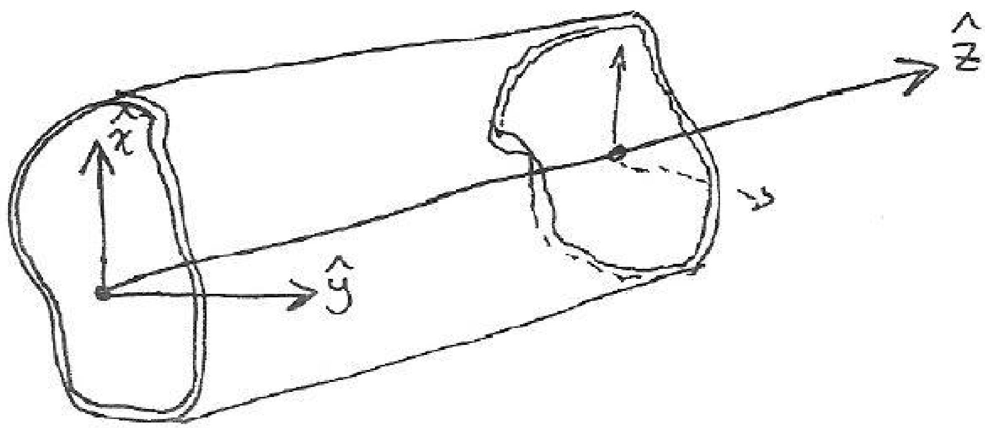
LECTURE NOTES 10

WAVE GUIDES and GUIDED *EM* WAVES

We consider/investigate the conditions under which *EM* waves can propagate when confined to the interior of some kind of "hollow" pipe – also know as a ***wave guide***. In the real world, ***wave guides*** consisting of *e.g.* rectangular, cylindrical, or arbitrarily cross-section shaped conducting and/or superconducting hollow metal pipes can be used to transport *EM* waves and *EM* energy in the radio and microwave region of the *EM* spectrum, whereas, *e.g.* glass or plastic optical fibers act as wave guides in the infrared, visible and even the UV portions of the *EM* spectrum.

We consider the simplest type of wave guide – a ***perfect*** conductor $\left(\sigma_C = \infty,\ \rho_C = 1/\sigma_C = 0\right)$ such that ***inside*** the walls of the perfect conductor: $\tilde{\vec{E}} = 0 \,\&\, \tilde{\vec{B}} = 0$.
n.b. in a ***perfect*** conductor $\tilde{\vec{E}}(\vec{r},t) = 0$ and by Faraday's Law, if $\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t) = 0 \Rightarrow \partial\tilde{\vec{B}}(\vec{r},t)/\partial t = 0$.
So if $\tilde{\vec{B}}(\vec{r},t=0) = 0$, it will remain $= 0\ \forall\, t$. A superconductor is a ***perfect*** conductor with $\tilde{\vec{B}}(\vec{r},t) = 0$ inside it (magnetic flux is ***expelled*** from a SC material – known as the Meissner effect).



The boundary conditions at/on the inner walls of a ***perfect*** conductor are:

$\boxed{\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t} = 0}$: $\boxed{\oint_C \tilde{\vec{E}}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \tilde{\vec{B}}\cdot d\vec{a} = 0}$ $\Rightarrow$ (1) Tangential $\vec{E}$ continuous: $\tilde{E}_{\parallel} = 0$ (since $\tilde{E}_{\parallel}^{inside} = 0$)

$\boxed{\vec{\nabla}\cdot\tilde{\vec{B}} = 0}$: $\boxed{\oint_S \tilde{\vec{B}}\cdot d\vec{a} = 0}$ $\Rightarrow$ (2) Normal $\tilde{\vec{B}}$ continuous: $\tilde{B}_{\perp} = 0$ (since $\tilde{B}_{\perp}^{inside} = 0$)

Note that free surface charges σ_{free} and free surface currents $\vec{K}_{free}$ ***will*** be induced on the inner surfaces of this ***perfectly*** conducting wave guide so as to "enforce" these boundary conditions:

$\boxed{\oint_S \tilde{\vec{D}}\cdot d\vec{a} = Q_{free}^{encl}}$: $\boxed{D_{\perp}^{outside} - D_{\perp}^{inside} = \sigma_{free}}$ and: $\boxed{\oint_C \tilde{\vec{H}}\cdot d\vec{\ell} = I_{free}^{encl}}$: $\boxed{H_{\parallel}^{outside} - H_{\parallel}^{inside} = \vec{K}_{free}\times\hat{n}}$

We assume {for the moment} that the wave guide has a rectangular cross section – hence we will use rectangular coordinates in the following discussions. Solutions for *E* and *B* must satisfy the wave equations: $\boxed{\nabla^2\tilde{\vec{E}} - \tfrac{1}{c^2}\partial^2\tilde{\vec{E}}/\partial t^2 = 0}$, $\boxed{\nabla^2\tilde{\vec{B}} - \tfrac{1}{c^2}\partial^2\tilde{\vec{B}}/\partial t^2 = 0}$ and the boundary conditions…

We are interested in/seek ***steady-state*** monochromatic/single-frequency *EM* traveling plane wave solutions - that propagate down the inside of the wave guide, *e.g.* in the $+\hat{z}$ direction of the above figure. Generically, these must be of the form:

$$\tilde{\vec{E}}(x,y,z,t)=\tilde{\vec{E}}_o(x,y)e^{i(k_z z-\omega t)}$$
$$\tilde{\vec{B}}(x,y,z,t)=\tilde{\vec{B}}_o(x,y)e^{i(k_z z-\omega t)}$$

n.b. for the cases of interest to us, the wave number k_z will turn out to be ***real***.

In the ***interior*** region of the wave guide, away from (*i.e.* not inside) the walls, Maxwell's equations must be satisfied, which, for empty space or *e.g.* air with $\varepsilon_{air}\simeq\varepsilon_o$ and $\mu_{air}\simeq\mu_o$ are:

(1) Gauss' Law: $\vec{\nabla}\cdot\tilde{\vec{E}}=0$ (2) No magnetic charges/monopoles: $\vec{\nabla}\cdot\tilde{\vec{B}}=0$

(3) Faraday's Law: $\vec{\nabla}\times\tilde{\vec{E}}=-\partial\tilde{\vec{B}}/\partial t$ (4) Ampere's Law: $\vec{\nabla}\times\tilde{\vec{B}}=\varepsilon_o\mu_o\,\partial\tilde{\vec{E}}/\partial t=\left(1/c^2\right)\partial\tilde{\vec{E}}/\partial t$

The question then is, what restrictions arising from the boundary conditions (1) $\tilde{E}^{\parallel}=0$ and (2) $\tilde{B}^{\perp}=0$ are imposed on $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ in satisfying Maxwell's equations (1) – (4) above?

Note also that ***confined*** *EM* waves (*e.g.* for propagation inside of wave guides) are ***not*** (in general) ***purely transverse waves***!

The boundary conditions (1) $\tilde{E}^{\parallel}=0$ and 2) $\tilde{B}^{\perp}=0$ will (in general, for confined waves) require ***longitudinal*** components: $\tilde{E}_{o_z}(x,y)$ and $\tilde{B}_{o_z}(x,y)$. Generically, our $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$- fields interior to the wave guide will thus be of the form(s):

$$\tilde{\vec{E}}(x,y,z,t)=\tilde{\vec{E}}_o(x,y)e^{i(k_z z-\omega t)} \quad \text{with: } \tilde{\vec{E}}_o(x,y)=\tilde{E}_{o_x}(x,y)\hat{x}+\tilde{E}_{o_y}(x,y)\hat{y}+\tilde{E}_{o_z}(x,y)\hat{z}$$

and: $$\tilde{\vec{B}}(x,y,z,t)=\tilde{\vec{B}}_o(x,y)e^{i(k_z z-\omega t)} \quad \text{with: } \tilde{\vec{B}}_o(x,y)=\tilde{B}_{o_x}(x,y)\hat{x}+\tilde{B}_{o_y}(x,y)\hat{y}+\tilde{B}_{o_z}(x,y)\hat{z}$$

If these expressions are inserted into (3) Faraday's Law and (4) Ampere's Law (above) we obtain:

(3) Faraday's Law:

(i) $$\frac{\partial\tilde{E}_{o_y}}{\partial x}-\frac{\partial\tilde{E}_{o_x}}{\partial y}=i\omega\tilde{B}_{o_z}$$

(ii) $$\frac{\partial\tilde{E}_{o_z}}{\partial y}-\frac{\partial\tilde{E}_{o_y}}{\partial z}=i\omega\tilde{B}_{o_x} \;\rightarrow\; \frac{\partial\tilde{E}_{o_z}}{\partial y}-ik_z\tilde{E}_{o_y}=i\omega\tilde{B}_{o_x}$$

(iii) $$\frac{\partial\tilde{E}_{o_x}}{\partial z}-\frac{\partial\tilde{E}_{o_z}}{\partial x}=i\omega\tilde{B}_{o_y} \;\rightarrow\; ik_z\tilde{E}_{o_x}-\frac{\partial\tilde{E}_{o_z}}{\partial x}=i\omega\tilde{B}_{o_y}$$

(4) Ampere's Law:

(iv) $$\frac{\partial\tilde{B}_{o_y}}{\partial x}-\frac{\partial\tilde{B}_{o_x}}{\partial y}=-\frac{i\omega}{c^2}\tilde{E}_{o_z}$$

(v) $$\frac{\partial\tilde{B}_{o_z}}{\partial y}-\frac{\partial\tilde{B}_{o_y}}{\partial z}=-\frac{i\omega}{c^2}\tilde{E}_{o_x} \;\rightarrow\; \frac{\partial\tilde{B}_{o_z}}{\partial y}-ik_z\tilde{B}_{o_y}=-\frac{i\omega}{c^2}\tilde{E}_{o_x}$$

(vi) $$\frac{\partial\tilde{B}_{o_x}}{\partial z}-\frac{\partial\tilde{B}_{o_z}}{\partial x}=-i\frac{\omega}{c^2}\tilde{E}_{o_y} \;\rightarrow\; ik_z\tilde{B}_{o_x}-\frac{\partial\tilde{B}_{o_z}}{\partial x}=-i\frac{\omega}{c^2}\tilde{E}_{o_y}$$

Note the cyclic permutations in *x, y, z* for (i)-(iii) and for (iv)-(vi).

We can use the four equations (ii), (iii), (v), and (vi) to solve for $\tilde{E}_{o_x}, \tilde{E}_{o_y}, \tilde{B}_{o_x}$ and $\tilde{B}_{o_y}$ in terms of $\tilde{E}_{o_z}$ and $\tilde{B}_{o_z}$, which, after some algebra yield:

(a) $$\tilde{E}_{o_x} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{E}_{o_z}}{\partial x} + \omega \frac{\partial \tilde{B}_{o_z}}{\partial y}\right)$$

(b) $$\tilde{E}_{o_y} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{E}_{o_z}}{\partial y} - \omega \frac{\partial \tilde{B}_{o_z}}{\partial x}\right)$$

(c) $$\tilde{B}_{o_x} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{B}_{o_z}}{\partial x} - \omega \frac{\partial \tilde{E}_{o_z}}{\partial y}\right)$$

(d) $$\tilde{B}_{o_y} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{B}_{o_z}}{\partial y} + \omega \frac{\partial \tilde{E}_{o_z}}{\partial x}\right)$$

We now insert (a) – (d) above into the other two Maxwell's equations:
(1) Gauss' Law: $\vec{\nabla}\bullet\tilde{\vec{E}} = 0$ and (2) No magnetic charges: $\vec{\nabla}\bullet\tilde{\vec{B}} = 0$

$$\frac{\partial \tilde{E}_{o_x}}{\partial x} + \frac{\partial \tilde{E}_{o_y}}{\partial y} + \frac{\partial \tilde{E}_{o_z}}{\partial z} = 0 \quad \text{and:} \quad \frac{\partial \tilde{B}_{o_x}}{\partial x} + \frac{\partial \tilde{B}_{o_y}}{\partial y} + \frac{\partial \tilde{B}_{o_z}}{\partial z} = 0$$

We obtain (after some more algebra): two ***decoupled*** wave equations for $\tilde{E}_{o_z}$ and $\tilde{B}_{o_z}$:

(α) $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k_z^2\right]\tilde{E}_{o_z} = 0$$

(β) $$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k_z^2\right]\tilde{B}_{o_z} = 0$$

For monochromatic *EM* traveling plane waves propagating in the $+\hat{z}$ direction:

Longitudinal component of $\tilde{\vec{E}} = 0$

Case I: $\tilde{E}_{o_z} = 0$ but: $\tilde{B}_{o_z} \neq 0$: **TE** (Transverse Electric) waves.

Longitudinal component of $\tilde{\vec{B}} = 0$

Case II: $\tilde{B}_{o_z} = 0$ but: $\tilde{E}_{o_z} \neq 0$: **TM** (Transverse Magnetic) waves.

Case III: ***Both*** $\tilde{E}_{o_z} = \tilde{B}_{o_z} = 0$: **TEM** (Transverse Electric & Magnetic) waves.

n.b. **TEM** waves ***cannot*** propagate in ***hollow*** wave guides*
{*unless the wavelength $\lambda \ll$ cross-sectional dimensions *a*, *b* of the waveguide}.
TEM waves ***can*** propagate *e.g.* in a ***coaxial*** waveguide structure with a ***center*** conductor.

Case I: $\boxed{\tilde{E}_{o_z}=0}$ {TE waves}, then Gauss' Law ($\vec{\nabla}\bullet\tilde{\vec{E}}=0$) becomes: $\boxed{\dfrac{\partial\tilde{E}_{o_x}}{\partial x}+\dfrac{\partial\tilde{E}_{o_y}}{\partial y}=0}$

Case II: $\boxed{\tilde{B}_{o_z}=0}$ {TM waves}, then Faraday's Law $\left(\vec{\nabla}\times\tilde{\vec{E}}\right)_z=-\dfrac{\partial\tilde{B}_z}{\partial t}$ becomes: $\boxed{\dfrac{\partial\tilde{E}_{o_y}}{\partial x}-\dfrac{\partial\tilde{E}_{o_x}}{\partial y}=0}$

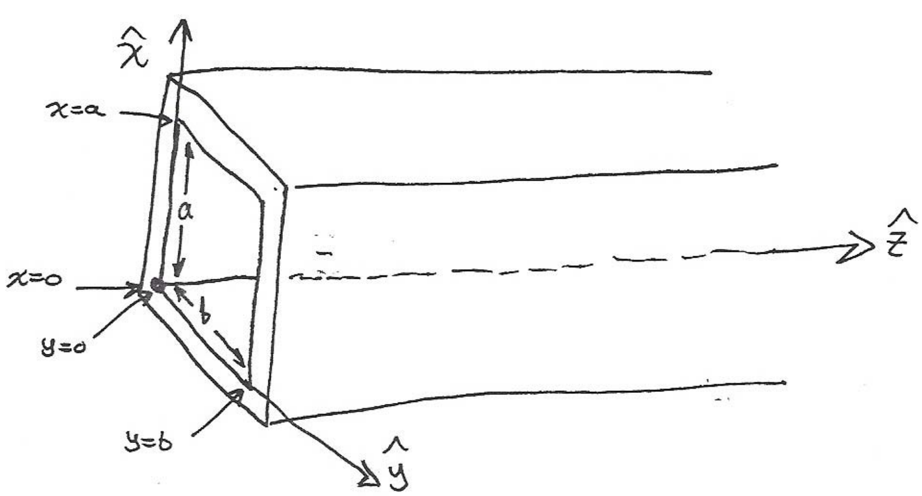
Case III: ***Both*** $\boxed{\tilde{E}_{o_z}=\tilde{B}_{o_z}=0}$ {TEM waves}, from (α) and (β) above, we see that $k=\omega/c$.
$\Rightarrow$ we must go back and fully solve equations (i) – (vi) on page 2 (above).

Note that $\tilde{\vec{E}}_o$ for TEM waves {with $\tilde{E}_{o_z}=0$} ***does*** satisfy $\vec{\nabla}\bullet\tilde{\vec{E}}=0$ and $\vec{\nabla}\times\tilde{\vec{E}}=0$.
i.e. $\tilde{\vec{E}}_o$ has zero divergence and zero curl. $\Rightarrow$ $\tilde{\vec{E}}_o=-\vec{\nabla}\tilde{V}_{scalar}$ $\Rightarrow$ Hence $\tilde{V}_{scalar}$ satisfies Laplaces' equation: $\vec{\nabla}\bullet\left(-\vec{\nabla}\tilde{V}\right)=-\nabla^2\tilde{V}=0$. However, the boundary condition (1): $\boxed{\tilde{\vec{E}}^{\parallel}=0}$ at the inner surface of waveguide $\Rightarrow$ the inner surface of the waveguide is an ***equipotential***, *i.e.* $\tilde{V}=$constant at/on the inner surface of the wave guide.

Since Laplace's equation does not allow local maxima or minima (extrema) anywhere ***except*** on the surfaces, then for a ***hollow*** waveguide, the potential $\tilde{V}$ interior to the wave guide ***must*** be a ***constant*** everywhere, hence: $\tilde{\vec{E}}_o=-\vec{\nabla}\tilde{V}=0$ everywhere inside the waveguide. $\Rightarrow$ No TEM wave propagation can occur in ***hollow*** wave guides* {*unless the wavelength $\lambda\ll$ cross-sectional dimensions *a*, *b* of the waveguide – then TEM waves are a special/limiting case of TE waves… *e.g. EM* light waves in an ***optical fiber*** = waveguide!!!}.

Case I: Propagation of TE Waves in a Perfectly Conducting Hollow Rectangular Waveguide

Consider a perfectly conducting, hollow rectangular waveguide of (inner) height *a* and width *b* as shown in the figure below {*n.b.* ***important***: $a\geq b$ by ***convention***!!!}:



For TE waves: $\boxed{\tilde{E}_{o_z}(x,y)=0}$ and: $\boxed{\tilde{B}_{o_z}(x,y)\neq 0}$, thus:

(α) $\boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{E}_{o_z}=0}$ But: $\boxed{\tilde{E}_{o_z}(x,y)=0}$ for TE waves.

i.e. 0 = 0 ⇒ ***this*** equation contains ***no*** information.

(β) $\boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{B}_{o_z}=0}$ But: $\boxed{\tilde{B}_{o_z}(x,y)\neq 0}$ for TE waves.

this equation ***does*** contain useful information.

The boundary condition for $\tilde{\vec{B}}_o(x,y)$ is $\boxed{\tilde{B}^{\perp}=0}$ on the inner walls of waveguide.

But: $\boxed{\tilde{\vec{B}}_o(x,y)=\tilde{B}_{o_x}(x,y)\hat{x}+B_{o_y}(x,y)\hat{y}+B_{o_z}(x,y)\hat{z}}$. Then, referring to the above figure:

$\boxed{B^{\perp}=0}$ in the $\hat{x}$-direction: $\boxed{\tilde{B}_{o_x}(x=0,y)=\tilde{B}_{o_x}(x=a,y)=0}$

$\boxed{B^{\perp}=0}$ in the $\hat{y}$-direction: $\boxed{\tilde{B}_{o_y}(x,y=0)=\tilde{B}_{o_y}(x,y=b)=0}$

But from equations (c) and (d) above:

(c) $\boxed{\tilde{B}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}-\omega\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial y}\right)}$

then: $\boxed{\tilde{B}_{o_x}(x=0,y)=\tilde{B}_{o_x}(x=a,y)=0}\Rightarrow\boxed{\frac{\partial\tilde{B}_{o_z}(x=0,y)}{\partial x}=\frac{\partial\tilde{B}_{o_z}(x=a,y)}{\partial x}=0}$

n.b. These terms = 0 because $E_{oz}(x,y)=0$ for TE waves.

(d) $\boxed{\tilde{B}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}+\omega\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial x}\right)}$

then: $\boxed{\tilde{B}_{o_y}(x,y=0)=\tilde{B}_{o_y}(x,y=b)=0}\Rightarrow\boxed{\frac{\partial\tilde{B}_{o_z}(x,y=0)}{\partial y}=\frac{\partial\tilde{B}_{o_z}(x,y=b)}{\partial y}=0}$

Now, to solve the wave equation for $\tilde{B}_{o_z}(x,y)$:

Namely (β) $\boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{B}_{o_z}(x,y)=0}$

Use ***separation of variables technique*** – try a ***product solution*** of the form: $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{X}(x)\cdot\tilde{Y}(y)}$

Inserting this into the above equation (β): $\boxed{\tilde{Y}(y)\frac{\partial^2\tilde{X}(x)}{\partial x^2}+\tilde{X}(x)\frac{\partial^2\tilde{Y}}{\partial y^2}+\left[\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{X}(x)\cdot\tilde{Y}(y)=0}$

Divide through by $\boxed{\tilde{X}((x)\cdot\tilde{Y}(y)}$:

$$\boxed{\underbrace{\frac{1}{\tilde{X}(x)}\frac{\partial^2\tilde{X}(x)}{\partial x^2}}_{fcn\ \text{of}\ x\ \text{only}}+\underbrace{\frac{1}{\tilde{Y}(y)}\frac{\partial^2\tilde{Y}(y)}{\partial y^2}}_{fcn\ \text{of}\ y\ \text{only}}=-\left[\left(\frac{\omega}{c}\right)^2-k_z^2\right]=\text{constant}}$$

The above relation can be true for arbitrary (x,y) points ***iff*** (if and only if):

$$(\gamma)\quad \boxed{\left(\frac{1}{\tilde{X}(x)}\frac{\partial^2\tilde{X}(x)}{\partial x^2}\right) = -k_x^2 = \text{constant}}$$

$$(\delta)\quad \boxed{\left(\frac{1}{\tilde{Y}(y)}\frac{\partial^2 Y(y)}{\partial y^2}\right) = -k_y^2 = \text{constant}' \ \left(\neq -k_x^2\right)}$$

The ***characteristic*** equation (*aka* the ***dispersion*** ***relation***) becomes:

$$\boxed{-k_x^2 - k_y^2 = -\left[\left(\frac{\omega}{c}\right)^2 - k_z^2\right] = \text{constant}''} \quad \text{or:} \quad \boxed{k_z^2(\omega) = \left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2} \Leftarrow \boxed{\textbf{n.b. } k_z(\omega) \text{ is frequency dependent!}}$$

We can rewrite the ***characteristic*** ***equation*** as: $\boxed{\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2(\omega) = k^2 = \left|\vec{k}\right|^2 = \vec{k}\bullet\vec{k}}$

The general solutions of the equations: $\boxed{\frac{\partial^2\tilde{X}(x)}{\partial x^2} + k_x^2\tilde{X}(x) = 0}$ and: $\boxed{\frac{\partial^2 Y(y)}{\partial y^2} + k_y^2\tilde{Y}(y) = 0}$

are of the form: $\boxed{\tilde{X}(x) = \tilde{A}_x\cos(k_x x) + \tilde{B}_x\sin(k_x x)}$ and: $\boxed{\tilde{Y}(y) = \tilde{A}_y\cos(k_y y) + \tilde{B}_y\sin(k_y y)}$

The boundary condition $\boxed{\tilde{B}^\perp = 0}$ requires ***not only***:

LHS (c): $\boxed{\tilde{B}_{o_x}(x=0,y) = \tilde{B}_{o_x}(x=a,y) = 0}$ ***but also*** RHS (c): $\boxed{\frac{\partial\tilde{B}_{o_z}(x=0,y)}{\partial x} = \frac{\partial\tilde{B}_{o_z}(x=a,y)}{\partial x} = 0}$

LHS (d): $\boxed{\tilde{B}_{o_y}(x,y=0) = \tilde{B}_{o_y}(x,y=b) = 0}$ ***but also*** RHS (d): $\boxed{\frac{\partial\tilde{B}_{o_z}(x,y=0)}{\partial y} = \frac{\partial\tilde{B}_{o_z}(x,y=b)}{\partial y} = 0}$

Since: $\boxed{\tilde{B}_{o_z}(x,y) = \tilde{X}(x)\cdot\tilde{Y}(y)}$, these LATTER boundary conditions require:

$$\boxed{\frac{\partial\tilde{X}(x=0)}{\partial x} = \frac{\partial\tilde{X}(x=a)}{\partial x} = 0} \quad \text{and:} \quad \boxed{\frac{\partial\tilde{Y}(y=0)}{\partial y} = \frac{\partial\tilde{Y}(y=a)}{\partial y} = 0}$$

So if: $\boxed{\tilde{X}(x) = \tilde{A}_x\cos(k_x x) + \tilde{B}_x\sin(k_x x)}$ and: $\boxed{\tilde{Y}(y) = \tilde{A}_y\cos(k_y y) + \tilde{B}_y\sin(k_y y)}$

Then: $\boxed{\frac{\partial\tilde{X}(x)}{\partial x} = -k_x\tilde{A}_x\sin(k_x x) + k_x\tilde{B}_x\cos(k_x x)}$ and: $\boxed{\frac{\partial\tilde{Y}(y)}{\partial y} = -k_y\tilde{A}_y\sin(k_y y) + k_y\tilde{B}_y\cos(k_x x)}$

Thus: $\boxed{\frac{\partial\tilde{X}(x=0)}{\partial x} = 0}$ ***requires***: $\boxed{\tilde{B}_x = 0}$ and: $\boxed{\frac{\partial\tilde{Y}(y=0)}{\partial y} = 0}$ ***requires***: $\boxed{\tilde{B}_y = 0}$

Hence: $\boxed{\tilde{X}(x) = \tilde{A}_x\cos(k_x x)}$ and: $\boxed{\tilde{Y}(y) = \tilde{A}_y\cos(k_y y)}$

Likewise:

$$\boxed{\frac{\partial \tilde{X}(x=a)}{\partial x}=0} \;\; \underline{\textbf{requires}}: \boxed{k_x a = m\pi,\;\; m=0,1,2,3,\ldots} \;\; \underline{\textbf{or}}: \boxed{k_x=\left(\frac{m\pi}{a}\right), m=0,1,2,3,\ldots}$$

and: $$\boxed{\frac{\partial Y(y=b)}{\partial y}=0} \;\; \underline{\textbf{requires}}: \boxed{k_y b = n\pi,\;\; n=0,1,2,3,\ldots} \;\; \underline{\textbf{or}}: \boxed{k_y=\left(\frac{n\pi}{b}\right),\;\; n=0,1,2,3,\ldots}$$

Then: $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{X}(x)\cdot\tilde{Y}(y)}$ becomes, after absorbing/re-defining the coefficients $\tilde{A}_x$ & $\tilde{A}_y$ into a ***single*** coefficient $\tilde{B}_o$:

$$\boxed{\tilde{B}_{o_z}(x,y)=\tilde{B}_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$$

The full (*x*, *y*, *z*, *t*) dependence is:

$$\boxed{\tilde{B}_z(x,y,z,t)=\tilde{B}_{o_z}(x,y)e^{i(k_z z-\omega t)}=\tilde{B}_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i(k_z z-\omega t)} \qquad \begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$$

The ***characteristic equation*** becomes:

$$\boxed{k_z^2(\omega)=\left(\frac{\omega}{c}\right)^2-k_x^2-k_y^2=\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2 \qquad \begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$$

Thus, having determined $\tilde{B}_{o_z}(x,y)$ and, since for the TE mode: $\boxed{\tilde{E}_{o_z}(x,y)\equiv 0}$, we can now determine $\tilde{E}_{o_x}$, $\tilde{E}_{o_y}$, $\tilde{B}_{o_x}$ and $\tilde{B}_{o_y}$ in terms of $\tilde{B}_o$ using equations (a) – (d) above:

(a) $$\boxed{\tilde{E}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial x}}+\omega\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}\right)=\frac{i\omega}{(\omega/c)^2-k_z^2}\left(\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}\right)}$$

(b) $$\boxed{\tilde{E}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial y}}-\omega\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}\right)=\frac{-i\omega}{(\omega/c)^2-k_z^2}\left(\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}\right)}$$

(c) $$\boxed{\tilde{B}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}-\omega\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial y}}\right)=\frac{ik}{(\omega/c)^2-k_z^2}\left(\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}\right)}$$

(d) $$\boxed{\tilde{B}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}+\omega\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial x}}\right)=\frac{ik}{(\omega/c)^2-k_z^2}\left(\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}\right)}$$

But: $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{B}_o\cos(k_x x)\cos(k_y y)}$ with: $\boxed{k_x=\left(\frac{m\pi}{a}\right)}$, $\boxed{k_y=\left(\frac{n\pi}{b}\right)}$ and: $\boxed{\begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$

Explicitly carrying out the spatial differentiation in (a)-(d) above, then for TE wave propagation:

(a) $\boxed{\tilde{E}_{o_x}(x,y) = \dfrac{-i\omega k_y}{(\omega/c)^2 - k_z^2}\tilde{B}_o \cos(k_x x)\sin(k_y y)}$ with: $\boxed{k_x = \left(\dfrac{m\pi}{a}\right)}$, $\boxed{k_y = \left(\dfrac{n\pi}{b}\right)}$, $\boxed{\begin{matrix} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \end{matrix}}$

(b) $\boxed{\tilde{E}_{o_y}(x,y) = \dfrac{+i\omega k_x}{(\omega/c)^2 - k_z^2}\tilde{B}_o \sin(k_x x)\cos(k_y y)}$ and: $\boxed{k_z^2 = \left(\dfrac{\omega}{c}\right)^2 - k_x^2 - k_y^2}$

(c) $\boxed{\tilde{E}_{o_z}(x,y) \equiv 0}$

(d) $\boxed{\tilde{B}_{o_x}(x,y) = \dfrac{-ik_z k_x}{(\omega/c)^2 - k_z^2}\tilde{B}_o \sin(k_x x)\cos(k_y y)}$

(e) $\boxed{\tilde{B}_{o_y}(x,y) = \dfrac{-ik_z k_y}{(\omega/c)^2 - k_z^2}\tilde{B}_o \cos(k_x x)\sin(k_y y)}$

(f) $\boxed{\tilde{B}_{o_z}(x,y) = \tilde{B}_o \cos(k_x x)\cos(k_y y)}$

> *n.b.* $\tilde{B}_o = \left|\tilde{B}_o\right| e^{i\varphi_B} \equiv B_o e^{i\varphi_B}$.
> However, we can always absorb/"rotate away" the phase φ_B *e.g.* by a global re-definition of the zero of time and/or a global translation of the coordinate system.
> Hence, let: $\tilde{B}_o \to B_o$.

The full (*x*, *y*, *z*, *t*) – dependence is:

$$\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \tilde{E}_z\hat{z} \begin{cases} \text{(a) } \tilde{E}_x(x,y,z,t) = \tilde{E}_{o_x}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-i\omega k_y}{(\omega/c)^2 - k_z^2} B_o \cos(k_x x)\sin(k_y y)e^{i(k_z z - \omega t)} \\ \text{(b) } \tilde{E}_y(x,y,z,t) = \tilde{E}_{o_y}(x,y)e^{i(k_z z - \omega t)} = \dfrac{+i\omega k_x}{(\omega/c)^2 - k_z^2} B_o \sin(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \\ \text{(c) } \tilde{E}_z(x,y,z,t) = \tilde{E}_{o_z}(x,y)e^{i(k_z z - \omega t)} = 0 \end{cases}$$

$$\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \tilde{B}_z\hat{z} \begin{cases} \text{(d) } \tilde{B}_x(x,y,z,t) = \tilde{B}_{o_x}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-ik_z k_x}{(\omega/c)^2 - k_z^2} B_o \sin(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \\ \text{(e) } \tilde{B}_y(x,y,z,t) = \tilde{B}_{o_y}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-ik_z k_y}{(\omega/c)^2 - k_z^2} B_o \cos(k_x x)\sin(k_y y)e^{i(k_z z - \omega t)} \\ \text{(f) } \tilde{B}_z(x,y,z,t) = \tilde{B}_{o_z}(x,y)e^{i(k_z z - \omega t)} = B_o \cos(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \end{cases}$$

Note that for the TE mode(s) of propagation of *EM* waves in a rectangular waveguide, the $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ -fields are ***in-phase*** with each other – the *x*, *y* and *z*-components of $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ ***all*** have the common phase factor $e^{i(k_z z - \omega t)}$.

The wave number $k_z(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2} = \frac{2\pi}{\lambda_z}$ with $k_x = \left(\frac{m\pi}{a}\right)$, $k_y = \left(\frac{n\pi}{b}\right)$ and $m = 0,1,2,3,\ldots$, $n = 0,1,2,3,\ldots$

Thus: $$k_z(\omega) = \frac{2\pi}{\lambda_z(\omega)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left\lfloor k_x^2 + k_y^2 \right\rfloor} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

We can define a so-called {angular} ***cutoff*** ***frequency*** for the $(m,n)^{th}$ TE mode as:

$$\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Thus, we can rewrite the above relation as:

$$k_z^{m,n}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{m,n}}{c}\right)^2} = \frac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2}$$

Note that for {angular} frequencies ***below*** the cutoff frequency: $\omega < \omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

Then: $\left(\omega^2 - \omega_{m,n}^2\right) < 0$ and: $k_z^{m,n}(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2}$ becomes ***imaginary***, hence: $e^{i(k_z z)} \rightarrow e^{-k_z z}$ which means that when $\omega < \omega_{m,n}$, the *EM* wave for the $(m,n)^{th}$ mode is ***exponentially*** ***damped***.

Note also that $m = n = 0$ corresponds to $k_x = k_y = 0$ with $k_z^{0,0}(\omega) = \omega/c$. But then, from the above $\tilde{\vec{E}}$- and $\tilde{\vec{B}}$-field relations on the previous page, we see that for ***this*** kind of TE wave, that:

$\tilde{E}_x = \tilde{E}_y = \tilde{E}_z = 0$ and: $\tilde{B}_x = \tilde{B}_y = 0$ with: $\tilde{B}_z = B_o e^{i\left(k_z^{0,0} z - \omega t\right)} \neq 0$.

$\Rightarrow$ This is ***not*** a proper kind of ***propagating*** *EM* wave, because $\tilde{\vec{E}} = 0$ everywhere, and hence *e.g.* Poynting's vector $\tilde{\vec{S}} = \frac{1}{\mu_o}\tilde{\vec{E}} \times \tilde{\vec{B}} = 0 \ \left(Watts/m^2\right)$ everywhere!!!

Thus, the ***lowest*** ***non-trivial*** propagating TE-type *EM* wave is the TE_{10} mode, where the notation TE_{mn} designates the $(m,n)^{th}$ mode of propagation. Note again, that ***by*** ***convention***, the index associated with the ***largest*** transverse dimension (here *a*) with corresponding integer index m is given ***first***.

Thus, for the ***lowest*** TE mode, $TE_{1,0}$: $k_z^{1,0}(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_{1,0}^2} = \frac{1}{c}\sqrt{\omega^2 - \left(\frac{\pi c}{a}\right)^2}$

We see that $k_z^{1,0}(\omega) \geq 0$ {*i.e.* is a purely ***real*** quantity} when: $\omega^2 - \omega_{1,0}^2 = \omega^2 - (\pi c/a)^2 > 0$

i.e. when: $\omega > \omega_{1,0} \equiv (\pi c/a)(radians/sec)$, or : $f > f_{1,0} = \left(\omega_{1,0}/2\pi\right) = c/2a \ \ (Hz)$.

A Numerical Example - TE Wave Propagation:

Suppose the rectangular wave guide's transverse internal dimensions are $a = 2\,cm$ and $b = 1\,cm$

Then: $\boxed{\omega_{1,0} = \pi c/a = 3\pi \times 10^8\, m/s/0.02m = 1.5\pi \times 10^{10}\, radians/sec = 4.71\times 10^{10}\, radians/sec}$

This corresponds to a cutoff frequency of: $\boxed{f_{1,0} = \omega_{1,0}/2\pi = \tfrac{3}{4}\times 10^{10}\, Hz = 7.5\, GHz}$ which is in the microwave portion of the *EM* spectrum, and corresponds to a wavelength of:

$$\boxed{\lambda_z^{1,0} = c/f_{1,0} = 3\times 10^8\, m/s/\tfrac{3}{4}\times 10^{10}\ /s = 4\times 10^{-2} m = 4.0\, cm} \quad i.e. \quad \boxed{\lambda_z^{1,0} = 2a}\ !!!$$

Thus, we see that if: $\lambda_z > \lambda_z^{1,0} = 2a$, we ***cannot*** propagate TE$_{1,0}$ waves because: $\boxed{f < f_{1,0} = \dfrac{c}{\lambda_z^{1,0}} = \dfrac{c}{2a}}$.

We also see that if: $\lambda_z < \lambda_z^{1,0} = 2a$, then ***can*** propagate TE$_{1,0}$ waves because: $\boxed{f > f_{1,0} = \dfrac{c}{\lambda_z^{1,0}} = \dfrac{c}{2a}}$.

Precisely ***at*** the angular cutoff frequency for the TE$_{1,0}$ mode, *i.e.* $\omega = \omega_{1,0} = \pi c/a$, we see that the wavenumber $k_z^{1,0}(\omega) = \tfrac{1}{c}\sqrt{\omega^2 - \omega_{1,0}^2} = 0 = 2\pi/\lambda_z^{1,0}(\omega)$ and thus $\lambda_z^{1,0}(\omega) = \infty$ for $\omega = \omega_{1,0}$,
where: $\lambda_z^{1,0}(\omega)$ = the wavelength of the *EM* wave ***in*** the waveguide for the TE$_{1,0}$ mode.

Now suppose that $\omega > \omega_{1,0}$ then: $\boxed{k_z^{m,n}(\omega) = \dfrac{2\pi}{\lambda_z^{m,n}(\omega)} = \dfrac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2} = \sqrt{\left(\dfrac{\omega}{c}\right)^2 - \left[\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2\right]}}$

The higher the angular frequency ω is, it then becomes possible to propagate TE$_{m,n}$ waves in ***more*** than just one mode.

$\Rightarrow$ There exists an angular cutoff frequency $\omega_{m,n}$ for ***each*** TE$_{m,n}$ mode: $\boxed{\omega_{m,n} \equiv c\sqrt{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2}}$

Another Numerical Example - TE Wave Propagation:

A rectangular wave guide's transverse internal dimensions are (again) $a = 2\,cm$ and $b = 1\,cm$. Suppose that: $f = 20\,GHz = 2\times 10^{10}\,Hz$, thus: $\omega = 2\pi f = 4\pi\times 10^{10} = 12.56\times 10^{10}\ radians/sec$ with corresponding vacuum wavelength $\lambda_o = c/f = 1.5\,cm$.

Which TE$_{m,n}$ modes are ***accessible*** ? $\boxed{\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$

$$\omega_{1,0} = c\sqrt{\left(\frac{1\pi}{a}\right)^2} = \left(\frac{\pi c}{a}\right) = 4.71\times 10^{10}\ radians/sec$$

$$\omega_{0,1} = c\sqrt{\left(\frac{1\pi}{b}\right)^2} = \left(\frac{\pi c}{b}\right) = 9.42\times 10^{10}\ radians/sec$$

$$\omega_{1,1} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \pi c\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 10.53\times 10^{10}\ radians/sec$$

$$\omega_{2,0} = c\sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{2\pi c}{a} = 9.42\times 10^{10}\ radians/sec$$

Degenerate, because $a = 2b$ ($\omega_{0,1}$ and $\omega_{2,0}$)

$$\omega_{3,0} = c\sqrt{\left(\frac{3\pi}{a}\right)^2} = \frac{3\pi c}{a} = 14.14\times 10^{10}\ radians/sec \leftarrow \text{TOO HIGH !!!}$$

$$\omega_{2,1} = c\sqrt{\left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \pi c\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \pi c\sqrt{2\left(\frac{1}{b}\right)^2} = 13.33\times 10^{10}\ radians/sec$$

Thus, for $f = 20\,GHz \Rightarrow \omega = 12.56\times 10^{10}$ radians/sec we ***can*** access/***can*** propagate TE$_{m,n}$ waves in the following 4 modes:

$TE_{1,0}$:	$\omega_{1,0} = 4.71\times 10^{10}\ radians/sec$	
$TE_{0,1}$:	$\omega_{0,1} = 9.42\times 10^{10}\ radians/sec$	← Degenerate, because $a = 2b$
$TE_{2,0}$:	$\omega_{2,0} = 9.42\times 10^{10}\ radians/sec$	← Degenerate, because $a = 2b$
$TE_{1,1}$:	$\omega_{1,1} = 10.53\times 10^{10}\ radians/sec$	

Note that if one operates a waveguide at an {angular} frequency ω that is above the cutoff frequenc(ies) $\omega_{m,n}$ *e.g.* of several (and/or ***many***) modes (m, n), ***all*** allowed modes will propagate in the waveguide – ***simultaneously*** – each mode propagates with their respective {frequency-dependent} phase and group speeds {see below}. If one is ***only*** interested in transporting *EM* ***energy***, this is {probably} ***fine***. However, operation of a ***multi***-***mode*** waveguide *e.g.* for ***telecommunication purposes*** can be seen to be ***problematic*** – ***single***-***mode*** operation avoids the dispersive smearing-out effects on information-carrying modulation associated with the ***total*** electric field (*e.g.* on the leading/trailing edges of digital 1's & 0's).

$TE_{m,n}$ Wavenumbers and Wavelengths Inside the Waveguide: $a = 2\,cm$ and $b = 1\,cm$

$f = 20\,GHz$
$= 2\times 10^{10}\,Hz$

$\omega = 12.56\times 10^{10}$ $radians/sec$

$$\begin{cases} TE_{1,0}:\ k_z^{1,0}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} = 388.31\,m^{-1},\ \lambda_z^{1,0}(\omega) = \frac{2\pi}{k_{1,0}(\omega)} = 1.620\,cm \\ TE_{0,1}:\ k_z^{0,1}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2} = 277.06\,m^{-1},\ \lambda_z^{0,1}(\omega) = \frac{\pi}{k_{0,1}(\omega)} = 2.268\,cm \\ TE_{2,0}:\ k_z^{2,0}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{2\pi}{a}\right)^2} = 277.06\,m^{-1},\ \lambda_z^{2,0}(\omega) = \frac{\pi}{k_{2,0}(\omega)} = 2.268\,cm \\ TE_{1,1}:\ k_z^{1,1}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} = 228.23\,m^{-1},\ \lambda_z^{1,1}(\omega) = \frac{\pi}{k_{1,1}(\omega)} = 2.750\,cm \end{cases}$$

Degenerate !!! ($TE_{0,1}$ and $TE_{2,0}$)

Compare these to vacuum wavenumber $k_o = \frac{2\pi}{\lambda_o} = 418.88\,m^{-1}$ and vacuum wavelength $\lambda_o = 1.5\,cm$. Note that the wavenumbers and wavelengths inside the wave guide will change when the frequency f (or $\omega = 2\pi f$) changes, because:

$$k_z^{m,n}(\omega) = \frac{2\pi}{\lambda_z^{m,n}(\omega)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

Physically, the ***phase*** speed $v_{\varphi_z}^{m,n}(\omega)$ is the speed of propagation of ***planes of constant phase*** $\Phi_{m,n}(\omega) \equiv \left[k_z^{m,n}(\omega)z - \omega t\right] = constant$ and is associated with the $e^{i(k_z z - \omega t)}$ phase-factor of the *EM* wave for each individual $TE_{m,n}$ mode.

If $\Phi_{m,n}(\omega) \equiv \left[k_z^{m,n}(\omega)z - \omega t\right] = constant$, then $\partial\Phi_{m,n}(\omega)/\partial t = 0$ which means that:

$$\frac{\partial\Phi_{m,n}(\omega)}{\partial t} = \frac{\partial}{\partial t}\left[k_z^{m,n}(\omega)z(t) - \omega t\right] = k_z^{m,n}(\omega)\frac{\partial z(t)}{\partial t} - \omega = 0$$

, or that: $k_z^{m,n}(\omega)\frac{\partial z(t)}{\partial t} = \omega$, or: $\frac{\partial z(t)}{\partial t} = \frac{\omega}{k_z^{m,n}(\omega)}$. The ***phase*** speed $v_{\varphi_z}^{m,n}(\omega) \equiv \frac{\partial z(t)}{\partial t} = \frac{\omega}{k_z^{m,n}(\omega)}$

Thus, the ***phase*** speed of a $TE_{m,n}$ wave for the $(m,n)^{th}$ mode is:

$$v_{\varphi_z}^{m,n}(\omega) \equiv \frac{\omega}{k_z^{m,n}(\omega)} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}$$

Since:

$$\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

then we see that the ***phase*** speed of a $TE_{m,n}$ wave is:

$$v_{\varphi_z}^{m,n}(\omega) \equiv \frac{\omega}{k_z^{m,n}(\omega)} = \frac{c}{\sqrt{1-(\omega_{m,n}/\omega)^2}} > c$$

for the $(m,n)^{th}$ allowed $TE_{m,n}$ mode!!!

For the $(m,n)^{th}$ TE$_{m,n}$ mode, *EM* ***energy*** in the waveguide propagates at the ***group*** speed:

$$v_{g_z}^{m,n}(\omega) \equiv 1 \Big/ \left(\frac{dk_z^{m,n}(\omega)}{d\omega}\right) = \left(\frac{dk_z^{m,n}(\omega)}{d\omega}\right)^{-1}$$

Let's explicitly determine $v_{g_z}^{m,n}(\omega)$:

$$\frac{dk_z^{m,n}(\omega)}{d\omega} = \frac{d}{d\omega}\left\{\frac{1}{c}\sqrt{\omega^2-\omega_{m,n}^2}\right\} = \frac{1}{c}\frac{\frac{1}{2}\cdot 2\omega}{\sqrt{\omega^2-\omega_{m,n}^2}} = \frac{\omega}{c}\cdot\frac{1}{\sqrt{\omega^2-\omega_{m,n}^2}} = \frac{\omega/c}{\sqrt{\omega^2-\omega_{m,n}^2}}$$

Thus: $$v_{g_z}^{m,n}(\omega) = \frac{1}{\dfrac{dk_z^{m,n}(\omega)}{d\omega}} = \frac{\sqrt{\omega^2-\omega_{m,n}^2}}{\omega/c} = c\sqrt{1-\left(\frac{\omega_{m,n}}{\omega}\right)^2}$$ where: $$\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

It can be seen from the above relation that $v_g^{m,n}(\omega) < c$ {***always***!}, as required by causality…

Note further that: $$v_{\varphi_z}^{m,n}(\omega)\cdot v_{g_z}^{m,n}(\omega) = \frac{c}{\sqrt{1-(\omega_{m,n}/\omega)^2}}\cdot c\sqrt{1-(\omega_{m,n}/\omega)^2} = c^2$$

The instantaneous free surface charge and current densities induced on the ***inner*** surfaces of the {***perfectly*** conducting} waveguide due to the *EM* fields within the waveguide can be obtained from:

$$\sigma_{surf}^{ind}(x,y,z,t) = \varepsilon_o \vec{E}_{surf}(x,y,z,t)\cdot\hat{n}_{surf}(x,y,z)$$

and:

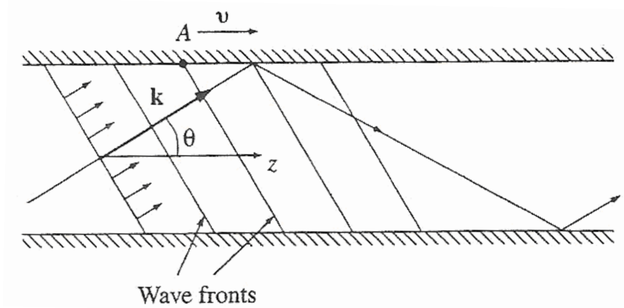
$$\vec{K}_{surf}^{ind}(x,y,z,t) = \tfrac{1}{\mu_o}\hat{n}_{surf}(x,y,z)\times\vec{B}_{surf}(x,y,z,t)$$

where $\hat{n}_{surf}(x,y,z)$ is the local {inward-pointing} unit normal at (x,y,z) associated with a given inner surface of the waveguide, and $\vec{E}_{surf}(x,y,z,t)$, $\vec{B}_{surf}(x,y,z,t)$ are the instantaneous electric, magnetic fields evaluated at (x,y,z,t) on that surface, *e.g.* $x=0,a$ and: $y=0,b$.

Note that $\vec{E}_{surf}(x,y,z,t)\cdot\hat{n}_{surf}(x,y,z)$ is the instantaneous local ***normal*** (*i.e.* $\perp$) component of the electric field at (x,y,z,t) on that surface, whereas $\hat{n}_{surf}(x,y,z)\times\vec{B}_{surf}(x,y,z,t)$ is the instantaneous local ***tangential*** (*i.e.* $\|$) component of the magnetic field at (x,y,z,t) on that surface.

The Physical Picture of *EM* Waves Propagating Inside a Wave Guide.

Consider an ***ordinary*** monochromatic *EM* plane wave ***initially*** propagating at speed $c = \omega/|\vec{k}|$ in the $\hat{k}$ -direction, making an angle θ with respect to the $\hat{z}$ -axis, as shown in the figure below:



Because the inner walls of the wave guide are ***perfectly*** conducting, they are lossless, *i.e.* ***perfectly*** reflecting. The *EM* waves are thus ***multiply-reflected*** {*n.b.* with π phase shift at ***each*** reflection} as they "bounce" down the wave guide – interfering with each other in such a way as to form ***transverse standing wave*** patterns of wavelength $\lambda_x = 2a/m$ in the $\hat{x}$ -direction and $\lambda_y = 2b/n$ in the $\hat{y}$ -direction!!!

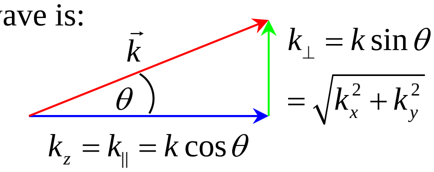
The *x*, *y* wavelengths respectively correspond to the *x*, *y*-wavenumbers $k_x = 2\pi/\lambda_x = m\pi/a$ in the $\hat{x}$ -direction, and $k_y = 2\pi/\lambda_y = n\pi/b$ in the $\hat{y}$ -direction. In the $\hat{z}$ -direction, the ensemble (*i.e.* **group**) of reflected waves results in a ***traveling wave***, with *z*-wavenumber:

$$k_z^{m,n}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(k_x^2 + k_y^2\right)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} = \frac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2}$$

where: $$\omega_{m,n} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$$

The propagation wave**vector** associated with the ***initial*** plane wave is:

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = \left(\frac{m\pi}{a}\right)\hat{x} + \left(\frac{n\pi}{b}\right)\hat{y} + k_z^{m,n}(\omega)\,\hat{z}$$



Thus, because *m*, *n* = 0, 1, 2, 3,… (*n.b.* both *m* = *n* = 0 simultaneously is ***not*** allowed), only ***certain*** angles $\theta_{m,n}$ will lead to one of the allowed standing wave patterns in *x* and *y*:

$$\cos\theta_{m,n} = \frac{k_z^{m,n}(\omega)}{|\vec{k}|} = \frac{\sqrt{\omega^2 - \omega_{m,n}^2}/c}{\omega/c} = \sqrt{1 - (\omega_{m,n}/\omega)^2}$$

where: $$\omega_{m,n} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$$

This "original" plane *EM* wave, traveling at angle $\theta_{m,n}$ with respect to the $\hat{z}$ -axis travels at speed $c = \omega/|\vec{k}|$ (*i.e.* we assume that the medium (*e.g.* air, or vacuum) inside the wave guide has $\varepsilon = \varepsilon_o$ and $\mu = \mu_o$).

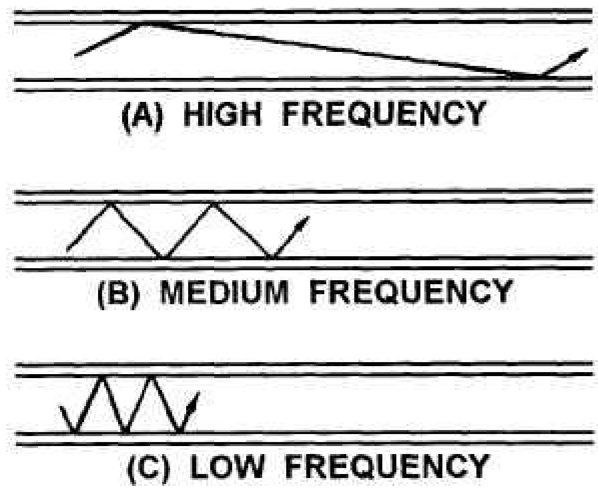
But because this plane *EM* wave makes an angle $\theta_{m,n}$ with respect to the $\hat{z}$-axis, the component of the initial wave's speed ***projected along*** the $\hat{z}$-axis is ***less*** than c:

$$v_z(\omega) = c\cos\theta_{m,n}(\omega) = c\sqrt{1-(\omega_{m,n}/\omega)^2} = v_{g_z}^{m,n}(\omega) = \text{group speed !!!}$$

The ***phase*** speed is the speed at which wavefronts (***planes of constant phase***) (*e.g.* point *A* in the figure on the previous page) propagate down the wave guide – these ***can*** move ***much*** faster than *c*, because:

$$v_{\varphi_z}^{m,n}(\omega) = \frac{c}{\cos\theta_{m,n}(\omega)} = \frac{c}{\sqrt{1-(\omega_{m,n}/\omega)^2}}$$

Note that if $\theta_{m,n} = 90^\circ$ (*i.e.* $\cos\theta_{m,n} = 0$), {*i.e.* when $\omega = \omega_{m,n}$}, for which $v_g^{m,n}(\omega) = 0$ and $v_{\varphi_z}^{m,n}(\omega) = \infty$!!! Physically, $\theta_{m,n} = 90^\circ$ corresponds to ***standing*** waves in (*x*,*y*), *i.e.* **NO** propagation along the $\hat{z}$-direction $\Rightarrow$ *i.e.* a 2-D resonant cavity!!!



Thus, the allowed solution(s) that we obtained on *p*. 8 above for the *x*, *y* and *z* components of the electric and magnetic fields for TE mode propagation of electromagnetic waves down a waveguide actually/physically represent the ***steady-state*** ensemble (*i.e.* ***group***) wave solution associated with the ***collective*** effect(s) of these ***multiply-reflected*** wave**s** interfering with each other as they propagate down the waveguide!

This ***group*** of ***multiply-reflected*** wave**s** for the $(m,n)^{th}$ TE mode, $TE_{m,n}$ propagates down the waveguide at the ***group*** speed $v_{g_z}^{m,n}(\omega) = c\sqrt{1-(\omega_{m,n}/\omega)^2} = c\cos\theta_{m,n}(\omega)$ (hence the origin of its name!).

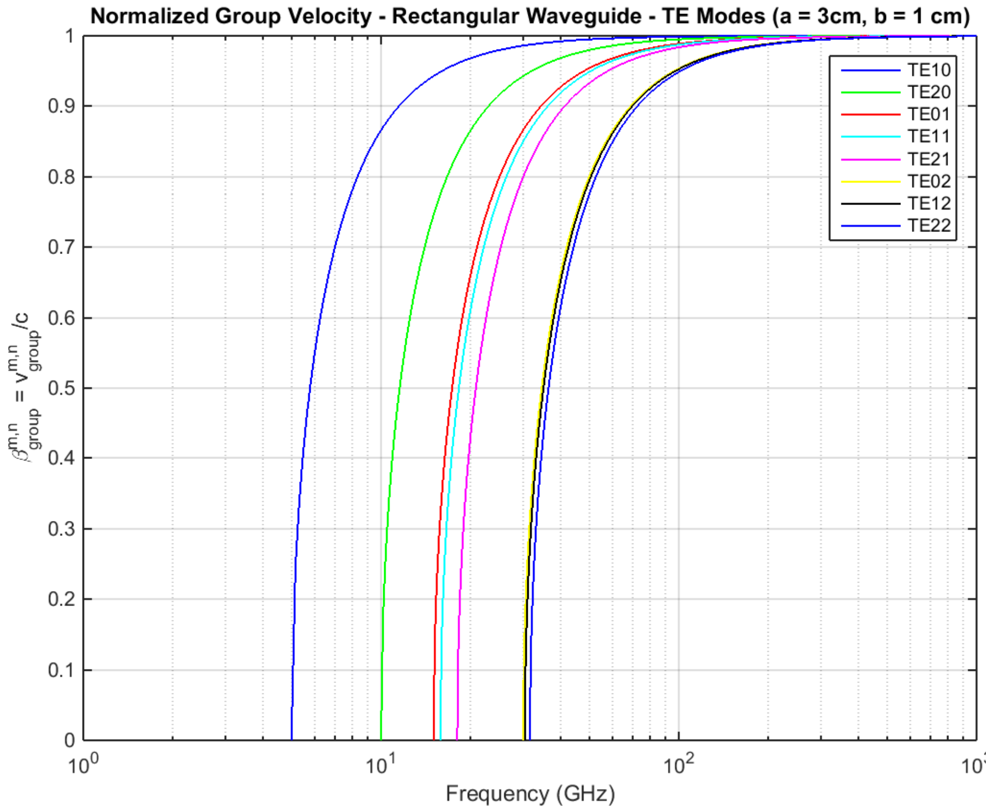
In the two figures below, we show plots of:

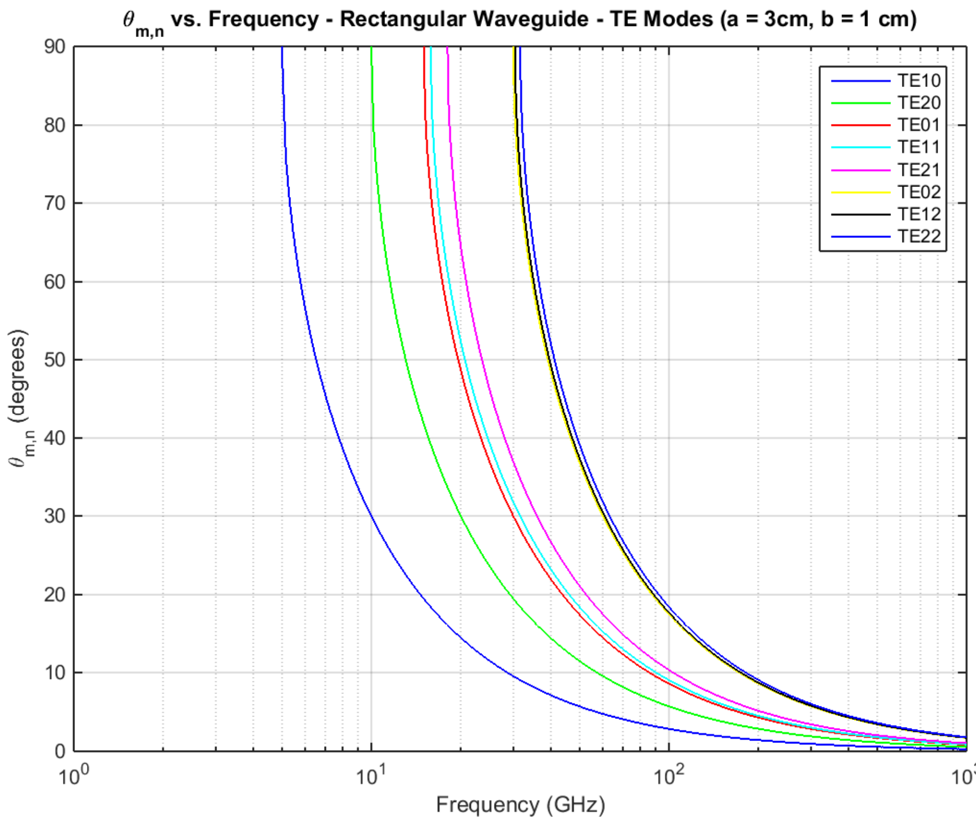
Normalized group speed: $\beta_{g_z}^{m,n}(f) \equiv v_{g_z}^{m,n}(f)/c = \sqrt{1-(f_{m,n}/f)^2}$ *vs. f*

and:

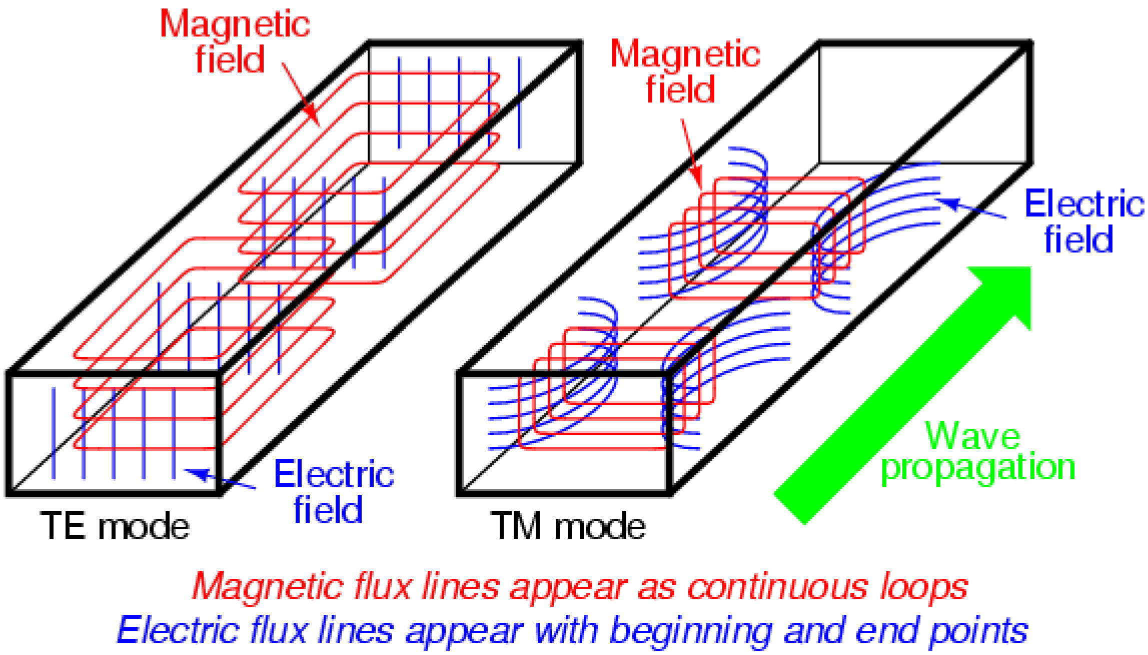
Propagation angle: $\theta_{m,n} = \cos^{-1}\left(\beta_{g_z}^{m,n}(f)\right) = \cos^{-1}\left(v_{g_z}^{m,n}(f)/c\right) = \cos^{-1}\left(\sqrt{1-(f_{m,n}/f)^2}\right)$ *vs. f*

for several of the lowest-lying $TE_{m,n}$ modes, for a perfectly-conducting waveguide of dimensions *a* = 3 *cm*, *b* = 1 *cm*.


Normalized Group Velocity - Rectangular Waveguide - TE Modes (a = 3cm, b = 1 cm)
$\beta^{m,n}_{group} = v^{m,n}_{group}/c$
Frequency (GHz)
TE10
TE20
TE01
TE11
TE21
TE02
TE12
TE22


$\theta_{m,n}$ vs. Frequency - Rectangular Waveguide - TE Modes (a = 3cm, b = 1 cm)
$\theta_{m,n}$ (degrees)
Frequency (GHz)
TE10
TE20
TE01
TE11
TE21
TE02
TE12
TE22

3-D Picture of $\vec{E}$ and $\vec{B}$-fields in a Rectangular Wave Guide for $TE_{1,0}$ & $TM_{1,1}$ Modes:



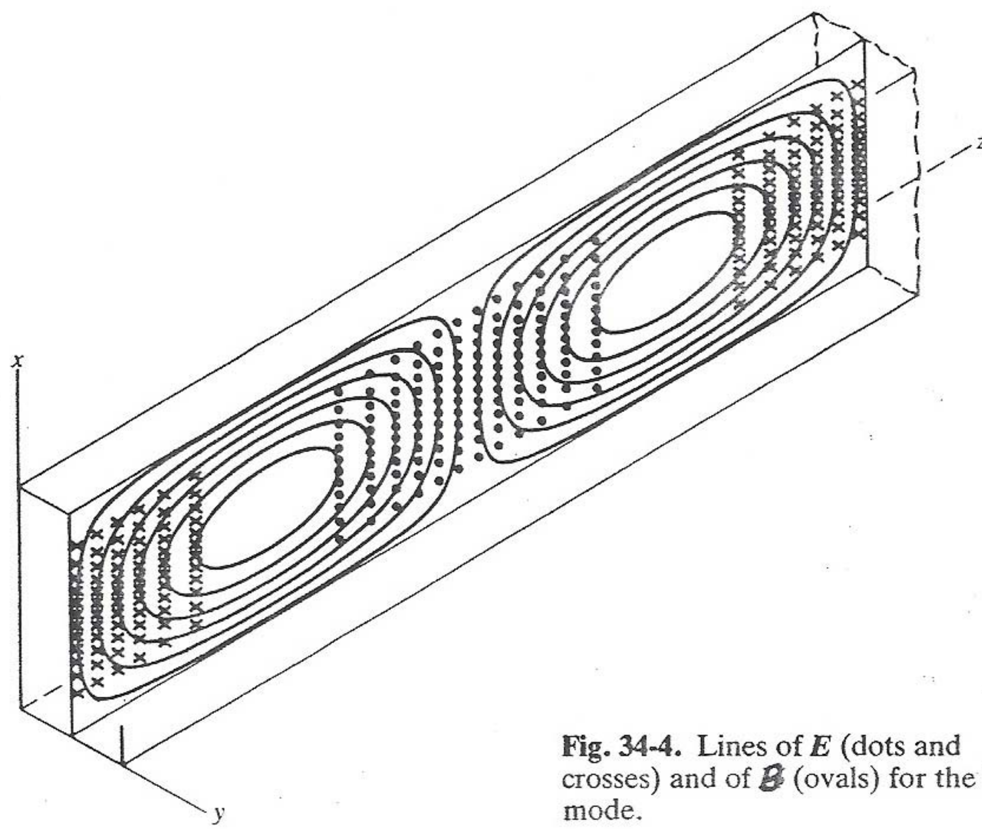
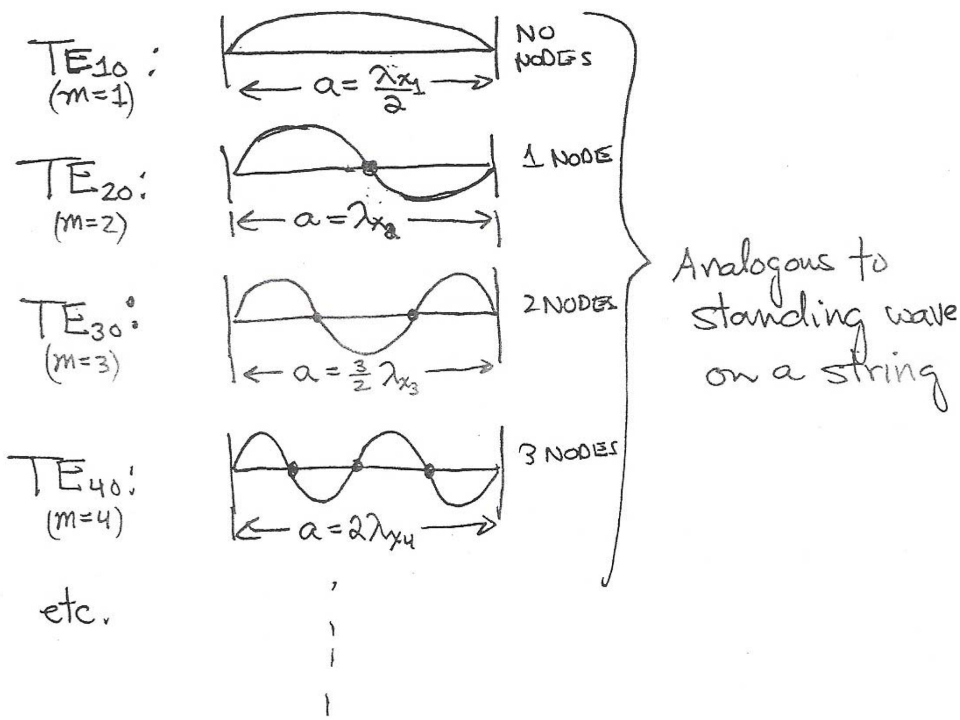


Fig. 34-4. Lines of E (dots and crosses) and of B (ovals) for the TE_{10} mode.

For $TE_{0,1}$ mode, rotate above pix by 90°

For $TE_{m,0}$ modes - ∃ nodes at the mid-plane:



Time-Averaged Power Transmitted Down a Rectangular Wave Guide in $TE_{m,n}$ Modes:

In order to calculate the time-averaged power transmitted down a rectangular wave guide {of cross-sectional area $A_\perp = ab\,(= h \times w)$ } we integrate the time-averaged Poynting vector, $\left\langle \vec{S}(\vec{r},t) \right\rangle_t$ over the cross-sectional area of the waveguide:

$\hat{n} = +\hat{z}$ direction (***here***)

$$\left\langle P_{m,n}^{trans}(z,t) \right\rangle = \int_{A_\perp} \left\langle \vec{S}_{m,n}(x,y,z,t) \right\rangle \cdot d\vec{a}_\perp = \int_{y=o}^{y=b} \int_{x=o}^{x=a} \left\langle \vec{S}_{m,n}(x,y,z,t) \right\rangle \cdot \hat{n} dxdy \qquad d\vec{a}_\perp = \hat{n}\ dxdy = \hat{z}\ dxdy$$

From Griffiths Problem 9.11 (*p*.382): $\left\langle \vec{S}_{m,n}(x,y,z,t) \right\rangle = \dfrac{1}{2\mu_o} \Re e\left\{ \tilde{\vec{E}}_{m,n}(x,y,z,t) \times \tilde{\vec{B}}_{m,n}^{*}(x,y,z,t) \right\}$

{Because $\left\langle f\, g \right\rangle = \tfrac{1}{2} \Re e\left\{ \tilde{f}\, \tilde{g}\,^* \right\}$, where * denotes complex conjugation}

For the TE$_{m,n}$ modes in a rectangular wave guide:

$$\boxed{\tilde{\vec{E}}_{m,n}(x,y,z,t)=\tilde{\vec{E}}_{o_{m,n}}(x,y)e^{i(k_z z-\omega t)}} \text{ and: } \boxed{\tilde{\vec{B}}^*_{m,n}(x,y,z,t)=\tilde{\vec{B}}^*_{o_{m,n}}(x,y)e^{-i(k_z z-\omega t)}} \quad \boxed{m,n=0,1,2,3,\ldots}$$

$$\text{with: } \boxed{k_z=k_z^{m,n}\equiv\sqrt{\left(\frac{\omega}{c}\right)^2-\left[k_{x_m}^2+k_{y_n}^2\right]}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}},\ \boxed{k_{x_m}\equiv\left(\frac{m\pi}{a}\right)},\ \boxed{k_{y_n}\equiv\left(\frac{n\pi}{b}\right)}$$

$$\text{and with: } \boxed{\tilde{\vec{E}}_{o_{mn}}(x,y)=\tilde{E}_{ox_{mn}}\hat{x}+\tilde{E}_{oy_{mn}}\hat{y}+\tilde{E}_{oz_{mn}}\hat{z}} \text{ and } \boxed{\tilde{\vec{B}}_{o_{mn}}(x,y)=\tilde{B}_{ox_{mn}}\hat{x}+\tilde{B}_{oy_{mn}}\hat{y}+\tilde{B}_{oz_{mn}}\hat{z}}$$

$$\tilde{E}_{ox_{mn}}(x,y)=\frac{i\omega\left(-k_{y_n}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\cos\left(k_{x_m}x\right)\sin\left(k_{y_n}y\right)=\frac{i\omega}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-n\pi}{b}\right)B_o\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\tilde{E}_{oy_{mn}}(x,y)=\frac{-i\omega\left(-k_{x_m}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\sin\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=\frac{-i\omega}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-m\pi}{a}\right)B_o\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\tilde{E}_{oz_{mn}}(x,y)=0$$

$$\tilde{B}^*_{ox_{mn}}(x,y)=\frac{-ik_{mn}\left(-k_{x_m}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\sin\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=\frac{ik_{mn}}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-m\pi}{a}\right)B_o\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\tilde{B}^*_{oy_{mn}}(x,y)=\frac{-ik_{mn}\left(-k_{y_n}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\cos\left(k_{x_m}x\right)\sin\left(k_{y_n}y\right)=\frac{ik_{mn}}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-n\pi}{b}\right)B_o\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\tilde{B}^*_{oz_{mn}}(x,y)=B_o\cos\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=B_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\text{Then: } \boxed{\left\langle\vec{S}(x,y,z,t)\right\rangle=\frac{1}{2\mu_o}\Re e\left\{\tilde{\vec{E}}(x,y,z,t)\times\tilde{\vec{B}}^*(x,y,z,t)\right\}} \Leftarrow$$

Note: All time dependence vanishes { $e^{i(k_z z-\omega t)}$ factor}

Very Useful Table:

$\hat{x}\times\hat{y}=\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$	$\hat{x}\times\hat{x}=0$
$\hat{y}\times\hat{z}=\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$	$\hat{y}\times\hat{y}=0$
$\hat{z}\times\hat{x}=\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$	$\hat{z}\times\hat{z}=0$

$$\begin{aligned}\text{Then: }&\left(\tilde{\vec{E}}=\tilde{E}_x\hat{x}+\tilde{E}_y\hat{y}+\tilde{E}_z\hat{z}\right)\times\left(\tilde{\vec{B}}^*=B_x^*\hat{x}+B_y^*\hat{y}+B_x^*\hat{z}\right)\\
&=\tilde{E}_x\tilde{B}_y^*(\hat{x}\times\hat{y})+\tilde{E}_x\tilde{B}_z^*(\hat{x}\times\hat{z})=\tilde{E}_x\tilde{B}_y^*\hat{z}-\tilde{E}_xB_z^*\hat{y}\\
&\quad+\tilde{E}_y\tilde{B}_x^*(\hat{y}\times\hat{x})+\tilde{E}_y\tilde{B}_z^*(\hat{y}\times\hat{z})-\tilde{E}_y\tilde{B}_x^*\hat{z}+\tilde{E}_yB_z^*\hat{x}\\
&\quad+\tilde{E}_z\tilde{B}_x^*(\hat{z}\times\hat{x})+\tilde{E}_z\tilde{B}_y^*(\hat{z}\times\hat{y})-\tilde{E}_z\tilde{B}_x^*\hat{y}-\tilde{E}_zB_y^*\hat{x}\\
&=\left(\tilde{E}_y\tilde{B}_z^*-\tilde{E}_z\tilde{B}_y^*\right)\hat{x}+\left(\tilde{E}_z\tilde{B}_x^*-\tilde{E}_x\tilde{B}_z^*\right)\hat{y}+\left(\tilde{E}_x\tilde{B}_y^*-\tilde{E}_y\tilde{B}_x^*\right)\hat{z}\end{aligned}$$

But $\boxed{E_{z_{mn}}=0}$ for TE$_{m,n}$ modes, and skipping (<u>much</u>) algebra:

Then:
$$\frac{1}{2\mu_o}\left(\tilde{\vec{E}}_{m,n}\times\tilde{\vec{B}}_{m,n}\right)=\frac{1}{2\mu_o}\left\{\frac{i\pi\omega B_o^2}{(\omega/c)^2-k_{z_{mn}}^2}\left(\frac{m}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\hat{x}\right.$$
$$+\frac{i\pi\omega B_o^2}{(\omega/c)^2-k_{z_{mn}}^2}\left(\frac{n}{b}\right)\cos^2\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{y}$$
$$\left.+\frac{\pi^2\omega k_{z_{m,n}}B_o^2}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{z}\right\}$$

Then: $\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle=\frac{1}{2\mu_o}\Re e\left\{\tilde{\vec{E}}_{m,n}(x,y,z,t)\times\tilde{\vec{B}}_{m,n}^*(x,y,z,t)\right\}}$

$$\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle=\frac{\pi^2\omega k_{z_{m,n}}B_o^2}{2\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{z}}$$

Note that: $\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle=\left\langle\left|\vec{S}_{m,n}(x,y,z,t)\right|\right\rangle\hat{z}}$ $\Leftarrow$ points in $+\hat{z}$ direction, as it should!!

Then: $\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\int_{A_\perp}\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle\bullet d\vec{a}_\perp=\int_{y=o}^{y=b}\int_{x=o}^{x=a}\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle\bullet\hat{n}dxdy}$ $\boxed{d\vec{a}_\perp=\hat{n}\ dxdy=\hat{z}\ dxdy}$

$$=\frac{\pi^2\omega k_{z_{m,n}}B_o^2}{2\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{n}{b}\right)^2\int_{y=o}^{y=b}\int_{x=o}^{x=a}\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy+\left(\frac{m}{a}\right)^2\int_{y=o}^{y=b}\int_{x=o}^{x=a}\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)dxdy\right]$$

But: $\boxed{\int_o^a\sin^2\left(\frac{m\pi x}{a}\right)dx=\int_o^a\cos^2\left(\frac{m\pi x}{a}\right)dx=\left(\frac{a}{2}\right)}$ and: $\boxed{\int_o^b\sin^2\left(\frac{n\pi y}{b}\right)dy=\int_o^b\cos^2\left(\frac{n\pi x}{b}\right)dx=\left(\frac{b}{2}\right)}$

$\therefore\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{\pi^2\omega k_{z_{m,n}}B_o^2ab}{8\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}$ (Watts) with: $m=0,1,2,\ldots$ (m,n not both = 0; $n=0,1,2,\ldots$ simultaneously!)

But: $\boxed{k_z(\omega)=k_{z_{m,n}}(\omega)=\sqrt{\left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}=\left(\frac{\omega}{c}\right)\sqrt{1-\left(\frac{c}{\omega}\right)^2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}}$

Now: $\boxed{\left(\frac{\omega}{c}\right)=k_o=\frac{2\pi}{\lambda_o}}$ where: $k_o=$ ***vacuum*** wavenumber $\boxed{k_{x_m}=\left(\frac{m\pi}{a}\right)}$, $\boxed{k_{y_n}=\left(\frac{n\pi}{b}\right)}$

$\Rightarrow$ $\boxed{\lambda_o=2\pi\left(\frac{c}{\omega}\right)}$ where: $\lambda_o=$ ***vacuum*** wavelength

Thus:

$$\left(\frac{c}{\omega}\right)^2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]=\left[2\pi\left(\frac{c}{\omega}\right)\right]^2\left[\left(\frac{m}{2a}\right)^2+\left(\frac{n}{2b}\right)^2\right]=\lambda_o^2\left[\left(\frac{m}{2a}\right)^2+\left(\frac{n}{2b}\right)^2\right]=\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]$$

Thus: $$k_{z_{m,n}}(\omega)=\sqrt{k_o^2-k_{x_m}^2-k_{y_n}^2}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}=k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}$$

∴ $$\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{\omega B_o^2 ab}{8\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}$$

or: $$\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{1}{2\mu_o\omega}\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}$$

But: $$\int_{A_\perp}\left\langle\left|\tilde{\vec{E}}_x(x,y,z,t)\right|^2\right\rangle dxdy=\frac{\omega^2 k_{y_n}^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}=\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left(\frac{n\pi}{b}\right)^2$$ <u>with</u>: $k_{y_n}\equiv\left(\frac{n\pi}{b}\right)$

$$\int_{A_\perp}\left\langle\left|\tilde{\vec{E}}_y(x,y,z,t)\right|^2\right\rangle dxdy=\frac{\omega^2 k_{x_m}^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}=\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left(\frac{m\pi}{a}\right)^2$$ <u>with</u>: $k_{x_m}\equiv\left(\frac{m\pi}{a}\right)$

Defining:

$$\left|\tilde{E}_{o_x}^{m,n}\right|\equiv\frac{\omega k_{y_n}B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}=\frac{\omega B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{n\pi}{b}\right)$$

$$\left|\tilde{E}_{o_y}^{m,n}\right|\equiv\frac{\omega k_{x_m}B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}=\frac{\omega B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{m\pi}{a}\right)$$

} Magnitudes of $\hat{x}$, $\hat{y}$ electric field amplitudes

Then: $$\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{k_o}{2\mu_o\omega}\left(\left|\tilde{E}_{o_x}^{mn}\right|^2+\left|\tilde{E}_{o_y}^{mn}\right|^2\right)\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2a}\right)^2\right]}$$

But: $\frac{k_o}{\omega}=\frac{1}{c}$ ⇒ $\frac{k_o}{\mu_o\omega}=\frac{1}{\mu_o}\left(\frac{k_o}{\omega}\right)=\frac{1}{\mu_o c}$ and: $c=\sqrt{\frac{1}{\varepsilon_o\mu_o}}$ ∴ $\frac{k_o}{\mu_o\omega}=\frac{1}{\mu_o c}=\frac{\sqrt{\varepsilon_o\mu_o}}{\mu_o}=\sqrt{\frac{\varepsilon_o}{\mu_o}}$

∴ $$\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2+\frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)ab\sqrt{1-\left[\left(\frac{m\lambda_o}{a}\right)^2+\left(\frac{n\lambda_o}{b}\right)^2\right]}$$

$\lambda_o\equiv$ vacuum wave length $=c/f$

⇒ The time-averaged power transported down the hollow rectangular waveguide for the TE_{mn}^{th} mode is proportional to the square of the *E*-field amplitudes in the $\hat{x}$ and $\hat{y}$ direction!!

The {scalar} *EM* **wave** **impedance** of **free** **space** is:

$$Z_o(\vec{r}) \equiv \left|\tilde{\vec{E}}_\perp(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_\perp(\vec{r})\big/\mu_o\right| = \mu_o c = \sqrt{\mu_o/\varepsilon_o} = 120\pi\,\Omega \simeq 377\Omega$$

(*n.b.* Z_o is a purely real, quantity - because ∃ no dissipation in the vacuum!).

For TE$_{mn}$ modes of *EM* wave propagation in a waveguide that has **perfectly** **conducting** **walls** (*i.e.* **no** dissipation/**no** losses), the *EM* wave impedance of the **waveguide** is **also** purely **real**:

$$Z_{TE}^{m,n}(\omega) \equiv \left|\tilde{\vec{E}}_{TE}^\perp(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_{TE}^\perp(\vec{r})\big/\mu_o\right| = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) = Z_o\left(k_o/k_{z_{m,n}}(\omega)\right).$$

Then since:

$$k_{z_{m,n}}(\omega) = \frac{2\pi}{\lambda_z^{m,n}(\omega)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2 + \left(\frac{n\lambda_o}{2b}\right)^2\right]} = \sqrt{k_o^2 - k_{x_m}^2 - k_{y_n}^2} < k_o$$

or equivalently $\lambda_z^{m,n}(\omega) > \lambda_o$ we see that: $Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) = Z_o\left(k_o/k_{z_{m,n}}(\omega)\right) > 377\Omega$ for TE$_{m,n}$ modes of *EM* wave propagation in a waveguide.

We can thus write the *EM* power transmitted down the waveguide for TE$_{mn}$ modes as:

$$\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \frac{1}{2Z_o}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2 + \frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)A_\perp\left(\frac{\lambda_o}{\lambda_z^{m,n}(\omega)}\right) = \frac{1}{2}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2 + \frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)A_\perp \Big/ Z_{TE}^{m,n}(\omega)$$

Where $A_\perp = ab$ = cross-sectional area of the rectangular waveguide.

Note also that this expression is analogous to $\langle P\rangle = \tfrac{1}{2}V_{peak}^2/R$ for electrical circuits, since $E^2 A_\perp \sim (Volts/m)^2 * m^2 = Volts^2$.

The Energy Density $\langle u_{m,n}(x,y,z,t)\rangle$ Stored in a Rectangular Waveguide - TE$_{m,n}$ Mode

Again, from Griffiths Problem 9.11 (*p*. 382) since $\langle f\,g\rangle = \tfrac{1}{2}\Re e\{\tilde f\,\tilde g^*\}$

Then: $\langle u(x,y,z,t)\rangle = \frac{1}{4}\Re e\left\{\varepsilon_o \tilde{\vec E}(x,y,z,t)\cdot\tilde{\vec E}^*(x,y,z,t)\right\} + \frac{1}{4}\Re e\left\{\frac{1}{\mu_o}\tilde{\vec B}(x,y,z,t)\cdot\tilde{\vec B}^*(x,y,z,t)\right\}$

Then in the $(m,n)^{th}$ TE mode:

$$\langle u_{m,n}(x,y,z,t)\rangle = \frac{\varepsilon_o}{4}\Re e\left\{\tilde{\vec E}_{m,n}(x,y,z,t)\cdot\tilde{\vec E}^*_{m,n}(x,y,z,t)\right\} + \frac{1}{4\mu_o}\Re e\left\{\tilde{\vec B}_{m,n}(x,y,z,t)\cdot\tilde{\vec B}^*_{m,n}(x,y,z,t)\right\}$$

0 for TE modes

Where: $\tilde{\vec E}_{m,n} = \tilde E_{x_{m,n}}\hat x + \tilde E_{y_{m,n}}\hat y + \tilde E_{z_{m,n}}\hat z$ *n.b.* $|A|^2 \equiv A\cdot A^*$

And: $\tilde{\vec B}_{m,n} = \tilde B_{x_{m,n}}\hat x + \tilde B_{y_{m,n}}\hat y + \tilde B_{z_{m,n}}\hat z$

0 for TE modes

Then: $\langle u_{m,n}(x,y,z,t)\rangle = \frac{\varepsilon_o}{4}\Re e\left\{\left[\left|E_{x_{m,n}}\right|^2 + \left|E_{y_{m,n}}\right|^2 + \left|E_{z_{m,n}}\right|^2\right]\right\} + \frac{1}{4\mu_o}\Re e\left\{\left[\left|B_{x_{m,n}}\right|^2 + \left|B_{y_{m,n}}\right|^2 + \left|B_{z_{m,n}}\right|^2\right]\right\}$

$$\begin{aligned}\langle u_{m,n}(x,y,z,t)\rangle = &\frac{\varepsilon_o}{4}\left(\frac{\omega\pi B_o}{\left[(\omega/c)^2 - k^2_{z_{m,n}}\right]}\right)^2\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\ &+\frac{1}{4\mu_o}\left\{\left(\frac{k_{z_{m,n}}\pi B_o}{\left[(\omega/c)^2 - k^2_{z_{m,n}}\right]}\right)^2\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right.\\ &\left.+ B_o^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right\}\end{aligned}$$

The time-averaged energy ***per*** ***unit*** ***length*** (*Joules*/*m*) in the waveguide for the $(m,n)^{th}$ TE mode is:

$$\langle U_{m,n}(z,t)\rangle/L \equiv \int_{A_\perp}\langle u_{m,n}(x,y,z,t)\rangle\,da_\perp = \int_{y=o}^{y=b}\int_{x=o}^{x=a}\langle u_{m,n}(x,y,z,t)\rangle\,dxdy$$

where L (meters) is the length of the waveguide.

$$\begin{aligned}\langle U_{m,n}(z,t)\rangle/L \equiv \int_{A_\perp}\langle u_{m,n}(x,y,z,t)\rangle\,da_\perp = &\frac{\varepsilon_o}{4}\left(\frac{\pi\omega B_o}{\left[(\omega/c)^2 - k^2_{z_{m,n}}\right]}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right]\\ &+\frac{1}{4\mu_o}\left\{\left(\frac{\pi k_{z_{m,n}} B_o}{\left[(\omega/c)^2 - k^2_{z_{m,n}}\right]}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)B_o^2\right\}\end{aligned}$$

$$\langle U_{m,n}(z,t)\rangle/L \equiv \int_{A_\perp}\langle u_{m,n}(x,y,z,t)\rangle da_\perp = \frac{\varepsilon_o}{4}\left(\frac{\pi\omega B_o}{\left[(\omega/c)^2 - k_{z_{m,n}}^2\right]}\right)^2\left(\frac{ab}{4}\right)\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right] + \frac{1}{4\mu_o}\left\{\left(\frac{\pi k_{z_{m,n}} B_o}{\left[(\omega/c)^2 - k_{z_{m,n}}^2\right]}\right)^2\left(\frac{ab}{4}\right)\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\left(\frac{ab}{4}\right)B_o^2\right\}$$

Now: $\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \int_{A_\perp}\left\langle \vec{S}_{m,n}(x,y,z,t)\right\rangle \bullet d\vec{a}_\perp = \dfrac{\omega k_{m,n} B_o^2 ab}{8\mu_o\left[(\omega/c)^2 - k_{z_{m,n}}^2\right]^2}\left[\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right]$

and: $k_{z_{m,n}}^2 = \left(\dfrac{\omega}{c}\right)^2 - \left[\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right] \Rightarrow \left[\left(\dfrac{\omega}{c}\right)^2 - k_{z_{m,n}}^2\right] = \left[\left(\dfrac{m\pi}{a}\right)^2+\left(\dfrac{n\pi}{b}\right)^2\right] \equiv \left(\dfrac{\omega_{m,n}^2}{c^2}\right)$

Thus: $\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \int_{A_\perp}\left\langle \vec{S}_{m,n}(x,y,z,t)\right\rangle \bullet d\vec{a}_\perp = \dfrac{\omega k_{z_{m,n}} ab}{8\mu_o\omega_{m,n}^2}c^2 B_o^2$

and: $\dfrac{\left\langle U_{m,n}(z,t)\right\rangle}{L} = \int_{A_\perp}\left\langle u_{m,n}(x,y,z,t)\right\rangle da_\perp = \dfrac{\omega^2 ab}{8\mu_o\omega_{m,n}^2}B_o^2$

Note that the ratio of: $\dfrac{\left\langle P_{m,n}^{trans}(z,t)\right\rangle}{\left\langle U_{m,n}(z,t)\right\rangle/L} = \dfrac{Watts}{Joules/m} = \dfrac{Joules/sec}{Joules/m} = \dfrac{m}{sec}$ (*i.e.* dimensions = speed)

$$\frac{\left\langle P_{m,n}^{trans}(z,t)\right\rangle}{\left\langle U_{m,n}(z,t)\right\rangle/L} = \frac{\dfrac{\omega k_{z_{m,n}} ab}{8\mu_o\omega_{m,n}^2}c^2 B_o^2}{\dfrac{\omega^2 ab}{8\mu_o\omega_{m,n}^2}B_o^2} = \frac{k_{m,n}c^2}{\omega} = \left(\frac{c}{\omega}\right)(k_{m,n}c)$$

But: $k_{z_{mn}}^2 c^2 = \omega^2 - \omega_{m,n}^2$ or: $k_{z_{mn}} c = \sqrt{\omega^2 - \omega_{m,n}^2}$

$$\therefore \frac{\left\langle P_{m,n}^{trans}(z,t)\right\rangle}{\left\langle U_{m,n}(z,t)\right\rangle/L} = \left(\frac{c}{\omega}\right)\sqrt{\omega^2-\omega_{m,n}^2} = c\sqrt{1-\left(\frac{\omega_{m,n}}{\omega}\right)^2} = v_{g_z}^{m,n}(\omega) \text{ !!!}$$

or: $v_{g_z}^{m,n}(\omega) = c\sqrt{1-\left(\dfrac{\omega_{mn}}{\omega}\right)^2} = \dfrac{\left\langle P_{m,n}^{trans}(z,t)\right\rangle}{\left\langle U_{m,n}(z,t)\right\rangle/L}$ or: $\left\langle P_{m,n}^{trans}(z,t)\right\rangle = v_{g_z}^{m,n}(\omega)\cdot\left\langle U_{m,n}(z,t)\right\rangle/L$

Case II: Propagation of TM Waves in a Perfectly Conducting Hollow Rectangular Waveguide:

For propagation TM waves in a ***perfectly*** conducting hollow waveguide: $\tilde{E}_z \neq 0$, but: $\tilde{B}_z = 0$.

$\therefore$ We need to solve the 3-D wave equation: $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]E_z=0$

subject to boundary conditions on the inner walls of rectangular waveguide: $\tilde{E}_{\parallel}=0$ and $\tilde{B}_{\perp}=0$

Following the same procedure that we developed for the TE mode case, let: $\tilde{\tilde{E}}_z(x,y)=\tilde{X}(x)\cdot\tilde{Y}(y)$

$$\tilde{X}(x)=\tilde{A}_x\cos k_x x+\tilde{B}_x\sin k_x x=0\ \{\text{at } x=0 \text{ and } x=a\} \Rightarrow \tilde{A}_x=0 \quad\Rightarrow\quad \tilde{X}(x)=\tilde{B}_x\sin k_x x$$

$$\tilde{Y}(y)=\tilde{A}_y\cos k_y y+\tilde{B}_y\sin k_y y=0\ \{\text{at } y=0 \text{ and } y=b\} \Rightarrow \tilde{A}_y=0 \quad\Rightarrow\quad \tilde{Y}(y)=\tilde{B}_y\sin k_y y$$

Because $m,n=0,1,2,3\ldots$ the ***lowest*** ***non-trivial*** $\text{TM}_{m,n}$ mode is TM_{11}

$$\begin{cases} k_x \equiv \left(\frac{m\pi}{a}\right), & m=1,2,3,\ldots \qquad \textit{n.b. } m=0 \text{ is NOT allowed here!!! } (\Rightarrow X(x)\equiv 0 \text{ everywhere!!!}) \\ k_y \equiv \left(\frac{n\pi}{a}\right), & n=1,2,3,\ldots \qquad \textit{n.b. } n=0 \text{ is NOT allowed here!!! } (\Rightarrow Y(y)\equiv 0 \text{ everywhere!!!}) \end{cases}$$

Then: $\tilde{E}_z(x,y)=\tilde{E}_o\sin(k_x x)\sin(k_y y)=\tilde{E}_o\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)$ with: $m,n=1,2,3,\ldots$

We can then determine the other components of E and B for the TM case, following the same procedure that we used for the TE case:

(a) $\tilde{E}_{o_x}(x,y)=\frac{+ik_zk_x}{(\omega/c)^2-k_z^2}\tilde{E}_o\cos(k_x x)\sin(k_y y)$ with: $k_x=\left(\frac{m\pi}{a}\right)$, $k_y=\left(\frac{n\pi}{b}\right)$, $m=0,1,2,3,\ldots$ $n=0,1,2,3,\ldots$

(b) $\tilde{E}_{o_y}(x,y)=\frac{+ik_zk_y}{(\omega/c)^2-k_z^2}\tilde{E}_o\sin(k_x x)\cos(k_y y)$ and: $k_z^2=\left(\frac{\omega}{c}\right)^2-k_x^2-k_y^2$

(c) $\tilde{E}_{o_z}(x,y)=\tilde{E}_o\sin(k_x x)\sin(k_y y)$

(d) $\tilde{B}_{o_x}(x,y)=\frac{-i\omega k_y}{(\omega/c)^2-k_z^2}\tilde{E}_o\sin(k_x x)\cos(k_y y)$

(e) $\tilde{B}_{o_y}(x,y)=\frac{+i\omega k_x}{(\omega/c)^2-k_z^2}\tilde{E}_o\cos(k_x x)\sin(k_y y)$

(f) $\tilde{B}_{o_z}(x,y)\equiv 0$

n.b. $\tilde{E}_o=|\tilde{E}_o|e^{i\varphi_E}\equiv E_oe^{i\varphi_E}$.
However, we can always absorb/"rotate away" the phase φ_E *e.g.* by a global re-definition of the zero of time and/or a global translation of the coordinate system.
Hence, let: $\tilde{E}_o\to E_o$.

The full (x, y, z, t) – dependence for the TM case is:

$$\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \tilde{E}_z\hat{z} \begin{cases} \text{(a) } \tilde{E}_x(x,y,z,t) = \tilde{E}_{o_x}(x,y)e^{i(k_z z-\omega t)} = \dfrac{+ik_z k_x}{(\omega/c)^2 - k_z^2} E_o \cos(k_x x)\sin(k_y y)e^{i(k_z z-\omega t)} \\ \text{(b) } \tilde{E}_y(x,y,z,t) = \tilde{E}_{o_y}(x,y)e^{i(k_z z-\omega t)} = \dfrac{+ik_z k_y}{(\omega/c)^2 - k_z^2} E_o \sin(k_x x)\cos(k_y y)e^{i(k_z z-\omega t)} \\ \text{(c) } \tilde{E}_z(x,y,z,t) = \tilde{E}_{o_z}(x,y)e^{i(k_z z-\omega t)} = E_o \sin(k_x x)\sin(k_y y)e^{i(k_z z-\omega t)} \end{cases}$$

$$\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \tilde{B}_z\hat{z} \begin{cases} \text{(d) } \tilde{B}_x(x,y,z,t) = \tilde{B}_{o_x}(x,y)e^{i(k_z z-\omega t)} = \dfrac{-i\omega k_y}{(\omega/c)^2 - k_z^2} E_o \sin(k_x x)\cos(k_y y)e^{i(k_z z-\omega t)} \\ \text{(e) } \tilde{B}_y(x,y,z,t) = \tilde{B}_{o_y}(x,y)e^{i(k_z z-\omega t)} = \dfrac{+i\omega k_x}{(\omega/c)^2 - k_z^2} E_o \cos(k_x x)\sin(k_y y)e^{i(k_z z-\omega t)} \\ \text{(f) } \tilde{B}_z(x,y,z,t) = \tilde{B}_{o_z}(x,y)e^{i(k_z z-\omega t)} = 0 \end{cases}$$

Note that {again} for the TM mode(s) of propagation of *EM* waves in a rectangular waveguide, the $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ -fields are ***in-phase*** with each other – the *x*, *y* and *z*-components of $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ ***all*** have the common phase factor $e^{i(k_z z-\omega t)}$.

All the rest is {nearly} the same as that for TE waves:

z-component wavenumber: $k_{z_{m,n}} = \sqrt{(\omega/c)^2 - (m\pi/a)^2 - (n\pi/b)^2}$ (same as before, for TE waves)

The **angular cutoff frequency**: $\omega_{m,n} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$ $\Rightarrow$ $k_{z_{m,n}} = \dfrac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2}$

Phase speed: $v_{\varphi_z}^{m,n} = c\dfrac{1}{\sqrt{1-(\omega_{m,n}/\omega)^2}}$ **Group speed**: $v_{g_z}^{m,n}(\omega) = c\sqrt{1-(\omega_{m,n}/\omega)^2}$ and: $v_{\varphi_z}^{m,n}(\omega)\cdot v_{g_z}^{m,n}(\omega) = c^2$

One ***difference*** for TM modes *vs*. TE modes is the *EM* wave ***impedance*** of the waveguide:

$$Z_{TM}^{m,n}(\omega) = Z_o\left(\lambda_o/\lambda_z^{m,n}(\omega)\right) = Z_o\left(k_{z_{m,n}}(\omega)/k_o\right) \quad vs. \quad Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) = Z_o\left(k_o/k_{z_{m,n}}(\omega)\right).$$

Since $\lambda_z^{m,n}(\omega) > \lambda_o$ and $Z_o \equiv \sqrt{\mu_o/\varepsilon_o} = 120\pi\ \Omega \simeq 377\Omega$ then we see that:

$$Z_{TM}^{m,n}(\omega) = Z_o\left(\lambda_o/\lambda_z^{m,n}(\omega)\right) < 377\Omega \quad \text{whereas: } Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) > 377\Omega$$

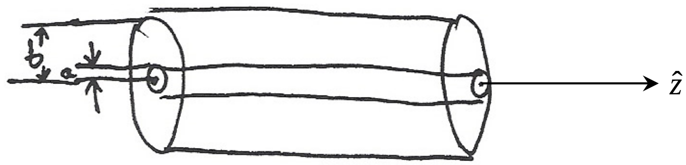
The ratio of the lowest TM mode to the lowest TE mode is:

$$\left(\omega_{m,n}^{TM}/\omega_{m,n}^{TE}\right) = \left(\omega_{11}^{TM}/\omega_{10}^{TE}\right) = \sqrt{(1/a)^2 + (1/b)^2}\Big/\sqrt{(1/a)^2} = \sqrt{1+(a/b)^2}$$

Use the above *E* and *B*-field components to compute TM $\langle u_{m,n}(x,y,z,t)\rangle, \langle P_{m,n}^{trans}(z,t)\rangle$, *etc.*

<u>Case III: Propagation of TEM Waves in a Perfectly Conducting Coaxial Transmission Line</u>:

- We have previously shown that a hollow waveguide ***cannot*** support TEM waves $\left\{\tilde{E}_z = \tilde{B}_z = 0\right\}$
- However, a ***coaxial*** transmission line, consisting of an inner, long straight wire of radius *a*, surrounded by a cylindrical conducting sheath of radius $b > a$ ***does*** support the propagation of TEM waves:



For TEM waves: $k = \omega/c$. TEM waves travel at the speed of light $c \Rightarrow$ ***non-dispersive***!

For TEM waves, Maxwell's equations give:

(1) <u>Gauss' Law</u>: $\vec{\nabla}\cdot\tilde{\vec{E}} = 0$

$$\frac{\partial \tilde{E}_{ox}}{\partial x} + \frac{\partial \tilde{E}_{oy}}{\partial_y} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{E}_{ox}}{\partial x} = -\frac{\partial \tilde{E}_{oy}}{\partial y}}$$

(2) <u>No monopoles</u>: $\vec{\nabla}\cdot\tilde{\vec{B}} = 0$

$$\frac{\partial \tilde{B}_{ox}}{\partial x} + \frac{\partial \tilde{B}_{oy}}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{B}_{ox}}{\partial x} = -\frac{\partial \tilde{B}_{oy}}{\partial y}}$$

(3) <u>Faraday's Law</u>: $\left(\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t}\right)$

(i) $\boxed{\frac{\partial \tilde{E}_{oy}}{\partial x} - \frac{\partial \tilde{E}_{ox}}{\partial y} = i\omega \underbrace{\tilde{B}_{oz}}_{=0} = 0}$

(ii) $\boxed{\underbrace{\frac{\partial \tilde{E}_{oz}}{\partial y}}_{=0} - ik\tilde{E}_{oy} = i\omega \tilde{B}_{ox}}$

(iii) $\boxed{ik\tilde{E}_{ox} - \underbrace{\frac{\partial \tilde{E}_{oz}}{\partial x}}_{=0} = i\omega \tilde{B}_{oy}}$

(4) <u>Ampere's Law</u>: $\left(\vec{\nabla}\times\tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial \tilde{\vec{E}}}{\partial t}\right)$

(iv) $\boxed{\frac{\partial \tilde{B}_{oy}}{\partial x} - \frac{\partial \tilde{B}_{ox}}{\partial y} = -\frac{i\omega}{c^2}\underbrace{\tilde{E}_{oz}}_{=0} = 0}$

(v) $\boxed{\underbrace{\frac{\partial \tilde{B}_{oz}}{\partial y}}_{=0} = -\frac{i\omega}{c^2}\tilde{E}_{ox}}$

(vi) $\boxed{ikB_{ox} - \underbrace{\frac{\partial \tilde{B}_{oz}}{\partial x}}_{=0} = -\frac{i\omega}{c^2}\tilde{E}_{oy}}$

which can be rewritten:

(i) $\boxed{\frac{\partial \tilde{E}_{oy}}{\partial x} = \frac{\partial \tilde{E}_{ox}}{\partial y}}$

(ii) $\boxed{\tilde{B}_{ox} = -\frac{k}{\omega}\tilde{E}_{oy} = -\frac{1}{c}\tilde{E}_{oy}}$

(iii) $\boxed{\tilde{B}_{oy} = \frac{k}{\omega}\tilde{E}_{ox} = +\frac{1}{c}\tilde{E}_{ox}}$

(iv) $\boxed{\frac{\partial \tilde{B}_{oy}}{\partial x} = \frac{\partial \tilde{B}_{ox}}{\partial y}}$

(v) $\boxed{\tilde{B}_{oy} = \frac{\omega}{c^2 k}\tilde{E}_{ox} = \frac{1}{c}\tilde{E}_{ox}}$

(vi) $\boxed{\tilde{B}_{ox} = -\frac{\omega}{c^2 k}\tilde{E}_{oy} = -\frac{1}{c}\tilde{E}_{oy}}$

Note that equations (iii) and (v) above give the same relation $\boxed{\tilde{B}_{oy} = \tfrac{1}{c}\tilde{E}_{ox}}$ as do equations (ii) and (vi), $\boxed{\tilde{B}_{ox} = -\tfrac{1}{c}\tilde{E}_{oy}}$.

Four of the following six relations are precisely the same equations of ***electrostatics*** and ***magnetostatics*** for empty space (*i.e.* the ***vacuum***) in ***two*** dimensions – for the infinite-length line charge, and the infinite-length line current problems, respectively:

$$\boxed{\tilde{B}_{oy} = \frac{1}{c}\tilde{E}_{ox}} \qquad \boxed{\tilde{B}_{ox} = -\frac{1}{c}\tilde{E}_{oy}}$$

$$\boxed{\frac{\partial \tilde{E}_{ox}}{\partial y} = \frac{\partial \tilde{E}_{oy}}{\partial x}} \qquad \boxed{\frac{\partial \tilde{B}_{ox}}{\partial y} = \frac{\partial \tilde{B}_{oy}}{\partial x}}$$

$$\boxed{\frac{\partial \tilde{E}_{ox}}{\partial x} = -\frac{\partial \tilde{E}_{oy}}{\partial y}} \qquad \boxed{\frac{\partial \tilde{B}_{ox}}{\partial x} = -\frac{\partial \tilde{B}_{oy}}{\partial y}}$$

Since a coaxial cable has ***cylindrical*** geometry/***cylindrical***/***axial*** symmetry, the TEM electric field (as in case of the infinite-length line charge problem) ***must*** be of the form:

$\boxed{\tilde{\vec{E}}_o(\rho,\varphi) = \dfrac{\tilde{A}}{\rho}\hat{\rho}}$ where: $\tilde{A} = \text{constant } (Volts)$, $\hat{\rho} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$, thus: $\boxed{\tilde{E}_{ox} = \dfrac{\tilde{A}}{\rho}\cos\varphi}$, $\boxed{\tilde{E}_{oy} = \dfrac{\tilde{A}}{\rho}\sin\varphi}$.

Similarly, the TEM magnetic field (as in the case of infinite-length line current) ***must*** be of the form:

$\boxed{\tilde{\vec{B}}_o(\rho,\varphi) = \dfrac{\tilde{A}}{\rho c}\hat{\varphi}}$, where: $\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}$, thus: $\boxed{\tilde{B}_{ox} = -\dfrac{\tilde{A}}{\rho c}\sin\varphi}$, $\boxed{\tilde{B}_{oy} = \dfrac{\tilde{A}}{\rho c}\cos\varphi}$.

Thus, for TEM wave propagation in a coaxial transmission line, the *E* and *B* fields are:

$$\boxed{\tilde{\vec{E}}(\rho,\varphi,z,t) = \tilde{\vec{E}}_o(\rho,\varphi)e^{i(kz-\omega t)} = \frac{\tilde{A}}{\rho}\, e^{i(kz-\omega t)}\hat{\rho}}$$

$$\boxed{\tilde{\vec{B}}(\rho,\varphi,z,t) = \tilde{B}_o(\rho,\varphi)e^{i(kz-\omega t)} = \frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\hat{\varphi}}$$

hence {again}: $\boxed{\tilde{\vec{B}}(\rho,\varphi,z,t) = \dfrac{1}{c}\hat{k}\times\tilde{\vec{E}}(\rho,\varphi,z,t)}$

Characteristic equation: $\boxed{k^2 = \left(\dfrac{\omega}{c}\right)^2}$ or: $\boxed{k = \left(\dfrac{\omega}{c}\right)}$, ***phase*** speed: $\boxed{v_\phi = \dfrac{\omega}{k} = c}$, ***group*** speed: $\boxed{v_g = 1\big/\left(dk(\omega)/d\omega\right) = 1\big/(1/c) = c}$, hence: $\boxed{v_g = v_\phi = c \neq fcn(\omega)}$ $\Rightarrow$ ***no*** dispersion for TEM waves!

Note also that there are ***no*** restrictions on the value of k / $\exists$ ***no*** mode cutoff frequencies for TEM waves propagating in a coaxial cable/waveguide/transmission line.

For TEM wave propagation in a ***hollow*** coaxial waveguide/transmission line that has ***perfectly conducting walls*** (*i.e.* ***no*** dissipation/***no*** losses), the *EM* wave impedance is (again) purely real:

$$\boxed{Z_{TEM}^{coax}(\omega) \equiv \left|\tilde{\vec{E}}_{TEM}^{\perp}(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_{TEM}^{\perp}(\vec{r})/\mu_o\right| = \sqrt{\mu_o/\varepsilon_o} = Z_o = 120\pi\,\Omega = 377\Omega}.$$

We explicitly show that the above E and B field TEM coaxial waveguide solutions do indeed satisfy Maxwell's equations and the boundary conditions:

$$\tilde{\vec{E}}(\rho,\varphi,z,t)=\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\hat{\rho} \text{ and: } \tilde{\vec{B}}(\rho,\varphi,z,t)=\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\hat{\varphi}$$

Gauss' law: $\vec{\nabla}\bullet\tilde{\vec{E}}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\tilde{A}e^{i(kz-\omega t)}\right)=0$ √

No Mag.Chgs: $\vec{\nabla}\bullet\tilde{\vec{B}}=\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)=0$ √

Faraday's law: $\vec{\nabla}\times\tilde{\vec{E}}=-\frac{\partial\tilde{\vec{B}}}{\partial t}$, note that {***here***}: $k=\left(\frac{\omega}{c}\right)$

$$\vec{\nabla}\times\tilde{\vec{E}}=\frac{\partial}{\partial z}\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\varphi}-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{z}=ik\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\varphi}=i\left(\frac{\omega}{c}\right)\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\varphi}$$

$$-\frac{\partial\tilde{\vec{B}}}{\partial t}=-\frac{\partial}{\partial t}\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\varphi}=+i\omega\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\varphi}=i\left(\frac{\omega}{c}\right)\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\varphi} \ \surd$$

Ampere's law: $\vec{\nabla}\times\tilde{\vec{B}}=\frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}$

$$\vec{\nabla}\times\tilde{\vec{B}}=-\frac{\partial}{\partial z}\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{z}=-ik\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\rho}=-i\left(\frac{\omega}{c}\right)\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\rho}$$

$$\frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\rho}=-i\omega\frac{1}{c^2}\left(\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\right)\hat{\rho}-i\left(\frac{\omega}{c}\right)\left(\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\right)\hat{\rho} \ \surd$$

Boundary Condition 1: Tangential $\tilde{E}_{\parallel}(\rho=(a,b),\varphi,z,t)=0 \Leftarrow$ *i.e.* in $\hat{z}$ and/or $\hat{\varphi}$ direction(s).

But {***here***}: $\tilde{\vec{E}}(\rho,\varphi,z,t)=\frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\hat{\rho}$. Hence ***this*** BC is satisfied! √

Boundary Condition 2: Normal $\tilde{B}_{\perp}(\rho=(a,b),\varphi,z,t)=0 \Leftarrow$ *i.e.* in $\hat{\rho}$ direction.

But {***here***}: $\tilde{\vec{B}}(\rho,\varphi,z,t)=\frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\hat{\varphi}$. Hence ***this*** BC is satisfied! √

Boundary Condition 3: Normal $\tilde{D}_{\perp}^{out}(\rho=(a,b),\varphi,z,t)=\sigma_{free}(\rho=(a,b),\varphi,z,t)$ $(\tilde{D}_{\perp}^{in}=0)$

At $\rho=a$: $\sigma_{free}(\rho=a,\varphi,z,t)=+\varepsilon_o\frac{\tilde{A}}{a}e^{i(kz-\omega t)}$

At $\rho=b$: $\sigma_{free}(\rho=b,\varphi,z,t)=-\varepsilon_o\frac{\tilde{A}}{b}e^{i(kz-\omega t)}$

n.b. the ***total*** free charge on inner *vs.* outer conducting surfaces ***must*** be ***equal***, but ***opposite*** in sign!

Boundary Condition 4: Tangential $\tilde{H}_{\parallel}^{out}\left(\rho=(a,b),\varphi,z,t\right)=\tilde{\vec{K}}_{free}\times\hat{n}$ $(\tilde{H}_{\parallel}^{in}=0)$, $\hat{n}\parallel\hat{\rho}$

At $\rho=a$: $\tilde{\vec{K}}_{free}\left(\rho=a,\varphi,z,t\right)=+\frac{1}{\mu_o}\frac{\tilde{A}}{ac}e^{i(kz-\omega t)}\hat{z}$

At $\rho=b$: $\tilde{\vec{K}}_{free}\left(\rho=b,\varphi,z,t\right)=-\frac{1}{\mu_o}\frac{\tilde{A}}{bc}e^{i(kz-\omega t)}\hat{z}$

n.b. the ***total*** free currents flowing on inner *vs*. outer conductors ***must*** be ***equal***, but ***opposite*** in sign!