

## LECTURE NOTES 7.5

### Dispersion: The Frequency-Dependence of the Electric Permittivity $\varepsilon = \varepsilon(\omega)$ and the Electric Susceptibility $\chi_e(\omega) = \varepsilon(\omega)/\varepsilon_o - 1$

Over the entire *EM* frequency interval  $\{0 \leq f \leq \infty \text{ Hz}\}$ , the speed of propagation  $v_{prop}$  of monochromatic (*i.e.* single-frequency) *EM* waves in matter is often not constant, not independent of frequency:  $v_{prop} \neq \text{constant}$ ;  $= fcn(\text{frequency}, f) = v_{prop}(f)$ , because matter - at the microscopic scale - is composite - comprised of atoms/molecules - which have resonances in energy/energy levels - which are governed by the laws of quantum mechanics...

The frequency-dependence of the wavelength  $\lambda = \lambda(\omega)$ , or wavenumber  $k(\omega) = 2\pi/\lambda(\omega)$ , and linear momentum  $p(\omega)$  associated with macroscopic *EM* waves propagating in a dispersive medium arises from the frequency-dependence of the macroscopic electric permittivity  $\varepsilon(\omega)$  (or equivalently the electric susceptibility  $\chi_e(\omega)$ ) since:

$$\varepsilon(\omega) = \varepsilon_o (1 + \chi_e(\omega)).$$

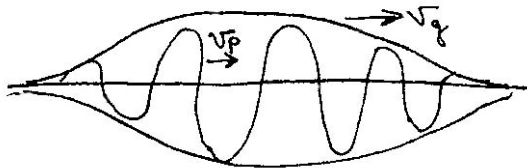
The frequency-dependence of the macroscopic electric permittivity  $\varepsilon(\omega)$  is known as dispersion; a medium that has  $\varepsilon = fcn(\omega)$  is known as a dispersive medium.

For non-magnetic/non-conducting linear/homogeneous/isotropic media, the index of refraction  $n = \sqrt{\varepsilon/\varepsilon_o}$ . Thus, if:  $\varepsilon = \varepsilon(\omega) = \varepsilon_o (1 + \chi_e(\omega))$  then:  $n(\omega) = \sqrt{\varepsilon(\omega)/\varepsilon_o}$ .

For a wave packet (= a group {= superposition/linear combination} of waves of many frequencies - as explained by Mssr. Fourier), the envelope of the wave packet travels with (in general, frequency-dependent) group speed = speed at which energy in the wave flows:

$$v_g(\omega) \equiv \frac{d\omega}{dk(\omega)} = \frac{1}{dk(\omega)/d\omega} = \left[ \frac{dk(\omega)}{d\omega} \right]^{-1}$$

A propagating wave packet:  $\tilde{f}(kz - \omega t)$



$$v_\phi(\omega) \equiv \frac{\omega}{k(\omega)}$$

$$v_g(\omega) \equiv \left[ \frac{dk(\omega)}{d\omega} \right]^{-1}$$

A space-point  $z(t)$  on the waveform moves - with constant phase  $\Phi \equiv (kz - \omega t)$  with (in general, frequency-dependent) phase speed  $v_\phi(\omega) \equiv \omega/k(\omega)$ .

Hence:  $z(t, \omega) = \Phi/k(\omega) + v_\phi(\omega)t$ . Note that {in general}:  $v_g(\omega) \neq v_\phi(\omega)$ .

If  $v_\phi(\omega) = \omega/k(\omega)$  is frequency-dependent, and  $v_g(\omega) = [dk(\omega)/d\omega]^{-1} \neq v_\phi(\omega)$  {e.g. as in the case for surface waves on water, where  $v_\phi = 2v_g$ } the relationship between  $v_\phi(\omega)$  and  $v_g(\omega)$  depends on the detailed physics of the medium (as we shall soon see. . .). Note that in certain circumstances,  $v_\phi$  can exceed  $c$  {= speed of light in the vacuum} but in these situations, **no** energy (and/or information) is transmitted at super-luminal speeds – energy/information is transmitted at  $v_g < c$  always, by causality...

**A physical/mechanical example**: calculate the **phase** speed of the **intersection** point of the two halves of a scissors as the blades of the scissors are closed. {Answer:  $v_\phi^{scissors} \rightarrow \infty$  !!!}



### **Dispersion Phenomena in Linear Dielectrics**

In a non-conducting, linear, homogeneous, isotropic medium there are no **free** electrons (i.e.  $\rho_{free}(\vec{r}) = 0$ ). Atomic electrons are permanently bound to nuclei of atoms comprising the medium.  $\Rightarrow \exists$  no preferential direction / no preferential directions in such an {isotropic} medium.

Suppose each atomic electron (charge  $-e$ ) in a dielectric is **displaced** by a small distance  $\vec{r}$  from its **equilibrium position**, e.g. by application of a **static** electric field  $\vec{E}(\vec{r}) \parallel \hat{r}$  direction.

The resulting **macroscopic** electric polarization (aka electric dipole moment per unit volume) is:

$$\boxed{\vec{P}(\vec{r}) = n_e^b \vec{p}(\vec{r})} \text{ where: } n_e^b = \text{bound atomic electron number density } (\#e/m^3)$$

and the {induced} atomic/molecular electric dipole moment is:  $\vec{p}(\vec{r}) = -e\vec{r}$  {here}, where  $\vec{r}$  is the **vector displacement** of the atomic electron from its **equilibrium**  $\{\vec{r} = 0\}$  position.

Thus: 
$$\boxed{\vec{P}(\vec{r}) = n_e^b \vec{p}(\vec{r}) = -n_e^b e \vec{r}}$$

The atomic electrons are each elastically bound to their equilibrium positions with a force constant  $k_e$  (N/m). The force equation for **each** atomic electron is thus:  $\vec{F}_e(\vec{r}) = -e\vec{E}(\vec{r}) = k_e \vec{r}$ .

Hence: 
$$\boxed{\vec{r} = -e\vec{E}(\vec{r})/k_e}$$

The **static** polarization is therefore given by: 
$$\boxed{\vec{P}(\vec{r}) = n_e^b \vec{p}(\vec{r}) = -n_e^b e \vec{r} = -n_e^b e \left( \frac{-e\vec{E}(\vec{r})}{k_e} \right) = + \frac{n_e^b e^2}{k_e} \vec{E}(\vec{r})}$$

However, if the  $\vec{E}$ -field e.g. **varies** harmonically with time, i.e. 
$$\boxed{\vec{E} = \vec{E}(\vec{r}, t; \omega) = \vec{E}_0 e^{i(kz - \omega t)} \hat{n}}$$
 due to a monochromatic EM plane wave incident on an atom, the above relation is **incorrect** !

A **more correct** {but “semi-classical”} approach to treat this situation is to consider the bound atomic electrons as classical, **damped, forced** harmonic oscillators (driven by the incident electric field), as mathematically described by the following differential equation:

$$\boxed{m_e \ddot{\vec{r}}(t; \omega) + m_e \gamma \dot{\vec{r}}(t; \omega) + k_e \vec{r}(t; \omega) = -e\vec{E}(\vec{r}, t; \omega)} \leftarrow \text{inhomogeneous 2}^{\text{nd}}\text{-order differential eqn.}$$

The damping constant  $\gamma$  (rads/sec) represents the effect of EM re-radiation by the atom {here}.

$$\underbrace{m_e \frac{\partial^2 \tilde{\vec{r}}(t, \omega)}{\partial t^2}}_{= m_e \ddot{\vec{a}}} + \underbrace{m_e \gamma \frac{\partial \tilde{\vec{r}}(t, \omega)}{\partial t}}_{\text{Velocity-dependent damping term } \gamma \equiv \text{damping constant}} + \underbrace{k_e \tilde{\vec{r}}(t, \omega)}_{\text{Potential Force (binding of atomic electrons to atom)}} = \underbrace{-e \tilde{\vec{E}}(\vec{r}, t, \omega)}_{\text{Driving Force}}$$

n.b. we have neglected the  $e \tilde{\vec{v}} \times \tilde{\vec{B}} (\ll e \tilde{\vec{E}})$  term here...

$m_e = \text{electron mass} = 9.1 \times 10^{-31} \text{ kg}$

Suppose the driving/forcing term varies sinusoidally/is harmonic/periodic with angular frequency  $\omega$ , i.e.  $\tilde{\vec{F}}_e(\vec{r}, t, \omega) = -e \tilde{\vec{E}}(\vec{r}, t, \omega) = -e \tilde{\vec{E}}_o e^{-i\omega t} \hat{r}$  because  $\tilde{\vec{E}}(\vec{r}, t, \omega) = \tilde{\vec{E}}_o e^{-i\omega t} \hat{r}$ .

n.b. The electric field  $\vec{E}$  is **complex**  $\tilde{\vec{E}}$  and **plane-polarized** in the  $\hat{r}$ -direction.

The **inhomogeneous** force equation becomes:  $m_e \ddot{\tilde{\vec{r}}} + m_e \gamma \dot{\tilde{\vec{r}}} + k_e \tilde{\vec{r}} = -e \tilde{\vec{E}}_o e^{-i\omega t} \hat{r}$  with complex **time-domain** vector displacement amplitude:  $\tilde{\vec{r}}(t, \omega) = \tilde{r}(t, \omega) \hat{r}$ . In the **steady state**, we have:

$$m_e \ddot{\tilde{r}} + m_e \gamma \dot{\tilde{r}} + k_e \tilde{r} = -e \tilde{E}_o e^{-i\omega t} \hat{r}$$

Since  $\tilde{r}(t, \omega)$  physically represents the complex vector spatial **displacement** of an atomic electron from its **equilibrium**  $\{\vec{r} = 0\}$  position, then:  $\tilde{\vec{r}}(t, \omega) = \tilde{r}(t, \omega) \hat{r} = \tilde{r}_o(\omega) e^{-i\omega t} \hat{r}$

Thus:

$$m_e \ddot{\tilde{r}} + m_e \gamma \dot{\tilde{r}} + k_e \tilde{r} = -e \tilde{E}_o e^{-i\omega t} \hat{r}$$

$$\begin{aligned}
 & m_e \frac{\partial^2 \tilde{r}(t, \omega)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t, \omega)}{\partial t} + k_e \tilde{r}(t, \omega) = -e \tilde{E}(\vec{r}, t, \omega) \\
 & -m_e \omega^2 \tilde{r}_o(\omega) e^{-i\omega t} - i\omega m_e \gamma \tilde{r}_o(\omega) e^{-i\omega t} + k_e \tilde{r}_o(\omega) e^{-i\omega t} = -e \tilde{E}_o e^{-i\omega t} \\
 & (m_e \omega^2 - k_e + i\omega m_e \gamma) \tilde{r}_o(\omega) = e \tilde{E}_o \quad \Leftarrow \text{characteristic equation}
 \end{aligned}$$

Divide this equation through by  $m_e$ :  $\left( \omega^2 - \left( \frac{k_e}{m_e} \right) + i\omega \gamma \right) \tilde{r}_o(\omega) = \frac{e}{m_e} \tilde{E}_o$

Define:  $\omega_0^2 \equiv \left( \frac{k_e}{m_e} \right)$  or:  $\omega_0 \equiv \sqrt{\frac{k_e}{m_e}}$  = characteristic/natural resonance {angular} frequency. Then:

$$\left[ (\omega^2 - \omega_0^2) + i\gamma\omega \right] \tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) \tilde{E}_o \quad \text{or:} \quad \tilde{r}_o(\omega) = \left( \frac{e}{m_e} \right) \tilde{E}_o \frac{1}{\left[ (\omega^2 - \omega_0^2) + i\gamma\omega \right]} \quad \Leftarrow \text{Bound atomic electron complex frequency-domain spatial displacement amplitude}$$

Note that the complexness of **frequency-domain**  $\tilde{r}_o(\omega)$  is in the denominator.

We can move it to the numerator using the following standard “trick”/procedure:

$$\text{If: } \tilde{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} \quad \text{where: } \Re\{\tilde{z}\} = \frac{x}{x^2+y^2} \quad \text{and: } \Im\{\tilde{z}\} = \frac{y}{x^2+y^2}$$

Thus:

$$\begin{aligned}
 \tilde{r}_o(\omega) &= \frac{\left(\frac{e}{m_e}\right)\tilde{E}_o}{\left[(\omega^2 - \omega_0^2) + i\gamma\omega\right]} = \frac{\left(\frac{e}{m_e}\right)\tilde{E}_o}{\left[(\omega^2 - \omega_0^2) + i\gamma\omega\right]} \cdot \frac{\left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]^*}{\left[(\omega^2 - \omega_0^2) + i\gamma\omega\right]^*} \\
 &= \frac{\left(\frac{e}{m_e}\right)\tilde{E}_o}{\left[(\omega^2 - \omega_0^2) + i\gamma\omega\right]} \cdot \frac{\left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]}{\left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]} \\
 &= \frac{\left(\frac{e}{m_e}\right)\tilde{E}_o \left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]}{\left[(\omega^2 - \omega_0^2)^2 + i\gamma\omega(\omega^2 - \omega_0^2) - i\gamma\omega(\omega^2 - \omega_0^2) + \gamma^2\omega^2\right]} \\
 &= \left(\frac{e}{m_e}\right)\tilde{E}_o \frac{\left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]}{\left[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2\right]}
 \end{aligned}$$

Note that:  $\tilde{E}_o = |\tilde{E}_o| e^{i\delta_E} \equiv E_o e^{i\delta_E}$ . However, we can always “rotate away”/absorb the phase  $\delta_E \rightarrow \delta_E = 0$  e.g. by a **global** redefinition of the zero of time, or a **global** redefinition/translation of our coordinate system. Hence, we can equivalently write, without any loss of generality:

$$\tilde{r}_o(\omega) = \left(\frac{e}{m_e}\right)E_o \frac{\left[(\omega^2 - \omega_0^2) - i\gamma\omega\right]}{\left[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2\right]} = r_o^{real}(\omega) + i r_o^{imag}(\omega) = |\tilde{r}_o(\omega)| e^{i\phi_r(\omega)}$$

The {frequency-dependent} **real** part of the **frequency-domain** displacement amplitude  $\tilde{r}_o(\omega)$  – i.e. the component of  $\tilde{r}_o(\omega)$  that is **in-phase** with the driving force  $-e\tilde{E}(\vec{r}, t; \omega) = -e\tilde{E}_o e^{-i\omega t} \hat{r}$  is:

$$r_o^{real}(\omega) = \Re\{\tilde{r}_o(\omega)\} = \left(\frac{e}{m_e}\right)\tilde{E}_o \frac{(\omega^2 - \omega_0^2)}{\left[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2\right]}$$

The {frequency-dependent} **imaginary** part of the **frequency-domain** displacement amplitude  $\tilde{r}_o(\omega)$  – i.e. the component of  $\tilde{r}_o(\omega)$  that is **90°-out-of-phase** with the driving force  $-e\tilde{E}(\vec{r}, t; \omega) = -e\tilde{E}_o e^{-i\omega t} \hat{r}$  is:

$$r_o^{imag}(\omega) = \Im\{\tilde{r}_o(\omega)\} = -\left(\frac{e}{m_e}\right)\tilde{E}_o \frac{\gamma\omega}{\left[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2\right]}$$

The {frequency-dependent} **phase** of the **frequency-domain** displacement amplitude  $\tilde{r}_o(\omega)$  is:

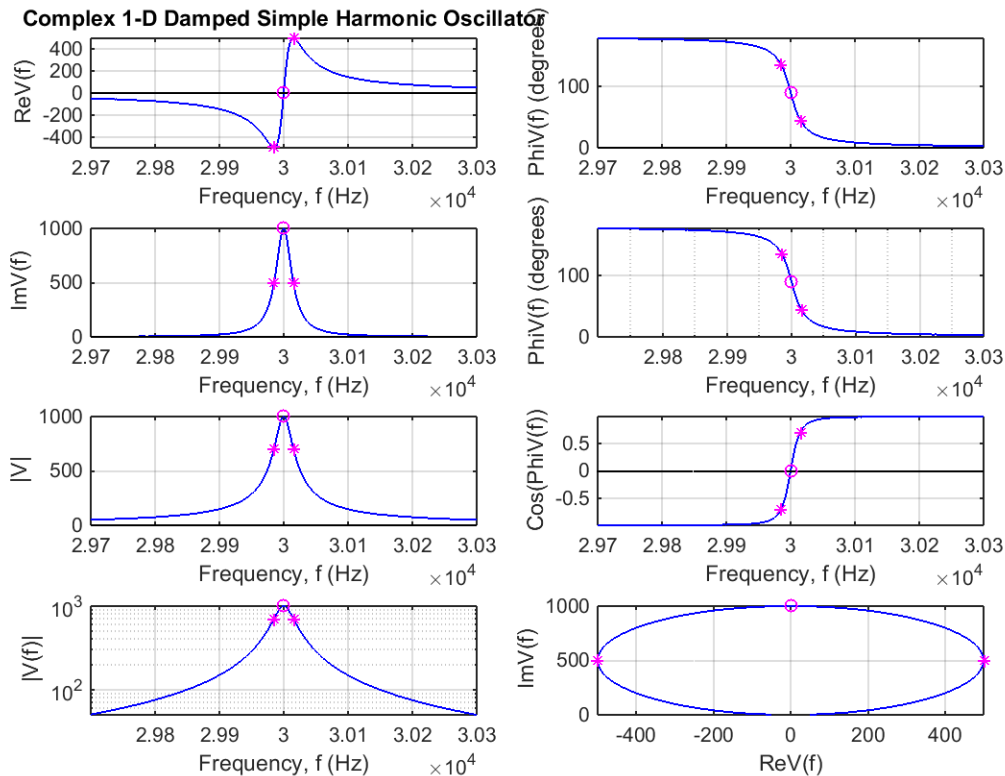
$$\phi_r(\omega) = \tan^{-1}\left(\frac{\Im\{\tilde{r}_o(\omega)\}}{\Re\{\tilde{r}_o(\omega)\}}\right) = \tan^{-1}\left(\frac{r_o^{imag}(\omega)}{r_o^{real}(\omega)}\right) = \tan^{-1}\left(\frac{-\gamma\omega}{(\omega^2 - \omega_0^2)}\right)$$

When:  $\omega < \omega_0 = \sqrt{\frac{k_e}{m_e}}$ ,  $\phi_r > 0 \Rightarrow \tilde{r}$  **lags**  $\tilde{E}$ . When:  $\omega > \omega_0 = \sqrt{\frac{k_e}{m_e}}$ ,  $\phi_r < 0 \Rightarrow \tilde{r}$  **leads**  $\tilde{E}$ .

The {frequency-dependent} **magnitude** of the **frequency-domain** displacement amplitude  $\tilde{r}_o(\omega)$  is:

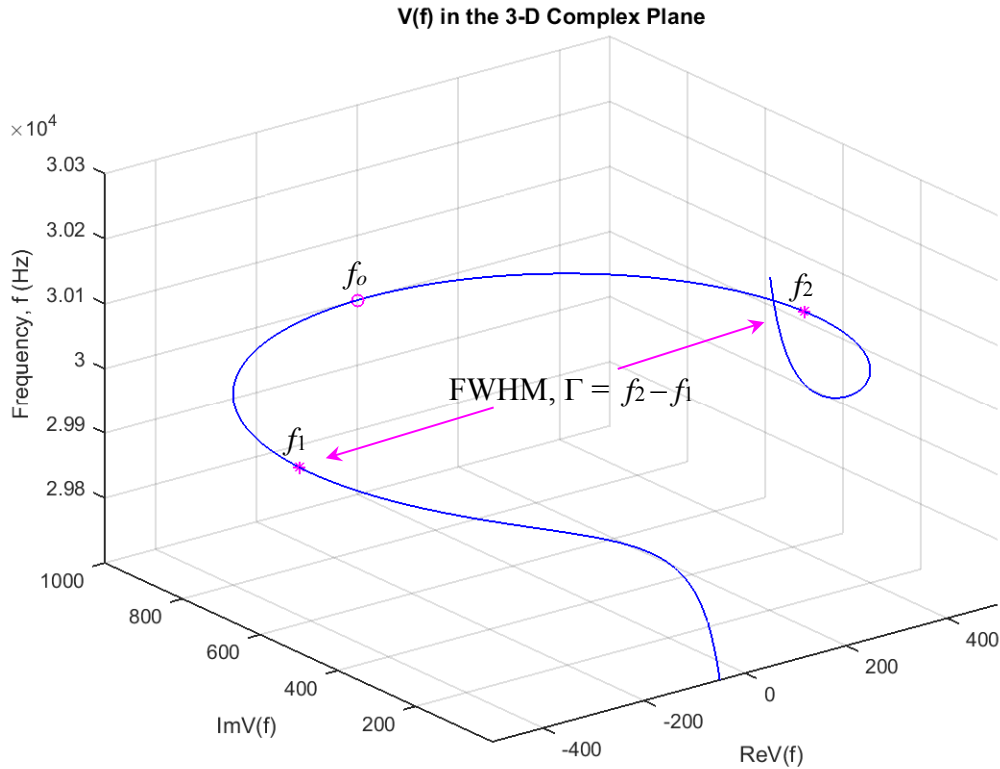
$$\begin{aligned}
 |\tilde{r}_o(\omega)| &= \sqrt{\tilde{r}_o(\omega) \cdot \tilde{r}_o^*(\omega)} = \sqrt{(r_o^{real}(\omega))^2 + (r_o^{imag}(\omega))^2} \\
 &= \left(\frac{e}{m_e}\right) E_o \sqrt{\frac{[(\omega^2 - \omega_0^2) - i\gamma\omega][(\omega^2 - \omega_0^2) + i\gamma\omega]}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]^2}} \\
 &= \left(\frac{e}{m_e}\right) E_o \sqrt{\frac{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]^2}} = \left(\frac{e}{m_e}\right) E_o \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}}
 \end{aligned}$$

The 8-figure plot below shows the behavior of a mechanical resonance at  $f_o = 30 \text{ KHz}$ , and FWHM  $\Gamma \equiv \gamma/2\pi = 30 \text{ Hz}$  in the vicinity of the resonance for unit amplitude  $\left(\frac{e}{m_e}\right) E_o = 1$ . The resonance frequency  $f_o$  and lo/hi frequencies  $f_1 < f_o < f_2$  that define the FWHM,  $\Gamma \equiv f_2 - f_1$  {i.e.  $\gamma \equiv 2\pi \Gamma \equiv 2\pi(f_2 - f_1) = (\omega_2 - \omega_1)$ } are also indicated in each plot:



Note that the so-called  $Q$  of {this} resonance {here} is:  $Q \equiv \omega_o/\gamma = f_o/\Gamma = 1000$ . For small damping  $\gamma \ll \omega_o$  ( $Q \gg 1$ ), it can be shown that  $f_1 \approx f_o - \frac{1}{2}(\gamma/2\pi)$ ,  $f_2 \approx f_o + \frac{1}{2}(\gamma/2\pi)$  and that  $\varphi_r(f_o) \approx \tan^{-1}(\infty) = 90^\circ$ ,  $\varphi_r(f_1) \approx \tan^{-1}(-1) = 135^\circ$  and  $\varphi_r(f_2) \approx \tan^{-1}(+1) = 45^\circ$ .

The 3-D plot below shows the CW path taken in the complex plane in passing through the resonance:



Now:  $\tilde{\tilde{\mathbf{P}}}(\vec{r}, t; \omega) = -n_e^b e \tilde{\tilde{\mathbf{r}}}(t; \omega) = -n_e^b e \tilde{\tilde{\mathbf{r}}}_o(\omega) e^{-i\omega t} \hat{r}$  where:  $\tilde{\tilde{\mathbf{r}}}_o(\omega) = \left( \frac{e}{m_e} \right) E_o \frac{[(\omega^2 - \omega_0^2) - i\gamma\omega]}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]}$

Thus:  $\tilde{\tilde{\mathbf{P}}}(\vec{r}, t; \omega) = -n_e^b e \left( \frac{e}{m_e} \right) E_o \frac{[(\omega^2 - \omega_0^2) - i\gamma\omega]}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]} e^{-i\omega t} \hat{r} = n_e^b \left( \frac{e^2}{m_e} \right) E_o \frac{[(\omega_0^2 - \omega^2) + i\gamma\omega]}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]} e^{-i\omega t} \hat{r}$

Since the complex time-domain  $\tilde{\tilde{\mathbf{P}}}(\vec{r}, t; \omega) = \tilde{\tilde{\mathbf{P}}}(\vec{r}; \omega) \cdot e^{-i\omega t} \hat{r}$ , we see that the complex frequency-

domain vector amplitude is:  $\tilde{\tilde{\mathbf{P}}}(\vec{r}; \omega) = n_e^b \left( \frac{e^2}{m_e} \right) E_o \frac{[(\omega_0^2 - \omega^2) + i\gamma\omega]}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]} \hat{r}$ .

In the static limit (*i.e.*  $\omega = 0$ ):

$$\tilde{\tilde{\mathbf{P}}}(\vec{r}; \omega = 0) = \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{\omega_0^2} E_o \hat{r} = \frac{n_e^b \left( \frac{e^2}{m_e} \right)}{\left( \frac{k_e}{m_e} \right)} E_o \hat{r} = \left( \frac{n_e^b e^2}{k_e} \right) E_o \hat{r} \leftarrow \begin{array}{l} \text{Static polarization } \tilde{\tilde{\mathbf{P}}}(\omega = 0) \\ \text{is } \underline{\text{in-phase}} \text{ with } \tilde{\tilde{\mathbf{E}}} \end{array}$$

Note that the **phase** of the **frequency-domain** vector amplitude  $\tilde{\tilde{\mathbf{P}}}(\vec{r}; \omega)$  is frequency dependent,  $\tilde{\tilde{\mathbf{P}}}(\omega)$  **lags** behind (or **leads**)  $\tilde{\tilde{\mathbf{E}}}(\omega)$  by a frequency-dependent phase angle of:

$$\phi_p(\omega) = \tan^{-1} \left[ \frac{\Im m \{ \tilde{\tilde{\mathbf{P}}}(\vec{r}, \omega) \}}{\Re e \{ \tilde{\tilde{\mathbf{P}}}(\vec{r}, \omega) \}} \right] = \tan^{-1} \left[ \frac{\gamma \omega}{(\omega_0^2 - \omega^2)} \right] \Leftarrow \begin{array}{l} \text{n.b. The **damping constant** } \\ \gamma \text{ has the same units as } \omega : \\ \text{radians/sec} \end{array}$$

When:  $\omega < \omega_0 = \sqrt{\frac{k_e}{m_e}}$ ,  $\phi_p > 0 \Rightarrow \tilde{\tilde{\mathbf{P}}}$  **lags**  $\tilde{\tilde{\mathbf{E}}}$ . When:  $\omega > \omega_0 = \sqrt{\frac{k_e}{m_e}}$ ,  $\phi_p < 0 \Rightarrow \tilde{\tilde{\mathbf{P}}}$  **leads**  $\tilde{\tilde{\mathbf{E}}}$ .

From the above formula, note that if the damping constant  $\gamma = 0$ , then  $\phi_p = 0$ , the polarization  $\tilde{\tilde{\mathbf{P}}}(\omega)$  is **always** in-phase with  $\tilde{\tilde{\mathbf{E}}}(\omega)$ , because if  $\gamma = 0$ , then  $\Im m \{ \tilde{\tilde{\mathbf{P}}}(\vec{r}, t) \} = 0$ ,

*i.e.* the polarization  $\tilde{\tilde{\mathbf{P}}}(\omega)$  is purely **real**! A damping constant of  $\gamma = 0$  also means that the **width** (FWHM)  $\Gamma \equiv \gamma/2\pi$  (Hz) of the atomic/molecular **resonance** is **infinitely narrow**, and thus there are **no dissipative** processes (*i.e.* energy **loss** mechanisms) present at the microscopic atomic/molecular level in this macroscopic medium! Note also that  $\gamma$  has physical/SI units of radians/second.

Note further that  $\tilde{\tilde{\mathbf{E}}}$  in the above expression is actually  $\tilde{\tilde{\mathbf{E}}}_{int}$  – the **internal** macroscopic electric field of the dielectric:  $\tilde{\tilde{\mathbf{E}}} = \tilde{\tilde{\mathbf{E}}}_{int} = \tilde{\tilde{\mathbf{E}}}_{ext} + \tilde{\tilde{\mathbf{E}}}_p$ , the **sum** of the macroscopic **external applied** electric field and the macroscopic electric field **due to the polarization of the dielectric medium**.

The electric field due to polarization of the medium is:  $\tilde{\tilde{\mathbf{E}}}_p = -\frac{1}{3\epsilon_0} \tilde{\tilde{\mathbf{P}}}$  See P435 Lect. Notes 10, p. 1-6. See also P435 Lect. Notes 9, p. 26.

Thus:  $\tilde{\tilde{\mathbf{E}}} = \tilde{\tilde{\mathbf{E}}}_{int} = \tilde{\tilde{\mathbf{E}}}_{ext} - \frac{1}{3\epsilon_0} \tilde{\tilde{\mathbf{P}}}$  Therefore:  $\tilde{\tilde{\mathbf{P}}} = n_e^b \left( \frac{e^2}{m_e} \right) \frac{[(\omega_0^2 - \omega^2) + i\gamma\omega]}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \left[ \tilde{\tilde{\mathbf{E}}}_{ext} - \frac{1}{3\epsilon_0} \tilde{\tilde{\mathbf{P}}} \right]$   $\omega_0^2 \equiv \left( \frac{k_e}{m_e} \right)$

Now solve for  $\tilde{\tilde{\mathbf{P}}}$ : Skipping writing out some {tedious} complex algebra, we obtain:

$$\tilde{\tilde{\mathbf{P}}} = n_e^b \left( \frac{e^2}{m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2} \tilde{\tilde{\mathbf{E}}}_{ext} \quad \text{where:} \quad \omega_1 \equiv \sqrt{\omega_0^2 - \left( \frac{n_e^b e^2}{3\epsilon_0 m_e} \right)} < \omega_0 = \text{effective angular resonance frequency of bound atomic electrons}$$

Note that this formula is essentially identical *e.g.* to the {complex} displacement amplitude formula for a driven harmonic oscillator, and/or that for the {complex} AC voltage amplitude in an LCR circuit, and for many other physical systems exhibiting a {damped} resonance-type behavior.

Now if  $\tilde{\tilde{\mathbf{E}}}_{ext} = \vec{E}$ -field associated with a monochromatic plane *EM* wave propagating in a dielectric medium:  $\tilde{\tilde{\mathbf{E}}}_{ext}(z, t) = \vec{E}_0 e^{i(kz - \omega t)}$ , then because of the **linear** relationship between the polarization  $\tilde{\tilde{\mathbf{P}}}$  and *e.g.*  $\tilde{\tilde{\mathbf{E}}}_{ext}(z, t) = \vec{E}_0 e^{i(kz - \omega t)} \hat{x}$ , Gauss' Law becomes (since  $\rho_{free}(\vec{r}) = 0$ ):

$$\vec{\nabla} \cdot \tilde{\tilde{\mathbf{E}}}_{ext} = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \tilde{\tilde{\mathbf{P}}} = \tilde{\tilde{\rho}}_{bound} = 0$$

The wave equation for a dielectric medium with  $\tilde{\rho}_{free}(\vec{r}) = 0$  and  $\tilde{\mathbf{J}}_{free} = 0$  becomes:

$$\nabla^2 \tilde{\mathbf{E}}_{ext} - \frac{1}{c^2} \frac{\partial^2 \tilde{\mathbf{E}}_{ext}}{\partial t^2} = \mu_o \frac{\partial^2 \tilde{\mathbf{P}}}{\partial t^2} = \mu_o n_e^b \left( \frac{e^2}{m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2]} \frac{\partial^2 \tilde{\mathbf{E}}_{ext}}{\partial t^2} \quad \text{with: } \frac{1}{c^2} = \epsilon_o \mu_o$$

Or: 
$$\nabla^2 \tilde{\mathbf{E}}_{ext} = \frac{1}{c^2} \left[ 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2]} \right] \frac{\partial^2 \tilde{\mathbf{E}}_{ext}}{\partial t^2} \quad \text{with: } \omega_1 \equiv \sqrt{\omega_0^2 - \left( \frac{n_e^b e^2}{3\epsilon_o m_e} \right)}$$

The general solution to this **dispersive** wave equation is of the form:

$$\tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega) = \tilde{\mathbf{E}}_o e^{i(\vec{k}z - \omega t)} \quad \text{with **complex** } \tilde{k} = k + i\kappa \quad \text{and: } \tilde{k}^2 = \frac{\omega^2}{c^2} \left[ 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2]} \right]$$

Thus, we also see that here {again} the complex wavenumber  $\tilde{k} = k + i\kappa$  is **explicitly** dependent on the angular frequency  $\omega$ , *i.e.*  $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$ .

We further see that monochromatic plane *EM* waves propagating in a **dispersive** dielectric medium are **exponentially** attenuated, because:  $\tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega) = \tilde{\mathbf{E}}_o e^{i(\vec{k}z - \omega t)} = \tilde{\mathbf{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}$ , *i.e.* the  $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$  term corresponds to **absorption/dissipation** in the macroscopic dielectric, and is physically related to/is proportional to the **damping constant**  $\gamma$ .

Note that we also have:  $\tilde{\mathbf{P}}(\vec{r}, t; \omega) \equiv \epsilon_o \tilde{\chi}_e(\omega) \tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega)$ , thus the susceptibility  $\tilde{\chi}_e(\omega)$  {here} is also complex, and frequency-dependent:  $\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$ . The  $\zeta_e(\omega) = \Im m\{\tilde{\chi}_e(\omega)\}$  term corresponds to **absorption/dissipation** in the dielectric, and is physically related to/is proportional to the **damping constant**  $\gamma$ . The corresponding dissipative energy **losses** at the microscopic, atomic/molecular level in the dielectric ultimately wind up as **heat!**

Since:

$$\begin{aligned} \tilde{\mathbf{P}}(\vec{r}, t; \omega) &\equiv \epsilon_o \tilde{\chi}_e(\omega) \tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega) = n_e^b \left( \frac{e^2}{m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2]} \tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega) \\ &= \epsilon_o \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2\omega^2]} \tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega) \end{aligned}$$

where: 
$$\omega_1 \equiv \sqrt{\omega_0^2 - \left( \frac{n_e^b e^2}{3\epsilon_o m_e} \right)}$$



We see that the complex susceptibility associated with a single resonance is:

$$\tilde{\chi}_e(\omega) = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{[(\omega_1^2 - \omega^2) + i\gamma\omega]}{[(\omega_1^2 - \omega^2)^2 + \gamma^2 \omega^2]} \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

Hence:

$$\chi_e(\omega) = \Re\{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{(\omega_1^2 - \omega^2)}{[(\omega_1^2 - \omega^2)^2 + \gamma^2 \omega^2]}$$

And:

$$\zeta_e(\omega) = \Im\{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \frac{\gamma\omega}{[(\omega_1^2 - \omega^2)^2 + \gamma^2 \omega^2]}$$

Now before we go much further with this, we need to discuss another aspect of our model – namely that in most linear dielectric materials, the atoms comprising the material are multi-electron atoms, and consequently there are many different binding energies – the outer shell atomic electrons are weakly bound, hence have small  $k_e$ , and thus small  $\omega_0 = \sqrt{k_e/m_e}$ , whereas the inner-shell electrons are much more tightly bound, hence have larger  $k_e$ , larger  $\omega_0 = \sqrt{k_e/m_e}$ .

Furthermore, in complex media, *i.e.* dielectrics with more than one kind of atom, electrons can be shared between atoms – *i.e.* they are bound to molecules *e.g.* the  $\pi$ -electrons in benzene ring / aromatic hydrocarbon-type compounds, which can be weakly bound in some molecules.

Thus, there can be also be {molecular} resonances *e.g.* in the microwave and infra-red regions of the *EM* spectrum – atomic resonances are typically in the optical and UV regions {for the outer-most shell electrons}, as well as in the far UV and x-ray regions {for the inner-shell electrons}!

Allowing for all such resonances, we can write the {complex} electric polarization  $\tilde{\mathbf{P}}$  as a summation over all of the resonances present in the linear dielectric as follows:

$$\tilde{\mathbf{P}}(\vec{r}, t; \omega) = \epsilon_o \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left( \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right) \tilde{\mathbf{E}}_{ext}(\vec{r}, t; \omega)$$

where:  $\omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \left( \frac{n_e^b e^2}{3\epsilon_o m_e} \right)}$  and:  $\omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$

and where:  $f_j^{osc} \equiv$  oscillator strength of  $j^{\text{th}}$  resonance, defined such that:  $\sum_{j=1}^n f_j^{osc} = 1$

Physically:  $f_j^{osc} =$  fractional strength of the  $j^{\text{th}}$  resonance and  $\gamma_j = 2\pi \times$  width  $\Gamma_j$  of the  $j^{\text{th}}$  resonance.

Thus, we see that the multi-resonance complex electric susceptibility  $\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$  is:

$$\tilde{\chi}_e(\omega) = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left( \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right) \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

Hence:

$$\chi_e(\omega) = \Re\{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left( \sum_{j=1}^n f_j^{osc} \frac{(\omega_{1j}^2 - \omega^2)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right)$$

And:

$$\zeta_e(\omega) = \Im\{\tilde{\chi}_e(\omega)\} = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left( \sum_{j=1}^n f_j^{osc} \frac{\gamma_j \omega}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right)$$

The complex electric permittivity  $\tilde{\epsilon}(\omega) = \epsilon_o(1 + \tilde{\chi}_e(\omega)) \equiv \epsilon(\omega) + i\varsigma(\omega)$  of a dispersive, linear dielectric medium is:

$$\tilde{\epsilon}(\omega) = \epsilon_o(1 + \tilde{\chi}_e(\omega)) = \epsilon_o \left( 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right] \right) \equiv \epsilon(\omega) + i\varsigma(\omega)$$

with the relations:

$$\begin{aligned} \epsilon(\omega) &= \Re\{\tilde{\epsilon}(\omega)\} = \Re\{\epsilon_o(1 + \tilde{\chi}_e(\omega))\} = \epsilon_o(1 + \chi_e(\omega)) \\ &= \epsilon_o \left( 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right] \right) \end{aligned}$$

and:

$$\begin{aligned} \varsigma(\omega) &= \Im\{\tilde{\epsilon}(\omega)\} = \Im\{\epsilon_o(1 + \tilde{\chi}_e(\omega))\} = \epsilon_o \zeta_e(\omega) \\ &= \epsilon_o \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left( \sum_{j=1}^n f_j^{osc} \frac{\gamma_j \omega}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right) \end{aligned}$$

Monochromatic plane *EM* wave solutions to the ***dispersive*** wave equation are of the form:

$$\boxed{\tilde{\vec{E}}(\vec{r}, t, \omega) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)}} \quad \text{with complex wavenumber } \boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega) \equiv \sqrt{\tilde{\epsilon}(\omega) \mu_o \omega}}$$

for a  $\cong$  ***non-magnetic*** medium, *i.e.*  $\mu \cong \mu_o$ .

Thus: 
$$\boxed{\tilde{\vec{E}}_{ext}(\vec{r}, t, \omega) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\vec{E}}_o \underbrace{e^{-\kappa z}}_{\substack{\text{exponential} \\ \text{damping} \\ \text{of EM wave}}} e^{i(kz - \omega t)}}$$

Introducing a {frequency-dependent} complex wavenumber  $\boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)}$  is ***equivalent***

to introducing a {frequency-dependent} complex index of refraction  $\boxed{\tilde{n}(\omega) = n(\omega) + i\eta(\omega)}$ .

For a dispersive dielectric, the complex index of refraction and complex wavenumber are

{simply} related to each other by: 
$$\boxed{\tilde{k}(\omega) = \left(\frac{\omega}{c}\right) \tilde{n}(\omega)}$$

$$\therefore \boxed{\left(k(\omega) + i\kappa(\omega)\right) = \left(\frac{\omega}{c}\right) \left(n(\omega) + i\eta(\omega)\right) = \left(\frac{\omega}{c}\right) n(\omega) + i\left(\frac{\omega}{c}\right) \eta(\omega)}$$

$$\Rightarrow \boxed{k(\omega) = \left(\frac{\omega}{c}\right) n(\omega)} \quad \text{and} \quad \boxed{\kappa(\omega) = \left(\frac{\omega}{c}\right) \eta(\omega)}$$

For a  $\cong$  ***non-magnetic*** medium ( $\mu \cong \mu_o$ ), the complex index of refraction  $\tilde{n}(\omega)$  is related to the complex electric permittivity  $\tilde{\epsilon}(\omega) = \epsilon_o (1 + \tilde{\chi}_e(\omega))$  and thus the complex electric susceptibility  $\tilde{\chi}_e(\omega)$  via the relation:

$$\boxed{\tilde{n}(\omega) = \sqrt{\frac{\tilde{\epsilon}(\omega)}{\epsilon_o}} = \sqrt{1 + \tilde{\chi}_e(\omega)}}$$

Squaring both sides: 
$$\boxed{\tilde{n}^2(\omega) = \frac{\tilde{\epsilon}(\omega)}{\epsilon_o} = 1 + \tilde{\chi}_e(\omega) = 1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left(\sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]}\right)}$$

But:

$$\boxed{\begin{aligned} \tilde{k}^2(\omega) &= \left(\frac{\omega}{c}\right)^2 \left[1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[\sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]}\right]\right] \\ &= (k(\omega) + i\kappa(\omega))^2 = k^2(\omega) + 2ik(\omega)\kappa(\omega) - \kappa^2(\omega) \end{aligned}}$$

Since:  $\tilde{n}(\omega) = \left(\frac{c}{\omega}\right) \tilde{k}(\omega)$  then:

$$\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega) = 1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]$$

$$= (n(\omega) + i\eta(\omega))^2 = n^2(\omega) + 2in(\omega)\eta(\omega) - \eta^2(\omega)$$

Equating the real and imaginary parts of the LHS & RHS of the above equation, we obtain:

$$n^2(\omega) - \eta^2(\omega) = 1 + \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]$$

$$2n(\omega)\eta(\omega) = \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]$$

2 equations and 2 unknowns:  
 $\{n(\omega) \& \eta(\omega)\}$   
 $\Rightarrow$  solve for  $n(\omega) \& \eta(\omega)$

First define:

$$\alpha_x(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]$$

$$\beta_x(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right] \quad (\text{n.b. } \beta_x(\omega) > 0, \text{ is always positive \{here\})$$

Then:  $n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)$  and  $2n(\omega)\eta(\omega) = \beta_x(\omega) \Rightarrow \eta(\omega) = \beta_x(\omega)/2n(\omega)$

Thus:  $n^2(\omega) - \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)^2 = (1 + \alpha_x(\omega)) \Leftarrow$  multiply equation through by  $n^2(\omega)$

$$n^4(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = (1 + \alpha_x(\omega))n^2(\omega)$$

Or:  $n^4(\omega) - (1 + \alpha_x)n^2(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = 0 \Leftarrow$  n.b. This may look like a quartic equation, but it is actually a quadratic equation !!!

Define:  $x \equiv n^2(\omega)$ . We temporarily suppress the  $(\omega)$ -dependence in the following:

Then:  $x^2 - (1 + \alpha_x)x - \left(\frac{\beta_x}{2}\right)^2 = 0 \Rightarrow ax^2 + bx + c = 0$  with:  $a = 1$ ,  $b = -(1 + \alpha_x)$ ,  $c = -\left(\frac{\beta_x}{2}\right)^2$

The solutions / roots of this quadratic equation are of the general form:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow x = \frac{+(1+\alpha_x) \pm \sqrt{(1+\alpha_x)^2 + 4\left(\frac{\beta_x}{2}\right)^2}}{2} = \frac{1}{2} \left[ (1+\alpha_x) \pm \sqrt{(1+\alpha_x)^2 + \beta_x^2} \right]$$

$$i.e. \quad x = \frac{1}{2}(1+\alpha_x) \left[ 1 \pm \sqrt{1 + \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2} \right] \quad n.b. \text{ the term: } \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2 > 0$$

$\Rightarrow$  **Must** select +ve root on **physical** grounds, since  $x \equiv n^2 > 0$ .

$$\therefore x = n^2 = \frac{1}{2}(1+\alpha_x) \left[ 1 + \sqrt{1 + \left(\frac{\beta_x}{(1+\alpha_x)}\right)^2} \right]$$

Finally, we obtain:

$$n(\omega) \equiv \Re\{\tilde{n}(\omega)\} = \sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right) \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2} \right]}$$

Complex index of refraction:

$$\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$$

$$\eta(\omega) \equiv \Im\{\tilde{n}(\omega)\} = \frac{\beta_x(\omega)}{2n(\omega)} = \frac{\beta_x(\omega)/2}{\sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right) \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1+\alpha_x(\omega))}\right)^2} \right]}}$$

Where:

$$\alpha_x(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right]$$

$$\beta_x(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_o m_e}\right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\gamma_j \omega)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right]$$

Obviously, **explicitly** writing out the full mathematical formulae for  $n(\omega)$  and  $\eta(\omega)$  is quite tedious – but these can be reasonably-easily coded up {i.e. a computer program} and plots of  $n(\omega)$  vs.  $\omega$  and  $\eta(\omega)$  vs.  $\omega$  can be obtained. We can also then obtain the following:

The complex relations:  $\tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega)$  and  $\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \left(\frac{\omega}{c}\right) \tilde{n}(\omega)$

and thus:  $k(\omega) = \left(\frac{\omega}{c}\right) n(\omega)$  and  $\kappa(\omega) = \left(\frac{\omega}{c}\right) \eta(\omega)$ .

The {frequency-dependent} **intensity/irradiance**  $I(\vec{r}, \omega) = \langle |\vec{S}(\vec{r}, t; \omega)| \rangle$  associated with a monochromatic plane *EM* wave propagating in a linear, dispersive dielectric is also exponentially decreased by a factor of  $1/e = e^{-1}$  of its original value in going a characteristic distance of:  $z = 1/\alpha(\omega) = 1/2\kappa(\omega) \equiv \ell_{\text{atten}}(\omega)$  i.e. defining:  $\ell_{\text{atten}}(\omega) \equiv 1/\alpha(\omega) = 1/2\kappa(\omega) =$  **intensity attenuation length** – which is  $\sim$  analogous to the skin depth,  $\delta_{\text{sc}} \equiv 1/\kappa$  for metals / conductors. However, note that  $\delta_{\text{sc}} \equiv 1/\kappa$  is associated with the attenuation of the  $\vec{E}$  and  $\vec{B}$  - fields, whereas attenuation effects in **intensity/irradiance**,  $I$  varies as the **square** of the  $\vec{E}$  -field:

$$I(\vec{r}, \omega) = \langle |\vec{S}(\vec{r}, t; \omega)| \rangle \propto \langle E_{\text{ext}}^2(\vec{r}, t; \omega) \rangle \quad \text{hence:} \quad I(\vec{r}, \omega) \propto E_o^2 e^{-2\kappa(\omega)z} = E_o^2 e^{-\alpha(\omega)z}$$

In the exponential  $z$ -dependent term  $e^{-2\kappa(\omega)z}$ , since the energy densit(ies)  $\langle u_{E,M}(\vec{r}, t; \omega) \rangle$  and intensity  $I(\vec{r}, \omega) = \langle |\vec{S}(\vec{r}, t; \omega)| \rangle$  are both proportional to  $E^2$  i.e. both proportional to  $e^{-2\kappa(\omega)z}$ ,

we define the {frequency-dependent} **absorption coefficient**:

$$\alpha(\omega) \equiv 2\kappa(\omega) = 1/\ell_{\text{atten}}(\omega).$$

Similarly, for the {frequency-dependent} complex index of refraction

$$\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$$

we can also define the {frequency-dependent} **extinction coefficient**:

$$\xi(\omega) \equiv 2\eta(\omega).$$

Since:  $\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \Rightarrow 2\kappa(\omega) = \left(\frac{\omega}{c}\right)2\eta(\omega)$  thus:  $\alpha(\omega) = \left(\frac{\omega}{c}\right)\xi(\omega) = 2\left(\frac{\omega}{c}\right)\eta(\omega)$ .

The **absorption coefficient**:  $\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)2\eta(\omega) = \left(\frac{\omega}{c}\right)\xi(\omega) = 1/\ell_{\text{atten}}(\omega)$  and

The **extinction coefficient**:  $\xi(\omega) \equiv 2\eta(\omega)$ .

Typical values of the (real) index of refraction  $n(\omega)$  for solids and liquids are  $n(\omega) \approx 1.3 - 1.7$  in the visible light region of *EM* spectrum, e.g.  $n_{\text{glass}}(\omega) \approx 1.5$ ,  $n_{\text{H}_2\text{O}}(\omega) \approx 1.3$ ,  $n_{\text{plastic}}(\omega) \approx 1.7$ .

Then if:  $n(\omega) = \sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right]} = 1.5 \quad \Leftarrow \quad \begin{matrix} \text{index of refraction of} \\ \text{glass in the visible} \\ \text{light region} \end{matrix}$

Then:  $n^2(\omega) = \left(\frac{1+\alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right] = (1.5)^2 = 2.25$

Thus:  $(1+\alpha_x(\omega)) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right] = 4.50 \quad \Leftarrow \quad \begin{matrix} \text{One equation \& two unknowns:} \\ \alpha_x(\omega) \text{ and } \beta_x(\omega) \end{matrix}$

$\Rightarrow$  Need another relation / independent constraint!!

Note that glass doesn't have significant absorption in the **visible light** region, typical such solid/liquid materials have (measured) absorption coefficients for visible light in the range of:

$$\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \approx 10^{-2} - 10^{-1} \text{ m}^{-1} \quad \Leftarrow \quad \begin{array}{l} \text{Intensity } I(z) \text{ decreases to } 1/e = e^{-1} = 0.368 \\ \text{of initial } I(z=0) \text{ value after light travels a} \\ \text{distance } z = 1/\alpha(\omega) \sim 10 - 100 \text{ m in glass.} \end{array}$$

So suppose:  $\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega) \approx 10^{-1} \text{ m}^{-1}$  in glass for visible light,  $\omega_{\text{vis}} \approx 10^{16}$  radians / sec

$$\Rightarrow \eta(\omega) = \left(\frac{c}{\omega}\right)\alpha(\omega) \approx \left(\frac{3 \times 10^8}{10^{16}}\right)10^{-1} = 3 \times 10^{-9} \ll 1$$

Now:  $\eta(\omega) = \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)$  and  $n(\omega) \approx 1.5$  for glass in visible light range of EM spectrum.

$$\Rightarrow \eta(\omega) = \frac{1}{3}\beta_x(\omega) \quad \text{or:} \quad \beta_x(\omega) = 3\eta(\omega) \approx 9 \times 10^{-9} \quad \ll 1 \text{ in the visible light range for glass}$$

Then:  $(1 + \alpha_x(\omega)) \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2} \right] = 4.50$  and:  $\beta_x(\omega) \approx 9 \times 10^{-9}$ .

Can now solve for  $\alpha_x(\omega)$ :

$$1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2} = \frac{4.50}{(1 + \alpha_x(\omega))} \Rightarrow 1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2 = \left[\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right]^2$$

$$\Rightarrow \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right) = \sqrt{\left[\left(\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right)^2\right]} - 1 \Rightarrow \beta_x(\omega) = (1 + \alpha_x(\omega)) \sqrt{\left[\left(\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right)^2\right]} - 1$$

This has a solution when:  $\alpha_x(\omega) \approx 1.25$  for:  $\beta_x(\omega) = 9 \times 10^{-9} \ll \alpha_x(\omega) \quad \Leftarrow \quad \begin{array}{l} \text{Obtained via numerical} \\ \text{methods using a computer} \end{array}$

Thus, for  $n(\omega) = 1.5$  for glass in the visible light region of the EM spectrum, with  $\alpha_x(\omega) \approx 1.25$  and  $\beta_x(\omega) = 9 \times 10^{-9}$  (i.e.  $\beta_x(\omega)/(1 + \alpha_x(\omega)) \ll 1$ ), as an explicit check, we see that:

$$\begin{aligned} n(\omega) &= \sqrt{\left(\frac{1 + \alpha_x(\omega)}{2}\right)} \left[ 1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2} \right] = \sqrt{\left(\frac{1 + 1.25}{2}\right)} \left[ 1 + \sqrt{1 + \left(\frac{9 \times 10^{-9}}{1 + 1.25}\right)^2} \right] \approx \sqrt{\left(\frac{1 + 1.25}{2}\right)} [1 + \sqrt{1}] \\ &= \sqrt{\left(\frac{1 + 1.25}{2}\right)} [2] = \sqrt{2.25} = 1.5 \quad \checkmark \end{aligned}$$

Thus we also see that: 
$$\alpha_x(\omega) \approx n^2(\omega) - 1 \equiv \left( \frac{n_e^b e^2}{\epsilon_0 m_e} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \quad i.e. \quad n^2(\omega) \approx 1 + \alpha_x(\omega)$$

for typical materials – glass, water, plastic – in the visible light region of the *EM* spectrum,  $\omega \approx 10^{16}$  radians/sec.

Whereas: 
$$\beta_x(\omega) \equiv \left( \frac{n_e^b e^2}{\epsilon_0 m_e} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \ll 1$$

for these same materials – glass, water, plastic – in the visible light region of the *EM* spectrum,  $\omega \approx 10^{16}$  radians/sec.

Our original equations were:  $n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)$  and  $2n(\omega)\eta(\omega) = \beta_x(\omega)$  or:  $\eta(\omega) = \beta_x(\omega)/2n(\omega)$  with:  $\alpha_x(\omega) \approx 1.25$  and  $\beta_x(\omega) \approx 9 \times 10^{-9}$  for  $n(\omega) = 1.5$  (for glass) with visible light and:  $\eta(\omega) = \beta_x(\omega)/2n(\omega) = 3 \times 10^{-9}$ .

We now see more clearly that:  $\eta(\omega) \ll n(\omega)$  in the visible light region of the *EM* spectrum for glass, *i.e.* the complex index of refraction  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega) \approx 1.25 + 9 \times 10^{-9}i$  for glass is predominantly real in the visible light region of the *EM* spectrum.

Thus, for glass in the visible light region of the *EM* spectrum:

$$n^2(\omega) - \eta^2(\omega) \approx n^2(\omega) = 1 + \alpha_x(\omega) = 1 + \left( \frac{n_e^b e^2}{\epsilon_0 m} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx (1.5)^2 = 2.25$$

and:

$$2n(\omega)\eta(\omega) = \beta(\omega)_x \equiv \left( \frac{n_e^b e^2}{\epsilon_0 m} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx 9 \times 10^{-9}$$

$$\alpha_x(\omega) \equiv \left( \frac{n_e^b e^2}{\epsilon_0 m} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx 1.25$$

Note that these results that we just obtained for glass in the visible light region of the *EM* spectrum do not hold true for all frequencies of *EM* waves {visible light region is in fact only a narrow portion of the *EM* spectrum}!!! In particular, these results do not hold at {or near} an atomic (or molecular) resonance!



Let us consider a “simplified” atomic/molecular system, that of having only a single resonance frequency (*i.e.* a single bound-state quantum energy level), then:

$$\omega_1 \equiv \sqrt{\omega_0^2 - \left(\frac{n_e^b e^2}{3\epsilon_0 m_e}\right)} = 2\pi f_1 \quad \text{with:} \quad \omega_0 \equiv \sqrt{\frac{k_e}{m_e}}$$

or:

$$f_1 = \frac{1}{2\pi} \omega_1 = \frac{1}{2\pi} \sqrt{\omega_0^2 - \left(\frac{n_e^b e^2}{3\epsilon_0 m_e}\right)}$$

**n.b.** Oscillator strength  $f_1^{osc} = 1$  {here} because have only a single resonance!

Then:

$$n(\omega) = \sqrt{\left(\frac{1 + \alpha_x^1(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x^1(\omega)}{1 + \alpha_x^1(\omega)}\right)^2}\right]}$$

$$\alpha_x^1(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_0 m_e}\right) \left[ \frac{f_1^{osc} (\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + \gamma_1^2 \omega_1^2} \right]$$

$$\eta(\omega) = \frac{\beta_x^1(\omega)}{2n(\omega)} = \frac{(\beta_x^1(\omega)/2)}{\sqrt{\left(\frac{1 + \alpha_x^1(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x^1(\omega)}{1 + \alpha_x^1(\omega)}\right)^2}\right]}}$$

$$\beta_x^1(\omega) \equiv \left(\frac{n_e^b e^2}{\epsilon_0 m_e}\right) \left[ \frac{f_1^{osc} \gamma_1 \omega}{(\omega_1^2 - \omega^2)^2 + \gamma_1^2 \omega_1^2} \right]$$

The figure on the left (immediately below) shows the behavior of the real and imaginary parts of the complex index of refraction of a dispersive, linear medium, *i.e.*  $\{n(\omega) \text{ vs. } \omega\}$  and  $\{\eta(\omega) \text{ vs. } \omega\}$  for a single atomic resonance. The figure on the right (immediately below) shows the behavior of  $\{n(\omega) - 1 \text{ vs. } \omega\}$  and the absorption coefficient  $\{\alpha(\omega) \equiv 2\kappa(\omega) \text{ vs. } \omega\}$  for a single atomic resonance.

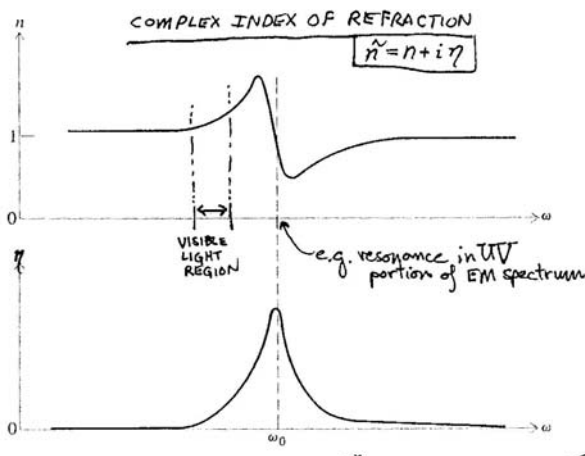
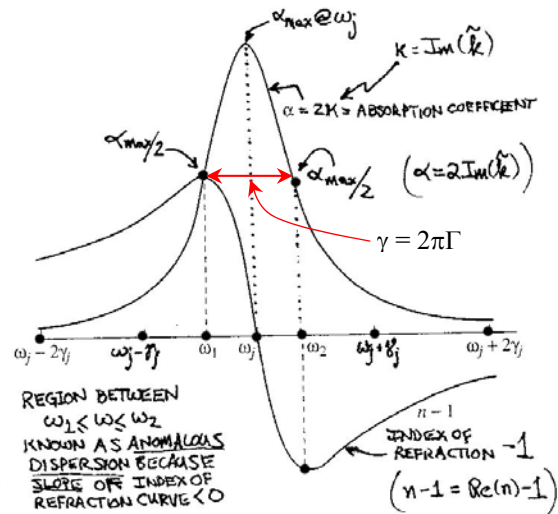
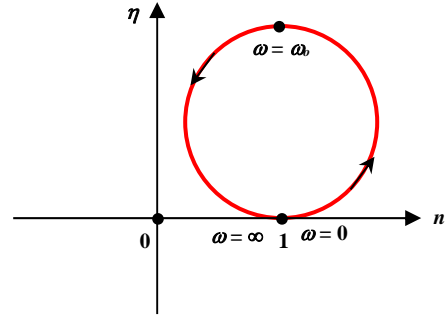


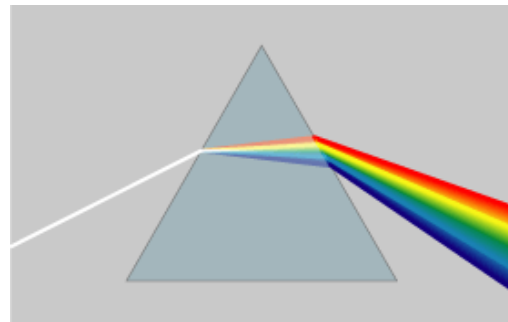
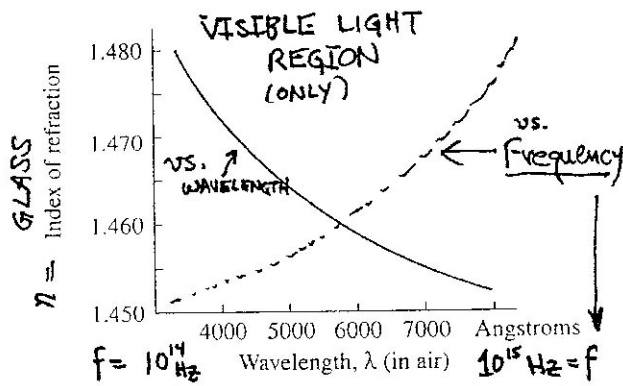
Figure 5.1. Graphs of the index of refraction and extinction coefficient versus frequency near a single resonance line.



*n.b.* The above curves are “classic” features of a complex resonance – with center / resonance frequency  $\omega_1$  and damping constant  $\gamma_j \equiv \omega_2 - \omega_1 = 2\pi\Gamma_j = 2\pi \times$  width  $\Gamma_j$  (= FWHM) of the  $j^{\text{th}}$  resonance. Note that the width (FWHM)  $\Gamma \equiv f_2 - f_1$  (Hz). In the  $\eta - n$  complex plane:



In the visible light region of the EM spectrum, the graph below shows both the frequency and wavelength behavior of the {real} index of refraction of glass, *i.e.*  $n(f)$  vs.  $f$  {dotted line} and  $n(\lambda)$  vs.  $\lambda$  {solid line}. Note that since  $v_{prop} = f\lambda \Rightarrow f = v_{prop}/\lambda$  or:  $\lambda = v_{prop}/f$

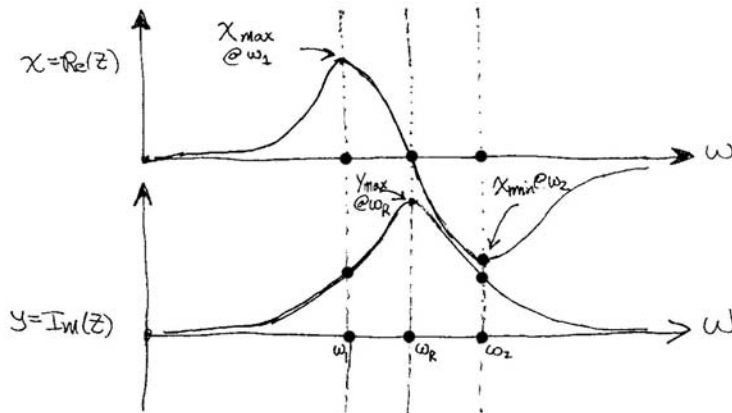


{1 Ångstrom =  $10^{-10}$  m = 0.1 nm}

*n.b.* Media which are very transparent *e.g.* in the visible light region are often almost (or are) opaque in the so-called anomalous dispersion region of a resonance,  $\omega_1 < \omega_R < \omega_2$  – *i.e.* in the FWHM region of the atomic resonance, where the extinction coefficient  $\eta(\omega)$  becomes very large – EM waves near the resonance frequency  $\omega_R$  are very rapidly exponentially attenuated!

**The General Behavior of “Classic” Complex Resonance:**

$$\tilde{z}(\omega) = x(\omega) + iy(\omega)$$

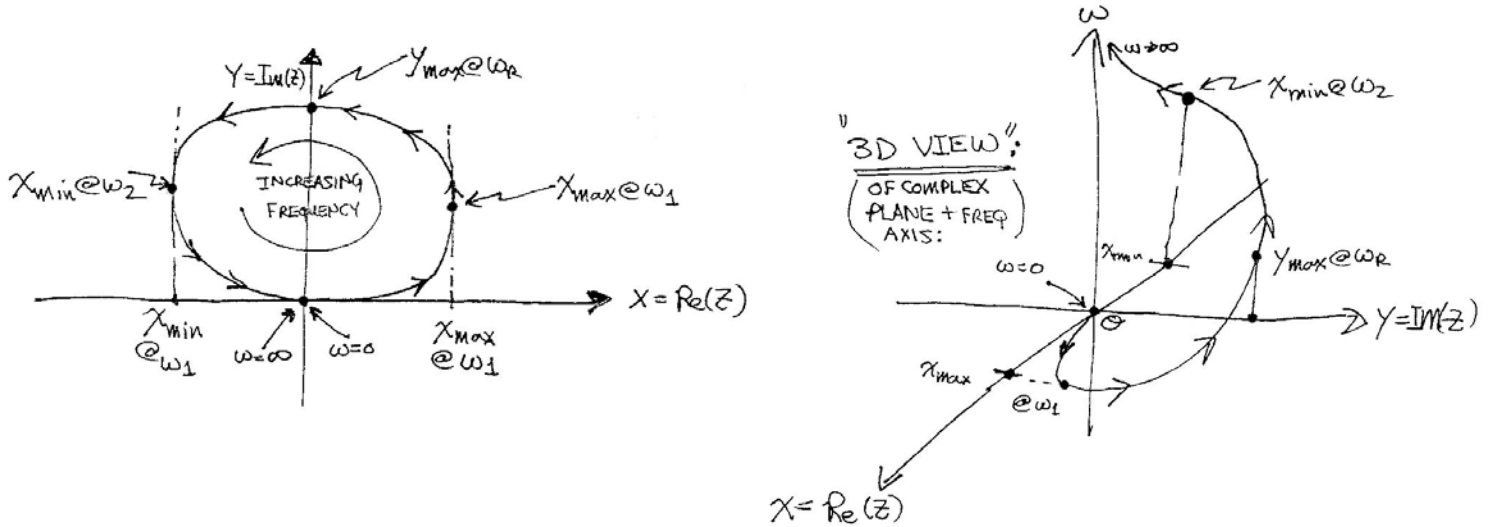


*n.b.* in some complex systems *e.g.* the resonance of a LCR circuit,  $\text{Re}\{\tilde{z}\}$  &  $\text{Im}\{\tilde{z}\}$  are interchanged from what is drawn here!

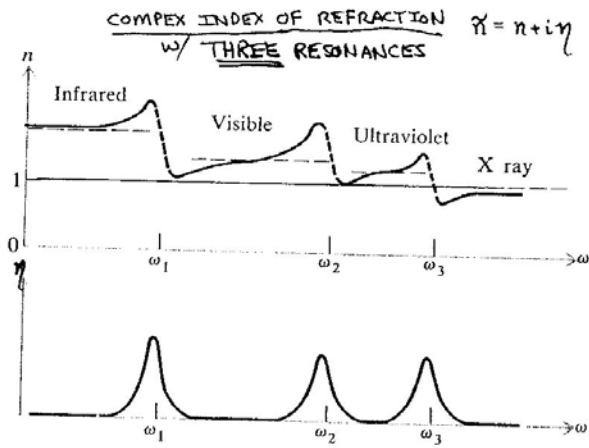
*i.e.*  $\text{Re}\{\tilde{z}\} \rightleftharpoons \text{Im}\{\tilde{z}\}$

Note that the **shape** of the curve for the **magnitude** of  $\tilde{z}$ ,  $|\tilde{z}(\omega)| = \sqrt{\tilde{z}(\omega) \cdot \tilde{z}^*(\omega)} = \sqrt{x^2(\omega) + y^2(\omega)}$ , is very similar to shape of the  $\Im m\{z(\omega)\}$  curve {as shown **here**}.

The **trajectory** of  $\tilde{z}(\omega)$  in the complex plane:



A more realistic microscopic picture of an atomic system – with many electrons with many quantum bound states  $\Rightarrow \exists$  **many** resonances in a dispersive, linear macroscopic dielectric!!!



Index of refraction and extinction coefficient for a hypothetical substance with absorption bands in the infrared, visible, and ultraviolet regions of the spectrum.

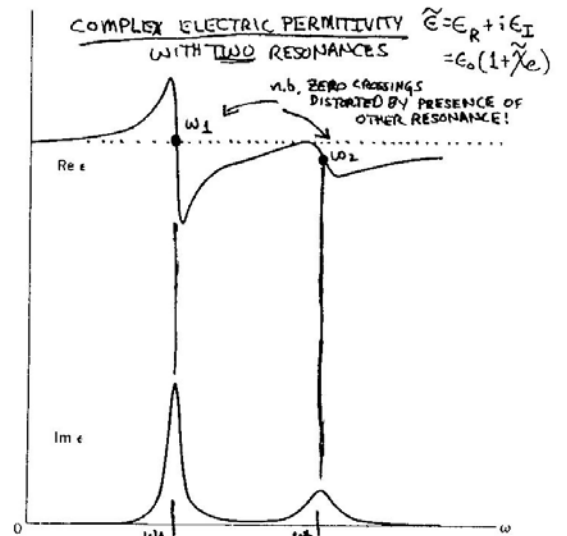


Figure 7.8 Real and imaginary parts of the dielectric constant  $\epsilon(\omega)/\epsilon_0$  in the neighborhood of two resonances. The region of anomalous dispersion is also the frequency interval where absorption occurs.

**Exercise(s):** Draw out the corresponding **trajectories** of complex  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$  and  $\tilde{\epsilon}(\omega) = \epsilon(\omega) + i\xi(\omega)$  for the above triple-resonance cases in the complex plane!

In the high-frequency region, above the highest resonant frequency (typically in UV or x-ray region), the index of refraction is predicted to be  $n(\omega) < 1.0$  (i.e. actually less than that of the vacuum). Indeed, this phenomenon has explicitly been observed/measured e.g. in quartz (SiO<sub>2</sub>) using x-rays:

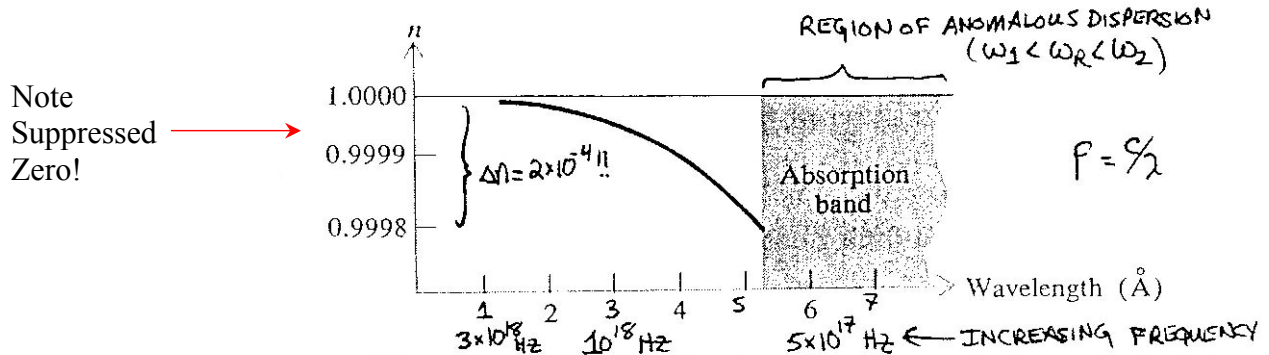


Figure 5.3. Measured index of refraction of quartz in the x-ray region.

Note that physically the damping constant  $\gamma_j =$  width of the  $j^{\text{th}}$  resonance is inversely related to the lifetime  $\tau_j$  associated with the corresponding excited state of the constituent atoms/molecules of the dispersive, linear dielectric, since at the microscopic level, the {real} photons associated with the monochromatic plane EM wave have energy  $E_\gamma = hf$  and {assuming the atoms/molecules of the dispersive, linear dielectric to all be in their ground state, with ground state energy  $E_1$ }, then if the monochromatic plane EM wave has {angular} frequency  $\omega = \omega_R = 2\pi f_R =$  the resonance frequency of the bound atomic electrons, then we see that  $\Delta E_{j1} = E_j - E_1 = E_\gamma = hf_R = \hbar\omega_R$  at resonance!

At a resonance, e.g. when  $\omega = \omega_{1j}$ , the {real} photons in the monochromatic plane EM wave easily stimulate the atomic electrons, causing them to resonate – the {real} photons are absorbed, enabling the atomic electron to make a transition from the ground state {with energy  $E_1$ } to the  $j^{\text{th}}$  excited state {with energy  $E_j$ } via an electric dipole transition, if so allowed by quantum-mechanical selection rules. The  $j^{\text{th}}$  excited atomic state has {mean} lifetime  $\tau_j$  associated with it, thus the atomic electron de-excites back to the ground state by emitting a {real} photon of this same frequency. The miracle of all of this is that {real} photons associated with the EM field are actually interacting simultaneously with all of the atoms in the dispersive linear dielectric (within the coherence length of the photon  $\xi_c^\gamma$ ) at any given instant in time, thus the resultant “scattered” photon that is {ultimately} emitted, actually must be summed over the response of the ensemble of many atoms – the miraculous result of which is forward scattering of the photons associated with the macroscopic EM wave, but with a {frequency-dependent} phase shift, which is related to the resonance lineshape {and the finite lifetime  $\tau_j$ } of the excited state of the atom!

At a resonance, *e.g.* when  $\omega = \omega_{1k}$ , a large, transitory/transient {complex and frequency-dependent} electric dipole moment  $\tilde{\vec{p}}(\vec{r}, \omega) = -e\tilde{\vec{r}}(\omega) = -e\tilde{r}(\omega)\hat{r}$  is induced in the atom, where:

$$\tilde{\vec{r}}(\omega) = r_o \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n f_j^{osc} \frac{[(\omega_{1j}^2 - \omega^2) + i\gamma_j \omega]}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right] \equiv r(\omega) + i\rho(\omega)$$

Note here we can also make a direct connection with quantum mechanics – the electric dipole moment **operator**  $\tilde{\vec{p}}(\vec{r}, \omega) = -e\tilde{\vec{r}}(\omega)$  and position **operator**  $\tilde{\vec{r}}(\omega)$  operating *e.g.* on the **ground state** wave function of the atom/molecule  $|\psi_1(\vec{r})\rangle$ , *i.e.*  $\tilde{\vec{p}}(\vec{r}, \omega)|\psi_1(\vec{r})\rangle$  and  $\tilde{\vec{r}}(\omega)|\psi_1(\vec{r})\rangle$ .

We can *e.g.* compute the **expectation value** of the **modulus squared** of the electric dipole moment  $\langle \psi_1(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_1(\vec{r}) \rangle$  of the atom/molecule. Inserting a **complete set of states**  $\sum_{j=1}^{\infty} \langle \psi_j(\vec{r}) | \psi_j(\vec{r}) \rangle = \sum_{j=1}^{\infty} |\psi_j(\vec{r})\rangle \langle \psi_j(\vec{r})| = 1$  into this expression, we can then obtain the quantum mechanical predictions for the {squares} the oscillator strengths  $f_j^{osc}$ :

$$\sum_{j=1}^{\infty} \langle \psi_1(\vec{r}) | \tilde{\vec{p}}^*(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle \langle \psi_j(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_1(\vec{r}) \rangle = \sum_{j=1}^{\infty} \left| \langle \psi_1(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle \right|^2$$

The transition rate  $\Gamma_{\vec{1}j}$  (= # atoms/molecules per second) from the ground state to the  $j^{th}$  excited state {via an **electric dipole transition**, as allowed by quantum mechanical **selection rules**} is proportional to  $\langle \psi_j(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_1(\vec{r}) \rangle$ , whereas the transition rate  $\Gamma_{j\vec{1}}$  (= # atoms/ molecules per second) from the  $j^{th}$  excited state to the ground state {via an **electric dipole transition**, as allowed by quantum mechanical **selection rules**} is proportional to  $\langle \psi_1(\vec{r}) | \tilde{\vec{p}}^*(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle$ .

Note that by the {microscopic} manifest **time-reversal invariance** of the electromagnetic interaction, the transition rates are **identical**, *i.e.*  $\Gamma_{\vec{1}j} \equiv \Gamma_{j\vec{1}} = \gamma_j / 2\pi$  = “damping constant” in our semi-classical model!

Note further that the **lifetimes**  $\tau_j$  of the excited states of atoms are {inversely} related to the **widths**  $\gamma_j$  of the  $j^{th}$  resonances/**widths** of the  $j^{th}$  excited states by the **Heisenberg uncertainty principle**:  $[\Delta E \Delta t \geq \hbar]$ , where  $\hbar \equiv h/2\pi$  and  $h$  = Planck’s constant. If we set this relation to its minimum, *i.e.*  $\Delta E \Delta t = \hbar$  then:

$$(\hbar \gamma_j) * \tau_j = \hbar \Rightarrow \hbar \gamma_j = \hbar / \tau_j \quad \text{or:} \quad 1/\tau_j = \Gamma_{\vec{1}j} \equiv \Gamma_{j\vec{1}} = \gamma_j / 2\pi$$

If one stays **well away/far** from **{all}** of the resonance frequencies of bound-state atomic electrons, the resonance factor becomes:

$$\frac{\left[ (\omega_{1j}^2 - \omega^2) + i\cancel{\gamma_j\omega} \right]}{\left[ (\omega_{1j}^2 - \omega^2)^2 + \cancel{\gamma_j^2\omega^2} \right]} \approx \frac{1}{(\omega_{1j}^2 - \omega^2)} \quad \text{i.e. **far** from a resonance: } \boxed{(\omega_{1j}^2 - \omega^2) \gg \gamma_j\omega}$$

Thus, **far** from a resonance / all resonances, relatively little absorption/dissipation occurs {i.e.  $\eta^2(\omega) \ll n^2(\omega)$ , such that  $\tilde{n}(\omega) = n(\omega) + i\tilde{\eta}(\omega) \simeq n(\omega)$  is predominantly **real**} and hence:

$$n^2(\omega) - \eta^2(\omega) \simeq n^2(\omega) \simeq 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2 - \omega^2} \right]$$

Now: 
$$\frac{1}{\omega_{1j}^2 - \omega^2} = \frac{1/\omega_{1j}^2}{\left[ 1 - (\omega^2/\omega_{1j}^2) \right]} \approx \frac{1}{\omega_{1j}^2} \left( 1 + \frac{\omega^2}{\omega_{1j}^2} \right) = \frac{1}{\omega_{1j}^2} + \frac{\omega^2}{\omega_{1j}^4}$$

Then: 
$$n^2(\omega) - \eta^2(\omega) \simeq n^2(\omega) \simeq 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_{1j}^2} \right) + \omega^2 \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_{1j}^4} \right) \right]$$

If  $\boxed{n^2 = 1 + \epsilon}$  and  $\boxed{\epsilon \ll 1} \Rightarrow \boxed{n = \sqrt{1 + \epsilon} \simeq 1 + \frac{1}{2}\epsilon}$

Thus, **far** from a resonance/resonances: 
$$n(\omega) \simeq 1 + \frac{1}{2} \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^2} \right) + \omega^2 \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^4} \right) \right]$$

But:  $\lambda_o = \frac{2\pi}{k_o} = \frac{2\pi c}{\omega}$  = **vacuum** wavelength, hence:  $\omega = \frac{2\pi c}{\lambda_o}$ , thus:

$$n(\lambda_o) \simeq 1 + \frac{1}{2} \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^2} \right) + \left( \frac{2\pi c}{\lambda_o} \right)^2 \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^4} \right) \right]$$

$\Rightarrow$  we obtain **Cauchy's Formula**: 
$$n(\lambda_o) \simeq 1 + A \left( 1 + \frac{B}{\lambda_o^2} \right)$$

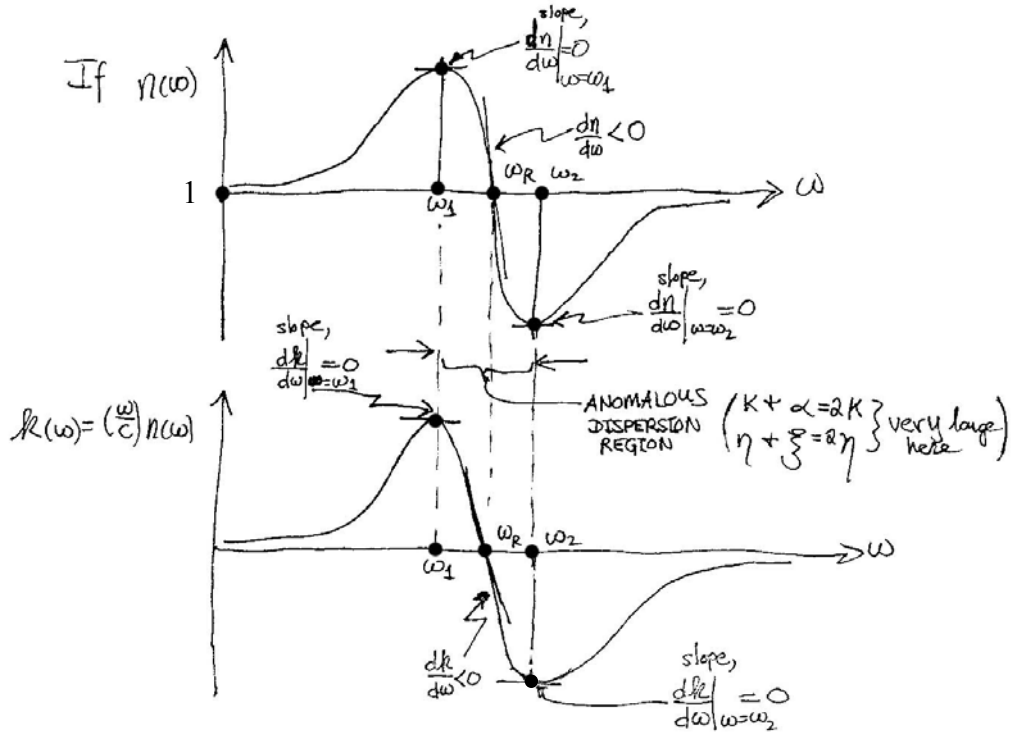
Where:  $A$  = Coefficient of **Refraction** and:  $B$  = Coefficient of **Dispersion**.

Comparing the 2 equations, we see that:

$$A = \frac{1}{2} \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^2} \right) \right] \quad \text{and:} \quad B = (2\pi c)^2 \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^4} \right) \right] / \left[ \sum_{j=1}^n \left( \frac{f_j^{osc}}{\omega_j^2} \right) \right]$$

Since:  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$  and/or:  $\tilde{k}(\omega) = k(\omega) + ik(\omega)$  and:  $\tilde{k}(\omega) = \left(\frac{\omega}{c}\right)\tilde{n}(\omega)$

Then:  $k(\omega) = \left(\frac{\omega}{c}\right)n(\omega)$  and:  $\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega)$ , thus:



**The phase speed:**  $v_\phi(\omega) = \frac{\omega}{k} = \frac{c}{n(\omega)} > c$  if  $n(\omega) < 1$

**The group speed:**  $v_g(\omega) = \left(\frac{d\omega}{dk}\right) = \left[\frac{dk(\omega)}{d\omega}\right]^{-1}$

Note that at the “turning points” of either the  $\{n(\omega) \text{ vs. } \omega\}$  or:  $\{k(\omega) \text{ vs. } \omega\}$  graphs, *i.e.* at  $\{\text{angular}\}$  frequencies  $\omega = \omega_1$  and/or  $\omega = \omega_2$  where the slope  $dk(\omega)/d\omega = 0 \Rightarrow v_g = \infty$  !!!

Note further that in the  $\{\text{angular}\}$  frequency region  $\omega_1 < \omega < \omega_2$  {the “**anomalous dispersion**” region}, since the slope  $dk(\omega)/d\omega < 0$  then the **group speed**  $v_g(\omega) = 1/(dk(\omega)/d\omega) < 0$  !!!  
{Hence the name **anomalous** dispersion...}

This phenomenon **has** been experimentally verified (see *e.g.* C.G.B. Garrett & D.E. McCumber, Phys. Rev. **A**, 1, p. 305 (1970)). If the dispersive medium is not **too** thick, a Gaussian pulse with a central frequency **near an absorption line** (*i.e.* **near a resonance**,  $\omega_R$ ) and with pulse width  $\Delta t \gg \tau_R = 1/\gamma_R$  propagates with **appreciable absorption**, but (more or less) retains its **shape**.

The **peak** of the Gaussian pulse propagates at  $v_g$  **even** when the group speed is **negative**!!!

$\Rightarrow$  Useful for **pulse re-shaping** applications - **leading** edge is **less** attenuated than **trailing** edge.

- ⇒ Can actually have the **peak** of a **greatly attenuated pulse** emerge **from** the absorber **before** the **peak** of the **incident** pulse **enters** the absorber (≡ definition of **negative group speed**)!!!  
 {i.e. microscopically, if the absorber is not **too** thick, then some photons **can** make it all the way through the absorber w/o interacting at all – this probability is **exponentially** suppressed.
- ⇒ Has applications/uses e.g. in **optical mammography**/breast cancer screening for women...}
- ⇒ See e.g. J.D. Jackson's **Electrodynamics**, 3<sup>rd</sup> Edition, pages 325-26 for more details!

Finally, if we set  $\omega = 0$ , then we obtain the **static** (i.e. zero-frequency) limit of {all of} these quantities. Note that they also {all} become **purely real** in this limit:

**Static Polarization:** 
$$\vec{P}(0) = \left( \frac{n_e^b e^2}{m} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} \quad \text{where} \quad \omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \left( \frac{n_e^b e^2}{3\epsilon_o m_e} \right)}$$
 and since  $\vec{P} = \epsilon_o \chi_e \vec{E}_{ext}$

**Static Electricity Susceptibility:** 
$$\chi_e(0) = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} \quad \text{and} \quad \omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$$

**Static Index of Refraction:** 
$$n(0) = \sqrt{1 + \alpha_x(0)} = \sqrt{1 + \chi_e(0)} = \sqrt{K_e(0)}$$

But: 
$$K_e(\omega) = \epsilon(\omega) / \epsilon_o = (1 + \chi_e(\omega)) \Rightarrow \epsilon(0) = \epsilon_o (1 + \chi_e(0))$$
 and thus:

**Static Dielectric Constant:** 
$$K_e(0) = \epsilon(0) / \epsilon_o = (1 + \chi_e(0)) = 1 + \alpha_x(0) = n^2(0)$$

But: 
$$\alpha_x(0) = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} = \chi_e(0)$$

∴ 
$$K_e(0) = 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} = n^2(0) \quad \text{and:} \quad \chi_e(0) = K_e(0) - 1 = \left( \frac{n_e^b e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2}$$

⇒ Note that the **static** dielectric constant {as measured at  $f = 0$  Hz/DC} is  $K_e(0) > 1.0$  **because** it contains information about **all** of the {quantum mechanical} resonances/excited states  $\omega_{1j} > 0$  present in the dispersive, linear medium, even into the x-ray region at  $\omega_{1j} \approx 10^{18-19}$  Hz and beyond !!!

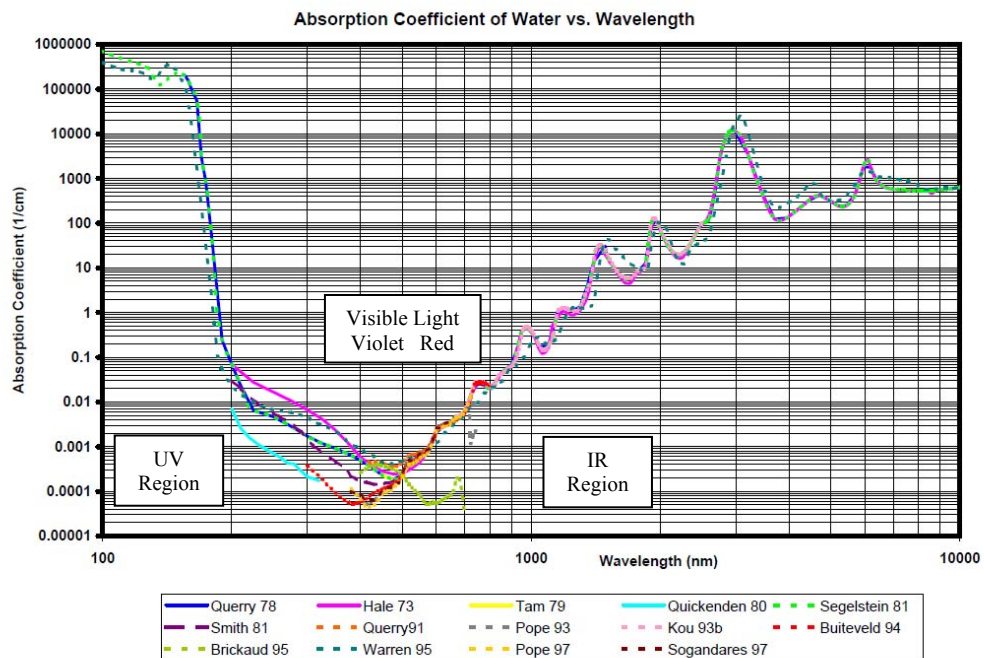
⇒ Equivalently, armed now with this knowledge of the microscopic behavior of a dispersive, linear medium, an electric susceptibility  $\chi_e(0) > 0$  {or equivalently, a dielectric constant  $K_e(0) > 1$ } **instantly** tells us that there are indeed {quantum mechanical} resonances/excited states present in the {composite} atoms/molecules that make up the **macroscopic** material of the dispersive, linear medium!!!



⇒ A **wonderful** macroscopic example of **dispersion** in nature is the **rainbow**. At the **microscopic** level, the frequency-dependence of the index of refraction of light  $n(\omega)$  arises as a consequence of the resonant behavior of quantum mechanical bound states of electrons in the atoms of the water molecule ( $H_2O$ ) responding to *EM* light waves {= visible light photons} coming from our sun.  
 ⇒ If **no** such composite behavior existed at the **microscopic** level, there would be **no** rainbows to enjoy in the **macroscopic** everyday world!



**Experimental measurements of the absorption coefficient of pure water:**



Absorption coefficient of **pure** water -  $\alpha_{H_2O}(\omega)$  is large in UV and IR regions due to **resonances!**