

LECTURE NOTES 8

A More Sophisticated Treatment of EM Wave Propagation in Conducting Media

In the previous P436 Lecture Notes 7, we discussed the propagation of *EM* waves in conducting media (*e.g.* metals), taking into account Ohm's Law $\vec{J}(\vec{r}, t) = \sigma_c \vec{E}(\vec{r}, t)$ and the continuity equation $\vec{\nabla} \cdot \vec{J}(\vec{r}, t) = -\partial \rho_{free}(\vec{r}, t) / \partial t$ in the conductor.

However, this treatment of conducting media neglected certain inertial effects associated with the finite mass of “free” / conduction electrons in the metal conductor – we need to consider more carefully the actual motion of the “free” / conduction electrons in the conductor, and their response *e.g.* to the application of a monochromatic plane *EM* wave of angular frequency ω .

Since the “free” / conduction electrons in a metal are not bound to individual atoms in a conductor, there are no “elastic” restoring forces acting on the “free” / conduction electrons *{i.e. “spring constant” $k_e = 0$ }*, as there was in the case of the polarization of bound atomic electrons, *e.g.* in a dispersive, linear non-conducting medium. Thus, the differential equation describing the motion of the “free” / conduction electrons in a metal is of the general form:

$$\boxed{m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) + 0 \vec{r}(t) = -e\vec{E}(\vec{r}, t) = \vec{F}_e(\vec{r}, t)}$$

where the driving force $\vec{F}_e(\vec{r}, t) = -e\vec{E}(\vec{r}, t)$ = the charge of the electron ($-e$) times the electric field $\vec{E}(\vec{r}, t)$ of the monochromatic plane *EM* wave propagating in the conducting metal.

Thus, we obtain a first-order inhomogeneous differential equation of the form:

$$\boxed{m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) = -e\vec{E}(\vec{r}, t)}$$

n.b. here again, we neglect the effects of the magnetic Lorentz force term,

$$\boxed{\left| \vec{F}_{Lorentz}(\vec{r}, t) \right| = \left| -e\vec{v}(\vec{r}, t) \times \vec{B}(\vec{r}, t) \right| \ll \left| e\vec{E}(\vec{r}, t) \right|} \text{ because: } \boxed{v = |\vec{v}| \approx v_D \ll v_{prop}}$$

Note also that m_e^* = mass of the “free” / conduction electron in the metal conductor which, depending on the microscopic/quantum mechanical details of the conductor is *{often}* not equal to the mass of a truly free, isolated electron $m_e = 9.1 \times 10^{-31}$ kg, but *{often}* $m_e^* \sim \mathcal{G}(m_e)$.

The quantity $m_e^* \gamma$ is known as the frictional / Joule dissipation constant, associated with electron energy / momentum losses – *i.e.* “free” / conduction electron scattering losses in the conductor. The damping constant γ *{n.b. units of angular frequency (radians/sec)}* is associated with dissipative/ energy loss mechanism(s) of the motion of “free” / conduction electrons in the metal – “free” / conduction electrons scattering off of each other, off of atoms, lattice vibrations, crystal defects and impurities in real metals. Thus, we can also define a relaxation time constant $\tau_{relax} \equiv 1/\gamma$ associated with the “free” / conduction electrons present in the metal conductor.

The free current density in the metal conductor is $\vec{J}_{free}(\vec{r}, t) = -n_e^f e \vec{v}(\vec{r}, t)$ {Amps/m²}, where:
 n_e^f = “free” / conduction electron number density in the metal conductor {#/unit volume = #/m³}.
 $q = -e$ = the electric charge of “free” / conduction electrons {Coulombs}.
 $\vec{v}(\vec{r}, t) = v_D$ = the drift velocity ($\ll c$) of “free” / conduction electrons in metal at (\vec{r}, t) {m/s}.
 {e.g. $v_D^{Cu} \sim 80 \mu\text{m}/\text{sec}$ in copper.}

Thus:
$$m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) = -e \vec{E}(\vec{r}, t) \quad \Leftarrow \text{multiply through by } -n_e^f e$$

$$\begin{aligned}
 & -n_e^f e m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} - n_e^f e m_e^* \gamma \vec{v}(\vec{r}, t) = +n_e^f e^2 \vec{E}(\vec{r}, t) \\
 = & m_e^* \left(-n_e^f e \frac{d\vec{v}(\vec{r}, t)}{dt} \right) - m_e^* \gamma \left(n_e^f e \vec{v}(\vec{r}, t) \right) = +n_e^f e^2 \vec{E}(\vec{r}, t) \\
 = & m_e^* \left(\frac{d}{dt} \left(-n_e^f e \vec{v}(\vec{r}, t) \right) \right) + m_e^* \gamma \left(-n_e^f e \vec{v}(\vec{r}, t) \right) = +n_e^f e^2 \vec{E}(\vec{r}, t)
 \end{aligned}$$

$$\therefore m_e^* \frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{J}_{free}(\vec{r}, t) = n_e^f e^2 \vec{E}(\vec{r}, t) \quad \text{or:} \quad \frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + \gamma \vec{J}_{free}(\vec{r}, t) = \left(\frac{n_e^f e^2}{m_e^*} \right) \vec{E}(\vec{r}, t)$$

which is also a first-order linear, **inhomogeneous** differential equation.

Recall that the solution to an **inhomogeneous** differential equation is the **sum** of the solution to the corresponding **homogeneous** differential equation, **plus** a **particular** solution satisfying the **inhomogeneous** differential equation. Thus, we first need to obtain the solution to the **homogeneous** differential equation, *i.e.* when no *EM* waves are present in the conducting material, *vis.* $\vec{E}(\vec{r}, t) = 0$.

a.) If $\vec{E}(\vec{r}, t) = 0$, the general solution to the first-order linear, **homogeneous** differential equation:

$$\frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + \gamma \vec{J}_{free}(\vec{r}, t) = 0$$

is of the form of a **damped** exponential (because $\gamma > 0$):

$$\vec{J}_{free}(\vec{r}, t) = \vec{J}_{o_{free}}(\vec{r}) e^{-\gamma t} = J_{o_{free}}(\vec{r}) e^{-t/\tau_{relax}} \quad \text{where:} \quad \tau_{relax} \equiv 1/\gamma \text{ (sec)}$$

\Rightarrow Transient free current(s) will decay to $1/e = e^{-1} = 0.368$ of their initial $t = 0$ value ($\vec{J}_{o_{free}}(\vec{r})$) in a characteristic relaxation time $t = \tau_{relax} \equiv 1/\gamma \text{ (sec)}$.

Using the **continuity equation** $\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\partial \rho_{free}(\vec{r}, t) / \partial t$ {expressing conservation of free electric charge} in the above **homogeneous** equation, with $\vec{E}(\vec{r}, t) = 0$:

$$\boxed{\frac{d}{dt} \left(\underbrace{\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t)} \right) + \gamma \left(\underbrace{\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t)} \right) = \left(\frac{n_e^f e^2}{m_e^*} \right) \left(\underbrace{\vec{\nabla} \cdot \vec{E}(\vec{r}, t)}_{=0} \right)}$$

$$= -\frac{\partial^2 \rho_{free}(\vec{r}, t)}{\partial t^2} - \gamma \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} = 0$$

thus: $\boxed{\frac{\partial^2 \rho_{free}(\vec{r}, t)}{\partial t^2} + \gamma \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} = 0} \Rightarrow \boxed{\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} + \gamma \rho_{free}(\vec{r}, t) = 0}$

$$\Rightarrow \boxed{\rho_{free}(\vec{r}, t) = \rho_{o_{free}}(\vec{r}) e^{-\gamma t} = \rho_{o_{free}}(\vec{r}) e^{-t/\tau_{relax}}}$$

But from P436 Lecture Notes 7, p. 1 we also obtained the relation $\rho_{free}(\vec{r}, t) = \rho_{o_{free}}(\vec{r}) e^{-\sigma_C t / \epsilon}$. Thus, comparing these two equations we see that: $\gamma = \sigma_C / \epsilon = 1 / \tau_{relax}$.

b.) For a **static** non-zero electric field $\vec{E}(\vec{r}) = \vec{E}_o$, then this **static** problem can have no explicit time dependence, hence $d\vec{J}_{free}(\vec{r}, t) / dt = 0$ and thus the first-order linear, **inhomogeneous** equation becomes:

$$\boxed{\gamma \vec{J}_{free}(\vec{r}) = \left(\frac{n_e^f e^2}{m_e^*} \right) \vec{E}(\vec{r}) = \left(\frac{n_e^f e^2}{m_e^*} \right) \vec{E}_o = \text{constant}} \quad \text{or:} \quad \boxed{\vec{J}_{free}(\vec{r}) = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right) \vec{E}(\vec{r}) = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right) \vec{E}_o = \text{constant}}$$

The **static** / DC continuity equation for free charge is: $\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}) = -\frac{\partial \rho_{free}(\vec{r})}{\partial t} = 0$

with **static** solution(s): $\vec{J}_{free}(\vec{r}) = \text{constant} \neq fcn(\vec{r}, \{\text{or } t\})$ and: $\rho_{free}(\vec{r}) \neq fcn(t)$.

However, Ohm's Law (for DC / steady free electric currents) is: $\vec{J}_{free}(\vec{r}) = \sigma_C \vec{E}(\vec{r})$ where: σ_C = the **static** conductivity of the metal (Siemens/m) = $1 / \rho_C \leftarrow 1 / \text{static resistivity} (\Omega\text{-m})$.

Thus, we see here that: $\boxed{\vec{J}_{free}(\vec{r}) = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right) \vec{E}(\vec{r}) = \sigma_C \vec{E}(\vec{r})}$ i.e. the **static** conductivity $\boxed{\sigma_C = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right)}$

or equivalently, the **static** resistivity: $\boxed{\rho_C \equiv \frac{1}{\sigma_C} = \left(\frac{m_e^* \gamma}{n_e^f e^2} \right)}$ {n.b. both purely **real** quantities}

The **static** / DC resistance e.g. of a long wire is thus: $\boxed{R = \frac{\rho_C \ell}{A_{\perp}} = \left(\frac{m_e^* \gamma}{n_e^f e^2} \right) \frac{\ell}{A_{\perp}} \text{ (Ohms)}}$

{again, a purely **real** quantity}

c.) For a **harmonically** time-dependent {in general, complex} electric field (e.g. associated with a monochromatic EM plane wave (or an AC current, using Ohm's law), with angular frequency $\omega = 2\pi f$:

$$\boxed{\tilde{E}(\vec{r}, t) = \tilde{E}_o(\vec{r}) e^{-i\omega t}} \quad \text{and:} \quad \boxed{\frac{d\tilde{J}_{free}(\vec{r}, t)}{dt} + \gamma \tilde{J}_{free}(\vec{r}, t) = \left(\frac{n_e^f e^2}{m_e^*} \right) \tilde{E}(\vec{r}, t)}$$

We assume that the {in general, complex} free current density solution to this linear **inhomogeneous** first-order differential equation is of the general form: $\tilde{J}_{free}(\vec{r}, t) = \tilde{J}_{o,free}(\vec{r}) e^{-i\omega t}$.

Then **the** solution to **this** linear, **inhomogeneous** first-order differential equation is:

$$\begin{aligned} & \boxed{(-i\omega + \gamma) \tilde{J}_{free}(\vec{r}, t) = \left(\frac{n_e^f e^2}{m_e^*} \right) \tilde{E}(\vec{r}, t) = \gamma \left(\frac{n_e^f e^2}{m_e^* \gamma} \right) \tilde{E}(\vec{r}, t) = \gamma \sigma_C \tilde{E}(\vec{r}, t)} \quad \text{since:} \quad \boxed{\sigma_C = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right)} \\ \Rightarrow & \boxed{\tilde{J}_{free}(\vec{r}, t) = \left(\frac{\gamma \sigma_C}{(-i\omega + \gamma)} \right) \tilde{E}(\vec{r}, t) = \left(\frac{\gamma \sigma_C}{(\gamma - i\omega)} \right) \tilde{E}(\vec{r}, t) = \left(\frac{\sigma_C}{(1 - i\omega/\gamma)} \right) \tilde{E}(\vec{r}, t) \equiv \tilde{\sigma}_C(\omega) \tilde{E}(\vec{r}, t)} \end{aligned}$$

Where:

The AC conductivity: $\tilde{\sigma}_C(\omega) \equiv \left(\frac{\sigma_C}{(1 - i\omega/\gamma)} \right) = \left(\frac{(n_e^f e^2 / m_e^* \gamma)}{(1 - i\omega/\gamma)} \right) \Leftarrow n.b. \text{ **complex** quantity}$

The AC resistivity: $\tilde{\rho}_C(\omega) \equiv \frac{1}{\tilde{\sigma}_C(\omega)} = \rho_C (1 - i\omega/\gamma) = \left(\frac{m_e^* \gamma}{n_e^f e^2} \right) (1 - i\omega/\gamma) \Leftarrow \text{**complex** quantity}$

Thus, we see that the {complex} AC resistance (aka **impedance**, Z) e.g. of a long wire is:

$$\boxed{\tilde{Z}(\omega) \equiv \tilde{R}_{AC}(\omega) = \frac{\tilde{\rho}_C(\omega) \ell}{A_{\perp}} = \left(\frac{\rho_C \ell}{A_{\perp}} \right) (1 - i\omega/\gamma) = \left(\frac{m_e^* \gamma}{n_e^f e^2} \right) \left(\frac{\ell}{A_{\perp}} \right) (1 - i\omega/\gamma) \Leftarrow n.b. \text{ **complex** quantity}$$

n.b. Can **hear** the effect of complex $\tilde{R}_{AC}(\omega)$ in {audiophile-type} **hi-fi stereo systems** – by comparing “good” (thick) vs. “bad” (thin/cheap) L & R loudspeaker cables!

The AC continuity equation is: $\boxed{\vec{\nabla} \cdot \tilde{J}_{free}(\vec{r}, t) = -\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t}}$ with: $\boxed{\vec{\nabla} \cdot \tilde{E}(\vec{r}, t) = \frac{\tilde{\rho}_{free}(\vec{r}, t)}{\epsilon}}$ Gauss' Law

But: $\boxed{\tilde{J}_{free}(\vec{r}, t) = \tilde{\sigma}_C(\omega) \tilde{E}(\vec{r}, t) = \left(\frac{\sigma_C}{(1 - i\omega/\gamma)} \right) \tilde{E}(\vec{r}, t) = \left(\frac{(n_e^f e^2 / m_e^* \gamma)}{(1 - i\omega/\gamma)} \right) \tilde{E}(\vec{r}, t)}$

Thus: $\boxed{\vec{\nabla} \cdot \tilde{J}_{free}(\vec{r}, t) = \tilde{\sigma}_C(\omega) \vec{\nabla} \cdot \tilde{E}(\vec{r}, t) = \left(\frac{\sigma_C}{(1 - i\omega/\gamma)} \right) \vec{\nabla} \cdot \tilde{E}(\vec{r}, t) = \left(\frac{(n_e^f e^2 / m_e^* \gamma)}{(1 - i\omega/\gamma)} \right) \vec{\nabla} \cdot \tilde{E}(\vec{r}, t)}$

\therefore $\boxed{-\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \left(\frac{\sigma_C}{(1 - i\omega/\gamma)} \right) \frac{\tilde{\rho}_{free}(\vec{r}, t)}{\epsilon}}$ or: $\boxed{-\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \frac{\sigma_C / \epsilon}{(1 - i\omega/\gamma)} \tilde{\rho}_{free}(\vec{r}, t)}$

The solution to the above differential equation for the free electric charge volume density is of the form: $\tilde{\rho}_{free}(\vec{r}, t) = \tilde{\rho}_o(\vec{r})e^{-i\omega t}$ provided that:

$$\frac{-\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \frac{\sigma_c / \epsilon_{free}}{(1 - i\omega / \gamma)} \tilde{\rho}_{free}(\vec{r}, t) \Rightarrow i\omega = \frac{\sigma_c / \epsilon}{(1 - i\omega / \gamma)} \text{ n.b. implies } \epsilon_{free} \rightarrow \tilde{\epsilon}_{free}(\omega) \{\text{complex!}\}$$

i.e. that: $\tilde{\epsilon}_{free}(\omega) = \frac{1}{i\omega} \frac{\sigma_c}{(1 - i\omega / \gamma)}$ but: $\tilde{\sigma}_c(\omega) \equiv \left(\frac{\sigma_c}{(1 - i\omega / \gamma)} \right) = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right) \left(\frac{1}{(1 - i\omega / \gamma)} \right)$

Thus, we see that: $\tilde{\epsilon}_{free}(\omega) = \frac{\tilde{\sigma}_c(\omega)}{i\omega} = \frac{1}{i\omega} \frac{\sigma_c}{(1 - i\omega / \gamma)}$ \Leftarrow \exists a connection between **complex** AC electric permittivity $\epsilon_{free}(\omega)$ and **complex** AC conductivity $\sigma_c(\omega)$ in a metal / conductor.

Technically-speaking, this relation **should** be:

$$\tilde{\epsilon}_{free}(\omega) = \underbrace{\epsilon_o}_{\text{vacuum}} + \underbrace{\frac{\tilde{\sigma}_c(\omega)}{i\omega}}_{\text{metal}} = \epsilon_o \left(\underbrace{1}_{\text{vacuum}} + \underbrace{\frac{\tilde{\sigma}_c(\omega)}{i\epsilon_o \omega}}_{\text{metal}} \right) = \epsilon_o \left(\underbrace{1}_{\text{vacuum}} + \underbrace{\frac{1}{i} \left(\frac{\tilde{\sigma}_c(\omega)}{\epsilon_o \omega} \right)}_{\text{metal}} \right) = \epsilon_o \left(\underbrace{1}_{\text{vacuum}} + \underbrace{\tilde{\chi}_e^{free}(\omega)}_{\text{metal}} \right)$$

because the vacuum pervades **all space** –the conducting medium and the vacuum **coexist** in the same region of space-time, invoking the **superposition principle**, the two electric permittivities (n.b. both **scalar** quantities) are **additive**.

However, it can be seen that because the {DC} conductivity of metals is so high {typically $\sigma_c \sim 10^7$ Siemens/m}, that even for {angular} frequencies in the optical region { $\omega \sim 10^{16}$ radians/sec} the electric permittivity of free space, ϵ_o is **dwarfed** by the second term, i.e. $\epsilon_o = 8.85 \times 10^{-12} \ll |\sigma_c(\omega) / i\omega| = |\sigma_c / i\omega(1 - i\omega / \gamma)|$ (Farads/m), so often the ϵ_o term is neglected/dropped in many textbooks... For technical correctness / completeness' sake, we will include it here. Thus, we see that even for metal conductors:

$$\frac{\tilde{\epsilon}_{free}(\omega)}{\epsilon_o} = 1 + \frac{1}{i} \left(\frac{\tilde{\sigma}_c(\omega)}{\epsilon_o \omega} \right) = 1 + \frac{1}{i} \left(\frac{\sigma_c(\omega)}{\epsilon_o \omega} \right) \frac{1}{(1 - i(\omega / \gamma))} = 1 + \frac{1}{i} \left(\frac{\sigma_c(\omega)}{\epsilon_o \omega} \right) \frac{[1 + i(\omega / \gamma)]}{[1 + (\omega / \gamma)^2]} = 1 + \tilde{\chi}_e^{free}(\omega)$$

$$\tilde{\chi}_e^{free}(\omega) = \frac{1}{i} \left(\frac{\tilde{\sigma}_c(\omega)}{\epsilon_o \omega} \right) = \frac{1}{i} \left(\frac{\sigma_c(\omega)}{\epsilon_o \omega} \right) \frac{1}{(1 - i\omega / \gamma)} = \frac{1}{i} \left(\frac{\sigma_c(\omega)}{\epsilon_o \omega} \right) \frac{[1 + i(\omega / \gamma)]}{[1 + (\omega / \gamma)^2]}$$

For pure copper metal at low frequencies, i.e. $\omega \rightarrow 0$, the **static** conductivity is $\sigma_c^{Cu} \approx 6 \times 10^7$ Siemens/m; the number density of free electrons in pure copper metal is $n_e^{Cu} \approx 8.5 \times 10^{28} / \text{m}^3$ and using $\sigma_c = (n_e^f e^2 / m_e^* \gamma) \Rightarrow \gamma_{Cu} = (n_e^{Cu} e^2 / m_e^* \sigma_c^{Cu}) \approx (n_e^{Cu} e^2 / m_e \sigma_c^{Cu}) \approx 4 \times 10^{13} \text{ sec}^{-1}$, which corresponds to a relaxation time for “free” / conduction electrons in pure copper of $\tau_{relax}^{Cu} \approx 1 / \gamma_{Cu} \approx 2.5 \times 10^{-14} \text{ sec}$.

For Conducting Metals with “Free” Electrons: $\tilde{\mathbf{J}}_{free}(\vec{r}, t) = \tilde{\sigma}_c(\omega) \tilde{\mathbf{E}}(\vec{r}, t)$

 Note that in the static limit ($\omega \rightarrow 0$):

$\tilde{\mathbf{E}}(\vec{r}, t) = \tilde{\mathbf{E}}_o(\vec{r}) e^{-i\omega t} \rightarrow \mathbf{E}_o(\vec{r})$	static / constant	}	$\tilde{\mathbf{J}}_{free} = \sigma_c \mathbf{E}$
$\tilde{\mathbf{J}}_{free}(\vec{r}, t) = \tilde{\mathbf{J}}_{o_{free}}(\vec{r}) e^{-i\omega t} \rightarrow \mathbf{J}_{o_{free}}(\vec{r})$	static / constant		$\vec{\nabla} \cdot \tilde{\mathbf{J}}_{free} = -\frac{\partial \rho_{free}}{\partial t}$
$\tilde{\rho}_{free}(\vec{r}, t) = \tilde{\rho}_{o_{free}}(\vec{r}) e^{-i\omega t} \rightarrow \rho_{o_{free}}(\vec{r})$	static / constant		
$\tilde{\sigma}_c(\omega) = \left(\frac{\sigma_c}{(1 - i\omega/\gamma)} \right) \rightarrow \sigma_c = 1/\rho_c$	static / constant		$\sigma_c = \left(\frac{n_e^f e^2}{m_e^* \gamma} \right)$
$\tilde{\rho}_c(\omega) = \frac{1}{\tilde{\sigma}_c(\omega)} = \rho_c (1 - i\omega/\gamma) \rightarrow \rho_c$	static / constant		
$\tilde{\epsilon}_{free}(\omega) = \epsilon_o + \frac{\tilde{\sigma}_c(\omega)}{i\omega} = \epsilon_o + \frac{\sigma_c}{i\omega(1 - i\omega/\gamma)} = \epsilon_o + \frac{\sigma_c}{\omega(\omega/\gamma + i)} \rightarrow \infty$			singular !!!

In the previous P436 Lecture Notes 7.5 on dispersion in non-conducting, non-magnetic linear / homogeneous / isotropic media, we showed / derived the complex electric permittivity (due to bound atomic electrons) of the dielectric medium to be:

$$\tilde{\epsilon}_{bnd}(\omega) = \epsilon_o \left(1 + \tilde{\chi}_e^{bnd}(\omega) \right) = \epsilon_o \left(1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right] \right)$$

with: $\tilde{K}_e^{bnd}(\omega) = \frac{\tilde{\epsilon}_{bnd}(\omega)}{\epsilon_o} = 1 + \tilde{\chi}_e^{bnd}(\omega)$ and: $\tilde{n}(\omega) = \sqrt{\tilde{\epsilon}_{bnd}(\omega)/\epsilon_o} = \sqrt{\tilde{K}_e^{bnd}(\omega)} = \sqrt{1 + \tilde{\chi}_e^{bnd}(\omega)}$.

If only one / a single resonance exists, this relation reduces to:

$$\tilde{\epsilon}_1(\omega) = \epsilon_o \left(1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2 \omega^2]} \right) \quad \text{with: } \omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \left(\frac{n_e^b e^2}{3\epsilon_o m_e} \right)} \quad \text{and: } \omega_{0j} \equiv \sqrt{\frac{k_{ej}^b}{m_e}}$$

Comparing this expression to that for conducting media with “free” / conduction electrons:

$\tilde{\epsilon}_{free}(\omega) = \epsilon_o + \frac{\sigma_c}{\omega(\omega/\gamma + i)}$	with <u>static</u> conductivity:	$\sigma_c \equiv \left(\frac{n_e^f e^2}{m_e^* \gamma} \right)$
\therefore	$\tilde{\epsilon}_{free}(\omega) = \epsilon_o + \left(\frac{n_e^f e^2}{m_e^*} \right) \left(\frac{1}{\gamma\omega} \right) \frac{1}{[\omega/\gamma + i]} = \epsilon_o + \left(\frac{n_e^f e^2}{m_e^*} \right) \frac{1}{[\omega^2 + i\gamma\omega]} = \epsilon_o + \left(\frac{n_e^f e^2}{m_e^*} \right) \frac{[\omega^2 - i\gamma\omega]}{[\omega^4 + \gamma^2 \omega^2]}$	
or:	$\tilde{\epsilon}_{free}(\omega) = \epsilon_o \left(1 - \left(\frac{n_e^f e^2}{\epsilon_o m_e^*} \right) \frac{(\omega_0^2 - \omega^2) + i\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2]} \right)$	with: $\omega_0 \equiv 0 \leftarrow \exists$ a resonance @ $\omega_0 \equiv 0!$

Note that the – sign in the above formula {relative to that for bound atomic electrons} arises because of fact that “free” / conduction electrons in metal are **not** bound (*i.e.* \exists phase shift of 180° @ $f = 0$ Hz).

Then for dispersive conducting media containing “free” electrons, Maxwell’s equations become:

$$\begin{array}{ll}
 1) \quad \boxed{\vec{\nabla} \cdot \tilde{\mathbf{D}} = \tilde{\rho}_{free}} \Rightarrow \boxed{\vec{\nabla} \cdot \tilde{\mathbf{E}} = \tilde{\rho}_{free} / \epsilon} & 5) \text{ Ohm's Law: } \boxed{\tilde{\mathbf{J}}_{free} = \tilde{\sigma}_c \tilde{\mathbf{E}} = \frac{\sigma_c}{1 - i\omega/\gamma} \tilde{\mathbf{E}}} \\
 2) \quad \boxed{\vec{\nabla} \cdot \tilde{\mathbf{B}} = 0} & 6) \text{ Continuity Equation: } \boxed{\vec{\nabla} \cdot \tilde{\mathbf{J}}_{free} = -\frac{\partial \tilde{\rho}_{free}}{\partial t}} \\
 3) \quad \boxed{\vec{\nabla} \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t}} & \\
 4) \quad \boxed{\vec{\nabla} \times \tilde{\mathbf{B}} = \mu_o \tilde{\mathbf{J}}_{free} + \mu_o \frac{\partial \tilde{\mathbf{D}}}{\partial t} = \mu_o \tilde{\mathbf{J}}_{free} + \mu_o \epsilon_{free} \frac{\partial \tilde{\mathbf{E}}}{\partial t}} \Leftarrow \boxed{\text{Assume dispersive conducting medium is } \underline{\text{non-magnetic}} \text{ } (\mu \approx \mu_o)}
 \end{array}$$

Again, we use the curl operator on equations 3) and 4) to obtain wave equation(s) for *EM* waves propagating in a dispersive metal / conductor, *e.g.* for equation 3):

$$\begin{array}{l}
 \boxed{\vec{\nabla} \times (\vec{\nabla} \times \tilde{\mathbf{E}}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \tilde{\mathbf{B}}) = -\mu_o \frac{\partial \tilde{\mathbf{J}}_{free}}{\partial t} - \mu_o \epsilon_{free} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}} \\
 \boxed{\vec{\nabla} \times (\cancel{\vec{\nabla} \cdot \tilde{\mathbf{E}}}) - \nabla^2 \tilde{\mathbf{E}} = -\mu_o \tilde{\sigma}_c \frac{\partial \tilde{\mathbf{E}}}{\partial t} - \mu_o \epsilon_{free} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}} \text{ and: } \boxed{\frac{n^2}{c^2} = \frac{1}{v^2} \approx \mu_o \epsilon_{free}} \Leftarrow \boxed{\text{here, for } \underline{\text{non-magnetic}} \text{ conductors } (\mu \approx \mu_o)} \\
 \boxed{= \tilde{\rho}_{free} / \tilde{\epsilon} \rightarrow 0} \text{ in } t < 10^{-12} \text{ sec} = 1 \text{ ps for } \underline{\text{charge-equilibrated}} \text{ conductors.}
 \end{array}$$

Thus, for times $t > 10^{-12}$ sec = 1 ps we obtain:
$$\boxed{\nabla^2 \tilde{\mathbf{E}} - \frac{1}{v^2} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} = \mu_o \tilde{\sigma}_c \frac{\partial \tilde{\mathbf{E}}}{\partial t} = \mu_o \frac{\sigma_c}{(1 - i\omega/\gamma)} \frac{\partial \tilde{\mathbf{E}}}{\partial t}}$$

Using equation 4) above, an identical wave equation is obtained for the magnetic field $\tilde{\mathbf{B}}(\vec{r}, t)$:

$$\boxed{\nabla^2 \tilde{\mathbf{B}} - \frac{1}{v^2} \frac{\partial^2 \tilde{\mathbf{B}}}{\partial t^2} = \mu_o \tilde{\sigma}_c \frac{\partial \tilde{\mathbf{B}}}{\partial t} = \mu_o \frac{\sigma_c}{(1 - i\omega/\gamma)} \frac{\partial \tilde{\mathbf{B}}}{\partial t}}$$

Solutions to above wave equation(s) for monochromatic *EM* plane waves propagating in the $+\hat{z}$ direction in a dispersive conducting media with “free” electrons will be of the form *e.g.*

$$\boxed{\tilde{\mathbf{E}}(\vec{r}, t) = \tilde{\mathbf{E}}_o e^{i(\vec{k}z - \omega t)}} \text{ and: } \boxed{\tilde{\mathbf{B}}(\vec{r}, t) = \frac{\tilde{k}}{\omega} \hat{k} \times \tilde{\mathbf{E}}(\vec{r}, t) = \tilde{\mathbf{B}}_o e^{i(\vec{k}z - \omega t)}}$$

Note that here {again} the wavenumber \tilde{k} is complex and frequency-dependent: $\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega)$ with $k(\omega) = \Re\{\tilde{k}(\omega)\} = 2\pi/\lambda(\omega)$ and $\kappa(\omega) = \Im\{\tilde{k}(\omega)\}$ {related to absorption/dissipation of the *EM* wave as it propagates in the dispersive conductive medium.}

We can associate the complex wavenumber $\tilde{k}(\omega)$ with a complex index of refraction $\tilde{n}(\omega)$:

$$\boxed{\tilde{k}(\omega) = \left(\frac{\omega}{c}\right) \tilde{n}(\omega)} \quad \text{or:} \quad \boxed{\tilde{n}(\omega) \equiv \left(\frac{c}{\omega}\right) \tilde{k}(\omega)} \quad \text{where the complex index of refraction:}$$

$$\boxed{\tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega)} \quad \text{thus:} \quad \boxed{n(\omega) = \Re\{ \tilde{n}(\omega) \} = \left(\frac{c}{\omega}\right) k(\omega)} \quad \text{and:} \quad \boxed{\eta(\omega) = \Im\{ \tilde{n}(\omega) \} = \left(\frac{c}{\omega}\right) \kappa(\omega)}$$

Plugging in the expression $\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$ for monochromatic plane *EM* waves propagating in a dispersive, ***non-magnetic*** conducting medium into the above wave equation, we obtain the so-called ***characteristic equation*** for this situation {Exercise - explicitly work this out yourselves!}:

$$\boxed{\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 + \frac{i\omega\mu_o\sigma_c}{(1-i\omega/\gamma)}} \quad \text{with:} \quad \boxed{c^2 = \frac{1}{\epsilon_o\mu_o}} \quad \text{and:} \quad \boxed{\sigma_c = \left(\frac{n_e^f e^2}{m_e^* \gamma}\right)} \quad \text{or:} \quad \boxed{\gamma = \left(\frac{n_e^f e^2}{m_e^* \sigma_c}\right)}$$

$$\text{Then:} \quad \boxed{\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega)} = (\text{complex index of refraction})^2 \Rightarrow \boxed{\tilde{n}^2(\omega) = 1 + i \left(\frac{\sigma_c}{\epsilon_o \omega}\right) \left[\frac{1 + i(\omega/\gamma)}{1 + (\omega/\gamma)^2}\right]}$$

The Low Frequency Behavior of a Dispersive, Non-Magnetic Conducting Medium:

$$\text{Definition of low frequency:} \quad \boxed{(\omega/\gamma) \ll 1} \quad \text{where:} \quad \boxed{\gamma = \left(\frac{n_e^f e^2}{m_e^* \sigma_c}\right)} \quad \text{and:} \quad \boxed{\omega \ll \left(\frac{\sigma_c}{\epsilon_o}\right)}$$

In this regime, the characteristic/mean/average time between successive collisions experienced by the “free” / conduction electrons in the dispersive, non-magnetic conducting medium $\langle \tau_{coll} \rangle$ is ***short*** in comparison to the oscillation period of *EM* waves $\tau = 1/f = 2\pi/\omega$, i.e. $\langle \tau_{coll} \rangle \ll \tau$.

$$\text{For } \omega \approx 0 \text{ (but not } \equiv 0 \text{!):} \quad \boxed{\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 + \frac{i\omega\mu_o\sigma_c}{(1-i\omega/\gamma)}} \Rightarrow \boxed{\tilde{k}^2(\omega) \approx \frac{i\omega\mu_o\sigma_c}{1} = i\omega\mu_o\sigma_c} \quad \text{keeping only terms } \underline{\text{linear}} \text{ in } \omega$$

$$\text{Now note that:} \quad \boxed{\sqrt{i} = \sqrt{-1} = \left(\frac{1+i}{\sqrt{2}}\right)} \quad \leftarrow \{ \text{Please work this out/derive this yourselves!} \}$$

$$\text{Thus for } \omega \approx 0 \text{ (i.e. } \omega/\gamma \ll 1 \text{):} \quad \boxed{\tilde{k}^2(\omega) \approx i\omega\mu_o\sigma_c} \Rightarrow \boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega) \approx \sqrt{i\omega\mu_o\sigma_c} = \sqrt{\frac{\mu_o\omega\sigma_c}{2}} (1+i)}$$

From which we see immediately that for monochromatic plane *EM* waves propagating in a dispersive, non-magnetic conducting medium, that:

$$\boxed{k(\omega) \approx \kappa(\omega) = \sqrt{\frac{\mu_o\omega\sigma_c}{2}}} \quad \text{and thus here}$$

{again} we see that the \vec{B} -field ***lags*** the \vec{E} -field by:
$$\boxed{\phi_k = \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa}{k}\right) \approx \tan^{-1}(1) = \frac{\pi}{4} = 45^\circ}$$

Similarly, for $\omega \approx 0$ (i.e. $\omega/\gamma \ll 1$) since: $\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega)$

Then: $\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega) \approx i\omega\mu_0\sigma_c \left(\frac{c}{\omega}\right)^2 = i\cancel{\omega}\cancel{\mu_0}\sigma_c \left(\frac{1}{\cancel{\epsilon_0}\cancel{\omega^2}}\right) = +\frac{i\sigma_c}{\epsilon_0\omega}$. Using: $\sqrt{i} = \left(\frac{1+i}{\sqrt{2}}\right)$

$\Rightarrow \tilde{n}(\omega) = n(\omega) + i\eta(\omega) = \sqrt{\frac{i\sigma_c}{\epsilon_0\omega}} = \sqrt{\frac{\sigma_c}{2\epsilon_0\omega}}(1+i)$ which again implies that: $n(\omega) \approx \eta(\omega) = \sqrt{\frac{\sigma_c}{2\epsilon_0\omega}}$.

Here again, we can define a **low-frequency skin depth**: $\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\mu_0\omega\sigma_c}}$ same as before, but with $\mu = \mu_0$

the characteristic distance that the \vec{E} and \vec{B} fields fall to $1/e = e^{-1} = 0.368$ of their $z = 0$ values.

We define a **low-frequency absorption coefficient**: $\alpha(\omega) \equiv 2\kappa(\omega) = 2/\delta_{sc}(\omega) \approx \sqrt{2\mu_0\omega\sigma_c}$

and corresponding **low-frequency extinction coefficient**: $\xi(\omega) \equiv 2\eta(\omega) \approx \sqrt{\frac{\sigma_c}{2\epsilon_0\omega}}$

The characteristic distance over which the {time-averaged} *EM* wave energy density $\langle u(\vec{r}) \rangle$ and intensity $\langle I(\vec{r}) \rangle$ fall to $1/e = e^{-1} = 0.368$ of their initial ($z = 0$) values, respectively are:

$$\langle u_{EM}(\vec{r}) \rangle = \langle u_{EM}(0) \rangle e^{-\alpha z} \quad \text{and} \quad \langle I(\vec{r}) \rangle = \langle I(0) \rangle e^{-\xi(\frac{\omega}{c})z} \quad \left[\text{Since both are proportional to } E^2 \right]$$

For pure copper metal, for which $\gamma_{Cu} \approx 4.1 \times 10^{13} \text{ sec}^{-1}$ and $\sigma_C^{Cu} \approx 6 \times 10^7 \text{ Siemens/m}$ and thus $\sigma_C^{Cu}/\epsilon_0 \approx 6.5 \times 10^{18} / \text{sec}$, then for low frequencies, e.g. $f = \omega/2\pi \leq 10^{12} \text{ Hz}$ we see that pure copper metal is in the low-frequency regime for vacuum wavelengths of:

$$\lambda_o = \frac{c}{f} \geq \frac{3 \times 10^8}{10^{12}} \approx 3 \times 10^{-4} \text{ m} = 300 \mu\text{m} \quad \text{cf w/} \quad \lambda_{Cu}(f) = \frac{2\pi}{k(f)} = 2\pi / \sqrt{\frac{\mu_0(2\pi f)\sigma_C^{Cu}}{2}} \geq 10^{-6} \text{ m} = 1 \mu\text{m}$$

In pure copper metal, the skin depth $\delta_{sc}(\omega)$ associated with monochromatic *EM* waves of frequency $f \sim 10^{12} \text{ Hz} = 1 \text{ THz} = 10^6 \text{ MHz}$ is:

$$\delta_{sc}^{cu}(f = 1 \text{ THz}, \lambda_o = 300 \mu\text{m}) \approx \sqrt{\frac{2}{\mu_0\omega\sigma_c}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 2\pi \times 10^{12} \times 6 \times 10^7}} \approx 6.5 \times 10^{-8} \text{ m} = 65 \text{ nm}$$

Thus at “low” frequencies, we see that:

$$\left\{ \delta_{sc}(10^{12} \text{ Hz}) = 65 \text{ nm} \right\} \ll \left\{ \lambda_{Cu}(10^{12} \text{ Hz}) = 1 \mu\text{m} \right\} \ll \left\{ \lambda_o = 300 \mu\text{m} \right\}$$

In pure copper metal, the skin depth $\delta_{sc}(\omega)$ associated with monochromatic *EM* waves of frequency $f \sim 10^{10} \text{ Hz} = 10 \text{ GHz} = 10^4 \text{ MHz}$ (e.g. cell phones) {which corresponds to a vacuum wavelength of $\lambda_o = c/f = 3 \times 10^8 / 10^{10} = 3 \times 10^{-2} \text{ m} = 3 \text{ cm}$ } is:

$$\delta_{sc}^{Cu}(10^{10} \text{ Hz}) \equiv \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\mu_o \omega \sigma_C}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 2\pi \times 10^{10} \times 6 \times 10^7}} \approx 6.5 \times 10^{-7} \text{ m} = 650 \text{ nm}$$

The wavelength of *EM* waves with frequency $f = 10^{10} \text{ Hz}$ propagating in the pure copper metal is:

$$\lambda_{Cu}(f) = \frac{2\pi}{k(f)} = 2\pi / \sqrt{\frac{\mu_o (2\pi f) \sigma_C^{Cu}}{2}} = 10^{-5} \text{ m} = 10 \mu\text{m}$$

Thus at “low” frequencies, we {again} see that:

$$\{\delta_{sc}(10^{10} \text{ Hz}) = 650 \text{ nm}\} \ll \{\lambda_{Cu}(10^{10} \text{ Hz}) = 10 \mu\text{m}\} \ll \{\lambda_o = 3.0 \text{ cm}\}$$

The High Frequency Behavior of a Dispersive, Non-Magnetic Conducting Medium:

Definition of high frequency: $\left(\frac{\omega}{\gamma}\right) \gg 1$ where: $\gamma = \left(\frac{n_e e^2}{m_e^* \sigma_C}\right)$ and: $\omega \gg \left(\frac{\sigma_C}{\epsilon_o}\right)$.

The square of the complex index of refraction for a metal is:

$$\tilde{n}^2(\omega) = 1 + i \left(\frac{\sigma_C}{\epsilon_o \omega}\right) \frac{1}{(1 - i(\omega/\gamma))} = 1 + i \left(\frac{\sigma_C}{\epsilon_o \omega}\right) \frac{[1 + i(\omega/\gamma)]}{[1 + (\omega/\gamma)^2]}$$

At **high** frequencies, the “free” / conduction electrons in a metal will undergo a great many oscillations before scattering – i.e. the period of oscillations $\tau = 1/f = 2\pi/\omega$ is **short** compared to the mean time between scatterings, essentially the relaxation time, $\tau_{relax} = 1/\gamma$.

In the high-frequency regime, the volume charge density of “free” electrons and the lattice of positive ions in metal together can be thought of as a **plasma** – whose electron charge density oscillates **longitudinally** { i.e. in the direction of propagation of the *EM* wave) at the {natural} angular resonance frequency known as the **plasma frequency**, defined as:

The **plasma frequency** in a metal/conductor: $\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$ (radians/sec) and: $f_p = \frac{\omega_p}{2\pi}$ (Hz).

For pure copper: $\omega_p^{Cu} = 1.644 \times 10^{16} \text{ rad/sec}$ \Rightarrow $f_p^{Cu} = \frac{\omega_p^{Cu}}{2\pi} \approx 2.616 \times 10^{15} \text{ Hz}$

The corresponding **plasma wavelength** in copper is: $\lambda_p^{Cu} = \frac{c}{f_p^{Cu}} = 1.147 \times 10^{-7} \text{ m} = 114.7 \text{ nm}$

\Rightarrow Operative in the optical \rightarrow UV portion of the *EM* spectrum – typical for many metals !!!

We can rewrite the (complex index of refraction)² in terms of the plasma frequency ω_p :

$$\tilde{n}^2(\omega) = 1 + \frac{i(\sigma_C/\epsilon_o)}{\omega(1-i\omega/\gamma)} = 1 + i \left(\frac{n_e^f e^2}{\epsilon_o m_e^*} \right) \left[\frac{1}{\gamma\omega(1-i\omega/\gamma)} \right] = 1 - \underbrace{\left(\frac{n_e^f e^2}{\epsilon_o m_e^*} \right)}_{=\omega_p^2} \left[\frac{1}{i\gamma\omega(1-i\omega/\gamma)} \right] \quad \begin{array}{l} \text{used:} \\ i \equiv \sqrt{-1} \\ i = -1/i \end{array}$$

Thus: $\tilde{n}^2(\omega) = 1 - \frac{\omega_p^2}{(\omega^2 + i\gamma\omega)}$ where: $\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$ and: $\gamma \equiv \left(\frac{n_e^f e^2}{m_e^* \sigma_C} \right) = \left(\frac{\epsilon_o}{\sigma_C} \right) \left(\frac{n_e^f e^2}{\epsilon_o m_e^*} \right) = \frac{\omega_p^2}{(\sigma_C/\epsilon_o)}$

Now: $\tilde{n}^2 = (n + i\eta) \cdot (n + i\eta) = n^2 + 2in\eta + \eta^2$ with: $\Re\{\tilde{n}^2\} = n^2 - \eta^2$ and: $\Im\{\tilde{n}^2\} = 2n\eta$

Equating the **real** and **imaginary** portions of $\tilde{n}^2 = (n^2 - \eta^2) + i(2n\eta)$ – use the following “trick”:

$$\frac{1}{\omega^2 + i\gamma\omega} = \frac{1}{(\omega^2 + i\gamma\omega)} \cdot \frac{\omega^2 - i\gamma\omega}{\omega^2 - i\gamma\omega} = \frac{\omega^2 - i\gamma\omega}{\omega^4 + \gamma^2\omega^2} = \frac{\omega^2 - \omega^2(i\gamma/\omega)}{\omega^2(\omega^2 + \gamma^2)} = \frac{\cancel{\omega^2}(1 - i\gamma/\omega)}{\cancel{\omega^2}(\omega^2 + \gamma^2)} = \frac{1 - i\gamma/\omega}{\omega^2 + \gamma^2}$$

$$\therefore \tilde{n}^2(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} = 1 - \frac{\omega_p^2(1 - i\gamma/\omega)}{\omega^2 + \gamma^2} = \underbrace{\left[1 - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right]}_{=(n^2 - \eta^2)} + i \underbrace{\left(\frac{(\gamma/\omega)\omega_p^2}{\omega^2 + \gamma^2} \right)}_{=2n\eta} = (n^2 - \eta^2) + i(2n\eta)$$

$$\therefore n^2(\omega) - \eta^2(\omega) = 1 - \frac{\omega_p^2}{(\omega^2 + \gamma^2)} \quad \text{and:} \quad 2n(\omega)\eta(\omega) = \left(\frac{(\gamma/\omega)\omega_p^2}{\omega^2 + \gamma^2} \right)$$


Now define: $\alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)}$ and: $\beta_x(\omega) \equiv (\gamma/\omega)$.

Then: $n^2(\omega) - \eta^2(\omega) = 1 - \alpha_x(\omega)$ and: $2n(\omega)\eta(\omega) = (\gamma/\omega)\alpha_x(\omega) = \beta_x(\omega)\alpha_x(\omega)$

Thus: $\eta = \frac{\beta_x \alpha_x}{2n} \Rightarrow n^2 - \eta^2 = 1 - \alpha_x \Rightarrow n^2 - \frac{\beta_x^2 \alpha_x^2}{4n^2} = (1 - \alpha_x)$ or: $n^4 - (1 - \alpha_x)n^2 - \left(\frac{\beta_x \alpha_x}{2} \right)^2 = 0$

Define: $x \equiv n^2$, then: $x^2 - (1 - \alpha_x)x - \left(\frac{\beta_x \alpha_x}{2} \right)^2 = 0$ which is a quadratic equation of the form:

$ax^2 + bx + c = 0$ with: $a = 1$, $b = -(1 - \alpha_x)$ and $c = -\left(\frac{\alpha_x \beta_x}{2} \right)^2$.

 **must** select +ve root on physical grounds!

The solution is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(1 - \alpha_x) \pm \sqrt{(1 - \alpha_x)^2 + \alpha_x^2 \beta_x^2}}{2}$

The Complex Index of Refraction: $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$

$$n^2(\omega) = \frac{(1 - \alpha_x(\omega))}{2} \left[1 + \sqrt{1 + \left(\frac{\alpha_x(\omega)\beta_x(\omega)}{1 - \alpha_x(\omega)} \right)^2} \right] \quad \alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)}$$

where: $n(\omega) = \sqrt{\frac{(1 - \alpha_x(\omega))}{2} \left[1 + \sqrt{1 + \left(\frac{\alpha_x(\omega)\beta_x(\omega)}{1 - \alpha_x(\omega)} \right)^2} \right]}$

and: $\eta(\omega) = \frac{\alpha_x(\omega)\beta_x(\omega)}{2n(\omega)}$ where: $\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$ and: $\gamma \equiv \left(\frac{n_e^f e^2}{m_e^* \sigma_c} \right) = \frac{\omega_p^2}{(\sigma_c / \epsilon_o)}$

Typical plasma frequencies for free electrons in metals are: $\omega_p \approx 10^{16}$ rad/sec.

Typical γ -factors for free electrons in metals are: $\gamma = 1/\tau_{relax} \sim 10^{13} - 10^{14}$ rad/sec.

Hence, in a typical metal/conductor we see that: $\omega_p/\gamma = \omega_p \tau_{relax} \sim 10^2 - 10^3 \gg 1$ i.e. $\omega_p \gg \gamma$.

Thus, at **high** frequencies ($\omega \gg \omega_p \gg \gamma$):

Then: $\alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)} \approx \left(\frac{\omega_p}{\omega} \right)^2 \ll 1$ and: $\beta_x(\omega) \equiv \left(\frac{\gamma}{\omega} \right) \ll 1$ at **high** frequencies.

And: $\frac{\alpha_x \beta_x}{1 - \alpha_x} \approx \frac{\left(\frac{\omega_p}{\omega} \right)^2 \left(\frac{\gamma}{\omega} \right)}{1 - \left(\frac{\omega_p}{\omega} \right)^2} = \left(\frac{\gamma}{\omega} \right) \cdot \frac{\left(\frac{\omega_p}{\omega} \right)^2}{1 - \left(\frac{\omega_p}{\omega} \right)^2} \approx \left(\frac{\gamma}{\omega} \right) \left(\frac{\omega_p}{\omega} \right)^2 \ll 1$

Then: $n(\omega) \approx \sqrt{1 - \alpha_x(\omega)} \approx \sqrt{1 - \left(\frac{\omega_p}{\omega} \right)^2} \approx 1 - \frac{1}{2} \left(\frac{\omega_p}{\omega} \right)^2 \leq 1$ {Using $\sqrt{1 - \epsilon} \approx 1 - \frac{1}{2}\epsilon$ for $\epsilon \ll 1$ }

And: $\eta(\omega) = \frac{\alpha_x(\omega)\beta_x(\omega)}{2n(\omega)} \approx \frac{\left(\frac{\gamma}{\omega} \right) \left(\frac{\omega_p}{\omega} \right)^2}{2 \left(1 - \frac{1}{2} \left(\frac{\omega_p}{\omega} \right)^2 \right)} \approx \frac{1}{2} \left(\frac{\gamma}{\omega} \right) \left(\frac{\omega_p}{\omega} \right)^2 \ll 1$ at **high** frequencies.

\Rightarrow At **high** frequencies, for $\omega \gg \omega_p$ the complex index of refraction $\tilde{n}(\omega)$ is predominantly **real**, i.e. the **imaginary** part $\eta(\omega) \ll$ **real** part $n(\omega)$ at **high** frequencies.

\Rightarrow Metals are \approx **transparent** to γ -rays and x-rays for $\omega \gg \omega_p$!!!

Some {alkali} metals are transparent / begin to be transparent in the UV region of the *EM* spectrum!

Metal	$\lambda_p = \frac{c}{f_p} = \frac{2\pi c}{\omega_p}$	$\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_0 m_e^*}}$
${}^7_3\text{Li}$	155.0 nm	
${}^{23}_{11}\text{Na}$	209.0 nm	
${}^{39}_{19}\text{K}$	287.0 nm	
${}^{63}_{29}\text{Cu}$	114.7 nm	
Alkali Metals ${}^{85}_{37}\text{Rb}$	322.0 nm ← UV	

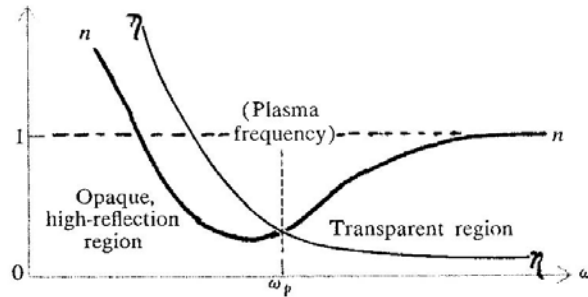


Figure 5.4. Index of refraction and extinction coefficient versus frequency for a metal.

$$\tilde{n} = n + i\eta$$

Well above the plasma frequency ($\omega \gg \omega_p$), then: $\kappa(\omega) \ll k(\omega)$:

$$\phi_k = \delta_B - \delta_E = \tan^{-1} \left(\frac{\kappa(\omega)}{k(\omega)} \right) \approx 0 \quad \text{for } \omega \gg \omega_p$$

\tilde{B} is {nearly} in-phase with \tilde{E} for *EM* waves propagating in metals/conductors with frequency well above the plasma frequency ($\omega > \omega_p$).

Well above the plasma frequency, we also have $\gamma \ll \omega$, such that:

$$m_e^* \frac{d\tilde{v}(\vec{r}, t)}{dt} + m_e^* \gamma \tilde{v}(\vec{r}, t) = -e\tilde{E}(\vec{r}, t) \approx m_e^* \tilde{a}(\vec{r}, t) = -e\tilde{E}(\vec{r}, t) = \tilde{F}_e(\vec{r}, t)$$

\Rightarrow Because $\tilde{E}(\vec{r}, t) = \tilde{E}_0 e^{i(\vec{k}z - \omega t)}$, and $\tilde{a}(\vec{r}, t) = \tilde{\ddot{r}}(t) = \partial^2 \tilde{r}(t) / \partial t^2$ we see that for $\omega > \omega_p$ that the “free” electrons in a metal oscillate {nearly} in-phase with the driving \tilde{E} of the monochromatic *EM* wave. Well above the plasma frequency ω_p , a metal’s complex index of refraction is real (i.e. absorption is small), the conductor becomes increasingly transparent as ω is increased beyond ω_p .

Note also that $n(\omega) < 1$ for $\omega > \omega_p$ {i.e. in the so-called anomalous dispersion region}!!!

$$n(\omega) \approx \sqrt{1 - \left(\frac{\omega_p}{\omega} \right)^2} \quad \text{for } (\omega > \omega_p) \quad \text{and:} \quad \eta(\omega) \approx \left(\frac{\gamma}{\omega} \right) \left(\frac{\omega_p}{\omega} \right)^2 \quad \text{for } (\omega > \omega_p)$$

Since $\tilde{n}^2(\omega) = \tilde{\epsilon}_{free}(\omega)/\epsilon_0 = 1 + \tilde{\chi}_e^{free}(\omega)$ for **non-magnetic** conductors, $\Rightarrow \tilde{\epsilon}_{free}(\omega)$ and $\tilde{\chi}_e^{free}(\omega)$ are {also} predominantly **real** at **high** frequencies ($\omega > \omega_p$), i.e. $\tilde{\epsilon}_{free}(\omega)/\epsilon_0 = \tilde{K}_e^{free}(\omega) \approx 1$ for $\omega > \omega_p$.

The skin depth for $\omega > \omega_p$ is:
$$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \left(\frac{c}{\omega}\right) \frac{1}{\eta(\omega)} \approx \left(\frac{c}{\gamma}\right) \left(\frac{\omega}{\omega_p}\right)^2$$

Below the plasma frequency ω_p , the index of refraction is complex, i.e. $\eta(\omega)$ is **large** and therefore **non-negligible**; Metals are thus **opaque** for $\omega < \omega_p$.

Well below the plasma frequency $\omega \ll \omega_p$: $k(\omega) \approx \kappa(\omega)$

and
$$\phi_k = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \approx \tan^{-1}(1) \approx \frac{\pi}{4} = 45^\circ \Rightarrow \tilde{B} \text{ lags } \tilde{E} \text{ by } 45^\circ \text{ for } \omega \ll \omega_p$$

The “free” / conduction electrons in a dispersive conductor also lag \vec{E} -field by \approx same phase lag.

For $\omega < \omega_p$, since the incident *EM* wave is not transmitted, much of the *EM* wave is reflected.
 \Rightarrow Metals have a high reflection coefficient $R(\omega)$ for $\omega < \omega_p$.

In certain situations, such as *EM* waves propagating in the Earth’s ionosphere, or e.g. in a **tenuous electronic plasma** in the laboratory, the electrons are {truly} free, hence damping is negligible ($\gamma \approx 0$) in such situations.

Then for situations with **negligible** damping, i.e. $\gamma \approx 0$, then $\tilde{n}^2(\omega) \rightarrow \approx n^2(\omega)$ and the above relations simply somewhat:

$$n^2(\omega) = \frac{\epsilon_{free}(\omega)}{\epsilon_0} = K_e^{free}(\omega) = (1 + \chi_e^{free}(\omega)) \approx \Re\{\tilde{n}^2(\omega)\} \Rightarrow n^2(\omega) \approx 1 - \left(\frac{\omega_p}{\omega}\right)^2 \text{ with: } \omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_0 m_e}}$$

The above relation holds {i.e. is valid} over a wide range of frequencies, including $\omega < \omega_p$.

For situations with $\gamma \approx 0$, the imaginary part of complex $\tilde{n}(\omega)$,

$$\eta(\omega) = \Im\{\tilde{n}(\omega)\} \approx \left(\frac{\gamma}{\omega}\right) \left(\frac{\omega_p}{\omega}\right)^2 \approx 0 \text{ because } \gamma \approx 0.$$

In **this** regime, where $\gamma \approx 0$, the wavenumber k is:
$$k \approx \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$

which is sometimes expressed as: $\omega^2 = \omega_p^2 + ck^2 \Leftarrow$ **dispersion relation** for $k(\omega)$.

{n.b. In **dispersive** conducting media, where γ is **not** small, the above formulae apply **only** for $\omega \gg \omega_p$.}

Note that in a **tenuous plasma** with $\omega < \omega_p$:
$$k = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$
 is **purely imaginary!!!**

\Rightarrow EM waves with $\omega < \omega_p$ incident on such a plasma are **reflected** for $\omega < \omega_p$, because the EM fields inside tenuous plasma are **exponentially** attenuated by the factor e^{-kz} .

It is **precisely** this fact that enables *e.g.* short-wave radio communication around the globe – the short-wave radio waves **reflect** off of the plasma in the earth’s ionosphere!

For **tenuous electron plasmas** in the **laboratory**:

$$n_e \approx 10^{18} - 10^{22} \text{ electrons/m}^3 \Rightarrow \omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}} \approx 6 \times 10^{10} - 6 \times 10^{12} \text{ radians/sec}$$

At $\omega = 0$, the **static** attenuation constant is: $\kappa = \Im m\{\tilde{k}\} \approx \omega_p/c$ at $\omega = 0$.

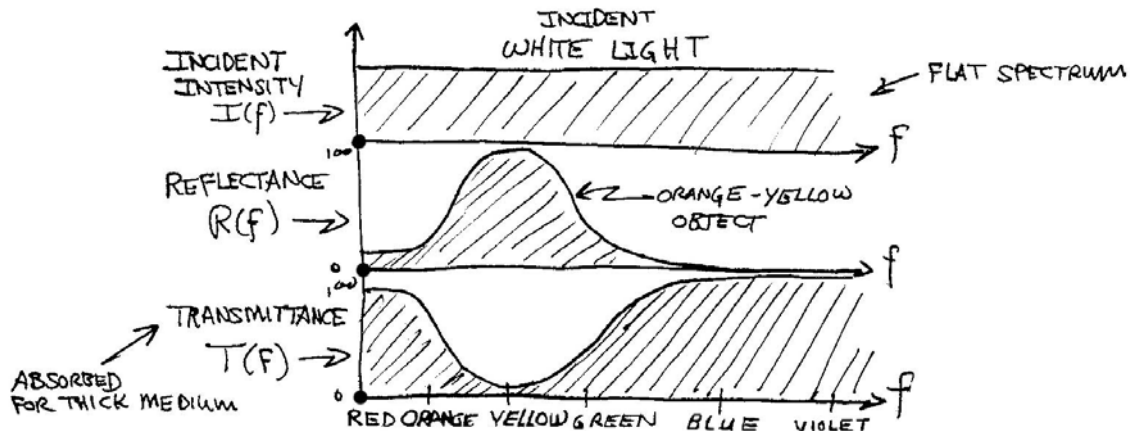
\Rightarrow Skin depth (at $\omega = 0$):

$$\delta_{sc}(\omega = 0) = \frac{1}{\kappa(\omega = 0)} = \frac{c}{\omega_p} \approx 0.5 \text{ cm} - 5 \times 10^{-3} \text{ cm} \text{ for } \textit{static} \text{ (or low-frequency) } EM \text{ fields}$$

Expulsion of EM fields within a plasma is well-known phenomena, *e.g.* in controlled thermo-nuclear processes, and can also be exploited *e.g.* in attempts at confinement of a hot plasma.

Electromagnetic Wave Propagation in a Linear, Homogeneous, Isotropic Dispersive Conductive Medium

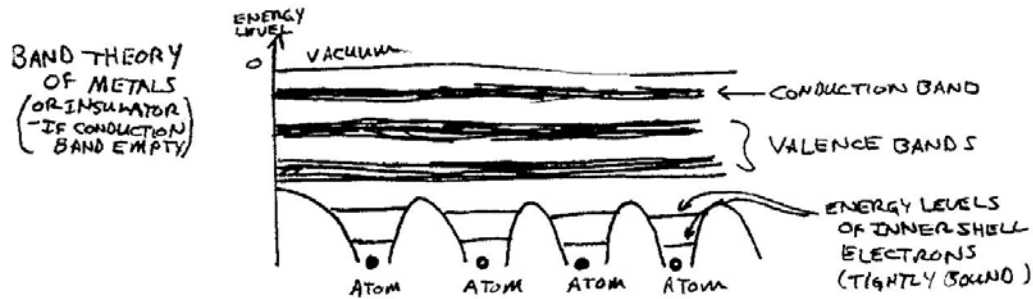
In a conducting medium, metals in particular, the optical / EM properties of many metals are dominated by the “free” electrons in the metal, resulting in high reflectance $R \sim 85\text{-}95\%$, nearly independent of the frequency / wavelength of the incident light, at least in the visible light region of the EM spectrum ($\sim 350 \text{ nm} < \lambda < 780 \text{ nm}$). These “free” electrons in such metals as aluminum, tin, sodium, potassium, cesium, vanadium, niobium, gadolinium, holmium, yttrium, scandium, osmium, and even iron have a silvery-gray appearance {due to visible light photons scattering off of the “free” electrons in the metal} – essentially these metals are colorless because of their wavelength-independent reflectance, but remember / realize that the physical color of an object illuminated *e.g.* by white light (= flat distribution in frequency) is reflected light – hence the illuminated object is a poor absorber of light at that wavelength – indeed, the object preferentially absorbs light at other frequencies / wavelengths, and not so much at the frequency / wavelength associated with the color we perceive it to be, from reflected light – see figure below:



Some conducting metals, such as gold and copper, as well as various alloys – *e.g.* bronze and brass (zinc, tin and copper) – do not have a silvery-gray appearance, but have a yellow / orange / red “tinge” to them. This is because these metals preferentially absorb (*i.e.* transmit) in the green-blue region and thus reflect more-so in the yellow / orange / red region of the visible light spectrum, giving these metals their characteristic hues / colors.

Gold-metalized safety glasses – with a very thin layer of gold deposited on them – have transmittance $T(f)$ peaked \sim in the green portion of the visible spectrum. Goggles/safety glasses coated with a thin gold layer are used by people who work around high-temperature furnaces. Visible light from the furnace {peaked \sim in the green} is transmitted, but infrared light (*i.e.* heat) is strongly reflected by gold!

Because “free” / conduction electrons in a metal have no resonances (except the one at $\omega = 0$) because there are no restoring forces acting on them. In such metals as gold and copper, bound electrons *e.g.* in the so-called valence bands of the metal also play a non-negligible role *e.g.* in the optical properties of the metal – *i.e.* in the visible light portion of the *EM* spectrum!



Since (*i.e.* we assume) the response of the “free” electrons in the conduction band is separable / independent of the response of the bound electrons in the valence band(s) to incident *EM* waves (valid as long as the amplitudes of \vec{E} / \vec{B} are not too large). Thus we {again} use the principle of linear superposition for the complex electric permittivity: $\tilde{\epsilon}_{tot}(\omega) = \epsilon_o (\tilde{\chi}_{vac}(\omega) + \tilde{\chi}_{bnd}(\omega) + \tilde{\chi}_e^{free}(\omega))$

where: $\tilde{\chi}_{vac}(\omega) \equiv 1$, $\tilde{\chi}_{bnd}(\omega) = \left(\frac{n_e^b e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n f_j^{bound} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]}$ with: $\sum_{j=1}^n f_j^{osc} = 1$

n.b. $n_e^b = \#$ density ($= \#/m^3$) of bound atomic electrons in the dispersive conducting medium

and where: $\tilde{\chi}_e^{free}(\omega) = - \left(\frac{n_e^f e^2}{\epsilon_o m_e^*} \right) \frac{[(\omega_0^2 - \omega^2) + i\gamma_0 \omega]}{[(\omega_0^2 - \omega^2)^2 + \gamma_0^2 \omega^2]} = +\omega_p^2 \frac{[\omega^2 - i\gamma_0 \omega]}{[\omega^4 + \gamma_0^2 \omega^2]}$ where: $\omega_0 = 0$

and: $n_e^f = \#$ density ($\#/m^3$) of “free” conduction-band electrons in the dispersive conducting medium.

The plasma frequency of free electrons in the dispersive conducting medium is: $\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$

Then:

$$\tilde{\epsilon}_{tot}(\omega) = \epsilon_o \left(1 + \tilde{\chi}_e^{tot}(\omega) \right) = \epsilon_o \left(1 + \underbrace{\left(\frac{n_e^b e^2}{\epsilon_o m_e} \sum_{j=1}^n f_j^{bound} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} + \omega_p^2 \frac{[\omega^2 - i\gamma_0 \omega]}{[\omega^4 + \gamma_0^2 \omega^2]} \right)}_{\tilde{\chi}_e^{tot}(\omega)} \right)$$

$$= \tilde{\chi}_e^{tot}(\omega) = \tilde{\chi}_e^{bnd}(\omega) + \tilde{\chi}_e^{free}(\omega)$$

$$\tilde{\chi}_e^{tot}(\omega) = \tilde{\chi}_e^{bnd}(\omega) + \tilde{\chi}_e^{free}(\omega) = \left(\frac{n_e^b e^2}{\epsilon_o m_e} \sum_{j=1}^n f_j^{bound} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} + \omega_p^2 \frac{[\omega^2 - i\gamma_0 \omega]}{[\omega^4 + \gamma_0^2 \omega^2]} \right)$$

If the dispersive conducting medium is \cong **non-magnetic** ($\mu \approx \mu_o$), the complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ is related to the complex permittivity $\tilde{\epsilon}(\omega) = \epsilon(\omega) + i\zeta(\omega)$ by:

$$\tilde{n}^2(\omega) = \left(\frac{\tilde{\epsilon}_{tot}(\omega)}{\epsilon_o} \right) = (1 + \tilde{\chi}_e^{tot}(\omega)) = \left[1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e} \sum_{j=1}^n f_j^{bound} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} + \omega_p^2 \frac{[\omega^2 - i\gamma_0 \omega]}{[\omega^4 + \gamma_0^2 \omega^2]} \right) \right]$$

with: $\sum_{j=1}^n f_j^{osc} = 1$, $\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$ and with: $\tilde{n}^2(\omega) = (n + i\eta)(n + i\eta) = (n^2 - \eta^2) + i(2n\eta)$

Complex $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ is related to complex $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$ by the relations:

$$\tilde{k}(\omega) = \left(\frac{\omega}{c} \right) \tilde{n}(\omega) \quad \text{or:} \quad \tilde{n}(\omega) = \left(\frac{c}{\omega} \right) \tilde{k}(\omega) \quad \text{with:} \quad n(\omega) = \left(\frac{c}{\omega} \right) k(\omega) \quad \text{and:} \quad \eta(\omega) = \left(\frac{c}{\omega} \right) \kappa(\omega)$$