left(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}\right)$

Now: $\vec\nabla\left(\vec A\cdot\vec B\right) = \vec A\times\left(\vec\nabla\times\vec B\right)+\vec B\left(\vec\nabla\times\vec A\right)+\left(\vec A\cdot\vec\nabla\right)\vec B+\left(\vec B\cdot\vec\nabla\right)\vec A$ {Griffiths Product Rule # 4}

$$\vec\nabla\left(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}\right) = \vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}}+\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}} = 2\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+2\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}}$$

$\therefore$ $\quad \vec\nabla\mathscr{r} = \frac{1}{2r}\vec\nabla\left(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}\right) = \frac{1}{2\mathscr{r}}\left[2\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+2\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}}\right] = \frac{1}{\mathscr{r}}\left[\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}}\right]$

But: $\left(\vec\nabla\times\vec{\mathscr{r}}\right) = \vec v(t_r)\times\vec\nabla t_r$ {from term (4) above} $\therefore$ $\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right) = \vec{\mathscr{r}}\times\left(\vec v(t_r)\times\vec\nabla t_r\right)$

And:

$$\begin{aligned}\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}} &= \vec{\mathscr{r}}\cdot\vec\nabla\left(\vec r-\vec r'(t_r)\right) = \vec{\mathscr{r}}\cdot\vec\nabla\left(\vec r-\vec{\mathrm{w}}(t_r)\right)\\ &= \left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec r-\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathrm{w}}(t_r)\\ &= \left\{\mathscr{r}_x\frac{\partial x}{\partial x}\hat x+\mathscr{r}_y\frac{\partial y}{\partial y}\hat y+\mathscr{r}_z\frac{\partial z}{\partial z}\hat z\right\}-\vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)\\ &= \vec{\mathscr{r}}-\vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)\end{aligned}$$

from (2) above: $\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathrm{w}}(t_r) = \vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)$

$\therefore$

$$\begin{aligned}-c\vec\nabla t_r &= \vec\nabla\mathscr{r} = \frac{1}{\mathscr{r}}\left[\vec{\mathscr{r}}\times\left(\vec\nabla\times\vec{\mathscr{r}}\right)+\left(\vec{\mathscr{r}}\cdot\vec\nabla\right)\vec{\mathscr{r}}\right]\\ &= \frac{1}{\mathscr{r}}\left[\underbrace{\vec{\mathscr{r}}\times\vec v(t_r)\times\vec\nabla t_r}+\vec{\mathscr{r}}-\vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)\right]\end{aligned}$$

BAC – CAB rule

Thus: $\quad -c\vec\nabla t_r = \frac{1}{r}\left[\vec{\mathscr{r}}-\cancel{\vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)}+\overbrace{\cancel{\vec v(t_r)\left(\vec{\mathscr{r}}\cdot\vec\nabla t_r\right)}-\vec\nabla t_r\left(\vec{\mathscr{r}}\cdot\vec v(t_r)\right)}\right]$ ***more*** cancellations occur!!

Or: $-c\vec{\nabla}t_r = \frac{1}{\mathcal{R}}\left[\vec{\mathcal{R}} - \left(\vec{\mathcal{R}}\bullet\vec{v}(t_r)\right)\vec{\nabla}t_r\right]$

Now solve for $\vec{\nabla}t_r$: $\left[-c + \frac{1}{\mathcal{R}}\left(\vec{\mathcal{R}}\bullet\vec{v}(t_r)\right)\right]\vec{\nabla}t_r = \frac{\vec{\mathcal{R}}}{\mathcal{R}} = \hat{\mathcal{R}}$ $\Rightarrow$ $\vec{\nabla}t_r = \frac{\vec{\mathcal{R}}}{\mathcal{R}}\frac{1}{\left[\frac{1}{\mathcal{R}}\left(\vec{\mathcal{R}}\bullet\vec{v}(t_r)\right)-c\right]} = \frac{\vec{\mathcal{R}}}{\left[\vec{\mathcal{R}}\bullet\vec{v}(t_r)-\mathcal{R}c\right]}$

$$\therefore \quad \vec{\nabla}t_r = \frac{\vec{\mathcal{R}}}{\left[\vec{\mathcal{R}}\bullet\vec{v}(t_r)-\mathcal{R}c\right]} = \frac{-\vec{\mathcal{R}}}{\left[\mathcal{R}c-\vec{\mathcal{R}}\bullet\vec{v}(t_r)\right]} = -\frac{\vec{\mathcal{R}}}{\mathcal{R}c}\frac{1}{\left[1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c\right]} = -\frac{1}{c}\frac{\hat{\mathcal{R}}}{\left[1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c\right]} = -\frac{\hat{\mathcal{R}}}{\kappa c}$$

Thus: $\vec{\nabla}t_r = -\frac{1}{c}\frac{\hat{\mathcal{R}}}{\left[1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c\right]} = -\frac{\hat{\mathcal{R}}}{\kappa c}$, $\hat{\mathcal{R}} = \frac{\vec{\mathcal{R}}}{\mathcal{R}}$ $\vec{\mathcal{R}} = \mathcal{R}\hat{\mathcal{R}}$ and $\kappa \equiv 1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c$ = ***retardation factor***

Then:
$$\vec{\nabla}V_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{(\kappa\mathcal{R})^2}\left\{\frac{\vec{v}(t_r)}{c} + \frac{1}{c}\left[c^2 - v^2(t_r) + \vec{\mathcal{R}}\bullet\vec{a}(t_r)\right]\vec{\nabla}t_r\right\}$$
$$= \frac{q}{4\pi\varepsilon_o}\frac{1}{(\kappa\mathcal{R})^2}\left\{\frac{\vec{v}(t_r)}{c} - \frac{1}{\kappa c^2}\left[c^2 - v^2(t_r) + \vec{\mathcal{R}}\bullet\vec{a}(t_r)\right]\hat{r}\right\}$$

Thus: $$-\vec{\nabla}V_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{(\kappa\mathcal{R})^2}\left\{\left[\frac{c^2 - v^2(t_r) + \vec{\mathcal{R}}\bullet\vec{a}(t_r)}{\kappa c^2}\right]\hat{\mathcal{R}} - \frac{\vec{v}(t_r)}{c}\right\}$$

Or: $$-\vec{\nabla}V_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{(\kappa\mathcal{R})^2}\left\{\frac{1}{\kappa}\left[1-\left(\frac{v(t_r)}{c}\right)^2 + \frac{\vec{\mathcal{R}}\bullet\vec{a}(t_r)}{c^2}\right]\hat{\mathcal{R}} - \frac{\vec{v}(t_r)}{c}\right\}$$

Now: $$\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\frac{q\vec{v}(t_r)}{\left(1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c\right)\mathcal{R}} = \frac{\mu_o}{4\pi}\frac{q\vec{v}(t_r)}{\kappa\mathcal{R}} \qquad \left\{n.b. = \frac{\vec{v}(t_r)}{c^2}V_r(\vec{r},t)\right\}$$

Then:
$$\frac{\partial\vec{A}_r(\vec{r},t)}{\partial t} = \frac{\mu_o}{4\pi}q\frac{\partial}{\partial t}\left(\frac{\vec{v}(t_r)}{\mathcal{R}}\frac{1}{\left(1-\hat{\mathcal{R}}\bullet\vec{v}(t_r)/c\right)}\right) = \frac{\mu_o}{4\pi}q\frac{\partial}{\partial t}\left(\frac{\vec{v}(t_r)}{\left(\mathcal{R}-\vec{\mathcal{R}}\bullet\vec{v}(t_r)/c\right)}\right)$$
$$= \frac{\mu_o}{4\pi}q\left\{\frac{1}{\left(\mathcal{R}-\vec{\mathcal{R}}\bullet\vec{v}(t_r)/c\right)}\frac{\partial\vec{v}(t_r)}{\partial t} + \vec{v}(t_r)\frac{\partial}{\partial t}\left(\frac{1}{\left(\mathcal{R}-\vec{\mathcal{R}}\bullet\vec{v}(t_r)/c\right)}\right)\right\}$$

Now: $\frac{\partial\vec{v}(t_r)}{\partial t} = \frac{d\vec{v}(t_r)}{dt_r}\frac{\partial t_r}{\partial t} = \vec{a}(t_r)\frac{\partial t_r}{\partial t}$ where: $\vec{a}(t_r) = \frac{d\vec{v}(t_r)}{dt_r}$

What is $\frac{\partial t_r}{\partial t}$ {***here***} ??

Now: $c\Delta t_r = c(t-t_r) = \mathscr{R}$

Or: $c^2\Delta t_r^2 = c^2(t-t_r)^2 = \mathscr{R}^2 = \vec{\mathscr{R}}\cdot\vec{\mathscr{R}}$ ⇐ differentiate this expression with respect to t.

$$\frac{\partial}{\partial t}\left(c^2\Delta t_r^2\right) = c^2\frac{\partial}{\partial t}(t-t_r)^2 = \frac{\partial}{\partial t}\left(\vec{\mathscr{R}}\cdot\vec{\mathscr{R}}\right) = 2c^2(t-t_r)\left(1-\frac{\partial t_r}{\partial t}\right) = 2\vec{\mathscr{R}}\cdot\frac{\partial\vec{\mathscr{R}}}{\partial t}$$

But: $\mathscr{R} = |\vec{\mathscr{R}}| = |\vec{r}(t)-\vec{r}'(t_r)| = |\vec{r}(t)-\vec{w}(t_r)| = c\Delta t_r = c(t-t_r)$ and also: $\vec{\mathscr{R}} = \vec{r}(t)-\vec{w}(t_r)$

Thus: $$\frac{\partial}{\partial t}\left(c^2\Delta t_r^2\right) = 2c\left(c(t-t_r)\right)\left(1-\frac{\partial t_r}{\partial t}\right) = 2c\mathscr{R}\left(1-\frac{\partial t_r}{\partial t}\right) = 2\vec{\mathscr{R}}\cdot\frac{\partial\vec{\mathscr{R}}}{\partial t} = 2\vec{\mathscr{R}}\cdot\frac{\partial}{\partial t}\left(\vec{r}(t)-\vec{w}(t_r)\right)$$

$$\therefore\quad c\mathscr{R}\left(1-\frac{\partial t_r}{\partial t}\right) = \vec{\mathscr{R}}\cdot\left[\overbrace{\frac{\partial\vec{r}}{\partial t}}^{=0} - \frac{\partial\vec{w}(t_r)}{\partial t}\right]$$

n.b. $\dfrac{\partial\vec{r}}{\partial t} \equiv 0$ Because $\vec{r}$ = vector to field point $P(\vec{r})$ is a ***constant*** vector – ***i.e.*** ***stationary*** observer!

Thus: $$c\mathscr{R}\left(1-\frac{\partial t_r}{\partial t}\right) = \mathscr{R}c\left(1-\frac{\partial t_r}{\partial t}\right) = -\vec{\mathscr{R}}\cdot\frac{\partial\vec{w}(t_r)}{\partial t} = -\vec{\mathscr{R}}\cdot\frac{d\vec{w}(t_r)}{dt_r}\frac{\partial t_r}{\partial t} = -\vec{\mathscr{R}}\cdot\vec{v}(t_r)\frac{\partial t_r}{\partial t}\qquad \vec{v}(t_r) = \frac{d\vec{w}(t_r)}{dt_r}$$

$$\therefore\quad \mathscr{R}c\left(1-\frac{\partial t_r}{\partial t}\right) = -\vec{\mathscr{R}}\cdot\vec{v}(t_r)\frac{\partial t_r}{\partial t}$$ ⇐ Solve for $\dfrac{\partial t_r}{\partial t}$

$$\mathscr{R}c = \mathscr{R}c\frac{\partial t_r}{\partial t} - \vec{\mathscr{R}}\cdot\vec{v}(t_r)\frac{\partial t_r}{\partial t} = \left[\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}(t_r)\right]\frac{\partial t_r}{\partial t} \Rightarrow \frac{\partial t_r}{\partial t} = \frac{\mathscr{R}c}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}(t_r)}$$

Thus: $$\frac{\partial t_r}{\partial t} = \frac{\mathscr{R}c}{\mathscr{R}c - \vec{\mathscr{R}}\cdot\vec{v}(t_r)} = \left(\frac{\mathscr{R}c}{\mathscr{R}c}\right)\frac{1}{\left(1-\hat{\mathscr{R}}\cdot\vec{v}(t_r)/c\right)} = \frac{1}{\left(1-\hat{\mathscr{R}}\cdot\vec{v}(t_r)/c\right)} = \frac{1}{\kappa}$$

$$\therefore\quad \frac{\partial t_r}{\partial t} = \frac{1}{1-\hat{\mathscr{R}}\cdot\vec{v}(t_r)/c} = \frac{1}{\kappa}$$ where: $\kappa \equiv 1-\hat{\mathscr{R}}\cdot\vec{v}(t_r)/c$ = ***retardation factor***

Then: $$\frac{\partial\vec{v}(t_r)}{\partial t} = \vec{a}(t_r)\frac{\partial t_r}{\partial t} = \frac{\vec{a}(t_r)}{\kappa} = \frac{\vec{a}(t_r)}{\left(1-\hat{\mathscr{R}}\cdot\vec{v}(t_r)/c\right)}$$

Using: $\dfrac{\partial\left(1/u(x)\right)}{\partial x}=\dfrac{\partial u^{-1}(x)}{\partial x}=-1\,u^{-2}\dfrac{\partial u(x)}{\partial x}=\dfrac{-1}{u^2}\dfrac{\partial u(x)}{\partial x}$

Then:

$$\frac{\partial}{\partial t}\left(\frac{1}{\left(\imath-\vec{\imath}\bullet\vec{v}(t_r)/c\right)}\right)=\frac{-1}{\left(\imath-\vec{\imath}\bullet\vec{v}(t_r)/c\right)^2}\frac{\partial}{\partial t}\left(\imath-\vec{\imath}\bullet\vec{v}(t_r)/c\right)$$
$$=\frac{-1}{(\kappa\imath)^2}\left[\frac{\partial\imath}{\partial t}-\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\imath}\bullet\vec{v}(t_r)\right)\right]$$
$$=\frac{-1}{(\kappa\imath)^2}\left[\frac{\partial\imath}{\partial t}-\frac{1}{c}\left(\vec{v}(t_r)\bullet\frac{\partial\vec{\imath}}{\partial t}+\vec{\imath}\bullet\frac{\partial\vec{v}(t_r)}{\partial t}\right)\right]$$

But: $\imath=c\Delta t_r=c(t-t_r)$

$$\therefore\quad \frac{\partial\imath}{\partial t}=\frac{\partial}{\partial t}c(t-t_r)=c\frac{\partial(t-t_r)}{\partial t}=c\left[\frac{\partial t}{\partial t}-\frac{\partial t_r}{\partial t}\right]=c\left[1-\frac{\partial t_r}{\partial t}\right]\quad\text{where: }\frac{\partial t_r}{\partial t}=\frac{1}{\kappa}=\frac{1}{1-\hat{\imath}\bullet\vec{v}(t_r)/c}$$

Thus: $$\frac{\partial\imath}{\partial t}=c\left[1-\frac{1}{\kappa}\right]=c\left(\frac{\kappa-1}{\kappa}\right)=c\frac{\left(\cancel{1}-\hat{\imath}\bullet\vec{v}(t_r)/c-\cancel{1}\right)}{\kappa}=-\frac{\hat{\imath}\bullet\vec{v}(t_r)}{\kappa}=-\frac{\vec{\imath}\bullet\vec{v}(t_r)}{\kappa\imath}$$

But: $$\frac{\partial\vec{\imath}}{\partial t}=\frac{\partial\left(\vec{r}-\vec{w}(t_r)\right)}{\partial t}=\overbrace{\frac{\partial\vec{r}}{\partial t}}^{=0}-\frac{\partial\vec{w}(t_r)}{\partial t}=-\frac{d\vec{w}(t_r)}{dt_r}\frac{\partial t_r}{\partial t}=-\frac{\vec{v}(t_r)}{\kappa}\quad\text{and: }\frac{\partial\vec{v}(t_r)}{\partial t}=\frac{d\vec{v}(t_r)}{dt_r}\frac{\partial t_r}{\partial t}=\vec{a}(t_r)\frac{1}{\kappa}$$

Thus:

$$\frac{\partial}{\partial t}\left(\frac{1}{\left(\imath-\vec{\imath}\bullet\vec{v}(t_r)/c\right)}\right)=-\frac{1}{(\kappa\imath)^2}\left[\frac{\partial\imath}{\partial t}-\frac{1}{c}\left(\vec{v}(t_r)\bullet\frac{\partial\vec{\imath}}{\partial t}+\vec{\imath}\bullet\frac{\partial\vec{v}(t_r)}{\partial t}\right)\right]$$
$$=-\frac{1}{(\kappa\imath)^2}\left\{-\frac{\vec{\imath}\bullet\vec{v}(t_r)}{\kappa\imath}-\frac{1}{c}\left[-\frac{v^2(t_r)}{\kappa}+\frac{\vec{\imath}\bullet\vec{a}(t_r)}{\kappa}\right]\right\}$$
$$=\frac{+1}{\kappa(\kappa\imath)^2}\left\{\hat{\imath}\bullet\vec{v}(t_r)+\frac{1}{c}\left[-v^2(t_r)+\vec{\imath}\bullet\vec{a}(t_r)\right]\right\}$$

Then:

$$\frac{\partial\vec{A}_r(\vec{r},t)}{\partial t}=\frac{\mu_o}{4\pi}q\left\{\frac{1}{(\kappa\imath)}\frac{\partial\vec{v}(t_r)}{\partial t}+\vec{v}(t_r)\frac{\partial}{\partial t}\left(\frac{1}{\left(\imath-\vec{\imath}\bullet\vec{v}(t_r)/c\right)}\right)\right\}$$
$$=\frac{\mu_o}{4\pi}q\left\{\frac{1}{(\kappa\imath)}\frac{\vec{a}(t_r)}{\kappa}+\frac{\vec{v}(t_r)}{\kappa(\kappa\imath)^2}\left(\hat{\imath}\bullet\vec{v}(t_r)+\frac{1}{c}\left[-v^2(t_r)+\vec{\imath}\bullet\vec{a}(t_r)\right]\right)\right\}$$

Or: $$\frac{\partial\vec{A}_r(\vec{r},t)}{\partial t}=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{c^2}\right)\frac{1}{\kappa(\kappa\imath)}\left\{\vec{a}(t_r)+\frac{\vec{v}(t_r)}{(\kappa r)}\left(\hat{\imath}\bullet\vec{v}(t_r)+\frac{1}{c}\left[-v^2(t_r)+\vec{\imath}\bullet\vec{a}(t_r)\right]\right)\right\}\quad\text{using }\mu_o=\frac{1}{c^2\varepsilon_o}$$

Does ***this*** expression = Griffiths result for $\dfrac{\partial \vec{A}_r(\vec{r},t)}{\partial t}$, eqn 10.63 bottom of page 437 ?? YES, it does!

$$\frac{\partial \vec{A}_r(\vec{r},t)}{\partial t} = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^2 c^2}\left\{\kappa \mathscr{r}\vec{a}(t_r) + \left(\hat{\mathscr{r}}\bullet\vec{v}(t_r)\right)\vec{v}(t_r) - \frac{v^2(t_r)\vec{v}(t_r)}{c} + \frac{\left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)}{c}\right\}$$

But: $$c(1-\kappa)\vec{v}(t_r) = c\left(\cancel{1} - \left(\cancel{1} - \hat{\mathscr{r}}\bullet\vec{v}(t_r)/c\right)\right)\vec{v}(t_r) = \left(\hat{\mathscr{r}}\bullet\vec{v}(t_r)\right)\vec{v}(t_r)$$

Thus:

$$\begin{aligned}
\frac{\partial \vec{A}_r(\vec{r},t)}{\partial t} &= \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^2 c^2}\left\{c(1-\kappa)\vec{v}(t_r) + \kappa \mathscr{r}\vec{a}(t_r) - \frac{v^2(t_r)\vec{v}(t_r)}{c} + \frac{\left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)}{c}\right\} \\
&= \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^2 c^2}\left\{-\kappa c\vec{v}(t_r) + \kappa \mathscr{r}\vec{a}(t_r) + c\vec{v}(t_r) - \frac{v^2(t_r)\vec{v}(t_r)}{c} + \frac{\left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)}{c}\right\} \\
&= \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^3 c^2}\left\{-\kappa \mathscr{r} c\vec{v}(t_r) + \kappa \mathscr{r}^2\vec{a}(t_r) + \frac{\mathscr{r}}{c}\left(c^2 - v^2(t_r) + \left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)\right)\right\} \\
&= \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^3 c^2}\left\{\kappa \mathscr{r} c\left(-\vec{v}(t_r) + \mathscr{r}\vec{a}(t_r)/c\right) + \frac{\mathscr{r}}{c}\left(c^2 - v^2(t_r) + \left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)\right)\right\}
\end{aligned}$$

But: $$\kappa \mathscr{r} c = \left(1 - \hat{\mathscr{r}}\bullet\vec{v}(t_r)/c\right)\mathscr{r} c = \left(\mathscr{r} - \vec{\mathscr{r}}\bullet\vec{v}(t_r)/c\right)c = \left(\mathscr{r} c - \vec{\mathscr{r}}\bullet\vec{v}(t_r)\right)$$

$$\therefore \quad \frac{\partial \vec{A}_r(\vec{r},t)}{\partial t} = \frac{1}{4\pi\varepsilon_o}\frac{q}{\kappa^3 \mathscr{r}^3 c^2}\left\{\left(\mathscr{r} c - \vec{\mathscr{r}}\bullet\vec{v}(t_r)\right)\left(-\vec{v}(t_r) + \mathscr{r}\vec{a}(t_r)/c\right) + \frac{\mathscr{r}}{c}\left(c^2 - v^2(t_r) + \vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)\right\}$$

Or:

$$\frac{\partial \vec{A}_r(\vec{r},t)}{\partial t} = \frac{1}{4\pi\varepsilon_o}\frac{qc}{\left(\mathscr{r} c - \vec{\mathscr{r}}\bullet\vec{v}(t_r)\right)^3}\left\{\left(\mathscr{r} c - \vec{\mathscr{r}}\bullet\vec{v}(t_r)\right)\left(-\vec{v}(t_r) + \mathscr{r}\vec{a}(t_r)/c\right) + \frac{\mathscr{r}}{c}\left(c^2 - v^2(t_r) + \vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)\right\}$$

≡ Griffiths Equation 10.63 on page 437

Then {"finally"!} the ***retarded*** electric field for a ***moving point*** electric charge q:

$$\vec{E}_r(\vec{r},t) = -\vec{\nabla}V_r(\vec{r},t) - \frac{\partial \vec{A}_r(\vec{r},t)}{\partial t}$$

$$\begin{aligned}
\vec{E}_r(\vec{r},t) = &\frac{q}{4\pi\varepsilon_o}\frac{1}{(\kappa\mathscr{r})^2}\left\{\frac{1}{\kappa}\left[1 - \left(\frac{v(t_r)}{c}\right)^2 - \frac{\vec{\mathscr{r}}\bullet\vec{a}(t_r)}{c^2}\right]\hat{\mathscr{r}} - \frac{\vec{v}(t_r)}{c}\right\} \\
&- \frac{q}{4\pi\varepsilon_0}\frac{1}{\kappa(\kappa\mathscr{r})}\left(\frac{1}{c^2}\right)\left\{\vec{a}(t_r) + \frac{\vec{v}(t_r)}{\kappa\mathscr{r}}\left(\hat{\mathscr{r}}\bullet\vec{v}(t_r) + \frac{1}{c}\left[-v^2(t_r) + \vec{\mathscr{r}}\bullet\vec{a}(t_r)\right]\right)\right\}
\end{aligned}$$

With some more algebra, more vector identities, and defining a ***retarded*** vector $\boxed{\vec{u}(t_r) \equiv c\hat{\mathscr{r}} - \vec{v}(t_r)}$, the ***retarded*** electric field for a ***moving point*** charge can equivalently be written as:

$$\boxed{\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^3}\left[\left(c^2 - v^2(t_r)\right)\vec{u}(t_r) + \vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right]}$$

Since: $\boxed{\vec{A}_r(\vec{r},t) = \vec{\beta}\left(\vec{\mathrm{w}}(t_r)\right)\dfrac{V_r(\vec{r},t)}{c}}$ with: $\boxed{\vec{\beta}\left(\vec{\mathrm{w}}(t_r)\right) \equiv \dfrac{\vec{v}\left(\vec{\mathrm{w}}(t_r)\right)}{c}}$ then: $\boxed{\vec{A}_r(\vec{r},t) = \dfrac{\vec{v}(t_r)}{c^2}V_r(\vec{r},t)}$

The ***retarded*** magnetic field of a ***moving point*** charge q can be written as:

$$\boxed{\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}_r(\vec{r},t) = \frac{1}{c^2}\vec{\nabla}\times\left(\vec{v}(t_r)V_r(\vec{r},t)\right) = \frac{1}{c^2}\left[V_r(\vec{r},t)\left(\vec{\nabla}\times\vec{v}(t_r)\right) - \vec{v}(t_r)\times\vec{\nabla}V_r(\vec{r},t)\right]}$$

n.b. The relation on the RHS used Griffiths Product Rule # 7: $\boxed{\vec{\nabla}\times\left(f\vec{A}\right) = f\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\times\vec{\nabla}f}$

We already calculated $\boxed{\vec{\nabla}\times\vec{v}(t_r) = -\vec{a}(t_r)\times\vec{\nabla}t_r}$ {term (3) above, page 11 of these lecture notes}

and we found that: $\boxed{\vec{\nabla}t_r = -\dfrac{1}{c}\dfrac{\hat{\mathscr{r}}}{\left[1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c\right]} = -\dfrac{\hat{\mathscr{r}}}{\kappa c}}$ {see page 14 of these lecture notes}:

Hence: $\boxed{\vec{\nabla}\times\vec{v}(t_r) = -\vec{a}(t_r)\times\vec{\nabla}t_r = -\vec{a}(t_r)\times\left(-\dfrac{\hat{\mathscr{r}}}{\kappa c}\right) = +\left(\dfrac{\vec{a}(t_r)\times\vec{\mathscr{r}}}{\kappa c\mathscr{r}}\right)}$ where: $\boxed{\kappa \equiv 1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c}$

$\boxed{V_r(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_o\kappa\mathscr{r}}}$ {from page 9} and $-\vec{\nabla}V_r(\vec{r},t)$ is as given above {on page 14}.

Thus: *n.b.* $\boxed{\vec{v}(t_r)\times\vec{v}(t_r) \equiv 0}$

$$\boxed{\begin{aligned}\vec{B}_r(\vec{r},t) = \vec{\nabla}\times\vec{A}_r(\vec{r},t) &= \frac{q}{4\pi\varepsilon_o}\frac{1}{c^2}\frac{1}{\kappa\mathscr{r}}\left(\frac{\vec{a}(t_r)\times\vec{\mathscr{r}}}{\kappa c\mathscr{r}}\right)\\ &+\frac{q}{4\pi\varepsilon_o}\frac{\vec{v}(t_r)}{(\kappa\mathscr{r})^2}\times\left\{\frac{1}{\kappa}\left[1-\left(\frac{v(t_r)}{c}\right)^2 - \frac{\vec{\mathscr{r}}\bullet\vec{a}(t_r)}{c}\right]\hat{\mathscr{r}} - \frac{\vec{v}(t_r)}{c}\right\}\end{aligned}}$$

After some more algebra, and again using the retarded vector: $\boxed{\vec{u}(t_r) \equiv c\hat{\mathscr{r}} - \vec{v}(t_r)}$ the ***retarded*** magnetic field of a ***moving point*** electric charge q is:

$$\boxed{\vec{B}_r(\vec{r},t) = -\frac{1}{c}\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^3}\hat{\mathscr{r}}\times\left[\left(c^2-v^2(t_r)\right)\vec{v}(t_r) + \left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r) + \left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)\vec{a}(t_r)\right]}$$

Compare this expression to that for the ***retarded*** electric field of a ***moving point*** electric charge:

$$\boxed{\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^3}\left[\left(c^2 - v^2(t_r)\right)\vec{u}(t_r) + \vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right]}$$

The terms in the ***square brackets*** have some striking similarities between $\vec{B}_r(\vec{r},t)$ *vs.* $\vec{E}_r(\vec{r},t)$. Because of the cross product of terms such as $\hat{\mathscr{r}}\times\vec{v}(t_r)$ contained in the square brackets of the expression for $\vec{B}_r(\vec{r},t)$, notice that since: $\boxed{\vec{u}(t_r)\equiv c\hat{\mathscr{r}}-\vec{v}(t_r)}$ or: $\boxed{\vec{v}(t_r)=c\hat{\mathscr{r}}-\vec{u}(t_r)}$

Then: $\boxed{\hat{\mathscr{r}}\times\vec{v}(t_r)=\hat{\mathscr{r}}\times\left(c\hat{\mathscr{r}}-\vec{u}(t_r)\right)=\underbrace{c\hat{\mathscr{r}}\times\hat{\mathscr{r}}}_{\equiv 0}-\hat{\mathscr{r}}\times\vec{u}(t_r)=-\hat{\mathscr{r}}\times\vec{u}(t_r)}$, *i.e.* $\boxed{\hat{\mathscr{r}}\times\vec{v}(t_r)=-\hat{\mathscr{r}}\times\vec{u}(t_r)}$.

Then, using the $\boxed{\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)}$ rule, and $\boxed{\hat{\mathscr{r}}\times\vec{v}(t_r)=-\hat{\mathscr{r}}\times\vec{u}(t_r)}$ we see that:

$$\begin{aligned}
&\hat{\mathscr{r}}\times\left[\left(c^2-v^2(t_r)\right)\vec{v}(t_r)+\left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{v}(t_r)+\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\vec{a}(t_r)\right)\right]\\
&=\hat{\mathscr{r}}\times\left[-\left(c^2-v^2(t_r)\right)\vec{u}(t_r)-\underbrace{\left(\vec{\mathscr{r}}\bullet\vec{a}(t_r)\right)\vec{u}(t_r)+\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\vec{a}(t_r)\right)}_{=\vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)}\right]\\
&=\hat{\mathscr{r}}\times\left[-\left(c^2-v^2(t_r)\right)\vec{u}(t_r)-\vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right]
\end{aligned}$$

Thus: $$\boxed{\vec{B}_r(\vec{r},t)=\frac{1}{c}\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^3}\hat{\mathscr{r}}\times\left[\left(c^2-v^2(t_r)\right)\vec{u}(t_r)+\vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right]}$$

But: $$\boxed{\vec{E}_r(\vec{r},t)=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^3}\left[\left(c^2-v^2(t_r)\right)\vec{u}(t_r)+\vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right]}$$

$\therefore$ We again see that: $\boxed{\vec{B}_r(\vec{r},t)=\tfrac{1}{c}\hat{\mathscr{r}}\times\vec{E}_r(\vec{r},t)}$ *i.e.* $\vec{B}_r$ is $\perp$ $\vec{E}_r$; note also that $\vec{B}_r$ is $\perp$ $\vec{\mathscr{r}}$ where $\boxed{\vec{\mathscr{r}}=\vec{r}-\vec{r}'(t_r)=\vec{r}-\vec{\mathrm{w}}(t_r)}$ = separation distance vector from ***retarded*** source point to field point.

- The first term in $\vec{E}_r(\vec{r},t)$ involving $\left(c^2-v^2(t_r)\right)\vec{u}(t_r)$ falls off / decreases as $\mathscr{r}^2$. If both the ***velocity*** and ***acceleration*** are ***zero***, then we obtain the ***static*** limit result: $\boxed{\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_o}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}}$
- The first term in $\vec{E}_r(\vec{r},t)$ is known as the "Generalized" Coulomb field, *a.k.a.* ***velocity field*** which describes the macroscopic, collective behavior of ***virtual*** photons!!
- The second term in $\vec{E}_r(\vec{r},t)$, involving the triple product $\vec{\mathscr{r}}\times\left(\vec{u}\times\vec{a}\right)$ falls off / decreases as $1/\mathscr{r}$ – *i.e.* this term dominates at large separation distances!
- The second term in $\vec{E}_r(\vec{r},t)$ is responsible for ***radiation*** – hence the 2nd term is known as the ***radiation field***. Since it is also proportional to $\vec{a}(t_r)$ it is also known as the ***acceleration field*** which describes the macroscopic, collective behavior of ***real*** photons!!
- The same terminology obviously also applies to the retarded magnetic field $\vec{B}_r(\vec{r},t)$.

The *EM* force exerted on a point ***test charge*** q_T at the observation/field point $\vec{r}$ at the present time, t due to the ***retarded*** electric and magnetic fields associated with a ***moving point*** charge q is given by the ***retarded*** Lorentz force law: $\boxed{\vec{F}_{\rm r}(\vec{r},t) = q_T\vec{E}_{\rm r}(\vec{r},t) + q_T\vec{\rm v}_T(\vec{r},t)\times\vec{B}_{\rm r}(\vec{r},t)}$ where $\vec{\rm v}_T(\vec{r},t)$ = the velocity of the point ***test charge*** q_T ***at*** the ***observation***/***field*** point $\vec{r}$ at the ***present*** time, t.

$$\boxed{\begin{aligned}\vec{F}_{\rm r}(\vec{r},t) &= q_T\vec{E}_{\rm r}(\vec{r},t) + q_T\vec{\rm v}_T(\vec{r},t)\times\vec{B}_{\rm r}(\vec{r},t)\\ &= \frac{qq_T}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\cdot\vec{u}(t_{\rm r})\right)^3}\Big\{\left[\left(c^2-v^2(t_{\rm r})\right)\vec{u}(t_{\rm r})+\vec{\mathscr{r}}\times\left(\vec{u}(t_{\rm r})\times\vec{a}(t_{\rm r})\right)\right]\\ &\qquad +\frac{\vec{\rm v}_T(t)}{c}\times\left[\hat{\mathscr{r}}\times\left[\left(c^2-v^2(t_{\rm r})\right)\vec{u}(t_{\rm r})+\vec{\mathscr{r}}\times\left(\vec{u}(t_{\rm r})\times\vec{a}(t_{\rm r})\right)\right]\right]\Big\}\end{aligned}}$$

where: $\boxed{\vec{\mathscr{r}} = \vec{r}-\vec{r}'(t_{\rm r}) = \vec{r}-\vec{\rm w}(t_{\rm r})}$ and: $\boxed{\vec{u}(t_{\rm r}) \equiv c\hat{\mathscr{r}}-\vec{v}(t_{\rm r})}$

The Lorentz force $\vec{F}_{\rm r}(\vec{r},t)$ is the net force acting on a point test charge q_T moving with velocity $\vec{\rm v}_T(\vec{r},t)$ {*n.b.* which is evaluated at the ***present*** time {*i.e.* the ***non***-***retarded***} time t }.

The Lorentz Force $\vec{F}_{\rm r}(\vec{r},t)$ is due to the retarded electric and magnetic fields $\vec{E}_{\rm r}(\vec{r},t)$ and $\vec{B}_{\rm r}(\vec{r},t)$ associated with the point charge q moving with {its own} retarded velocity $\vec{v}(t_{\rm r})$ and acceleration $\vec{a}(t_{\rm r})$.

n.b. The ***lower***-***case*** quantities $\vec{\mathscr{r}}(t_{\rm r}) = \vec{r}-\vec{\rm w}(t_{\rm r})$, $\vec{v}(t_{\rm r})$, $\vec{u}(t_{\rm r}) \equiv c\hat{\mathscr{r}}-\vec{v}(t_{\rm r})$ and $\vec{a}(t_{\rm r})$ are ***all*** evaluated at the ***retarded*** time $t_{\rm r} = t-\mathscr{r}/c$.

<u>Griffiths Example 10.4</u>:

Calculate the electric and magnetic fields of a point charge q moving with ***<u>constant</u>*** velocity.

Constant velocity $\Rightarrow$ $\vec{a}(t_r) \equiv 0$ (***<u>no</u>*** acceleration).

Then: $$\vec{E}_r(\vec{r},t) \equiv \frac{q}{4\pi\varepsilon_o}\frac{\scriptstyle{\mathcal{R}}}{\left(\vec{\mathcal{R}}\bullet\vec{u}(t_r)\right)^3}\left[\left(c^2 - v^2(t_r)\right)\vec{u}(t_r)\right]$$ with: $\vec{u}(t_r) \equiv c\hat{\mathcal{R}} - \vec{v}(t_r)$

Trajectory: $\vec{w}(t_r) = \vec{v}(t_r)t_r = \vec{v}t_r$ for ***<u>constant</u>*** velocity.

The retarded time: $t_r = t - \mathcal{R}/c$ with: $\vec{\mathcal{R}}(t_r) = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}(t_r)t_r$ and: $\mathcal{R} = c\Delta t_r = c(t - t_r)$

Since: $\vec{u}(t_r) \equiv c\hat{\mathcal{R}} - \vec{v}(t_r)$, then:

$$\begin{aligned}
\mathcal{R}\vec{u}(t_r) &= \mathcal{R}\left(c\hat{\mathcal{R}} - \vec{v}(t_r)\right) \\
&= c\vec{\mathcal{R}} - \mathcal{R}\vec{v}(t_r) \\
&= c\left(\vec{r} - \vec{r}'(t_r)\right) - \mathcal{R}\vec{v}(t_r) \\
&= c\vec{r} - c\vec{w}(t_r) - \mathcal{R}\vec{v}(t_r) \\
&= c\vec{r} - c\vec{v}(t_r)t_r - c(t - t_r)\vec{v}(t_r) \\
&= c\vec{r} - \cancel{ct_r\vec{v}(t_r)} - ct\vec{v}(t_r) + \cancel{ct_r\vec{v}(t_r)} \\
&= c\left(\vec{r} - \vec{v}(t_r)t\right)
\end{aligned}$$

Note also: $$\vec{\mathcal{R}}\bullet\vec{u}(t_r) = \vec{\mathcal{R}}\bullet\left(c\hat{\mathcal{R}} - \vec{v}(t_r)\right) = c\mathcal{R} - \vec{\mathcal{R}}\bullet\vec{v}(t_r)$$

It can be shown that, since {***<u>here</u>***} $\vec{v}(t_r) = \vec{v}$ = ***<u>constant</u>*** velocity vector, then: $\vec{v}(t_r) = \vec{v}(t) = \vec{v}$, and thus:

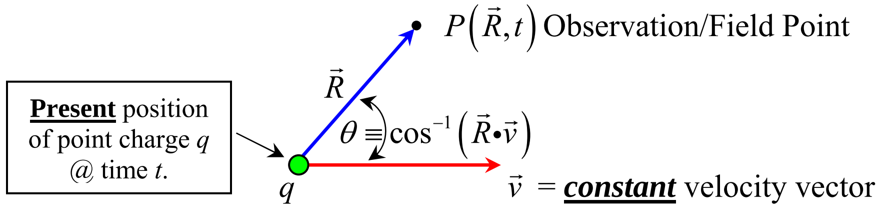
$$\begin{aligned}
\vec{\mathcal{R}}\bullet\vec{u}(t_r) &= \sqrt{\left(c^2 t - \vec{r}\bullet\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2t^2\right)} \\
&= Rc\sqrt{1 - v^2\sin^2\theta/c^2} \\
&= Rc\sqrt{1 - \beta^2\sin^2\theta} \quad \text{where } \beta \equiv v/c
\end{aligned}$$

(See *p*. 8 above, ***i.e***. Griffiths Example 10.3, *p*. 433)

(See Griffiths Problem 10.14, *p*. 434)

where $\vec{R} \equiv \vec{r} - \vec{v}t$ = vector from the ***<u>present</u>*** location of the point electric charged particle to the observation/field point $P(\vec{r}(t))$ at the ***<u>present</u>*** time t moving with ***<u>constant</u>*** velocity $\vec{v}$.

The angle $\theta \equiv \cos^{-1}(\vec{R}\bullet\vec{v})$ is the opening angle between $\vec{R}$ and the ***constant*** velocity vector $\vec{v}$, as shown in the figure below:



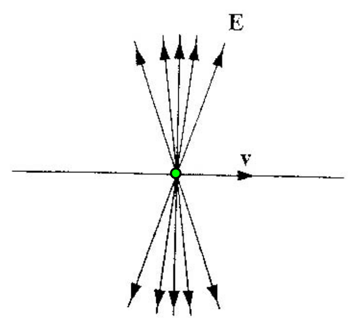
Thus: $$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}\left(\frac{\hat{R}}{R^2}\right)$$ with: $\beta \equiv \left(\frac{v}{c}\right)$ = ***constant***

This expression for $\vec{E}(\vec{r},t)$ shows that $\vec{E}(\vec{r},t)$ points along the line from the ***present*** position of the point charged particle to the observation/field point $P(\vec{R},t)$ – which is ***strange***, since the "*EM* news" came from the ***retarded*** position of the point charge. We will see/learn that the explanation for this is due to {special} relativity – peek ahead in Physics 436 Lect. Notes 18.5, *p*. 10-17.

Due to the $\beta^2\sin^2\theta$ term in the denominator of this expression, the $\vec{E}$-field of a fast-moving point charged particle is flattened/compressed into a "pancake" $\perp$ to the direction of motion, ***increasing*** the $\vec{E}$-field strength in the $\perp$ direction by a factor of $\gamma \equiv 1/\sqrt{1-\beta^2}$, whereas in the forward/backward directions { *i.e.* || and/or anti-|| to the direction of motion} the strength of the $\vec{E}$-field is ***reduced*** by a factor of $1-\beta^2$ relative to that of the $\vec{E}$-field strength when the point electric charge is at rest, as shown in the figure:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}\left(\frac{\hat{R}}{R^2}\right)$$

with: $\beta \equiv \left(\frac{v}{c}\right)$ = ***constant***



Lines of $\vec{E}$ are compressed into a "pancake" $\perp$ to the direction of motion as $v \to c$

For $\vec{B}(\vec{r},t)$ we have: $\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{\vec{r}-\vec{v}t_r}{\mathscr{r}} = \frac{(\vec{r}-\vec{v}t)+(t-t_r)\vec{v}}{\mathscr{r}} = \frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}}{c}$ since $t_r = t - \mathscr{r}/c$ or $\mathscr{r} = c\Delta t_r$

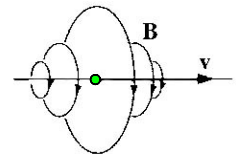
Thus:

$$\vec{B}(\vec{r},t) = \frac{1}{c}\left(\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)\right) = \frac{1}{c}\left(\left(\frac{\vec{R}}{\mathscr{r}}+\frac{\vec{v}}{c}\right)\times\vec{E}(\vec{r},t)\right) = \frac{1}{c\mathscr{r}}\overbrace{\vec{R}\times\vec{E}(\vec{r},t)}^{=0} + \frac{1}{c}\left(\frac{\vec{v}}{c}\right)\times\vec{E}(\vec{r},t)$$ $\Leftarrow$ *n.b.* $\vec{R} \parallel \vec{E}$

Or: $$\vec{B}(\vec{r},t) = \frac{1}{c}\left(\frac{\vec{v}}{c}\right)\times\vec{E}(\vec{r},t) = \frac{1}{c}\vec{\beta}\times\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o c}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}\left(\frac{\vec{\beta}\times\hat{R}}{R^2}\right)$$

These expressions for E and B were first obtained by Oliver Heaviside in 1888.

The lines of $\vec{B}$ circle around the point charge q as shown in the figure:

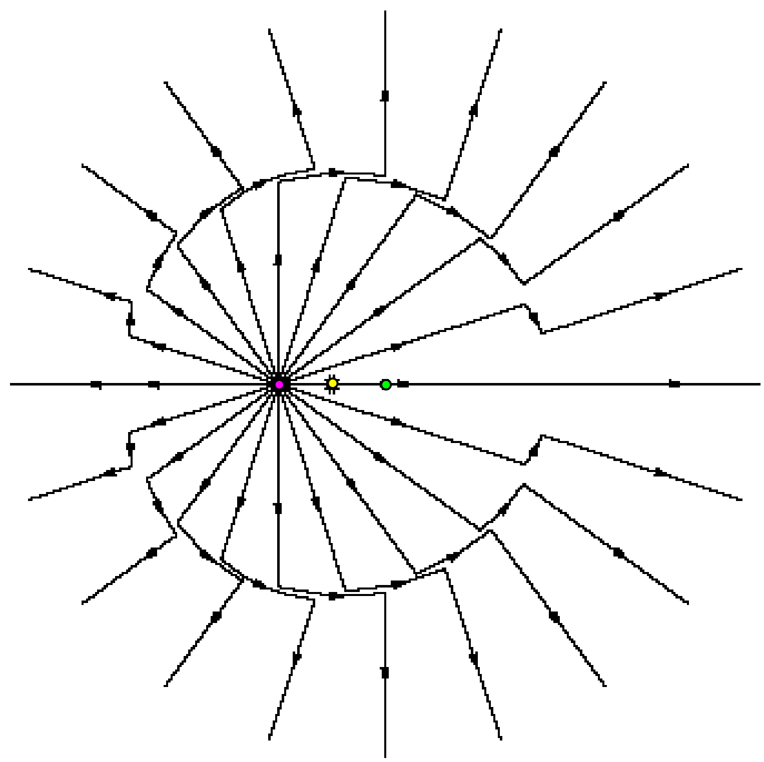


When $v \ll c$ (*i.e.* $\beta \ll 1$) the Heaviside expressions for $\vec{E}$ and $\vec{B}$ reduce to:

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\frac{q}{R^2}\hat{R}$$ ⇐ essentially Coulomb's law for a point electric charge

$$\vec{B}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right)\frac{q}{R^2}\left(\vec{v}\times\hat{R}\right)$$ ⇐ essentially Biot-Savart law for a point electric charge

The figure below shows a "snapshot-in-time" at time $t_2\,(sec)$ of the classical/macroscopic electric field lines associated with a point electric charge *q*, initially at rest {at $t_o = 0\,(sec)$, where the **green** dot is located}, that undergoes an abrupt, momentary acceleration {*i.e.* a short impulse lasting $\Delta t = t_1 - t_o = t_1\,(sec)$, where the yellow dot is located} in the ***horizontal*** direction, to the ***left*** in the figure. After the impulse has been applied, the charge continues to move to the ***left*** with ***constant*** velocity *v*, at time $t_2\,(sec)$ the charge is where the **pink** dot is located.



The classical/macroscopic electric field lines associated with one "epoch" in time must connect to their counterparts in another "epoch" of time. Here, in this situation, the spatial slopes of the *E*-field lines are discontinuous due to the abrupt, momentary nature of the acceleration. The spherical shell associated with the discontinuity(ies) in the electric field lines in the "transition" region between the two "epochs" expands at the speed of light, as the *EM* "news" propagates outward/away from the accelerated charge.

Note that this picture also meshes in nicely (and naturally!) with the ***microscopic*** perspective – namely that, when a point charge is accelerated it radiates ***real*** photons, which subsequently propagate away from the electric charge at the speed of light. Real photons have a ***transverse*** electric field relative to their propagation direction (whereas ***virtual*** photons associated with the static/Coulomb field are ***longitudinally*** polarized). The spherical shell associated with the discontinuity(ies) in the electric field lines of the "transition" region is precisely where the ***real*** photons are located in this "snapshot-in-time" picture, having propagated that far out from the charge after application of the abrupt, momentary impulse-type acceleration of the electric charge.

