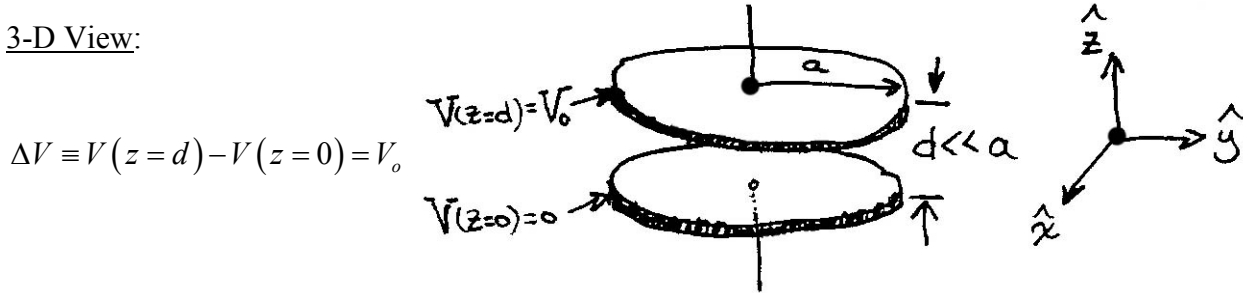


LECTURE NOTES 9

AC Electromagnetic Fields Associated with a Parallel-Plate Capacitor

Let's investigate the nature of AC electromagnetic fields associated with a parallel-plate capacitor, e.g. with circular plates of radius a separated by a small distance $d \ll a$ as shown in the figure below – we will neglect edge effects here:

3-D View:



At DC ($f=0$ Hz), we know the static solution to this problem, namely that the {free} charge Q_{free} on the capacitor is related to the potential difference ΔV across the capacitor's plates by: $Q_{free} = C\Delta V$ where the capacitance of the capacitor is: $C = \epsilon_0 A/d$ (Farads) for $d \ll a$; the area of one plate of the parallel plate capacitor is $A = \pi a^2$.

Since there is no free electric charge between the plates of the parallel plate capacitor, then for $d \ll a$, the solution to Laplace's Equation $\nabla^2 V(\vec{r}) = 0$ {derived from Gauss' Law $\vec{\nabla} \cdot \vec{E}(\vec{r}) = \rho_{free}(\vec{r})/\epsilon_0 = 0$, with $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ } yields:

$$\Delta V \equiv V(z=d) - V(z=0) = -\int_{z=0}^{z=d} \vec{E}(\vec{r}) \cdot d\vec{\ell}$$

But: $\vec{E}(\vec{r}) = -E_0 \hat{z}$ between the plates of the parallel plate capacitor for $d \ll a$

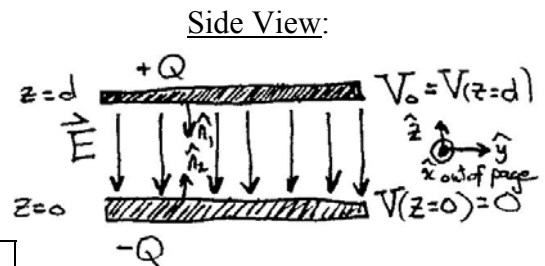
$$\therefore \Delta V \equiv V(z=d) - V(z=0) = (V_0 - 0) = V_0 = -E_0 d$$

$$\Rightarrow \vec{E}(\vec{r}) = -(V_0/d) \hat{z} = (\sigma_{free}/\epsilon_0) \hat{n}_1$$

where: $\sigma_{free} = Q_{free}/A$

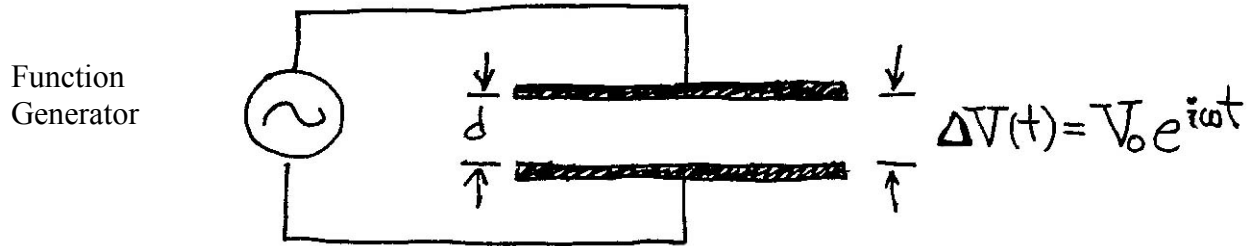
$$\Rightarrow \sigma_{free}(z=d) = +\frac{Q_{free}}{A} = \frac{C\Delta V}{A} = \frac{\epsilon_0 A V_0}{A d} = \epsilon_0 \frac{V_0}{d} = \epsilon_0 E_0$$

And:
$$\sigma_{free}(z=0) = -\frac{Q_{free}}{A} = -\frac{C\Delta V}{A} = -\frac{\epsilon_0 A V_0}{A d} = -\epsilon_0 \frac{V_0}{d} = -\epsilon_0 E_0$$



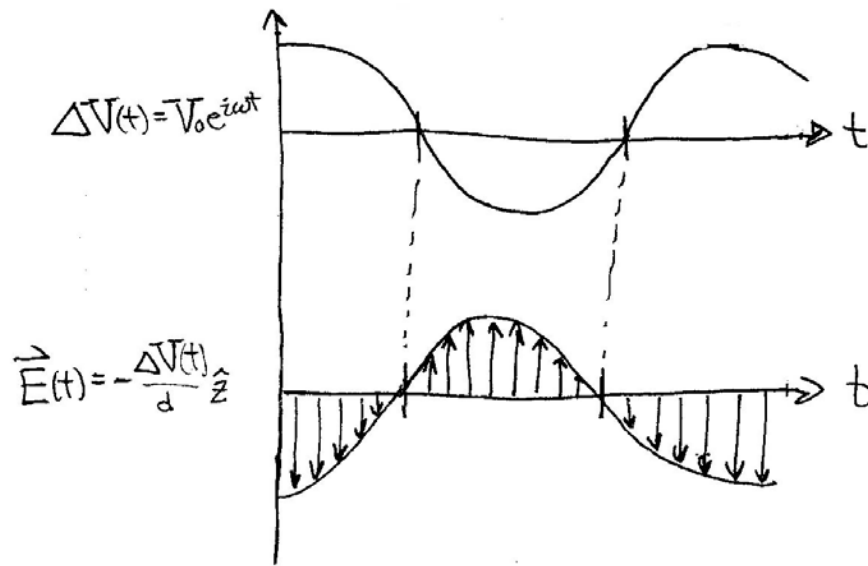
We ask:

What happens when we slowly raise the frequency from $f = 0$ Hz (static E -field) to $f > 0$?
 e.g. Apply a sinusoidally time-varying potential difference across the plates of the capacitor of the form: $\Delta\tilde{V}(t) = V_o e^{i\omega t} = V_o [\cos \omega t + i \sin \omega t] \leftarrow$ single frequency, $f = \omega/2\pi$ e.g. using a sine-wave function generator, as shown in the figure below:



For $d \ll a$: $\tilde{\vec{E}}(\vec{r}, t) = -\frac{\Delta\tilde{V}(t)}{d} \hat{z} = -\frac{V_o e^{i\omega t}}{d} \hat{z} = E_o e^{i\omega t} \hat{z}$ with: $E_o \equiv -V_o/d$

The potential difference $\Delta\tilde{V}(t)$ and electric field $\tilde{\vec{E}}(t)$ vs. time t : $|\tilde{\vec{E}}(t)| = \tilde{E}(t) = E_o e^{i\omega t}$



Maxwell's Equations must be obeyed in the gap-region between the parallel plates of capacitor, where: $\tilde{\rho}_{free}(\vec{r}, t) = \tilde{\rho}_{bound}(\vec{r}, t) = 0$ and: $\tilde{\vec{J}}_{free}(\vec{r}, t) = \tilde{\vec{J}}_{bound}(\vec{r}, t) = 0$:

- 1) Coulomb's Law: $\tilde{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t) = 0$
- 2) No magnetic charges / monopoles: $\tilde{\nabla} \cdot \tilde{\vec{B}}(\vec{r}, t) = 0$
- 3) Faraday's Law: $\tilde{\nabla} \times \tilde{\vec{E}}(\vec{r}, t) = -\partial \tilde{\vec{B}}(\vec{r}, t) / \partial t$ Maxwell's Displacement Current:
- 4) Ampere's Law: $\tilde{\nabla} \times \tilde{\vec{B}}(\vec{r}, t) = \mu_o \epsilon_o \partial \tilde{\vec{E}}(\vec{r}, t) / \partial t = \mu_o \tilde{\vec{J}}_D(\vec{r}, t)$ where: $\tilde{\vec{J}}_D(\vec{r}, t) \equiv \epsilon_o \partial \tilde{\vec{E}}(\vec{r}, t) / \partial t$

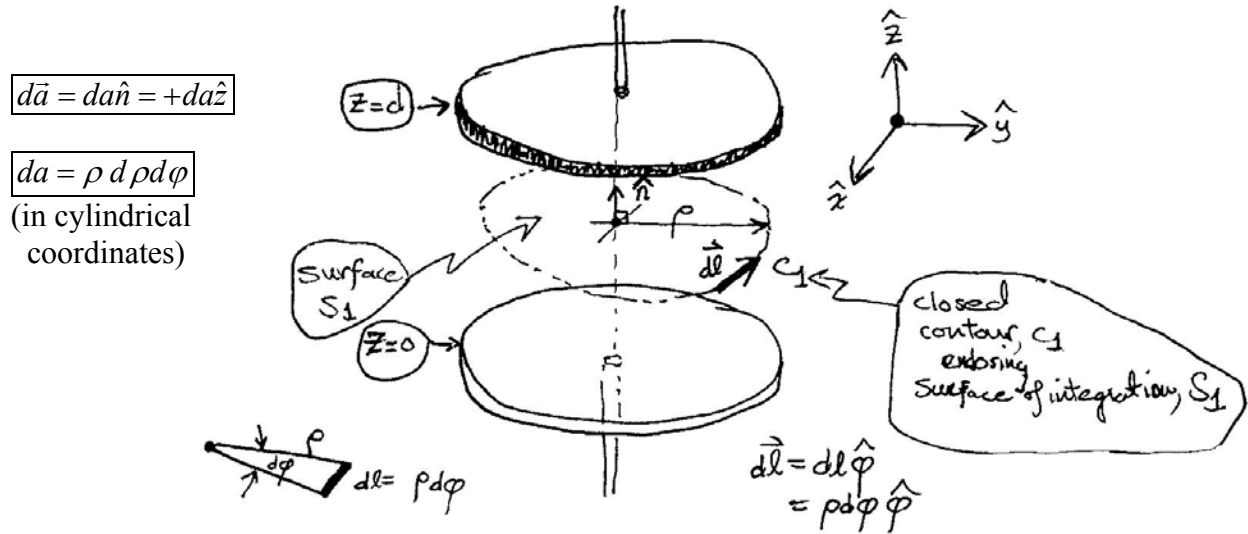
Ampere's Law (with Maxwell's Displacement Current) in integral form tells us that:

$$\int_S (\nabla \times \tilde{\vec{B}}(\vec{r}, t)) \cdot d\vec{a} = \int_S \mu_0 \tilde{\vec{J}}_D(\vec{r}, t) \cdot d\vec{a} \quad \leftarrow \text{n.b. not a closed surface!}$$

$$\oint_C \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{\ell} = \mu_0 \int_S \tilde{\vec{J}}_D(\vec{r}, t) \cdot d\vec{a} = \mu_0 \epsilon_0 \int_S \frac{\partial \tilde{\vec{E}}(\vec{r}, t)}{\partial t} \cdot d\vec{a} \quad \left[\text{Using Stokes' Theorem} \right]$$

$$\therefore \oint_C \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{\ell} = \frac{1}{c^2} \frac{\partial}{\partial t} \left(\int_S \frac{\partial \tilde{\vec{E}}(\vec{r}, t)}{\partial t} \cdot d\vec{a} \right) \quad \text{where: } c^2 = 1/\epsilon_0 \mu_0$$

Let us consider a contour path of integration C_1 enclosing the surface S_1 as shown in the figure below:



$$\tilde{\vec{E}}(\vec{r}, t) = E_o e^{i\omega t} \hat{z} \quad \text{and:} \quad \oint_{C_1} \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{\ell} = \frac{1}{c^2} \frac{\partial}{\partial t} \left(\int_{S_1} \frac{\partial \tilde{\vec{E}}(\vec{r}, t)}{\partial t} \cdot d\vec{a} \right)$$

Note that: $\tilde{\vec{B}}(\vec{r}, t) = \tilde{B}(\rho, t) \hat{\phi}$ due to the circular/azimuthal symmetry associated with this problem.

$d\vec{\ell} = d\ell \hat{\phi} = \rho d\phi \hat{\phi}$, and $d\vec{a} = da \hat{n} = +da \hat{z}$ by the right-hand rule, $d\vec{a} = \rho d\rho d\phi \hat{z}$, $da = \rho d\rho d\phi$ in cylindrical coordinates, thus $\tilde{\vec{B}} \parallel d\vec{\ell}$, and $\tilde{\vec{E}} \parallel d\vec{a}$, and:

$$\therefore B(\rho, t) 2\pi\rho = \frac{1}{c^2} \frac{\partial}{\partial t} E(t) \pi\rho^2 \Rightarrow \tilde{\vec{B}}(\rho, t) = \frac{1}{c^2} \frac{\partial \tilde{E}(t)}{\partial t} \hat{\phi} = \frac{\rho}{2c^2} \left(\frac{\partial \tilde{E}(t)}{\partial t} \right) \hat{\phi}$$

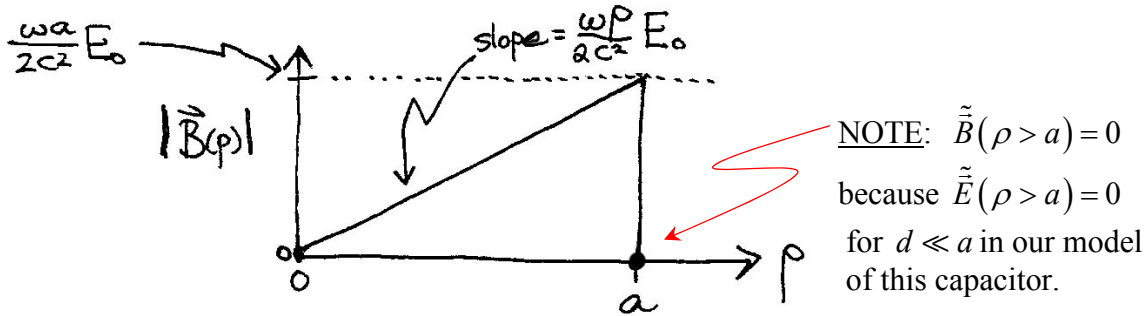
$$\text{But: } \tilde{\vec{E}}(t) = E_o e^{i\omega t} \Rightarrow \frac{\partial \tilde{E}(t)}{\partial t} = i\omega E_o e^{i\omega t} = i\omega \tilde{E}(t)$$

$$\therefore \tilde{\vec{B}}(\rho, t) = \frac{i\omega\rho}{2c^2} \tilde{E}(t) \hat{\phi} = \underbrace{\frac{i\omega\rho}{2c^2} E_o}_{=B_o(\rho)} e^{i\omega t} \hat{\phi} = B_o(\rho) e^{i\omega t} \hat{\phi}$$

$$\Rightarrow \tilde{\vec{B}}(\rho, t) = B_o(\rho) e^{i\omega t} \hat{\phi} = \left[\frac{i\omega\rho}{2c^2} E_o \right] e^{i\omega t} \hat{\phi} = i \left[\frac{\omega\rho}{2c^2} E_o \right] e^{i\omega t} \hat{\phi}$$

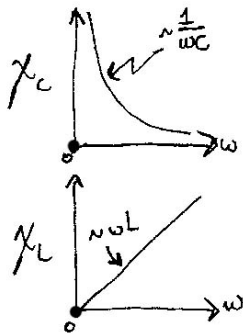
n.b. $\tilde{\vec{B}}(\rho, t)$ also oscillates sinusoidally like $\tilde{\vec{E}}(t)$ but is 90° out-of-phase with $\tilde{\vec{E}}(t)$.

Note also that $\vec{\tilde{B}}(\rho, t)$ is **linearly** proportional to ρ (the radial distance from the axis of capacitor) and that $\vec{\tilde{B}}(\rho = 0) = 0$ at the center of the capacitor:



Thus, we see that for $\omega > 0$, \exists (i.e. there exists) an azimuthal, ρ -dependent and time-varying magnetic field $\vec{\tilde{B}}(\rho, t) = \left[\frac{i\omega\rho E_0}{2c^2} \right] e^{i\omega t} \hat{\phi}$ in the gap region of the parallel-plate capacitor, for $d \ll a$. Note also that the azimuthal magnetic field is also linearly proportional to $\omega = 2\pi f$, thus as the frequency increases, this magnetic field also increases in strength. Note that for $\omega = 0$, $\vec{\tilde{B}}(\omega) = 0$ as we obtained for the static limit case!

Furthermore, because the capacitor now has a non-zero magnetic field associated with it, for $\omega > 0$, the complex, frequency-dependent impedance $\tilde{Z}(\omega) \equiv R(\omega) + i\chi(\omega)$ (Ohms) {where $R(\omega) = AC$ **resistance** and $\chi(\omega) = AC$ **reactance**} of the parallel-plate capacitor is no longer just: $\tilde{Z}_C(\omega) = i\chi_C(\omega) = i(1/\omega C)$ (Ohms) where $\chi_C(\omega) = 1/\omega C =$ the AC capacitive reactance of the capacitor (Ohms), with (complex) AC Ohm's Law: $\Delta\tilde{V}(\omega) = \tilde{I}(\omega) \cdot \tilde{Z}(\omega)$



Because of the existence of the magnetic field in gap-region of \parallel -plate capacitor, *EM* energy can **also** be/is stored in the magnetic field of \parallel -plate capacitor due to the **inductance**, L_C (Henry) associated with the parallel-plate capacitor and hence it has an inductive **reactance** of $\chi_L(\omega) = \omega L$ and hence has an inductive complex **impedance** associated with it, of $Z_L(\omega) = i\chi_L(\omega) = i\omega L_C$ (Ohms). Since the inductance associated with this capacitor is in **series** with its capacitance, we **add** the two impedances:

$$\tilde{Z}_C^{TOT}(\omega) = \tilde{Z}_C(\omega) + \tilde{Z}_L(\omega) = i\chi_C(\omega) + i\chi_L(\omega) = i\left(\frac{1}{\omega C}\right) + i\omega L_C$$

$$\tilde{Z}_C^{TOT}(\omega) = i\left(\frac{1}{\omega C} + \omega L_C\right)$$

The {complex} form Ohm's Law {here} is thus: $\Delta\tilde{V}(\omega) = \tilde{I}(\omega) \tilde{Z}_C^{TOT}(\omega)$

Note that at low frequencies ($\omega \approx 0$) for the parallel-plate capacitor with $d \ll a$, the capacitive reactance $\chi_C(\omega) = 1/\omega C \gg \chi_L(\omega) = \omega L_C$ and thus $\tilde{Z}_C^{TOT}(\omega \approx 0) \approx \tilde{Z}_C(\omega \approx 0)$. However, at very high frequencies ($\omega \rightarrow \infty$), $\chi_C(\omega) \ll \chi_L(\omega) \Rightarrow \tilde{Z}_C^{TOT}(\omega \rightarrow \infty) \approx \tilde{Z}_L(\omega \rightarrow \infty)$, i.e. in the very high frequency limit, this capacitor instead behaves like a pure inductor!!!

Note also that the electric, magnetic and total EM energy densities in the gap-region of the parallel plate capacitor, respectively are:

$$u_E(\vec{r}, t) = \frac{1}{2} \epsilon_0 |\tilde{\vec{E}}(\vec{r}, t)|^2, \quad u_M(\vec{r}, t) = \frac{1}{2\mu_0} |\tilde{\vec{B}}(\vec{r}, t)|^2 \quad \text{and} \quad u_{TOT}^{EM}(\vec{r}, t) = u_E(\vec{r}, t) + u_M(\vec{r}, t)$$

Now because the capacitor has a non-zero time-varying magnetic field: $\tilde{\vec{B}}(\rho, t) = \frac{i\omega\rho}{2c^2} E_0 e^{i\omega t} \hat{\phi}$

Faraday's Law $\vec{\nabla} \times \tilde{\vec{E}}(\vec{r}, t) = -\partial \tilde{\vec{B}}(\vec{r}, t) / \partial t$ tells us that there will be an **additional** {induced} electric field, because $\tilde{\vec{B}}(\vec{r}, t)$ is **also** varying in time!!!

$$\text{Faraday's Law in integral form is: } \int_S (\vec{\nabla} \times \tilde{\vec{E}}(\vec{r}, t)) \cdot d\vec{a} = -\frac{\partial}{\partial t} \left(\int_S \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a} \right) = -\frac{\partial \tilde{\Phi}_m(t)}{\partial t}$$

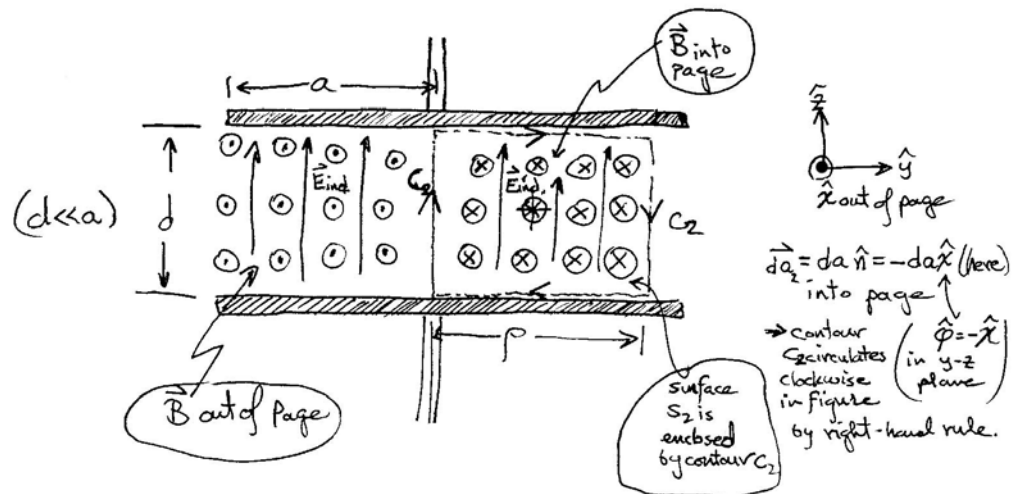
where $\tilde{\Phi}_m(t) \equiv \int_S \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}$ is the magnetic flux (*Webers = Tesla-m²*) enclosed by the surface S

at time t . Applying Stokes' Theorem, we have: $\oint_C \tilde{\vec{E}}(\vec{r}, t) \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \left(\int_S \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a} \right) = -\frac{\partial \tilde{\Phi}_m(t)}{\partial t}$

where the contour C around a closed path of integration encloses the surface S through which magnetic flux $\tilde{\Phi}_m(t) \equiv \int_S \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}$ passes.

Now $\tilde{\vec{B}} = \tilde{B} \hat{\phi}$ (i.e. points in the $\hat{\phi}$ {azimuthal} direction) and thus here we need $\tilde{\vec{B}} \parallel d\vec{a}$ hence $d\vec{a}_2 = da \hat{\phi}$ also, and thus we take the closed contour C_2 line-integral path around the surface S_2 as shown in the side-view figure below:

Side-View:

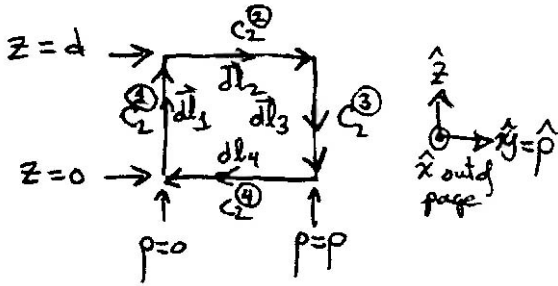


The **induced** electric field, as created by the time-varying magnetic field is:

$$\oint_{C_2} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \left(\int_{S_2} \vec{B}(\vec{r}, t) \cdot d\vec{a}_2 \right) = -\frac{\partial \Phi_m(t)}{\partial t}$$

where $\Phi_m(t) \equiv \int_{S_2} \vec{B}(\vec{r}, t) \cdot d\vec{a}_2$ = magnetic flux enclosed by contour C_2 passing through surface S_2

Then:
$$\oint_{C_2} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = \int_{(1)} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_1 + \int_{(2)} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_2 + \int_{(3)} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_3 + \int_{(4)} \vec{E}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_4$$



$$d\vec{\ell}_1 = dl\hat{z}$$

$$d\vec{\ell}_2 = dl\hat{\rho} \quad (\hat{\rho} = \hat{y} \text{ here in } \hat{y}-\hat{z} \text{ plane})$$

$$d\vec{\ell}_3 = dl(-\hat{z}) = -dl\hat{z}$$

$$d\vec{\ell}_4 = dl(-\hat{\rho}) = -dl\hat{\rho}$$

Now $\vec{\nabla} \times \vec{E}_{ind}(\vec{r}, t) = -\partial \vec{B}(\vec{r}, t) / \partial t$ tells us that if $\vec{B} = B\hat{\phi}$ direction, then in cylindrical coordinates:

$$\vec{\nabla} \times \vec{E}_{ind}(\vec{r}, t) = \underbrace{\left[\frac{1}{\rho} \frac{\partial \tilde{E}_z}{\partial \phi} - \frac{\partial \tilde{E}_\phi}{\partial z} \right]}_{=0} \hat{\rho} + \left[\frac{\partial \tilde{E}_\rho}{\partial z} - \frac{\partial \tilde{E}_z}{\partial \rho} \right] \hat{\phi} + \underbrace{\frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho \tilde{E}_\phi) - \frac{\partial \tilde{E}_\rho}{\partial \phi} \right]}_{=0} \hat{z} = -\frac{\partial \tilde{E}_z(\vec{r}, t)}{\partial \rho} \hat{\phi}$$

Thus, we see that $\vec{\nabla} \times \vec{E}(\vec{r}, t) = \left| \vec{\nabla} \times \vec{E}(\vec{r}, t) \right| \hat{\phi}$ only, for all points (ρ, ϕ, z) in the gap region of \parallel -plate capacitor and for all times t . However, we see that due to the azimuthal / rotational symmetry associated with the cylindrical \parallel -plate capacitor, neither $\vec{E}_{ind}(\vec{r}, t)$ nor $\vec{B}(\vec{r}, t)$ can have any explicit ϕ -dependence, thus $\partial \tilde{E}_z / \partial \phi = 0$ and $\partial \tilde{E}_\rho / \partial \phi = 0$, which in turn respectively imply that $\partial \tilde{E}_\phi / \partial z = 0$ and $\partial(\rho \tilde{E}_\phi) / \partial \rho = 0$. Note further that Faraday's Law tells us that we must also have $\vec{E}_{ind}(\vec{r}, t) \perp \vec{B}(\vec{r}, t)$.

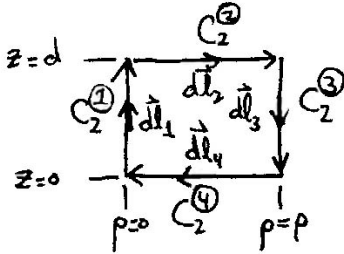
For $d \ll a$, the electric field in the gap region of the \parallel -plate capacitor **cannot** explicitly depend on z either. Thus, $\partial E_\rho / \partial z = 0 \Rightarrow \therefore$ the only surviving term in $\vec{\nabla} \times \vec{E}_{ind}(\vec{r}, t)$ is:

$$\vec{\nabla} \times \vec{E}_{ind}(\vec{r}, t) = -\frac{\partial \tilde{E}_z(\vec{r}, t)}{\partial \rho} \hat{\phi}$$

$\therefore \vec{E}_{ind}(\vec{r}, t) = \tilde{E}_{ind}(\vec{r}, t) \hat{z}$ i.e. the induced \vec{E} -field points in the \hat{z} direction (must be $\perp \vec{B} = \tilde{B}\hat{\phi}$) which is satisfied because $\hat{z} \perp \hat{\phi}$.

Thus, if the induced electric field $\tilde{\vec{E}}_{ind}(\vec{r}, t) = \tilde{E}_{ind}(\vec{r}, t)\hat{z}$ {only}, then we see that:

$$\begin{aligned} \oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell} &= \int_{(1)} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_1 + \int_{(2)} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_2 + \int_{(3)} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_3 + \int_{(4)} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell}_4 \\ &= \int_{(1)} \tilde{E}_{ind}\hat{z} \cdot dz\hat{z} + \underbrace{\int_{(2)} \tilde{E}_{ind}\hat{z} \cdot d\rho\hat{\rho}}_{=0 \text{ } (\hat{z} \perp \hat{\rho})} + \int_{(3)} \tilde{E}_{ind}\hat{z} \cdot (-dz\hat{z}) + \underbrace{\int_{(4)} \tilde{E}_{ind}\hat{z} \cdot (-d\rho\hat{\rho})}_{=0 \text{ } (\hat{z} \perp \hat{\rho})} \\ &= \int_{(1)} \tilde{E}_{ind}\hat{z} \cdot dz\hat{z} + \int_{(3)} \tilde{E}_{ind}\hat{z} \cdot (-dz\hat{z}) \\ &= \int_{z=0}^{z=d} \tilde{E}_{ind}(\rho=0) dz - \int_{z=0}^{z=d} \tilde{E}_{ind}(\rho=\rho) dz \end{aligned}$$



But $\tilde{\vec{E}}_{ind}(\vec{r}, t)$ has no explicit z -dependence, thus:

$$\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = \tilde{E}_{ind}(\rho=0, t) \int_{z=0}^{z=d} dz - \tilde{E}_{ind}(\rho=\rho, t) \int_{z=0}^{z=d} dz = \tilde{E}_{ind}(\rho=0, t) * d + \tilde{E}_{ind}(\rho=\rho, t) * d$$

Or: $\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = [\tilde{E}_{ind}(\rho=0, t) - \tilde{E}_{ind}(\rho=\rho, t)] d$ where: $\tilde{\vec{E}}_{ind}(\rho, t) = \tilde{E}_{ind}(\rho, t)\hat{z}$

But: $\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{S_2} \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}_2$ where $S_2 =$ surface enclosed by contour C_2
and $d\vec{a}_2 = da\hat{n}_2 = da\hat{\phi}$ (i.e. S_2 lies in the y - z plane) $= -da\hat{x}$ {here} and $da = dydz = d\rho dz$

Now: $\tilde{\vec{B}}(\rho, t) = B_o(\rho) e^{i\omega t} \hat{\phi} = \left(\frac{i\omega\rho}{2c^2}\right) E_o e^{i\omega t} \hat{\phi}$ and $\hat{\phi} = -\hat{x}$ (S_2 lies in the y - z plane)

$$\begin{aligned} \therefore \int_{S_2} \tilde{\vec{B}}(\rho, t) \cdot d\vec{a}_2 &= \int_{\rho=0}^{\rho=\rho} \int_{z=0}^{z=d} \left(\frac{i\omega\rho}{2c^2}\right) E_o e^{i\omega t} \hat{\phi} \cdot d\rho dz \hat{\phi} \text{ but: } \hat{\phi} \cdot \hat{\phi} = 1 \\ &= \left(\frac{i\omega}{2c^2}\right) E_o e^{i\omega t} \int_{\rho=0}^{\rho=\rho} \int_{z=0}^{z=d} \rho d\rho dz = \left(\frac{i\omega d}{2c^2}\right) E_o e^{i\omega t} \int_{\rho=0}^{\rho=\rho} \rho d\rho \text{ and: } \int \rho d\rho = \frac{1}{2} \rho^2 \end{aligned}$$

$$\therefore \int_{S_2} \tilde{\vec{B}}(\rho, t) \cdot d\vec{a}_2 = \left(\frac{i\omega\rho^2 d}{4c^2}\right) E_o e^{i\omega t}$$

Then: $-\frac{\partial}{\partial t} \int_{S_2} \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}_2 = -\frac{\partial}{\partial t} \left[\left(\frac{i\omega\rho^2 d}{4c^2}\right) E_o e^{i\omega t} \right] = -i\omega \left[\left(\frac{i\omega\rho^2 d}{4c^2}\right) E_o e^{i\omega t} \right]$

And: $(-i\omega)(i\omega) = (-i * i)\omega^2 = +1\omega^2 = \omega^2$ {since $i = \sqrt{-1}$ and $-i = -\sqrt{-1}$ }

$$\therefore -\frac{\partial}{\partial t} \int_{S_2} \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}_2 = \frac{\omega^2 \rho^2 d}{4c^2} E_0 e^{i\omega t}$$

Then: $\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r}, t) \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{S_2} \tilde{\vec{B}}(\vec{r}, t) \cdot d\vec{a}_2$ yields:

$$\cancel{d} \left[\tilde{\vec{E}}_{ind}(\rho=0, t) - \tilde{\vec{E}}_{ind}(\rho=\rho, t) \right] = \frac{\omega^2 \rho^2 \cancel{d}}{4c^2} E_0 e^{i\omega t} \hat{z}$$

Note that the d 's cancel on both sides of the above equation. Note also that because of the explicit ρ^2 dependence on the RHS of above equation, we see that $\tilde{\vec{E}}_{ind}(\rho=0, t) = 0$.

Hence:
$$\tilde{\vec{E}}_{ind}(\rho, t) = -\frac{\omega^2 \rho^2}{4c^2} E_0 e^{i\omega t} \hat{z} = -\left(\frac{\omega\rho}{2c}\right)^2 E_0 e^{i\omega t} \hat{z}$$

Thus the **total** $\tilde{\vec{E}}$ -field in the capacitor gap is:

$$\tilde{\vec{E}}_{TOT}(\rho, t) = \tilde{\vec{E}}(t) + \tilde{\vec{E}}_{ind}(\rho, t) = E_0 e^{i\omega t} \hat{z} - \left(\frac{\omega\rho}{2c}\right)^2 E_0 e^{i\omega t} \hat{z} = \left(1 - \left(\frac{\omega\rho}{2c}\right)^2\right) E_0 e^{i\omega t} \hat{z}$$

Thus, we see here that the **induced** electric field caused by the time-varying magnetic field points in the direction **opposite** to the initial/original $\tilde{\vec{E}}$ -field, **reducing** the overall $\tilde{\vec{E}}$ -field for $\rho > 0$, as we would expect from Lenz's Law.

However, note that we now **also** have an **additional** contribution to the $\tilde{\vec{B}}$ -field inside the gap-region of the parallel plate capacitor, due to the presence of the **induced** $\tilde{\vec{E}}$ -field contribution, $\tilde{\vec{E}}_{ind}(\rho, t)$.

Before we proceed further on this discussion, it would be best for us change our notation: Call our original time-dependent $\tilde{\vec{E}}$ -field, $\tilde{\vec{E}}(\vec{r}, t) = E_0 e^{i\omega t} = \tilde{\vec{E}}_1(\vec{r}, t)$.

This $\tilde{\vec{E}}$ -field in turn creates a time-dependent $\tilde{\vec{B}}$ -field by **Ampere's Law**:

$$\vec{\nabla} \times \tilde{\vec{B}}_1(\vec{r}, t) = \mu_0 \epsilon_0 \frac{\partial \tilde{\vec{E}}_1(\vec{r}, t)}{\partial t} = \frac{1}{c^2} \frac{\partial \tilde{\vec{E}}_1(\vec{r}, t)}{\partial t}$$

However, because $\tilde{\vec{B}}_1(\vec{r}, t)$ **also** varies in time, it turn creates **another** induced time-dependent electric field by **Faraday's Law**:

$$\vec{\nabla} \times \tilde{\vec{E}}_2(\vec{r}, t) = -\frac{\partial \tilde{\vec{B}}_1(\vec{r}, t)}{\partial t}$$

But $\tilde{\vec{E}}_2(\vec{r}, t)$ is **also** time-varying, and so it in turn produces **another** time-varying contribution to the magnetic field $\tilde{\vec{B}}_2(\vec{r}, t)$.

But because $\tilde{\vec{B}}_2(\vec{r}, t)$ is **also** time-varying, it in turn will induce **another** contribution to the electric field $\tilde{\vec{E}}_3(\vec{r}, t)$ and so on... *i.e.*:

$$\tilde{\vec{E}}_1(\vec{r}, t) \xrightarrow{A.L.} \tilde{\vec{B}}_1(\vec{r}, t) \xrightarrow{F.L.} \tilde{\vec{E}}_2(\vec{r}, t) \xrightarrow{A.L.} \tilde{\vec{B}}_2(\vec{r}, t) \xrightarrow{F.L.} \tilde{\vec{E}}_3(\vec{r}, t) \xrightarrow{A.L.} \tilde{\vec{B}}_3(\vec{r}, t) \xrightarrow{F.L.} \tilde{\vec{E}}_4(\vec{r}, t) \xrightarrow{A.L.} \tilde{\vec{B}}_4(\vec{r}, t) \xrightarrow{F.L.} \dots$$

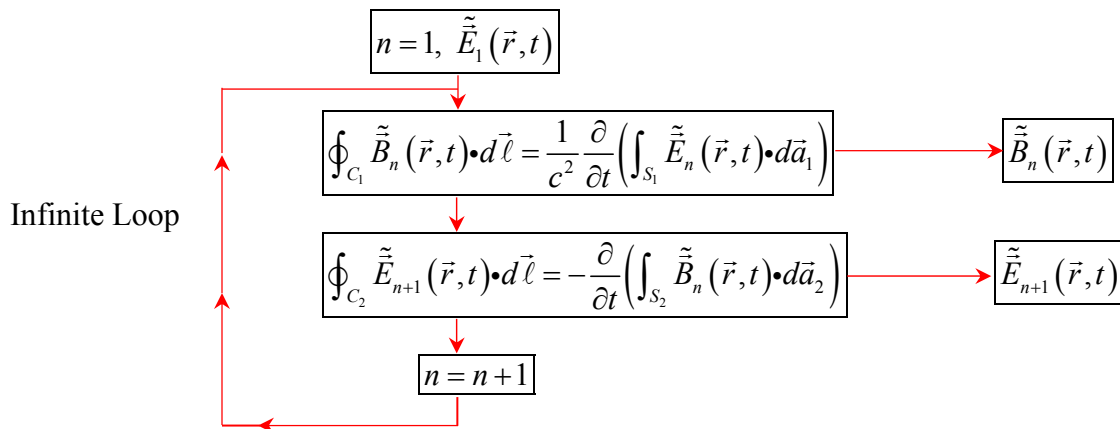
Then:
$$\tilde{\vec{E}}_{TOT}(\vec{r}, t) = \tilde{\vec{E}}_1(\vec{r}, t) + \tilde{\vec{E}}_2(\vec{r}, t) + \tilde{\vec{E}}_3(\vec{r}, t) + \tilde{\vec{E}}_4(\vec{r}, t) + \tilde{\vec{E}}_5(\vec{r}, t) + \dots = \sum_{n=1}^{\infty} \tilde{\vec{E}}_n(\vec{r}, t)$$

And:
$$\tilde{\vec{B}}_{TOT}(\vec{r}, t) = \tilde{\vec{B}}_1(\vec{r}, t) + \tilde{\vec{B}}_2(\vec{r}, t) + \tilde{\vec{B}}_3(\vec{r}, t) + \tilde{\vec{B}}_4(\vec{r}, t) + \tilde{\vec{B}}_5(\vec{r}, t) + \dots = \sum_{n=1}^{\infty} \tilde{\vec{B}}_n(\vec{r}, t)$$

So thus we see that:

$$\begin{aligned} \oint_{C_1} \tilde{\vec{B}}_1(\vec{r}, t) \cdot d\vec{\ell} &= \frac{1}{c^2} \frac{\partial}{\partial t} \left(\int_{S_1} \tilde{\vec{E}}_1(\vec{r}, t) \cdot d\vec{a}_1 \right) \\ \oint_{C_2} \tilde{\vec{E}}_2(\vec{r}, t) \cdot d\vec{\ell} &= -\frac{\partial}{\partial t} \left(\int_{S_2} \tilde{\vec{B}}_1(\vec{r}, t) \cdot d\vec{a}_2 \right) \\ \oint_{C_1} \tilde{\vec{B}}_2(\vec{r}, t) \cdot d\vec{\ell} &= \frac{1}{c^2} \frac{\partial}{\partial t} \left(\int_{S_1} \tilde{\vec{E}}_2(\vec{r}, t) \cdot d\vec{a}_1 \right) \\ \oint_{C_2} \tilde{\vec{E}}_3(\vec{r}, t) \cdot d\vec{\ell} &= -\frac{\partial}{\partial t} \left(\int_{S_2} \tilde{\vec{B}}_2(\vec{r}, t) \cdot d\vec{a}_2 \right) \\ \oint_{C_1} \tilde{\vec{B}}_3(\vec{r}, t) \cdot d\vec{\ell} &= \frac{1}{c^2} \frac{\partial}{\partial t} \left(\int_{S_1} \tilde{\vec{E}}_3(\vec{r}, t) \cdot d\vec{a}_1 \right) \\ \oint_{C_2} \tilde{\vec{E}}_4(\vec{r}, t) \cdot d\vec{\ell} &= -\frac{\partial}{\partial t} \left(\int_{S_2} \tilde{\vec{B}}_3(\vec{r}, t) \cdot d\vec{a}_2 \right) \\ \dots \text{ etc.} \end{aligned}$$

Algorithmically, this **infinite** sequence can be written as:



Where contour C_1 enclosing surface S_1 and area element $d\vec{a}_1$ are associated with the figure drawn on page 3 of these lecture notes, and where contour C_2 enclosing surface S_2 and area element $d\vec{a}_2$ are associated with the figure drawn on page 5 of these lecture notes.

It can thus be shown for the parallel-plate capacitor with $d \ll a$ that:

$$\begin{aligned} \tilde{\tilde{E}}_{TOT}(\rho, t) &= \left[1 - \frac{1}{(1!)^2} \left(\frac{\omega\rho}{2c} \right)^2 + \frac{1}{(2!)^2} \left(\frac{\omega\rho}{2c} \right)^4 - \frac{1}{(3!)^2} \left(\frac{\omega\rho}{2c} \right)^6 + \dots \right] E_o e^{i\omega t} \hat{z} && \text{with: } E_o = - \left(\frac{V_o}{d} \right) \\ \tilde{\tilde{B}}_{TOT}(\rho, t) &= \left[1 - \frac{1}{(1!)^2} \left(\frac{\omega\rho}{2c} \right)^2 + \frac{1}{(2!)^2} \left(\frac{\omega\rho}{2c} \right)^4 - \frac{1}{(3!)^2} \left(\frac{\omega\rho}{2c} \right)^6 + \dots \right] B_o e^{i\omega t} \hat{\phi} && \text{with: } B_o = \frac{i\omega\rho}{2c^2} E_o \end{aligned}$$

and where: $n! \equiv n \cdot (n-1) \cdot (n-2) \dots 2 \cdot 1$, $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc.

We also see that: $B_n = \frac{i\omega\rho}{2c^2} E_n$ and: $E_{n+1} = \left(\frac{i\omega\rho}{2} \right) B_n = \left(\frac{i\omega\rho}{2} \right) \left(\frac{i\omega\rho}{2c^2} \right) E_n = - \left(\frac{\omega\rho}{2c} \right)^2 E_n$

and: $B_{n+1} = \left(\frac{i\omega\rho}{2c^2} \right) E_{n+1} = \left(\frac{i\omega\rho}{2c^2} \right) \left(\frac{i\omega\rho}{2} \right) B_n = - \left(\frac{\omega\rho}{2c} \right)^2 B_n$

Due to the **cylindrical geometry** / **azimuthal symmetry** associated with this problem, it should not come as a surprise that:

Defining: $x \equiv \frac{\omega\rho}{c} = k\rho$ where $k = \frac{\omega}{c}$ = wavenumber

$$\begin{array}{|c|} \hline k = \frac{2\pi}{\lambda} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \lambda = \frac{c}{f} \\ \hline \end{array} \quad \begin{array}{|c|} \hline f = \frac{\omega}{2\pi} \\ \hline \end{array}$$

Then the quantity in square brackets on the previous page becomes:

$$1 - \frac{1}{(1!)^2} \left(\frac{\omega\rho}{2c} \right)^2 + \frac{1}{(2!)^2} \left(\frac{\omega\rho}{2c} \right)^4 - \frac{1}{(3!)^2} \left(\frac{\omega\rho}{2c} \right)^6 + \dots = 1 - \frac{1}{(1!)^2} \left(\frac{x}{2} \right)^2 + \frac{1}{(2!)^2} \left(\frac{x}{2} \right)^4 - \frac{1}{(3!)^2} \left(\frac{x}{2} \right)^6 + \dots$$

The so-called “ordinary” Bessel function of the first kind, of order zero has a series expansion of the form:

$$\begin{aligned} J_0(x) &= 1 - \frac{1}{(1!)^2} \left(\frac{x}{2} \right)^2 + \frac{1}{(2!)^2} \left(\frac{x}{2} \right)^4 - \frac{1}{(3!)^2} \left(\frac{x}{2} \right)^6 + \dots \\ J_0(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+1)} \left(\frac{x}{2} \right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot k!} \left(\frac{x}{2} \right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2} \right)^{2k} \\ k! &\equiv k \cdot (k-1) \cdot (k-2) \dots 3 \cdot 2 \cdot 1 && \text{where: } \Gamma(k+1) = k! \quad (\text{for } k = \text{integer}) \end{aligned}$$

In general, the series expansion of the ordinary Bessel functions of the first kind, of order n are:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{x}{2} \right)^{n+2k}$$

Thus, for the cylindrical ||-plate capacitor with $d \ll a$ the electric and magnetic fields in the gap region are of the form:

$$\tilde{\vec{E}}(\rho, t) = J_0\left(\frac{\omega\rho}{c}\right) E_o e^{i\omega t} \hat{z} = J_0(k\rho) E_o e^{i\omega t} \hat{z} \quad \text{with} \quad k = \frac{\omega}{c} \quad \text{and} \quad E_o = -\frac{V_o}{d}$$

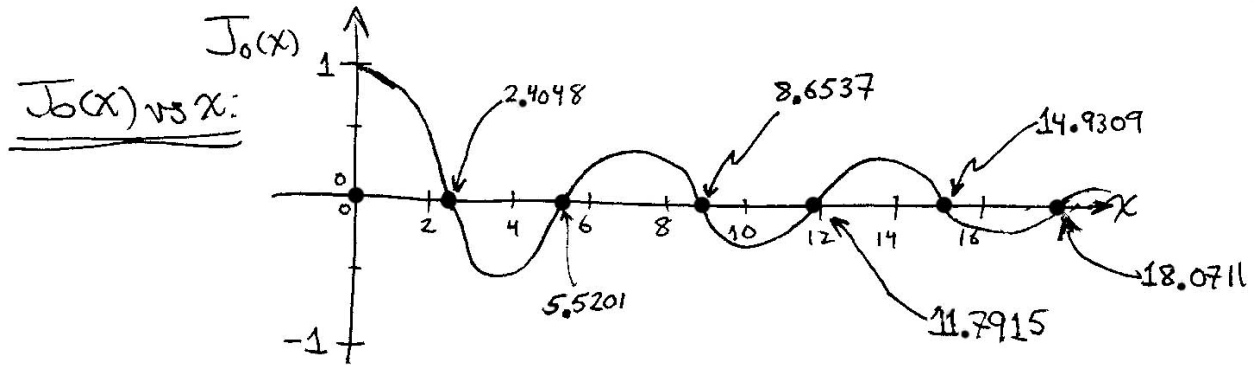
\vec{B} is 90° out-of-phase with \vec{E} {here}:

$$\tilde{\vec{B}}(\rho, t) = J_0\left(\frac{\omega\rho}{c}\right) B_o(\rho) e^{i\omega t} \hat{\phi} = J_0(k\rho) B_o(\rho) e^{i\omega t} \hat{\phi} \quad \text{with} \quad B_o(\rho) = \frac{i\omega\rho}{2c^2} E_o = i\left(\frac{\omega\rho}{2c^2}\right) E_o$$

Note that for $\rho = 0$ that: $\tilde{\vec{E}}(\rho = 0, t) = E_o e^{i\omega t}$ but: $\tilde{\vec{B}}(\rho = 0, t) = 0$.

The {Radial} Zeroes of $J_0(x)$:

$$x = k\rho = \left(\frac{\omega}{c}\right)\rho$$



n.b. the zeroes of $J_n(x)$ are not integer related!!

Since $\tilde{\vec{E}}(\rho, t) = J_0(k\rho) E_o e^{i\omega t} \hat{z}$ and $\tilde{\vec{B}}(\rho, t) = J_0(k\rho) B_o(\rho) e^{i\omega t} \hat{\phi}$ we see that the zeroes x_n of $J_0(x)$ are physically where the electric and magnetic fields vanish (!!!), *i.e.* $\vec{E}(\rho, t) = 0$ and $\vec{B}(\rho, t) = 0$ when $\rho_n = x_n/k = cx_n/\omega$ with $x_1 = 2.4048, x_2 = 5.5201, x_3 = 8.6537, \text{etc.!!!}$

So let's now examine the frequency-dependence of the \vec{E} and \vec{B} fields of the ||-plate capacitor:

$$\tilde{\vec{E}}(\rho, t) = J_0\left(\frac{\omega\rho}{c}\right) E_o e^{i\omega t} \hat{z} = J_0(k\rho) E_o e^{i\omega t} \hat{z} \quad \text{with:} \quad k = \frac{\omega}{c} \quad \text{and:} \quad E_o = -\frac{V_o}{d}$$

$$\tilde{\vec{B}}(\rho, t) = J_0\left(\frac{\omega\rho}{c}\right) B_o(\rho) e^{i\omega t} \hat{\phi} = J_0(k\rho) B_o(\rho) e^{i\omega t} \hat{\phi} \quad \text{with:} \quad B_o(\rho) = \frac{i\omega\rho}{2c^2} E_o = \frac{ik\rho}{2c} E_o$$

a.) When: $\omega = 0, f = 0$ then: $k = \frac{\omega}{c} = 0 \Rightarrow \lambda = \infty$ (static case). Then: $x = k\rho = 0$ and $J_0(0) = 1$

$$\tilde{\vec{E}}(\rho, t) = E_o \hat{z} = -\frac{V_o}{d} \hat{z} \quad \text{with:} \quad E_o = -\frac{V_o}{d} \quad \text{and:} \quad \tilde{\vec{B}}(\rho, t) = 0 \quad \leftarrow \quad \text{n.b. Same result as original static calculation}$$

b.) When: $\omega \geq 0$, e.g. $f = 60 \text{ Hz} \rightarrow \omega = 2\pi f = 120\pi \text{ rad/sec}$

$$\text{Then: } k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{120\pi \text{ rad/sec}}{3 \times 10^8 \text{ m/sec}} = 1.257 \times 10^{-6} \text{ rad/meter}$$

Suppose the radius of the capacitor is $a = 1 \text{ cm} = 10^{-2} \text{ m}$ (reasonable/typical diameter)

$$\text{Then: } ka = 1.257 \times 10^{-8} = x \text{ (dimensionless)}$$

$$\text{And: } J_0(ka) = J_0(1.257 \times 10^{-8}) \approx 1.0 \text{ (n.b. see/refer to above graph of } J_0(x) \text{ vs. } x)$$

Thus, we see that at $f = 60 \text{ Hz}$, the \vec{E} -field is \approx that of the DC \vec{E} -field, and e.g. if $V_o = 10 \text{ V}$ and $d = 0.1 \text{ mm} \ll a = 1 \text{ cm}$, i.e. $d = 10^{-4} \text{ m} \ll a = 10^{-2} \text{ m}$, then: $E_o = -\frac{V_o}{d} = -\frac{10 \text{ V}}{10^{-4} \text{ m}} = -10^5 \text{ Volts/m}$

$$\text{and: } |B_o(\rho = a)| = \frac{\omega a}{2c^2} E_o = \frac{ka}{2c} E_o = \frac{1.257 \times 10^{-8}}{2 \times 3 \times 10^8} \times 10^5 = 2.1 \times 10^{-12} \text{ Tesla } \{i.e. \text{ is very small}\}.$$

$$\underline{\text{Another way to see this:}} \quad c |B_o(\rho = a)| = 6.3 \times 10^{-4} \text{ Volts/m} \ll |E_o| = 10^5 \text{ Volts/m}$$

c.) Now suppose: $f = 1 \text{ MHz} = 10^6 \text{ Hz}$ and $\omega = 2\pi f = 2\pi \times 10^6 \text{ rads/sec}$

$$\text{Then: } k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \frac{2\pi \times 10^6}{3 \times 10^8} \approx 2.1 \times 10^{-2} = 0.021 \text{ radians/m} \text{ and if } a = 1 \text{ cm} = 0.01 \text{ m}$$

$$\text{Then: } (ka) = 0.021 \times 0.01 = 2.1 \times 10^{-4} \text{ and } J_0(ka) = J_0(2.1 \times 10^{-4}) \approx 1 \text{ (still).}$$

$$\rightarrow E_o = -\frac{V_o}{d} \text{ (constant), and: } |B_o(\rho = a)| = \frac{ka}{2c} E_o = 3.5 \times 10^{-8} \text{ Tesla} = 35 \text{ nT (still very small)}$$

$$\text{for } E_o = -10^5 \text{ Volts/meter and \{still\} } c |B_o(\rho = a)| = 10.5 \text{ Volts/m} \ll |E_o| = 10^5 \text{ V/m}$$

$$\text{for } V_o = 10 \text{ Volts, } d = 0.1 \text{ mm and } a = 1 \text{ cm} = 10^{-2} \text{ m.}$$

d.) Now suppose: $f = 100 \text{ GHz} = 10^{11} \text{ Hz}$ and $\omega = 2\pi f = 6.3 \times 10^{11} \text{ rads/sec}$

$$\text{Then: } k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \frac{6.3 \times 10^{11}}{3 \times 10^8} = 2.1 \times 10^3 \text{ radians/m}$$

$$\text{Then: } (ka) = 2.1 \times 10^3 \times 10^{-2} = 21 \rightarrow J_0(k\rho) \text{ has 5 zeroes in it !!!}$$

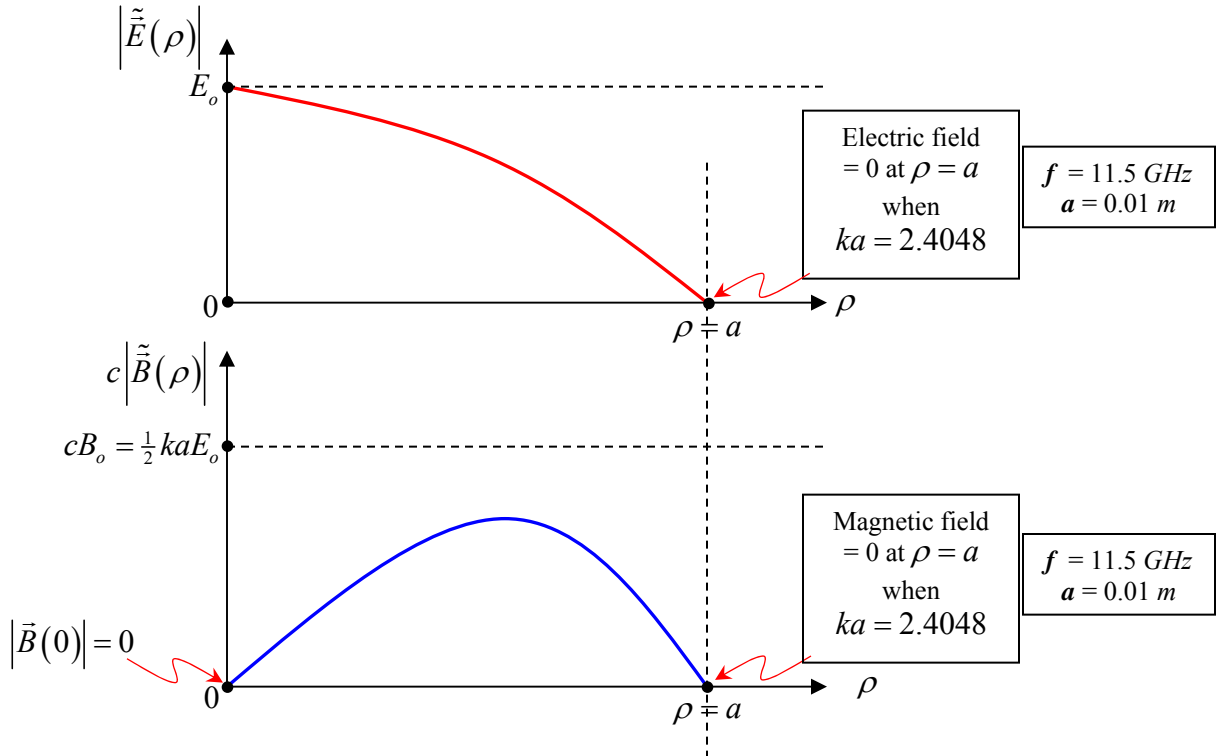
EEK!! \rightarrow the \vec{E} -field points in the reverse direction depending on $0 \leq \rho \leq a$ value !!!

(see above graph of $J_0(x)$ vs. x on page 11 of these lecture notes)

Suppose instead that we pick: $ka = 2.4048 = x_1 = 1^{\text{st}}$ zero of $J_0(x) = J_0(k\rho)$ ($a = 10^{-2} \text{ m} = 1 \text{ cm}$)

$$\text{Then: } k = 240.48 \text{ radians/m} = \frac{\omega}{c} \rightarrow f = 1.15 \times 10^{10} \text{ Hz} = 11.5 \text{ GHz (in the microwave region)}$$

$$\text{Then: } \boxed{\vec{E}(\rho, t) = J_0(k\rho) E_o e^{i\omega t} \hat{z}} \text{ and } \boxed{\lambda = \frac{2\pi}{k} = 2.61 \text{ cm}}, \quad \boxed{\vec{B}(\rho, t) = J_0(k\rho) B_o e^{i\omega t} \hat{\phi}}, \quad \boxed{B_o = \frac{ik\rho}{2c} E_o}$$



The Inductance of a Parallel-Plate Capacitor

Equate: $W_m = \frac{1}{2}LI^2 = \frac{1}{2\mu_0} \int_v |\tilde{B}|^2 d\tau$ $\Delta\tilde{V} = \tilde{I}\tilde{Z}_{ToT}$ $\tilde{I} = \Delta\tilde{V}/\tilde{Z}_{ToT}$ $\Delta\tilde{V} = V_0 e^{i\omega t}$

Capacitance: $C = \frac{\epsilon_0 A}{d}$ (for $d \ll a$) $\tilde{Z}_{ToT} = \tilde{Z}_C + \tilde{Z}_L = i \left(\frac{1}{\omega C} + \omega L \right)$

$\tilde{I} = \frac{V_0 e^{i\omega t}}{i \left(\frac{1}{\omega C} + \omega L \right)}$ $\tilde{I}^* = \frac{V_0 e^{-i\omega t}}{-i \left(\frac{1}{\omega C} + \omega L \right)}$ then: $|I|^2 = \tilde{I}\tilde{I}^* = \frac{V_0^2}{\left(\frac{1}{\omega C} + \omega L \right)^2}$

Thus: $W_m = \frac{1}{2}L|I|^2 = \frac{1}{2\mu_0} L \frac{V_0^2}{\left(\frac{1}{\omega C} + \omega L \right)^2} = \frac{1}{2\mu_0} \int_v |\tilde{B}|^2 d\tau = \int J_0^2(k\rho) \frac{k^2 \rho^2}{4c^2} E_0^2 \rho d\rho d\phi dz$

$= \frac{1}{2} L \frac{d^2 (V_0^2/d^2)}{\left(\frac{1}{\omega C} + \omega L \right)^2} = \frac{2\pi d}{2\mu_0} \frac{k^2}{4c^2} E_0^2 \int_0^a J_0^2(k\rho) \rho^3 d\rho$

$= L \frac{E_0^2 d}{\left(\frac{1}{\omega C} + \omega L \right)^2} = \frac{\pi \epsilon_0 \cancel{\mu_0} k^2 E_0^2}{2\cancel{\mu_0}} \int_0^a J_0^2(k\rho) \rho^3 d\rho$ with $\frac{1}{c^2} = \epsilon_0 \mu_0$, $\omega = ck$, $k = \omega/c$

$$\Rightarrow \frac{L}{\left(\frac{1}{\omega C} + \omega L\right)^2} = \left[\frac{\pi \epsilon_0 \omega^2}{2c^2 d} \int_0^a J_0^2(k\rho) \rho^3 d\rho \right] \equiv \mathcal{A}$$

$$\Rightarrow L = \mathcal{A} \left(\frac{1}{\omega C} + \omega L\right)^2 = \mathcal{A} \left(\frac{1}{\omega C}\right)^2 + 2\mathcal{A} \frac{\omega L}{\omega C} + \mathcal{A} \omega^2 L^2 = \mathcal{A} \left(\frac{1}{\omega C}\right)^2 + 2\mathcal{A} \left(\frac{L}{C}\right) + \mathcal{A} \omega^2 L^2$$

$$\text{or: } \underbrace{\mathcal{A} \omega^2 L^2}_{=a} + \underbrace{\left(\frac{2\mathcal{A}}{C} - 1\right)L}_{=b} + \underbrace{\mathcal{A} \left(\frac{1}{\omega C}\right)^2}_{=c} = 0$$

$$\Rightarrow \text{Quadratic equation of the form: } \boxed{aL^2 + bL + c = 0}, \text{ solve for } L: \boxed{L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$L = \frac{\left(1 - 2\mathcal{A}/C\right) \pm \sqrt{\left(1 - 2\mathcal{A}/C\right)^2 - 4\mathcal{A}^2 \omega^2 / \omega^2 C^2}}{2\mathcal{A} \omega^2} = \frac{\left(1 - 2\mathcal{A}/C\right) \pm \sqrt{\left(1 - 2\mathcal{A}/C\right)^2 - \left(2\mathcal{A}/C\right)^2}}{2\mathcal{A} \omega^2}$$

$$L = \frac{\left(1 - 2\mathcal{A}/C\right) \pm \sqrt{1 - 4\mathcal{A}/C + \cancel{4\mathcal{A}^2/C^2} - \cancel{4\mathcal{A}^2/C^2}}}{2\mathcal{A} \omega^2} = \frac{\left(1 - 2\mathcal{A}/C\right) \pm \sqrt{1 - 4\mathcal{A}/C}}{2\mathcal{A} \omega^2}$$

Physically, we want $L \rightarrow 0$ when $\omega \rightarrow 0$ \therefore must choose $-$ (negative) sign in above formula!

$$\therefore L = \frac{\left(1 - 2\mathcal{A}/C\right) - \sqrt{1 - 4\mathcal{A}/C}}{2\mathcal{A} \omega^2} \quad \text{Now: } \sqrt{1 - \epsilon} \approx 1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \dots \quad \text{for } \epsilon \equiv \frac{4\mathcal{A}}{C} \ll 1, \text{ thus:}$$

$$L \approx \frac{1}{8} \left(\frac{4\mathcal{A}}{C}\right)^2 \frac{1}{2\mathcal{A} \omega^2} = \frac{\mathcal{A}}{\omega^2 C^2} = \frac{\pi \cancel{\omega^2} \cancel{\omega^2}}{2c^2 \cancel{d}} \int_0^a J_0^2(k\rho) \rho^3 d\rho = \frac{\pi d}{2c^2 \epsilon_0 A^2} \int_0^a J_0^2(k\rho) \rho^3 d\rho = \frac{\mu_0 \pi d}{2A^2} \int_0^a J_0^2(k\rho) \rho^3 d\rho$$

Where the capacitance and inductance of the parallel-plate capacitor, for $d \ll a$ are:

$$\boxed{C = \frac{\epsilon_0 A}{d}} \quad \text{and} \quad \boxed{L \approx \frac{\mu_0 \pi d}{2A^2} \int_0^a J_0^2(k\rho) \rho^3 d\rho} \quad \text{for } \epsilon \equiv \left(\frac{4\mathcal{A}}{C}\right) = \left[\frac{2\pi\omega^2}{Ac^2} \int_0^a J_0^2(k\rho) \rho^3 d\rho\right] \ll 1.$$

Note that for $ka =$

- 2.4048 – 1st
- 5.5201 – 2nd
- 8.6537 – 3rd
- 11.7915 – 4th
- 14.9309 – 5th
- 18.0711 – 6th
- .
- .
- .

} zeroes of $J_0(ka)$

The electric field $\tilde{E}(\rho = a) = 0$ for these values of ka , corresponding to wavelengths $\lambda = \frac{2\pi}{k}$

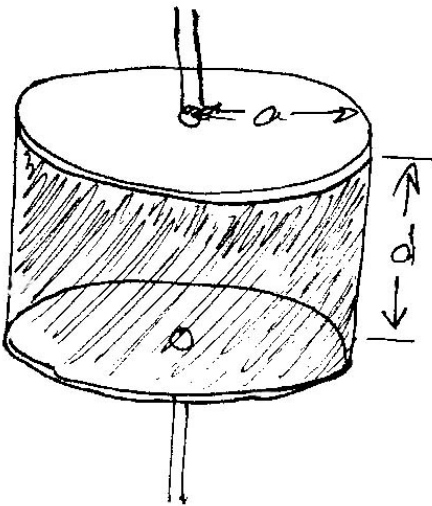
and frequencies $f = \frac{c}{\lambda} = \frac{\omega}{2\pi} = \frac{ck}{2\pi}$

Compare with radius $a = 1.0 \text{ cm}$ and diameter $D = 2a = 2.0 \text{ cm}$ of \parallel -plate cylindrical capacitor, as well as the gap dimension of $d = 0.1 \text{ mm} = 0.01 \text{ cm}$

$$\left\{ \begin{array}{l} \lambda_1 = 2.61 \text{ cm} \leftrightarrow f_1 = 1.15 \times 10^{10} \text{ Hz} = 11.5 \text{ GHz} \\ \lambda_2 = 1.14 \text{ cm} \leftrightarrow f_2 = 2.64 \times 10^{10} \text{ Hz} = 26.4 \text{ GHz} \\ \lambda_3 = 0.73 \text{ cm} \leftrightarrow f_3 = 4.13 \times 10^{10} \text{ Hz} = 41.3 \text{ GHz} \\ \lambda_4 = 0.53 \text{ cm} \leftrightarrow f_4 = 5.63 \times 10^{10} \text{ Hz} = 56.3 \text{ GHz} \\ \lambda_5 = 0.42 \text{ cm} \leftrightarrow f_5 = 7.13 \times 10^{10} \text{ Hz} = 71.3 \text{ GHz} \\ \lambda_6 = 0.35 \text{ cm} \leftrightarrow f_6 = 8.63 \times 10^{10} \text{ Hz} = 86.3 \text{ GHz} \end{array} \right.$$

Note that because the electric field $\tilde{E}(\rho = a) = 0$ for these specific frequencies (corresponding to the zeroes of $J_0(x) = J_0(ka)$), this means that physically, we could actually **short out** the capacitor at $\rho = a$ and it wouldn't make **any** difference to the behavior / physics of this "capacitor" at these specific frequencies f_1, f_2, f_3, \dots !!!

For $ka = \text{zero of } J_0(ka)$ (i.e. $J_0(ka) = 0$), we can short out the capacitor by wrapping it e.g. with sheet metal at $\rho = a$, thus turning it into a **cylindrical, fully-enclosed can** with $d \ll a$!



→ No change in physics for frequencies f_1, f_2, f_3, \dots because $\tilde{E}(\rho = a) = 0$ for these frequencies!

Thus, at these frequencies $f_1, f_2, f_3, \dots, f_n$ corresponding to the zeroes of the Bessel Function $J_0(ka)$ (i.e. $J_0(ka) = 0$), a cylindrical conducting metal can of radius a and height $d \ll a$ is actually a **resonant cavity** with electric field:

$$\tilde{E}(\rho, t) = J_0(k_n \rho) E_o e^{i\omega_n t} \hat{z} \text{ and magnetic field:}$$

$$\tilde{B}(\rho, t) = J_0(k_n \rho) B_o(\rho) e^{i\omega_n t} \hat{\phi}$$

subject to the **boundary conditions** that:

$$\tilde{E}_{\parallel}(\rho = a, t) = \tilde{E}_z(\rho = a, t) = 0 \text{ and also that:}$$

$$\tilde{B}_{\perp}(\rho = a, t) = \tilde{B}_{\rho}(\rho = a, t) = 0$$

$$\text{for: } k_n = \frac{\omega_n}{c}, \quad \omega_n = 2\pi f_n, \quad n = 1, 2, 3, \dots$$

$$\text{with: } B_o = \frac{ik\rho}{2c} E_o \text{ and: } E_o = -V_o/d$$

We will see shortly in the next set of P436 Lecture Notes (# 10) that the resonant frequencies of a resonant cavity and the allowed modes of EM wave propagation in wave guides can be derived directly from the wave equation for EM waves in these structures, as determined by the boundary conditions imposed on the EM waves by the conducting walls of these devices and also the allowed polarization states of these EM waves.

Here in these lecture notes, we obtained harmonic EM wave solutions for $\tilde{\tilde{E}}$ and $\tilde{\tilde{B}}$ in the gap region of a parallel plate capacitor (and cylindrical can capacitor, subject to boundary condition $E = 0$ at $\rho = a$) via a **perturbative technique**, analogous to what we did last semester in P435 for the \tilde{E} -field associated with a dielectric sphere immersed in an initially uniform external \tilde{E} -field and the \tilde{B} -field associated with a magnetizable sphere immersed in an initially uniform external \tilde{B} -field. (See/work Griffiths Problems 4.23 and 6.18).

“Homework” Exercises:

- 1.) Calculate the electric, magnetic and total energy densities $u_E(\rho, \varphi, z, t)$, $u_m(\rho, \varphi, z, t)$ and $u_{Tot}(\rho, \varphi, z, t)$ and their time averages; make e.g. plots of these vs. ρ . Investigate/plot their behavior for low frequencies ($\omega \approx 0$) and at higher frequencies, when $\omega_n = ck_n = c(x_n/a)$ where $x_n = k_n a =$ zeroes of $J_0(x_n) = 0$.
- 2.) Calculate Poynting's vector $\tilde{\tilde{S}}(\rho, \varphi, z, t) = \frac{1}{\mu_0} \tilde{\tilde{E}}(\rho, \varphi, z, t) \times \tilde{\tilde{B}}(\rho, \varphi, z, t)$ and its time average; make plots of $|\tilde{\tilde{S}}(\rho)|$ vs. ρ , investigate/plot its behavior for low frequencies ($\omega \approx 0$) and when $\omega_n = ck_n = c(x_n/a)$.
- 3.) Calculate the linear EM momentum density, $\tilde{\tilde{\rho}}_{EM}(\rho, \varphi, z, t) = \epsilon_0 \mu_0 \tilde{\tilde{S}}(\rho, \varphi, z, t)$ and angular momentum density, $\tilde{\tilde{\ell}}_{EM}(\rho, \varphi, z, t) = \vec{r} \times \tilde{\tilde{\rho}}_{EM}(\rho, \varphi, z, t)$, and their time averages; make plots of $\tilde{\tilde{\rho}}_{EM}(\rho)$ and $\tilde{\tilde{\ell}}_{EM}(\rho)$ vs. ρ ; investigate/plot their behavior for low frequencies ($\omega \approx 0$) and when $\omega_n = ck_n = c(x_n/a)$.