LECTURE NOTES 7

EM WAVE PROPAGATION IN CONDUCTORS

Inside a conductor, free charges can move/migrate around in response to *EM* fields contained therein, as we saw for the case of the longitudinal $\vec{E}$-field inside a current-carrying wire that had a static potential difference ΔV across its ends. Even in the static case of electric charge residing on the surface of a conductor, we saw that $\vec{E}_{inside}(\vec{r})=0$, but recall that this actually means (as we showed last semester) that the ***NET*** electric field inside the conductor is zero, *i.e.* $\vec{E}_{inside}^{NET}(\vec{r})=0$.

n.b. here, we assume {for simplicity's sake} that the conductor is linear/homogeneous/isotropic – *i.e.* no crystalline structure/no anisotropies/no inhomogenities/voids/defects…

From Ohm's Law, we know that the ***free*** current density $\vec{J}_{free}(\vec{r},t)$ is proportional to the (ambient) electric field inside the conductor: $\boxed{\vec{J}_{free}(\vec{r},t)=\sigma_C\vec{E}(\vec{r},t)}$ where:

$\sigma_C=$ ***conductivity*** of the metal conductor $\left(Siemens/m=Ohm^{-1}/m\right)$ and $\boxed{\sigma_C=1/\rho_C}$

$\rho_C=$ ***resistivity*** of the metal conductor $\left(Ohm\text{-}m\right)$.

Thus inside such a conductor, we can assume that the linear/homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ. Maxwell's equations inside such a conductor {with $\vec{J}_{free}(\vec{r},t)\neq 0$} are thus:

Using Ohm's Law: $\vec{J}_{free}(\vec{r},t)=\sigma_C\vec{E}(\vec{r},t)$

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=\rho_{free}(\vec{r},t)/\varepsilon}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\vec{J}_{free}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

Electric charge is (always) conserved, thus the continuity equation inside the conductor is:

$\boxed{\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)=-\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ but: $\boxed{\vec{J}_{free}(\vec{r},t)=\sigma_C\vec{E}(\vec{r},t)}$

$\therefore$ $\boxed{\sigma_C\left(\vec{\nabla}\cdot\vec{E}(\vec{r},t)\right)=-\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ but: $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=\dfrac{1}{\varepsilon}\rho_{free}(\vec{r},t)}$

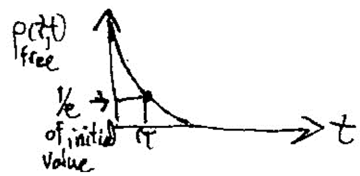
thus: $\boxed{\dfrac{\sigma_C\rho_{free}(\vec{r},t)}{\varepsilon}=-\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ or: $\boxed{\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}+\left(\dfrac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t)=0}$ $\Leftarrow$ 1st order linear, homogeneous differential equation

The general solution of this differential equation for the free charge density is of the form:

$$\boxed{\rho_{free}(\vec{r},t)=\rho_{free}(\vec{r},t=0)e^{-\sigma_C t/\varepsilon}=\rho_{free}(\vec{r},t=0)e^{-t/\tau_{relax}}}$$ *i.e.* a ***damped*** exponential !!!

Characteristic ***damping*** time: $\boxed{\tau_{relax}\equiv\varepsilon/\sigma_C}$ = ***charge relaxation time*** {*aka* ***time constant***}.

Thus, the continuity equation $\vec{\nabla}\bullet\vec{J}_{free}(\vec{r},t) = -\partial\rho_{free}(\vec{r},t)/\partial t$ inside a conductor tells us that any free charge density $\rho_{free}(\vec{r},t=0)$ initially present at time $t = 0$ is ***exponentially*** damped / dissipated in a characteristic time $\boxed{\tau_{relax} \equiv \varepsilon/\sigma_C}$ = ***charge relaxation time*** {*aka* ***time constant***}, such that at when: $t = \tau_{relax} \equiv \varepsilon/\sigma_C$: $\boxed{\rho_{free}(\vec{r},t=\tau_{relax}) = \rho_{free}(\vec{r},t=0)e^{-1} = 0.369\cdot\rho_{free}(\vec{r},t=0)}$



Calculation of the Charge Relaxation Time for Pure Copper:

$$\boxed{\rho_{Cu} = 1/\sigma_{Cu} = 1.68\times10^{-8}\,\Omega\text{-m}} \Rightarrow \boxed{\sigma_{Cu} = 1/\rho_{Cu} = 5.95\times10^{7}\text{ Siemens/m}}$$

If we assume $\varepsilon_{Cu} \approx 3\varepsilon_o = 3\times8.85\times10^{-8}$ F/m for copper metal, then:

$$\boxed{\tau_{Cu}^{relax} = \varepsilon_{Cu}/\sigma_{Cu} = \rho_{Cu}\,\varepsilon_{Cu} = 4.5\times10^{-19}\text{ sec}} \text{ !!!}$$

However, the characteristic (*aka* mean} ***collision time*** of free electrons in pure copper is $\tau_{Cu}^{coll} \simeq \lambda_{Cu}^{coll}/v_{thermal}^{Cu}$ where $\lambda_{Cu}^{coll} \simeq 3.9\times10^{-8}\,m$ = mean free path (between successive collisions) in pure copper, and $v_{thermal}^{Cu} \simeq \sqrt{3k_BT/m_e} \simeq 12\times10^5\,m/sec$ and thus we obtain: $\tau_{coll}^{Cu} \simeq 3.2\times10^{-13}$ sec.

Hence we see that the calculated charge relaxation time in pure copper, $\tau_{Cu}^{relax} \simeq 4.5\times10^{-19}$ sec is $\ll$ than the calculated collision time in pure copper, $\tau_{coll}^{Cu} \simeq 3.2\times10^{-13}$ sec.

Furthermore, the ***experimentally measured*** charge relaxation time in pure copper is $\tau_{Cu}^{relax}(\text{exp't}) \simeq 4.0\times10^{-14}$ sec, which is ≈ 5 orders of magnitude ***larger*** than the ***calculated*** charge relaxation time $\tau_{Cu}^{relax} \simeq 4.5\times10^{-19}$ sec. The problem here is that {the ***macroscopic***} Ohm's Law is simply out of its range of validity on such short time scales! Two ***additional*** facts here are that ***both*** ε and σ_C are ***frequency-dependent*** quantities { *i.e.* $\varepsilon = \varepsilon(\omega)$ and $\sigma_C = \sigma_C(\omega)$ }, which becomes ***increasingly*** important at the higher frequencies $(f = 2\pi/\omega \sim 1/\tau_{relax})$ associated with short time-scale, transient-type phenomena!

So in reality, if we are willing to wait a short time (*e.g.* $\Delta t \sim 1\text{ ps} = 10^{-12}$ sec) then, any initial free charge density $\rho_{free}(\vec{r},t=0)$ accumulated inside a ***good*** conductor at $t = 0$ will have dissipated away/damped out, and from that time onwards, $\rho_{free}(\vec{r},t) = 0$ ***can*** be safely assumed.

Note: For a ***poor*** conductor $(\sigma_C \to 0)$, then: $\tau_{relax} \equiv \varepsilon/\sigma_C \to \infty$!!! Please keep this in mind…

After ***many*** charge relaxation time constants, *e.g.* $20\tau_{relax} \le \Delta t \simeq 1\ ps = 10^{-12}\,sec$, Maxwell's ***steady-state*** equations for a ***good*** conductor become {with $\rho_{free}(\vec{r}, t \ge \Delta t) = 0$ from then onwards}:

Maxwell's equations for a ***charge-equilibrated*** conductor

1) $\vec{\nabla} \bullet \vec{E}(\vec{r},t) = 0$ 2) $\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$ 4) $\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu\sigma_C \vec{E}(\vec{r},t) + \mu\varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t} = \mu\left(\sigma_C \vec{E}(\vec{r},t) + \varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}\right)$

These equations are different from the previous derivation(s) of monochromatic plane *EM* waves propagating in free space/vacuum and/or in linear/homogeneous/isotropic non-conducting materials {*n.b.* only equation 4) has changed}, hence we re-derive {***steady-state***} wave equations for $\vec{E}$ & $\vec{B}$ from scratch. As before, we apply $\vec{\nabla} \times (\ )$ to equations 3) and 4):

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$
$$= \vec{\nabla}(\cancel{\vec{\nabla} \bullet \vec{E}}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma_C \vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$
$$= \nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\left(\sigma_C(\vec{\nabla} \times \vec{E})\right) + \varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$
$$= \vec{\nabla}(\cancel{\vec{\nabla} \bullet \vec{B}}) - \nabla^2 \vec{B} = -\mu\sigma_C \frac{\partial \vec{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$
$$= \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{B}}{\partial t}$$

Again: $\nabla^2 \vec{E}(\vec{r},t) = \mu\varepsilon \dfrac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}$ and: $\nabla^2 \vec{B}(\vec{r},t) = \mu\varepsilon \dfrac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

Note that the {***steady-state***} 3-D wave equations for $\vec{E}$ and $\vec{B}$ in a conductor have an additional term that has a single time derivative – which is analogous *e.g.* to a velocity-dependent ***damping term*** associated with the motion of a 1-D mechanical harmonic oscillator.

The general solution(s) to the above {***steady-state***} wave equations are usually in the form of an oscillatory function × a damping term (*i.e.* a decaying exponential) – in the direction of the propagation of the *EM* wave, complex plane-wave type solutions for $\vec{E}$ and $\vec{B}$ associated with the above wave equation(s) are of the general form:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)} \quad \text{and:} \quad \tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \frac{1}{\tilde{v}}\hat{k} \times \tilde{\vec{E}}(\vec{r},t) = \left(\frac{\tilde{k}}{\omega}\right)\hat{k} \times \tilde{\vec{E}}(\vec{r},t)$$

n.b. with {frequency-dependent} ***complex*** wave number: $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$

where: $k(\omega) = \Re e\{\tilde{k}(\omega)\}$ and $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$ and corresponding ***complex*** wave vector $\tilde{\vec{k}}(\omega) = \tilde{k}(\omega)\hat{k} = \tilde{k}(\omega)\hat{z}$ (for *EM* wave propagating in the $\hat{k} = +\hat{z}$ direction, ***here***).

Physically, $k(\omega) = \Re e\{\tilde{k}(\omega)\}$ is associated with wave ***propagation***, and $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$ is associated with wave ***attenuation*** (*i.e.* ***dissipation***).

We plug $\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}$ and $\tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z-\omega t)}$ into their respective wave equations above, and obtain from each wave equation the same/identical ***characteristic*** ***equation*** – {*aka* a ***dispersion*** ***relation***} between complex $\tilde{k}(\omega)$ and ω {please work this out yourselves!}:

$$\boxed{\tilde{k}^2(\omega) = \mu\varepsilon\omega^2 + i\mu\sigma_C\omega}$$

Thus, since $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$, then:

$$\boxed{\tilde{k}^2(\omega) = \left(k(\omega)+i\kappa(\omega)\right)^2 = k^2(\omega) - \kappa^2(\omega) + 2ik(\omega)\kappa(\omega) = \mu\varepsilon\omega^2 + i\mu\sigma_C\omega}$$

If we {temporarily} suppress the ω-dependence of complex $\tilde{k}(\omega)$, this relation becomes:

$$\boxed{\tilde{k}^2 = (k+i\kappa)^2 = k^2 - \kappa^2 + 2ik\kappa = \mu\varepsilon\omega^2 + i\mu\sigma_C\omega}$$

We can re-write this expression as: $\boxed{\left[\left(k^2-\kappa^2\right)-\mu\varepsilon\omega^2\right] + i\left[2k\kappa - \mu\sigma_C\omega\right] = 0}$, which ***must*** be true for ***any/all*** values of {any of} the parameters involved. The only in-general way that this relation can hold is if ***both*** $\left[\left(k^2-\kappa^2\right)-\mu\varepsilon\omega^2\right] = 0$ ***.and.*** $\left[2k\kappa-\mu\sigma_C\omega\right] = 0$. Then:

$$\boxed{k^2 - \kappa^2 = \mu\varepsilon\omega^2} \quad \text{and:} \quad \boxed{2k\kappa = \mu\sigma_C\omega}$$

Thus, we have ***two*** separate/independent equations: $k^2 - \kappa^2 = \mu\varepsilon\omega^2$ and: $2k\kappa = \mu\sigma_C\omega$. We have ***two*** unknowns: k and κ. Hence, we solve these equations ***simultaneously*** to determine k and κ!

From the ***latter*** relation, we see that: $\boxed{\kappa = \tfrac{1}{2}\mu\sigma_C\omega/k}$. Plug ***this*** result into the ***other*** relation:

$$\boxed{k^2 - \kappa^2 = k^2 - \left(\tfrac{1}{2}\mu\sigma_C\omega/k\right)^2 = k^2 - \frac{1}{k^2}\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2 = \mu\varepsilon\omega^2}$$

Then multiply by k^2 and rearrange the terms to obtain the following relation:

$$\boxed{k^4 - \left(\mu\varepsilon\omega^2\right)k^2 - \left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2 = 0}$$

This may ***look*** like a scary equation to try to solve (*i.e.* a ***quartic*** equation - *eeekkk*!), but it's actually just a ***quadratic*** equation! {So, it's really just a ***leprechaun***, masquerading as a ***unicorn***!}

Define: $x \equiv k^2$, $a \equiv 1$, $b \equiv -\left(\mu\varepsilon\omega^2\right)$ and $c \equiv -\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2$, this equation then becomes "the usual" quadratic equation, of the form: $ax^2 + bx + c = 0$, with solution(s)/root(s):

$$\boxed{x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}} \quad \text{or:} \quad \boxed{k^2 = \frac{1}{2}\left[+\left(\mu\varepsilon\omega^2\right) \mp \sqrt{\left(\mu\varepsilon\omega^2\right)^2 + 4\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2}\right]}$$

This relation can be re-written as:

$$k^2 = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1 \mp \sqrt{1 + \cancel{4}\frac{\left(\cancel{\mu^2}\sigma_C^2\cancel{\omega^2}\right)}{\cancel{4}\left(\cancel{\mu^2}\varepsilon^2\omega^{\cancel{4}2}\right)}}\right] = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1 \mp \sqrt{1 + \frac{\left(\sigma_C^2\right)}{\left(\varepsilon^2\omega^2\right)}}\right] = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1 \mp \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]$$

On ***physical*** grounds ($k^2 > 0$), we ***must*** select the + sign, hence:

$$k^2 = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]$$ and thus: $$k = \sqrt{k^2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$

Having thus solved for k (or equivalently, k^2), we can use ***either*** of our original ***two*** relations to solve for κ, *e.g.* $k^2 - \kappa^2 = \mu\varepsilon\omega^2$, thus:

$$\kappa^2 = k^2 - \mu\varepsilon\omega^2 = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right] - \mu\varepsilon\omega^2 = \frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]$$

Hence {finally}, we obtain:

$$k(\omega) = \Re e\left\{\tilde{k}(\omega)\right\} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$ and: $$\kappa(\omega) = \Im m\left\{\tilde{k}(\omega)\right\} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

The above two relations ***clearly*** show the frequency dependence of ***both*** the ***real*** and ***imaginary*** components of the complex wavenumber $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$. This physically means that *EM* wave propagation in a conductor is ***dispersive*** (*i.e.* *EM* wave propagation is ***frequency dependent***).

Note also that the ***imaginary*** part of $\tilde{k}(\omega)$, $\kappa(\omega) = \Im m\left\{\tilde{k}(\omega)\right\}$ results in an ***exponential attenuation/damping*** of the monochromatic plane *EM* wave with increasing *z*:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}$$ where: $$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

and: $$\tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz - \omega t)} = \frac{\tilde{k}}{\omega}\hat{k} \times \tilde{\vec{E}}(z,t) = \frac{\tilde{k}}{\omega}\hat{k} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}$$

The ***characteristic distance*** *z* over which $\vec{E}$ and $\vec{B}$ are attenuated/reduced to $1/e = e^{-1} = 0.368$ of their initial values (at *z* = 0) is known as the ***skin depth***, $\delta_{sc}(\omega) \equiv 1/\kappa(\omega)$ (<u>SI units</u>: meters).

i.e. $$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}} \Rightarrow \begin{array}{l} \tilde{\vec{E}}(z = \delta_{sc}, t) = \tilde{\vec{E}}_o e^{-1} e^{i(kz - \omega t)} \\ \tilde{\vec{B}}(z = \delta_{sc}, t) = \tilde{\vec{B}}_o e^{-1} e^{i(kz - \omega t)} \end{array}$$

The ***real*** part of $\tilde{k}(\omega)$, *i.e.* $k(\omega)=\Re e\{\tilde{k}(\omega)\}$ determines the ***spatial*** wavelength $\lambda(\omega)$, the ***phase*** speed $v_\phi(\omega)$ and also the ***group*** speed $v_g(\omega)$ of the monochromatic *EM* plane wave in the conductor:

$$\lambda(\omega)=\frac{2\pi}{k(\omega)}=\frac{2\pi}{\Re e\{\tilde{k}(\omega)\}}$$

$$v_\phi(\omega)\equiv\frac{\omega}{k(\omega)}=\frac{\omega}{\Re e\{\tilde{k}(\omega)\}}$$

$v_\phi(\omega)$ = propagation speed of a ***point*** on waveform that has ***constant phase*** Φ.

Phase $\Phi \equiv (kz-\omega t)$ = ***constant***. A constant phase ***point*** on the waveform moves: $z(t)=\Phi/k+v_\phi t$.

$$v_g(\omega)\equiv\frac{1}{dk(\omega)/d\omega}=\left[\frac{dk(\omega)}{d\omega}\right]^{-1}$$

$v_g(\omega)$ = propagation speed of ***energy / information***.

We will discuss ***phase*** speed $v_\phi(\omega)$ and the ***group*** speed $v_g(\omega)$ more – later…

The above plane wave solutions satisfy the above *EM* wave equations(s) for ***any*** choice of $\tilde{\vec{E}}_o$. As we have also seen before, it can similarly be shown here that Maxwell's equations 1) and 2) $(\vec{\nabla}\bullet\vec{E}=0$ and $\vec{\nabla}\bullet\vec{B}=0)$ rule out the presence of any {***longitudinal***} *z*-components for $\vec{E}$ and $\vec{B}$.

$\Rightarrow$ For *EM* waves propagating in a conductor, $\vec{E}$ and $\vec{B}$ are {still} ***purely transverse***!

If we consider *e.g.* a ***linearly polarized*** monochromatic plane *EM* wave propagating in the $\hat{k}=+\hat{z}$-direction in a conducting medium, *e.g.* $\tilde{\vec{E}}(\vec{r},t)=\tilde{E}_o e^{-\kappa z}e^{i(kz-\omega t)}\hat{x}$, then:

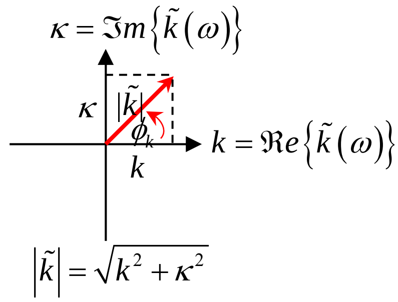
$$\tilde{\vec{B}}(\vec{r},t)=\left(\frac{\tilde{k}}{\omega}\right)\hat{k}\times\tilde{\vec{E}}(\vec{r},t)=\left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z}e^{i(kz-\omega t)}\hat{y}=\left(\frac{k+i\kappa}{\omega}\right)\tilde{E}_o e^{-\kappa z}e^{i(kz-\omega t)}\hat{y}$$

$\Rightarrow\quad \tilde{\vec{E}}(\vec{r},t)\perp\tilde{\vec{B}}(\vec{r},t)\perp(\hat{k}=+\hat{z})$ ($\hat{k}=+\hat{z}$ = propagation direction of *EM* wave, ***here***)

The complex wavenumber: $\tilde{k}(\omega)=k(\omega)+i\kappa(\omega)=|\tilde{k}(\omega)|e^{i\phi_k(\omega)}$

where: $|\tilde{k}(\omega)|=\sqrt{\tilde{k}(\omega)\tilde{k}^*(\omega)}=\sqrt{k^2(\omega)+\kappa^2(\omega)}$ and: $\phi_k(\omega)=\tan^{-1}(\kappa(\omega)/k(\omega))$

In the complex $\tilde{k}$-plane:



Then we see that: $\vec{\tilde{E}}(\vec{r},t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}$ has: $\tilde{E}_o = E_o e^{i\delta_E}$ $\tilde{k} = |\tilde{k}| e^{i\phi_k}$

and that: $\vec{\tilde{B}}(\vec{r},t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y} = \frac{\tilde{k}}{\omega} \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}$ has: $\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega} \tilde{E}_o = \frac{|\tilde{k}| e^{i\phi_k}}{\omega} E_o e^{i\delta_E}$

Thus, we see that: $B_o e^{i\delta_B} = \frac{|\tilde{k}| e^{i\phi_k}}{\omega} E_o e^{i\delta_B} = \frac{|\tilde{k}|}{\omega} E_o e^{i(\delta_E+\phi_k)} = \frac{\sqrt{k^2+\kappa^2}}{\omega} E_o e^{i(\delta_E+\phi_k)}$

i.e., inside a conductor, $\vec{E}$ and $\vec{B}$ are ***no longer in phase with each other***!!!

Phases of $\vec{E}$ and $\vec{B}$: $\delta_B = \delta_E + \phi_k$

With phase ***difference***: $\Delta\varphi_{B-E} \equiv \delta_B - \delta_E = \phi_k$ ⇐ magnetic field ***lags*** behind electric field!!!

We also see that: $\frac{B_o}{E_o} = \frac{|\tilde{k}|}{\omega} = \left[\varepsilon\mu \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega} \right)^2} \right]^{1/2} \neq \frac{1}{c}$

The real/physical $\vec{E}$ and $\vec{B}$ fields associated with linearly polarized monochromatic plane *EM* waves propagating in a conducting medium are ***exponentially*** damped:

$$\vec{E}(\vec{r},t) = \Re e\left\{ \vec{\tilde{E}}(\vec{r},t) \right\} = E_o e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{x} \qquad \delta_B = \delta_E + \phi_k$$

$$\vec{B}(\vec{r},t) = \Re e\left\{ \vec{\tilde{B}}(\vec{r},t) \right\} = B_o e^{-\kappa z} \cos(kz - \omega t + \delta_B) \hat{y} = B_o e^{-\kappa z} \cos\left(kz - \omega t + \{\delta_E + \phi_k\} \right) \hat{y}$$

$\frac{B_o}{E_o} = \frac{|\tilde{k}(\omega)|}{\omega} = \left[\varepsilon\mu \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega} \right)^2} \right]^{\frac{1}{2}}$ where: $\left| \tilde{k}(\omega) \right| = \sqrt{k^2(\omega) + \kappa^2(\omega)} = \omega \left[\varepsilon\mu \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega} \right)^2} \right]^{1/2}$

$\delta_B = \delta_E + \phi_k$, $\phi_k(\omega) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)} \right)$ and: $\vec{\tilde{k}}(\omega) = \left(k(\omega) + i\kappa(\omega) \right) \hat{z}$, $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$

Definition of the ***skin depth*** $\delta_{sc}(\omega)$ in a conductor:

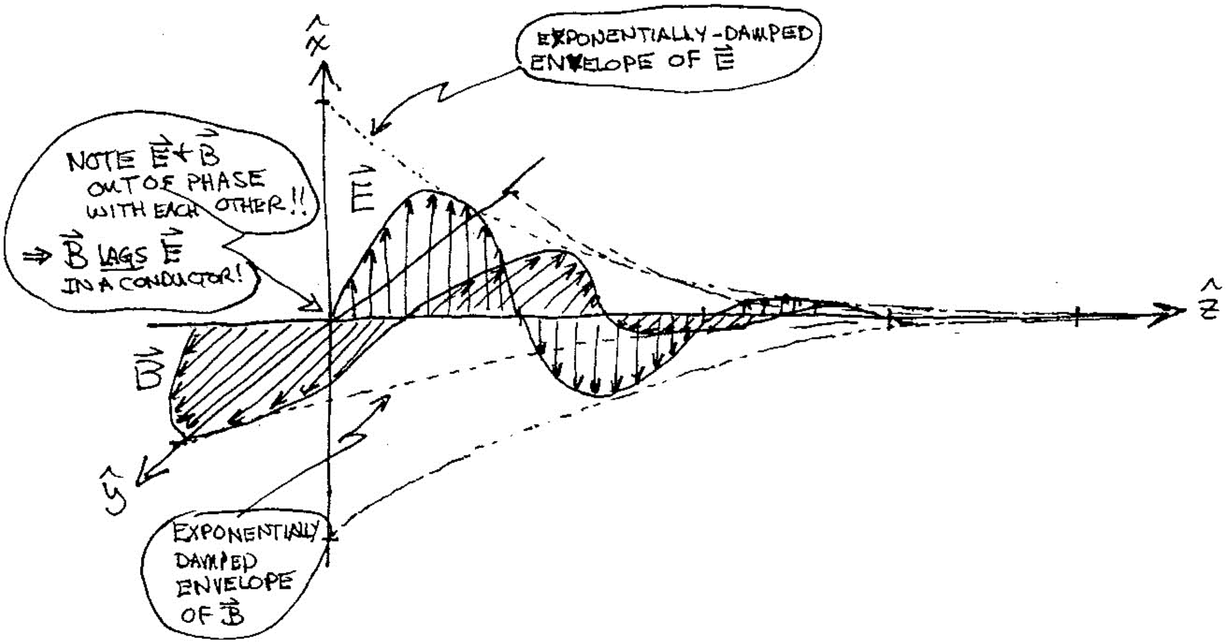
$$\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} = \frac{1}{\omega \sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega} \right)^2} - 1 \right]^{1/2}} =$$

Distance z over which the $\vec{E}$ and $\vec{B}$ fields fall to $1/e = e^{-1} = 0.368$ of their initial values.

The ***instantaneous*** power ***per unit volume*** in the conductor {ultimately dissipated as ***heat***!} is:

$$p(\vec{r},t) = \vec{J}(\vec{r},t) \cdot \vec{E}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t) \cdot \vec{E}(\vec{r},t) = \sigma_C E^2(\vec{r},t) = E_o^2 e^{-2\kappa z} \cos^2(kz - \omega t + \delta_E) \quad \left(Watts/m^3 \right)$$

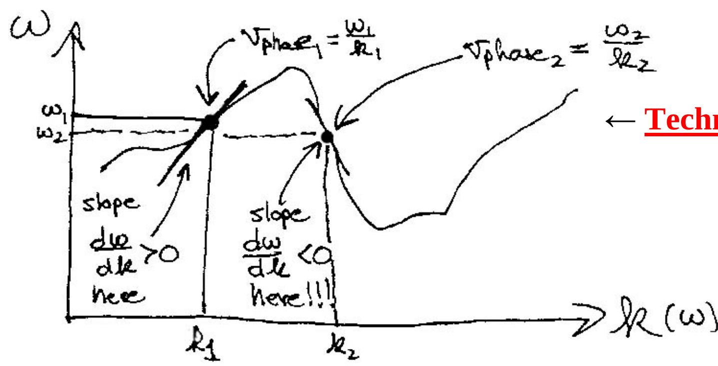
The ***time-averaged*** power ***per unit volume*** in the conductor is: $\left\langle p(\vec{r},t) \right\rangle_t = \tfrac{1}{2} E_o^2 e^{-2\kappa z} \equiv p_o e^{-\alpha z}$



Phase Speed *vs.* Group Speed of a Wave:

The ***phase*** speed = ***numerical value*** of $v_\phi(\omega) \equiv \omega / k(\omega)$ at a ***point*** on the ω *vs.* $k(\omega)$ curve,

The ***group*** speed $v_g(\omega) \equiv {}^{"} d\omega/dk(\omega)^{"}$ = the ***local slope*** at a ***point*** on the ω *vs.* $k(\omega)$ curve:



← **Technically, *making* this plot is wrong !!!**

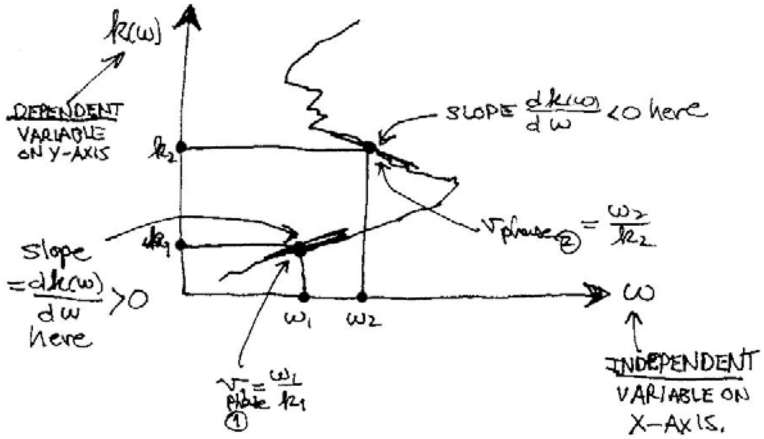
***Why* is this plot technically wrong ???** **Because:** $k(\omega) = fcn(\omega)$

i.e. ω is the ***independent*** variable {***always*** plotted on the ***axis of abscissas*** (*i.e.* the ***x-axis***)}
$k(\omega)$ is the ***dependent*** variable {***always*** plotted on the ***axis of ordinates*** (*i.e.* the ***y-axis***)}

Thus, the ***technically correct*** way ***is*** to plot $k(\omega)$ *vs.* ω: {because k depends on ω, ***not*** *vice-versa*!}

Then: $v_g(\omega) \equiv 1 \Big/ \left(\frac{dk(\omega)}{d\omega}\right) = \left[\frac{dk(\omega)}{d\omega}\right]^{-1}$ *i.e.* $v_g(\omega) \equiv 1/\text{slope of } \{k(\omega) \text{ vs. } \omega\}$ graph

See below:



Another way to think about this issue is to remember that the angular frequency ω and wavenumber $k(\omega)$ are ***Fourier*** ***transforms*** of time *t* and position $z(t)$, respectively. In the ***space-time*** ***domain***, clearly the space position $z(t)$ is the ***dependent*** variable, time *t* is the ***independent*** variable. The Fourier transform of the ***dependent*** variable $z(t)$ is the ***dependent*** variable $k(\omega)$, the Fourier transform of the ***independent*** variable *t* is the ***independent*** variable ω.

Thus ***here***, for the physics associated with propagation of *EM* plane waves in a conductor, with frequency-dependent ***real***-component wavenumber $k(\omega)$:

$$k(\omega)=\Re e\left\{\tilde{k}(\omega)\right\}=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

The ***phase*** speed: $$v_\phi(\omega)\equiv\frac{\omega}{k(\omega)}=\frac{1}{\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

The ***group*** speed: $$v_g(\omega)\equiv\frac{1}{dk(\omega)/d\omega}=\left[\frac{dk(\omega)}{d\omega}\right]^{-1}=\left[\frac{d}{d\omega}\left\{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}\right\}\right]^{-1}$$

So let's work out what the ***group*** ***speed*** $v_g(\omega)$ is for an *EM* plane wave propagating in a conductor. Using the chain rule of differentiation:

$$\frac{dk(\omega)}{d\omega}=\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}+\omega\sqrt{\frac{\varepsilon\mu}{2}}\frac{d}{d\omega}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

$$=\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}+\omega\sqrt{\frac{\varepsilon\mu}{2}}\cdot\frac{\frac{1}{2}\cdot\cancel{\frac{1}{2}}\cdot\cancel{2}\cdot\left(\frac{\sigma_C}{\varepsilon}\right)^2\left(-\frac{1}{\omega^3}\right)}{\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}}$$

$$=\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}-\frac{1}{2}\sqrt{\frac{\varepsilon\mu}{2}}\cdot\frac{\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}{\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}}$$

$$=\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}\cdot\left\{1-\frac{1}{2}\frac{\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}{\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}}\right\}$$

The ***group*** ***speed*** of an *EM* plane wave propagating in a conductor is: (eeek!!!)

$$v_g(\omega)=\left[\frac{dk(\omega)}{d\omega}\right]^{-1}=\frac{1}{\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}\cdot\frac{1}{\left\{1-\frac{1}{2}\frac{\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}{\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}}\right\}}$$

The relation between ***phase speed*** vs. ***group*** ***speed*** of an *EM* plane wave propagating in a conductor is:

$$v_\phi(\omega)\equiv\frac{\omega}{k(\omega)}=\frac{1}{\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

$$v_g(\omega)=v_\phi(\omega)\cdot\frac{1}{\left\{1-\frac{1}{2}\frac{\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}{\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}}\right\}}$$

***EM* Wave Complex Impedance in a Conductor**:

The complex vector impedance associated with an *EM* wave propagating in a conductor is:

$$\tilde{\vec{Z}}(\vec{r},t;\omega) \equiv \tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^{-1}(\vec{r},t;\omega) = \frac{\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^{*}(\vec{r},t;\omega)}{\left|\tilde{\vec{H}}(\vec{r},t;\omega)\right|^2} = \mu\frac{\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{B}}^{*}(\vec{r},t;\omega)}{\left|\tilde{\vec{B}}(\vec{r},t;\omega)\right|^2}$$

If the electric and magnetic fields associated with the *EM* wave propagating in the conductor are:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x} \quad \text{and:} \quad \tilde{\vec{B}}(\vec{r},t) = \left(\frac{\tilde{k}}{\omega}\right)\hat{k}\times\tilde{\vec{E}}(\vec{r},t) = \left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}$$

Then:

$$\tilde{\vec{Z}}(\vec{r},t;\omega) = \mu\frac{\tilde{E}_o e^{-\kappa z}\cancel{e^{i(kz-\omega t)}}\hat{x}\times\left(\frac{\tilde{k}^*(\omega)}{\omega}\right)\tilde{E}_o^* e^{-\kappa z}\cancel{e^{-i(kz-\omega t)}}\hat{y}}{\left|\left(\frac{\tilde{k}(\omega)}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}\right|^2}$$

$$= \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\frac{\cancel{|\tilde{E}_o|^2}\,\cancel{e^{-2\kappa z}}}{\cancel{|\tilde{E}_o|^2}\,\cancel{e^{-2\kappa z}}}\hat{z} = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z}\ \ (Ohms)$$

Note that {again}: $\tilde{\vec{Z}}(\vec{r},t;\omega) = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z}$ has ***no*** explicit time dependence

$$\tilde{\vec{Z}}(\vec{r},\omega) = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\omega\left(\frac{k(\omega)-i\kappa(\omega)}{k^2(\omega)+\kappa^2(\omega)}\right)\hat{z}\ \ (Ohms).$$

Complex impedance is manifestly a complex ***frequency-domain*** quantity

Note that since: $\tilde{k}^*(\omega) = k(\omega) - i\kappa(\omega) = \left|\tilde{k}^*(\omega)\right|e^{-i\varphi_k(\omega)} = \left|\tilde{k}(\omega)\right|e^{-i\varphi_k(\omega)}$

and: $\varphi_k(\omega) = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) = \delta_B(\omega) - \delta_E(\omega)$

We can also equivalently write this expression as:

$$\tilde{\vec{Z}}(\vec{r},\omega) = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\omega\left(\frac{\left|\tilde{k}(\omega)\right|e^{-i\varphi_k(\omega)}}{|\tilde{k}(\omega)|^2}\right)\hat{z}$$

$$= \mu\left(\frac{\omega}{|\tilde{k}(\omega)|}\right)e^{-i(\delta_B(\omega)-\delta_E(\omega))}\hat{z} = \mu\left(\frac{\omega}{|\tilde{k}(\omega)|}\right)e^{+i(\delta_E(\omega)-\delta_B(\omega))}\hat{z}$$

EM Wave Propagation in a Conductor – Special/Limiting Cases:

a) **Good conductors:** $\sigma_C \gg \varepsilon\omega$ Conductivity of a ***good*** conductor: $\sigma_C \to \infty \;\left(i.e.\;\; \rho_C = 1/\sigma_C \to 0\right)$.

Since: $\tilde{k} = k + i\kappa$ and: $\sigma_C \gg \varepsilon\omega$, *i.e.* $\left(\dfrac{\sigma_C}{\varepsilon\omega} \gg 1\right)$ for a ***good*** conductor. Then:

$$k \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right]^{1/2} = \omega\sqrt{\frac{\not{\varepsilon}\,\mu\sigma_C}{2\not{\varepsilon}\,\omega}} = \sqrt{\frac{\omega\mu\sigma_C}{2}}$$

and:

$$\kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right]^{1/2} = \omega\sqrt{\frac{\not{\varepsilon}\,\mu\sigma_C}{2\not{\varepsilon}\,\omega}} = \sqrt{\frac{\omega\mu\sigma_C}{2}}$$

$\therefore$ In a ***good*** conductor $\left(\dfrac{\sigma_C}{\varepsilon\omega} \gg 1\right)$:

$$k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\frac{\omega\mu\sigma_C}{2}} \quad \text{and skin depth:} \quad \delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} \simeq \sqrt{\frac{2}{\omega\mu\sigma_C}}.$$

FORMULAS FOR *EM* WAVE PROPAGATION IN A *GOOD* CONDUCTOR $\left(\dfrac{\sigma_C}{\varepsilon\omega} \gg 1\right)$

$$k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\frac{\omega\mu\sigma_C}{2}} \quad \text{and:} \quad \delta_{sc}(\omega) = \text{ skin depth } \equiv \frac{1}{\kappa(\omega)} \simeq \sqrt{\frac{2}{\omega\mu\sigma_C}}$$

Wavenumber, $k(\omega) \equiv \dfrac{2\pi}{\lambda(\omega)} \Rightarrow \lambda(\omega) = \dfrac{2\pi}{k(\omega)} \simeq \dfrac{2\pi}{\kappa(\omega)} = 2\pi\delta_{sc}(\omega) = 2\pi\sqrt{\dfrac{2}{\omega\mu\sigma_C}}$

n.b. in a ***perfect*** conductor: $\sigma_C = \infty$

$$\Rightarrow k(\omega) \simeq \kappa(\omega) = \sqrt{\frac{\omega\mu\sigma_C}{2}} = \infty$$

$$\Rightarrow \lambda(\omega) = \frac{2\pi}{k(\omega)} = 0$$

$$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} \simeq \sqrt{\frac{2}{\omega\mu\sigma_C}} = 0$$

$$\phi_k(\omega) \equiv (\delta_B - \delta_E) = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \simeq \tan^{-1}(1)$$

But: $\tan^{-1}(1) = 45^\circ = \dfrac{\pi}{4}$

$$\Rightarrow \phi = \delta_B - \delta_E = 45^\circ = \frac{\pi}{4}$$

$\Rightarrow \vec{B}$ ***lags*** $\vec{E}$ by $\simeq 45^\circ$ in a ***good*** conductor.

n.b. In a ***perfect*** conductor: $\sigma_C = \infty,\; \phi \equiv 45^\circ = \frac{\pi}{4}$

In a typical *good* conductor (*e.g.* gold/silver/copper/…): $(\sigma_C/\varepsilon\omega) \gg 1$

For ***optical*** frequencies/visible light region: $\omega \simeq 10^{16}\ radians/sec$. A ***good*** conductor typically has $\sigma_C \simeq 10^7$ Siemens/m and $\varepsilon \simeq 3\varepsilon_o$, and at optical frequencies: $(\sigma_C/\varepsilon\omega) \simeq 37.7 \gg 1$ is satisfied.

If the conductor is ≅ ***non-magnetic*** (*e.g.* copper, aluminum, gold, silver, platinum… *etc.*)
$\Rightarrow \quad \mu \simeq \mu_o = 4\pi\times10^{-7}\ Henrys/m$.

Then: $k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}} \approx \sqrt{\dfrac{\omega\mu_o\sigma_C}{2}} = \left[\dfrac{10^{16}\times4\pi\times10^{-7}\times10^7}{2}\right]^{1/2} \simeq 2.51\times10^8\ radians/m$

And: $\lambda(\omega) = 2\pi/k(\omega)$ = wavelength in good conductor $\simeq 2.51\times10^{-8}\,m = 25.1\,nm$

cf w/ vacuum wavelength: $\lambda_o = \dfrac{2\pi}{k_o} = \dfrac{2\pi c}{\omega} = \dfrac{c}{f} \simeq \dfrac{2\pi\times3\times10^8}{10^{16}} = 1.885\times10^{-7}\,m = 188.5\,nm$

$\Rightarrow \quad \lambda(\omega) \simeq 25.1\,nm\left(\text{good conductor}\right) \ll \lambda_o = 188.5\,nm\left(\text{vacuum wavelength}\right)$

Vacuum/conductor λ-ratio: $\left(\dfrac{\lambda_o}{\lambda(\omega)}\right) = \dfrac{188.5\,nm}{25.1\,nm} \simeq 7.52$ at ***optical*** frequencies, $\omega \approx 10^{16}$ rad/sec.

Skin depth: $\delta_{sc}(\omega) = \dfrac{1}{\kappa(\omega)} \simeq \dfrac{\lambda(\omega)}{2\pi} \simeq 4.0\times10^{-9}\,m = 4.0\,nm$!!!

$\Rightarrow$ This explains why metals are ***opaque*** at optical frequencies, $\omega \approx 10^{16}\ radians/sec$
{and ***also*** *e.g.* explains why/how silvered sunglasses work!}

Compare these results for *EM* waves propagating in conductors at ***optical*** frequencies to those for *EM* waves propagating in conductors, but instead at ***very low*** frequencies – *e.g.* the AC line frequency, $f_{AC} = 60\,Hz \Rightarrow \omega_{AC} = 2\pi f_{AC} = 120\pi$ rad/sec, where the criterion for a ***good*** conductor, $(\sigma_C/\varepsilon\omega) \simeq 10^{15} \gg 1$ is certainly well-satisfied:

At $f = 60\ Hz$:

$$k_{AC} \simeq \kappa_{AC} \simeq \sqrt{\frac{\omega\mu\sigma_C}{2}} = \left[\frac{120\pi\times4\pi\times10^{-7}\times10^7}{2}\right] = 48.7\ radians/m$$

$$\lambda_{AC} = \frac{2\pi}{k} = 0.129\,m = 12.9\,cm$$

$$\lambda_{o_{AC}} = 5\times10^6\,m!!$$

$$\frac{\lambda_{o_{AC}}}{\lambda_{AC}} = \frac{5\times10^6\,m}{0.129\,m} \simeq 3.87\times10^7!!$$

$$60\ Hz\ \text{AC skin depth:}\ \delta_{sc}^{AC} = \frac{\lambda_{AC}}{2\pi} \simeq 2.05\times10^{-2}\,m = 2.05\,cm!!$$

$\Rightarrow$ Need ***at least*** 3-4$\times\delta_{sc} \approx$ several $\rightarrow 10\ cm$ to screen out unwanted 60 *Hz* AC signals !!!

Phase speed *vs*. group speed in a *Good* conductor:

Given that: $k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\frac{\omega\mu\sigma_C}{2}}$ in a ***good*** conductor, where: $\left(\frac{\sigma_C}{\varepsilon\omega}\right) \gg 1$

The ***phase*** speed and ***group*** speed in a ***good*** conductor are respectively:

$$v_\phi(\omega) \equiv \frac{\omega}{k(\omega)} \simeq \frac{\omega}{\sqrt{\frac{\omega\mu\sigma_C}{2}}}$$ and: $$v_g(\omega) = \left[\frac{dk(\omega)}{d\omega}\right]^{-1} \simeq \frac{1}{\sqrt{\frac{\omega\mu\sigma_C}{2}}} \cdot \frac{1}{\left\{1-\frac{1}{2}\right\}} = \frac{2}{\sqrt{\frac{\omega\mu\sigma_C}{2}}} = 2v_\phi(\omega) \text{ !!!}$$

Complex impedance in a *Good* conductor:

$$\tilde{\tilde{Z}}(\vec{r},\omega) = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\omega\left(\frac{|\tilde{k}(\omega)|\, e^{-i\varphi_k(\omega)}}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\left(\frac{\omega}{|\tilde{k}(\omega)|}\right)e^{-i\varphi_k(\omega)}\hat{z} \ (Ohms)$$

Since: $k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\frac{\mu\omega\sigma_C}{2}}$ Then: $|\tilde{k}(\omega)| = \sqrt{k^2(\omega)+\kappa^2(\omega)} = \sqrt{2}k(\omega) \simeq \sqrt{\mu\omega\sigma_C}$

And: $\varphi_k(\omega) = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \simeq \tan^{-1}(1) = 45° = \delta_B(\omega) - \delta_E(\omega)$ $\Rightarrow \vec{B}$ ***lags*** $\vec{E}$ by $\simeq 45°$ in a ***good*** conductor.

Hence in a ***good*** conductor:

$$\tilde{\tilde{Z}}_{cond}^{good}(\vec{r},\omega) \simeq \mu\left(\frac{\omega}{\sqrt{\mu\omega\sigma_C}}\right)e^{-i(\delta_B(\omega)-\delta_E(\omega))}\hat{z} = \sqrt{\frac{\mu\omega}{\sigma_C}}e^{+i(\delta_E(\omega)-\delta_B(\omega))}\hat{z} = \sqrt{\frac{\varepsilon}{\varepsilon}}\sqrt{\frac{\mu\omega}{\sigma_C}}e^{+i(\delta_E(\omega)-\delta_B(\omega))}\hat{z}$$
$$= \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\varepsilon\omega}{\sigma_C}}e^{+i(\delta_E(\omega)-\delta_B(\omega))}\hat{z} = \left(\sqrt{\frac{\mu}{\varepsilon}}\Big/\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right)e^{+i(\delta_E(\omega)-\delta_B(\omega))}\hat{z} \ (Ohms)$$

Define {real scalar}: $Z_{med}^{lin} \equiv \sqrt{\frac{\mu}{\varepsilon}}$ Then {real scalar} $Z_{cond}^{good} \equiv \left(\sqrt{\frac{\mu}{\varepsilon}}\Big/\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right) = Z_{med}^{lin}\Big/\left(\frac{\sigma_C}{\varepsilon\omega}\right)^{\frac{1}{2}} \ll Z_{med}^{lin}$

since $\left(\frac{\sigma_C}{\varepsilon\omega}\right) \gg 1$ in a ***good*** conductor!

Since {in general} the complex impedance is: $\tilde{\tilde{Z}}(\vec{r},\omega) = \left|\tilde{\tilde{Z}}(\vec{r},\omega)\right|e^{i\varphi_Z(\omega)}\hat{z} = \left|\tilde{\tilde{Z}}(\vec{r},\omega)\right|e^{i(\delta_E(\omega)-\delta_B(\omega))}\hat{z}$

We see the ***phase*** of the complex wave impedance for a ***good*** conductor is:

$$\varphi_Z(\omega) = \tan^{-1}\left(\frac{-\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}(-1) = -45° = \delta_E(\omega) - \delta_B(\omega) = -(\delta_B(\omega) - \delta_E(\omega)) = -\varphi_k(\omega)$$

***Instantaneous EM* Wave Energy Densities in a *Good* Conductor:**

$$\left(\frac{\sigma_C}{\varepsilon\omega}\right) \gg 1$$

$$u_{EM} = u_E^{EM} + u_M^{EM} = \left(\frac{1}{2}\varepsilon E^2\right) + \left(\frac{1}{2\mu}B^2\right) = \left(\frac{1}{2}\varepsilon \vec{E}\cdot\vec{E}\right) + \left(\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)$$

$$\phi_k \equiv (\delta_B - \delta_E) \simeq \tfrac{\pi}{4} = 45^\circ \quad \{\text{in a } good \text{ conductor}\}$$

$$\vec{E}(\vec{r},t) = E_o e^{-\kappa z}\cos(kz - \omega t + \delta_E)\,\hat{x} \quad \text{and} \quad \vec{B}(\vec{r},t) = B_o e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\,\hat{y}$$

Where: $B_o = \dfrac{|\tilde{k}(\omega)|}{\omega} E_o = \left[\varepsilon\mu\sqrt{1+\left(\dfrac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} E_o \simeq \sqrt{\dfrac{\cancel{\varepsilon}\,\mu\sigma_C}{\cancel{\varepsilon}\,\omega}}\,E_o = \sqrt{\dfrac{\mu\sigma_C}{\omega}}\,E_o$ for a ***good*** conductor,

And: $k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}}$,

$v_\phi(\omega) = \dfrac{\omega}{k(\omega)} \simeq \dfrac{\omega}{\sqrt{\dfrac{\omega\mu\sigma_C}{2}}} = \sqrt{\dfrac{2\omega}{\omega\sigma_C}}$ for a ***good*** conductor.

Then:

$$u_E^{EM}(\vec{r},t) = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \vec{E}\cdot\vec{E} = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E) \quad \text{and:}$$

$$u_M^{EM}(\vec{r},t) = \frac{1}{2\mu}B^2 = \frac{1}{2\mu}\vec{B}\cdot\vec{B} = \frac{1}{2\mu}B_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E + \phi_k) \simeq \frac{\sigma_C}{2\omega}E_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E + \phi_k)$$

Time-averaging these quantities over one complete cycle: $\langle u(\vec{r},t)\rangle \equiv \dfrac{1}{\tau}\int_0^\tau u(\vec{r},t)\,dt$

$$\langle u_E^{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\underbrace{\frac{1}{\tau}\int_0^\tau \cos^2(kz - \omega t + \delta_E)\,d\tau}_{=\frac{1}{2}} = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}$$

$$\langle u_M^{EM}(\vec{r},t)\rangle = \frac{\sigma_C}{2\omega}E_o^2 e^{-2\kappa z}\underbrace{\frac{1}{\tau}\int_0^\tau \cos^2(kz - \omega t + \delta_E + \phi_k)\,d\tau}_{=\frac{1}{2}} = \frac{1}{4}\left(\frac{\sigma_C}{\omega}\right)E_o^2 e^{-2\kappa z} = \left(\frac{\sigma_C}{\varepsilon\omega}\right)\cdot\frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}$$

$$\therefore \quad \langle u_{Tot}^{EM}(\vec{r},t)\rangle = \langle u_E^{EM}(\vec{r},t)\rangle + \langle u_M^{EM}(\vec{r},t)\rangle = \frac{1}{4}\varepsilon\left(1+\frac{\sigma_C}{\varepsilon\omega}\right)E_o^2 e^{-2\kappa z}$$

n.b. Exponentially attenuated in *z* !!!

But: $\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \gg 1$ for a ***good*** conductor, $\Rightarrow \langle u_{Tot}^{EM}(\vec{r},t)\rangle \simeq \dfrac{1}{2}\left(\dfrac{\sigma_C}{\varepsilon\omega}\right)\left[\dfrac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\right]$

i.e. the ***ratio***: $\dfrac{\langle u_M^{EM}(\vec{r},t)\rangle}{\langle u_E^{EM}(\vec{r},t)\rangle} = \left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \gg 1$ or: $\langle u_M^{EM}(\vec{r},t)\rangle \gg \langle u_E^{EM}(\vec{r},t)\rangle$ for a ***good*** conductor.

$\Rightarrow$ Vast majority of *EM* wave energy is carried by the ***magnetic field*** in a ***good*** conductor !!!

Poynting's Vector: $\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$ $\Rightarrow$ $\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{\mu}\langle\vec{E}\times\vec{B}\rangle = \frac{1}{2\mu}E_o B_o e^{-2\kappa z}\cos\phi_k\ \hat{z}$ ← $\phi_k = \frac{\pi}{4}$

EM wave intensity (*aka* ***irradiance***): $I(\vec{r}) = \langle|\vec{S}(\vec{r},t)|\rangle = \frac{1}{2\mu}E_o B_o e^{-2\kappa z}\cos\phi_k = \frac{1}{2\mu}E_o^2 e^{-2\kappa z}\left(\frac{|\tilde{k}|}{\omega}\cos\phi_k\right)$

But: $\frac{|\tilde{k}|\cos\phi_k}{\omega} = \frac{k}{\omega} \simeq \frac{\sqrt{\frac{\omega\mu\sigma_C}{2}}}{\omega} = \sqrt{\frac{\mu\sigma_C}{2\omega}}$ $\therefore$ $I(\vec{r}) = \langle|\vec{S}(\vec{r},t)|\rangle = \frac{1}{2\mu}\left(\frac{k}{\omega}\right)E_o^2 e^{-2\kappa z} = \frac{1}{2}\sqrt{\frac{\sigma_C}{2\mu\omega}}E_o^2 e^{-2\kappa z}$

b.) **Special/Limiting Case of a *Fair* Conductor:** $\left(\frac{\sigma_C}{\varepsilon\omega}\right)\approx 1$ $\Rightarrow$ Must use ***exact*** formulae!

c.) **Special/Limiting Case of a *Poor* Conductor:** (*i.e.* ≅ an ***insulator***):

Here: $\left(\frac{\sigma_C}{\varepsilon\omega}\right)\ll 1$. Conductivity of ***poor*** conductor: $\sigma_C \to 0\ \ (i.e.\ \ \rho_C = 1/\sigma_C \to \infty)$.

Complex wavenumber: $\tilde{k} = k + i\kappa$, with: $k = k(\omega) = \Re e\{\tilde{k}(\omega)\}$ and: $\kappa = \kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$.

Noting that to 1st order in the Taylor series expansion: $\sqrt{1+\epsilon} \simeq 1+\frac{1}{2}\epsilon$ for $\epsilon \ll 1$, thus:

$$k(\omega) \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2+1\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[2+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2\right]^{1/2} \simeq \omega\sqrt{\varepsilon\mu}$$

$$\kappa(\omega) \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\not{1}+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2-\not{1}\right]^{1/2} = \omega\sqrt{\frac{\not{\varepsilon}\mu\sigma_C^2}{4\varepsilon^{\not{2}}\omega^2}} \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}$$

$\therefore$ $k(\omega) \simeq \omega\sqrt{\varepsilon\mu}$ and: $\kappa(\omega) \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}$ for a ***poor*** conductor.

In a ***poor*** conductor $\left(\frac{\sigma_C}{\varepsilon\omega}\right)\ll 1$, the ratio: $\left(\frac{\kappa(\omega)}{k(\omega)}\right) \simeq \frac{\frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}{\omega\sqrt{\varepsilon\mu}} = \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)\ll 1$ *i.e.* $\kappa(\omega) \ll k(\omega)$.

$\Rightarrow$ The complex wavenumber $\tilde{k} \equiv k + i\kappa$ is primarily ***real***, because $\kappa \ll k$ in a ***poor*** conductor.

Phase angle in a ***poor*** conductor: $\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}\left(\frac{1}{2}\left(\frac{\omega_C}{\varepsilon\omega}\right)\right) \simeq \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right) \sim 0 \ll 1$

$\Rightarrow \delta_B = \delta_E + \phi_k \simeq \delta_E$, *i.e.* $\vec{B}$ and $\vec{E}$ are ***nearly*** in phase with each other in a ***poor*** conductor (*i.e.* dissipation/losses ***very*** small in a ***poor*** conductor).

In a *typical* ***poor*** conductor, *e.g.* pure water:

Water has a ***huge*** **static** electric permittivity (due to the permanent electric dipole moment of water molecule): $\varepsilon_{H_2O} \simeq 81\varepsilon_o \left(@\ f = 0\ Hz\right)$ $\left(\text{at } P = 1\,ATM \text{ and } T = 20^\circ C\right)$, however, at ***optical*** frequencies $\left(\omega \approx 10^{16}\ \text{rad/sec}\right)$: $\varepsilon_{H_2O}(\omega) \approx 1.777\varepsilon_o$, where $\varepsilon_o = 8.85\times 10^{-12}\ Farads/m$.
Since water is ≅ ***non-magnetic***: $\mu_{H_2O} \simeq \mu_o = 4\pi\times 10^{-7}\ Henrys/m$

$\Rightarrow$ index of refraction: $n_{H_2O}(\omega) = \sqrt{\varepsilon_{H_2O}(\omega)\mu_{H_2O}/\varepsilon_o\mu_o} \simeq 1.333$ at ***optical*** frequencies.

The conductivity of pure water is: $\sigma_C^{H_2O} = 1/\rho_C^{H_2O} \simeq 1/2.5\times 10^5\,\Omega\text{-}m = 4.0\times 10^{-6}\ Siemens/m$ $\left(\text{at } P = 1\,ATM \text{ and } T = 20^\circ C\right)$. Thus, the criteria for a ***poor*** conductor $\left(\sigma_C/\varepsilon\omega\right) \simeq 2.54\times 10^{-11} \ll 1$ is certainly satisfied at ***optical*** frequencies.

The wavenumber in pure H_2O at ***optical*** frequencies is:

$$k_{H_2O}(\omega) \simeq \omega\sqrt{\varepsilon\mu} \approx \omega\sqrt{\varepsilon\mu_o} = 10^{16}\sqrt{1.777\times 8.85\times 4\pi\times 10^{-7}} \simeq 4.45\times 10^7\ radians/m$$

The wavelength in pure H_2O is: $\lambda_{H_2O} = 2\pi/k_{H_2O} = 1.413\times 10^{-7}\,m = 141.3\,nm$ at ***optical*** frequencies.
cf w/ the ***vacuum*** wavelength: $\lambda_o = c/f = 2\pi c/\omega = 1.885\times 10^{-7}\,m = 188.5\,nm$

Note that the ***optical*** wavelength ratio: $\left(\dfrac{\lambda_o}{\lambda_{H_2O}}\right) = \dfrac{188.5\,nm}{141.3\,nm} = 1.333 = n_{H_2O}$ for a ***poor*** conductor.

Skin depth: $\delta_{sc}(\omega) \equiv \dfrac{1}{\kappa(\omega)} \simeq \dfrac{1}{\frac{1}{2}\sigma_C\sqrt{\mu/\varepsilon}}$ for a ***poor*** conductor $\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1$.

For pure H_2O at ***optical*** frequencies:

$$\kappa_{H_2O}(\omega) \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}} \approx \frac{1}{2}\sigma_C\sqrt{\frac{\mu_o}{\varepsilon}} = \frac{1}{2}\left(\frac{1}{2.5\times 10^5}\right)\sqrt{\frac{4\pi\times 10^{-7}}{1.777\times 8.85\times 10^{-12}}} \simeq 5.65\times 10^{-4}\ rad/m$$

$$\delta_{sc}^{H_2O}(\omega) \equiv \frac{1}{\kappa_{H_2O}} = 1.7688\times 10^3\,m = 1.77\,km$$

n.b. neglects/ignores Rayleigh scattering process – visible light photons elastically scattering off of H_2O molecules. $\lambda_{atten}^{vis} \simeq 10\,m$

Ratio: $$\left(\frac{\kappa_{H_2O}(\omega)}{k_{H_2O}(\omega)}\right) = \frac{\frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}{\omega\sqrt{\varepsilon\mu}} = \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right) = \frac{1}{2}\left(\frac{1}{2.5\times 10^5}\right)\frac{1}{1.777\times 8.85\times 10^{-12}\times 10^{16}} = 1.27\times 10^{-11} \ll 1$$

Phase difference: $\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\dfrac{\kappa_{H_2O}}{k_{H_2O}}\right) \simeq 1.27\times 10^{-11}\,\text{radians}\ (\ll 1)$ *i.e.* $\delta_B = \delta_E + \phi_k \simeq \delta_E$

$\Rightarrow \vec{B}$ and $\vec{E}$ are ***nearly*** in phase with each other in pure H_2O at ***optical*** frequencies.

For ***pure*** H_2O *at low* frequencies – *e.g.* 60 Hz AC line frequency $\left(\omega_{AC} = 2\pi f_{AC} = 120\pi\, rad/sec\right)$:

The electric permittivity at $f = 60$ *Hz* is $\varepsilon_{H_2O}^{AC}\left(f \simeq 60Hz\right) \simeq 80\varepsilon_o = 80\times 8.85\times 10^{-12}\ Farads/m$ and $\mu_{H_2O}^{AC} \simeq \mu_o = 4\pi\times 10^{-7}$ Henrys/m. Conductivity of pure H_2O: $\sigma_C^{H_2O} = 4.0\times 10^{-6}\ Siemens/m$

Note that the criteria for a ***poor*** conductor: $\left(\dfrac{\sigma_C}{\varepsilon_{H_2O}^{AC}\omega_{AC}}\right) \simeq \dfrac{4.0\times 10^{-6}}{80\times 8.85\times 10^{-12}\cdot 120\pi} \simeq 15 \ll 1$

is ***not*** satisfied at the 60 *Hz* AC line frequency – *i.e.* at ***low enough*** frequencies, even ***poor*** conductors such as pure water are actually quite ***good*** conductors !!!

Thus, for the following, we ***must*** use the ***good*** conductor approximations:

$$k_{AC}^{H_2O}\left(\omega\right) \simeq \kappa_{AC}^{H_2O}\left(\omega\right) \simeq \sqrt{\frac{\omega_{AC}\mu_{AC}^{H_2O}\sigma_C}{2}} \approx \sqrt{\frac{\omega_{AC}\mu_o\sigma_C}{2}} = \sqrt{\frac{120\pi\cdot 4\pi\times 10^{-7}\cdot 4\times 10^{-6}}{2}} = 3.08\times 10^{-5}\ \text{rads/m}$$

$$\lambda_{AC}^{H_2O}\left(\omega\right) \simeq \frac{2\pi}{k_{AC}^{H_2O}\left(\omega\right)} = 2.04\times 10^{5}\,\text{m}$$

cf *w/* vacuum wavelength: $\lambda_o = c/f_{AC} = \dfrac{2\pi c}{\omega_{AC}} = 5.00\times 10^{6}\,\text{m}$

Vacuum/good conductor wavelength ratio: $\left(\dfrac{\lambda_o}{\lambda_{AC}^{H_2O}}\right) = \dfrac{5.00\times 10^{6}\,m}{2.04\times 10^{5}\,m} \simeq 24.495$

Skin depth for pure H_2O at 60 *Hz* AC line frequency: $\delta_{H_2O}^{AC} \equiv 1/\kappa_{H_2O}^{AC} \simeq 3.25\times 10^{4}\,m = 32.5\,km$

This may seem like a large distance scale associated with the attenuation of the 60 Hz *EM* waves propagating in pure water, however compare the skin depth to the wavelength at this frequency: $\delta_{H_2O}^{AC} = 32.5\,km$ *vs.* $\lambda_{AC}^{H_2O} = 1.77\times 10^{6}\,m$, *i.e.* we see that $\delta_{H_2O}^{AC} \ll \lambda_{H_2O}^{AC}$, as we expect for the case of a ***good*** conductor !!!

The ratio $\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \simeq 1$ for pure H_2O at 60 *Hz* AC line frequency, which is what we expect for a ***good*** conductor {this ratio should be $\ll 1$ for a ***poor*** conductor!}.

Thus, the phase difference is: $\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \simeq \tan^{-1}\left(1\right) = \dfrac{\pi}{4} = 45^{\circ}$

which again is what we expect for a ***good*** conductor, *i.e.* $\vec{B}$ ***lags*** $\vec{E}$ by 45° !

<u>Phase speed *vs*. group speed in a *Poor* conductor</u>:

Given that: $k(\omega) \simeq \omega\sqrt{\varepsilon\mu}$ and: $\kappa(\omega) \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}$ in a ***poor*** conductor, where: $\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1$

$\Rightarrow$ The complex wavenumber $\tilde{k} \equiv k + i\kappa$ is primarily ***real***, because $\kappa \ll k$ in a ***poor*** conductor.

The ***phase*** speed in a ***poor*** conductor is:

$$v_\phi(\omega) \equiv \frac{\omega}{k(\omega)} = \frac{\omega}{\omega\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon\mu}}$$ compare to vacuum/free space: $$c = \frac{1}{\sqrt{\varepsilon_o\mu_o}}$$

The ***group*** speed in a ***poor*** conductor is:

$$v_g(\omega) = \left[\frac{dk(\omega)}{d\omega}\right]^{-1} \simeq \frac{1}{\sqrt{\varepsilon\mu}} = v_\phi(\omega) \text{ !!!}$$

<u>Complex impedance in a *Poor* conductor</u>:

$$\tilde{\bar{Z}}(\vec{r},\omega) = \mu\omega\left(\frac{\tilde{k}^*(\omega)}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\omega\left(\frac{|\tilde{k}(\omega)|e^{-i\varphi_k(\omega)}}{|\tilde{k}(\omega)|^2}\right)\hat{z} = \mu\left(\frac{\omega}{|\tilde{k}(\omega)|}\right)e^{-i\varphi_k(\omega)}\hat{z}\ (Ohms)$$

But: $k(\omega) \simeq \omega\sqrt{\varepsilon\mu}$ and: $\kappa(\omega) \simeq \frac{1}{2}\sqrt{\mu\omega\sigma_C}$ in a ***poor*** conductor, where: $\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1$

And: $$\phi_k(\omega) = (\delta_B - \delta_E) = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}\left(\frac{1}{2}\left(\frac{\omega_C}{\varepsilon\omega}\right)\right) \simeq \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right) \sim 0 \ll 1$$

E and *B* are $\cong$ ***in-phase*** with each other for a ***poor*** conductor.

Hence in a ***poor*** conductor: $\tilde{\bar{Z}}^{poor}_{cond}(\vec{r},\omega) \simeq \sqrt{\frac{\mu}{\varepsilon}}\,\hat{z} = Z^{poor}_{cond}\,\hat{z}$. Define {real scalar}: $Z^{lin}_{med} \equiv \sqrt{\frac{\mu}{\varepsilon}}$.

Then {real scalar} characteristic longitudinal *EM* wave impedance for a ***poor*** conductor is:

$$Z^{poor}_{cond} = Z^{lin}_{med} = \sqrt{\frac{\mu}{\varepsilon}}\,(Ohms)$$ *n.b*. Compare to free space: $$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \simeq 376.8\,(Ohms)$$

Since {in general} the complex impedance is: $\tilde{\bar{Z}}(\vec{r},\omega) = \left|\tilde{\bar{Z}}(\vec{r},\omega)\right|e^{i\varphi_Z(\omega)}\hat{z} = \left|\tilde{\bar{Z}}(\vec{r},\omega)\right|e^{i(\delta_E(\omega)-\delta_B(\omega))}\hat{z}$.

The ***phase*** of the complex wave impedance for a ***poor*** conductor is:

$$\varphi_Z(\omega) = \tan^{-1}\left(\frac{-\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}\left(-\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)\right) \simeq 0° = \delta_E(\omega) - \delta_B(\omega)$$

Instantaneous EM **energy densities in a *poor* conductor:** $\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1$

$$u_{EM}(\vec{r},t) = u_E^{EM}(\vec{r},t) + u_M^{EM}(\vec{r},t) = \left(\frac{1}{2}\varepsilon E^2\right) + \left(\frac{1}{2\mu}B^2\right) = \left(\frac{1}{2}\varepsilon\vec{E}\cdot\vec{E}\right) + \left(\frac{1}{2\mu}\vec{B}\cdot\vec{B}\right)$$

The physical/instantaneous purely real ***time-domain*** $\vec{E}$ and $\vec{B}$ fields are:

$$\vec{E}(\vec{r},t) = E_o e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{x} \quad \text{and:} \quad \vec{B}(\vec{r},t) = B_o e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi_k)\hat{y}$$

where: $B_o = \dfrac{|\tilde{k}|}{\omega}E_o = \left[\varepsilon\mu\sqrt{1+\left(\dfrac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} E_o \approx \sqrt{\varepsilon\mu}\,E_o$ for a ***poor*** conductor, $\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1$.

$k \simeq \omega\sqrt{\varepsilon\mu} = \dfrac{\omega}{v_\phi}$ where: $v_\phi = \dfrac{\omega}{k(\omega)} = \dfrac{1}{\sqrt{\varepsilon\mu}}$ for a ***poor*** conductor.

and: $\kappa \simeq \dfrac{1}{2}\sigma_c\sqrt{\dfrac{\mu}{\varepsilon}} \ll k \simeq \omega\sqrt{\varepsilon\mu}$, $|\tilde{k}| \simeq \omega\sqrt{\varepsilon\mu}$ for a ***poor*** conductor.

then: $u_E^{EM}(\vec{r},t) = \dfrac{1}{2}\varepsilon E^2 = \dfrac{1}{2}c\vec{E}\cdot\vec{E} = \dfrac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E)$ and:

$$u_M^{EM}(\vec{r},t) = \frac{1}{2\mu}B^2 = \frac{1}{2\mu}\vec{B}\cdot\vec{B} = \frac{1}{2\mu}B_o^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E+\phi_k)$$

Time-averaging these quantities:

$$\langle u_E^{EM}(\vec{r},t)\rangle = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z} \quad \text{and:} \quad \langle u_M^{EM}(\vec{r},t)\rangle = \frac{1}{4\mu}B_o^2 e^{-2\kappa z} \simeq \frac{1}{4\cancel{\mu}}(\varepsilon\cancel{\mu})E_o^2 e^{-2\kappa z} = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}$$

$$\therefore \quad \langle u_{Tot}^{EM}(\vec{r},t)\rangle = \langle u_E^{EM}(\vec{r},t)\rangle + \langle u_M^{E}(\vec{r},t)\rangle \simeq \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z} + \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z} = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}$$

Thus: $\langle u_{Tot}^{EM}(\vec{r},t)\rangle = \dfrac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}$ for a ***poor*** conductor, $\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1$.

The ***ratio*** of {time-averaged} electric/magnetic energy densities for a ***poor*** conductor:

$$\frac{\langle u_E^{EM}(\vec{r},t)\rangle}{\langle u_M^{EM}(\vec{r},t)\rangle} \simeq \frac{\frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}}{\frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}} = 1 \qquad \phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa_{H_2O}}{k_{H_2O}}\right) \ll 1 \qquad \kappa_{H_2O} \simeq \frac{1}{2}\sigma_c\sqrt{\frac{\mu_o}{\varepsilon}} \ll k_{H_2O} \simeq \omega\sqrt{\varepsilon\mu_o}$$

$\Rightarrow$ *EM* wave energy is shared $\approx$ equally by the $\vec{E}$ and $\vec{B}$ fields in a ***poor*** conductor!

Instantaneous* Poynting's Vector** for *EM* waves propagating in a ***poor conductor:

$$\vec{S}(\vec{r},t) = \frac{1}{\mu}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) \Rightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{1}{\mu}\langle\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\rangle \simeq \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_o}}E_o^2 e^{-2\kappa z}\underbrace{\cos\phi_k}_{\approx 1}\hat{z}$$

Intensity of *EM* waves propagating in a ***poor*** conductor: $$I(\vec{r}) = \langle|\vec{S}(\vec{r},t)|\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_o}}E_o^2 e^{-2\kappa z}$$

Reflection of *EM* Waves at Normal Incidence from a Conducting Surface:

In the presence of free surface charges σ_{free} and/or free surface currents, $\vec{K}_{free}$ the boundary conditions obtained from (the integral forms of) Maxwell's equations for reflection and refraction at *e.g.* a dielectric-conductor interface become:

BC 1): (normal $\vec{D}$ at interface): $\varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma_{free}$

BC 2): (tangential $\vec{E}$ at interface): $E_1^\parallel - E_2^\parallel = 0 \Rightarrow E_1^\parallel = E_2^\parallel$

BC 3): (normal $\vec{B}$ at interface): $B_1^\perp - B_2^\perp = 0 \Rightarrow B_1^\perp = B_2^\perp$

BC 4): (tangential $\vec{H}$ at interface): $\frac{1}{\mu_1}B_1^\parallel - \frac{1}{\mu_2}B_2^\parallel = \vec{K}_{free} \times \hat{n}_{\vec{21}}$

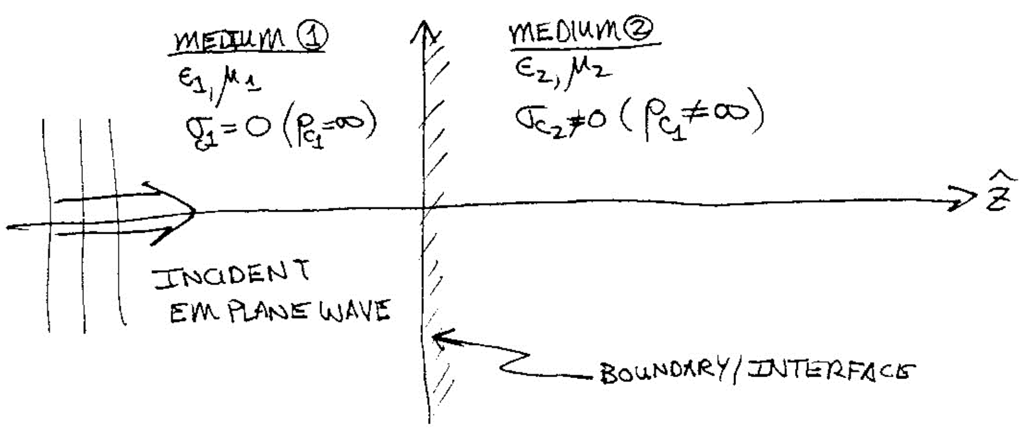
$\perp$ = normal to plane of interface
$\parallel$ = parallel to plane of interface

where $\hat{n}_{\vec{21}}$ is a unit vector $\perp$ to the interface, pointing ***from*** medium (2) ***into*** medium (1).

{*n.b.* do ***not*** confuse $\hat{n}_{\vec{21}}$ with the *EM* wave ***polarization vector*** $\hat{n}$!!!}

<u>Note</u>: For ***Ohmic*** conductors (*i.e.* "normal" conductors obeying Ohm's law $\vec{J}_{free} = \sigma_C \vec{E}$) there can be ***no free surface*** currents - *i.e.* $\vec{K}_{free} = 0$, because $\vec{K}_{free} \neq 0$ would require an ***infinite*** $\vec{E}$-field at the boundary/interface! ($\vec{J}_{free} \neq 0$ ***inside*** the conductor is fine/OK…)

Suppose $\exists$ a boundary/interface (located in the *x-y* plane at *z* = 0) between a non-conducting linear/homogeneous/isotropic medium (1) and a conductor (2). A monochromatic plane *EM* wave is incident on the interface, linearly polarized in $+\hat{x}$ direction, traveling in the $+\hat{z}$ direction, approaches the interface/boundary from the left {in medium (1)} as shown in the figure below:



$$\tilde{\vec{B}} = \frac{1}{v}\left(\hat{k}\times\tilde{\vec{E}}\right)$$

Incident *EM* wave {medium (1)}: $\tilde{\vec{E}}_{inc}(\vec{r},t) = \tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{x}$ and: $\tilde{\vec{B}}_{inc}(\vec{r},t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{y}$

Reflected *EM* wave {medium (1)}: $\tilde{\vec{E}}_{refl}(\vec{r},t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z=\omega t)}\hat{x}$ and: $\tilde{\vec{B}}_{refl}(\vec{r},t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z-\omega t)}\hat{y}$

Transmitted *EM* wave {medium (2)}: $\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z-\omega t)}\hat{x}$ and: $\tilde{\vec{B}}_{trans}(\vec{r},t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z-\omega t)}\hat{y}$

n.b. ***complex*** wavenumber in {conducting} medium (2): $\tilde{k}_2 = k_2 + i\kappa_2$

In medium (1) *EM* fields are: $\tilde{\vec{E}}_{Tot_1}(\vec{r},t) = \tilde{\vec{E}}_{inc}(\vec{r},t) + \tilde{\vec{E}}_{refl}(\vec{r},t)$ and: $\tilde{\vec{B}}_{Tot_1}(\vec{r},t) = \tilde{\vec{B}}_{inc}(\vec{r},t) + \tilde{\vec{B}}_{refl}(\vec{r},t)$

In medium (2) *EM* fields are: $\tilde{\vec{E}}_{Tot_2}(\vec{r},t) = \tilde{\vec{E}}_{trans}(\vec{r},t)$ and: $\tilde{\vec{B}}_{Tot_2}(\vec{r},t) = \tilde{\vec{B}}_{trans}(\vec{r},t)$

Apply BC's at the *z* = 0 interface in the *x*-*y* plane:

BC 1): $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}$ but: $E_1^{\perp} = \tilde{E}_{1_z} = 0$ and: $E_2^{\perp} = \tilde{E}_{2_z} = 0$ $\therefore$ $0 - 0 = \sigma_{free}$ $\Rightarrow$ $\sigma_{free} = 0$

BC 2): $E_1^{\parallel} = E_2^{\parallel}$ $\therefore$ $\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$

BC 3): $B_1^{\perp} = B_2^{\perp}$ but: $B_1^{\perp} = B_{1_z} = 0$ and: $B_2^{\perp} = B_{2_z} = 0$ $\Rightarrow$ $0 = 0$

BC 4): $\frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = \vec{K}_{free}\times\hat{n}_{\overrightarrow{21}}$ but: $\vec{K}_{free} = 0$ $\therefore$ $\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) - \frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{o_{trans}} = 0$

or: $\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta}\tilde{E}_{o_{trans}}$ with: $\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2$

Thus we obtain: $\tilde{E}_{o_{refl}} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{o_{inc}}$ and: $\tilde{E}_{o_{trans}} = \frac{2}{\left(1+\tilde{\beta}\right)}\tilde{E}_{o_{inc}}$

or: $\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right) = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)$ and: $\left(\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}}\right) = \frac{2}{\left(1+\tilde{\beta}\right)}$ with: $\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2$

Note that these "Fresnel" relations for reflection/transmission of *EM* waves at normal incidence on a non-conductor/conductor boundary/interface are ***identical*** to those obtained for reflection / transmission of *EM* waves at normal incidence on a boundary/interface between two ***linear*** non-conductors, ***except*** for the replacement of β with a now complex $\tilde{\beta}$ for the present situation.

Note also that ***here***, $\tilde{\beta}$ is ***frequency-dependent***, *i.e.* $\tilde{\beta} = \tilde{\beta}(\omega) \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2(\omega)$.

For the case of a ***perfect*** conductor, the conductivity $\sigma_C = \infty$ {thus resistivity, $\rho_C = 1/\sigma_C = 0$ }

$\Rightarrow$ *both* $k_2 \simeq \kappa_2 \simeq \sqrt{\dfrac{\mu_2\omega\sigma_C}{2}} = \infty$ and since: $\tilde{k}_2 = k_2 + i\kappa_2$ then: $\tilde{k}_2 = \infty + i\infty = \infty(1+i)$

and since: $\tilde{\beta} \equiv \left(\dfrac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2 \Rightarrow \tilde{\beta} = \infty$

Thus, for a ***perfect*** conductor, we see that: $\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}$ and: $\tilde{E}_{trans} = 0$

Thus, for a ***perfect*** conductor, the reflection and transmission coefficients are:

$$R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = 1 \quad \text{and: } T = 1 - R = 0$$

We also see that for a ***perfect*** conductor, for normal incidence, the ***reflected*** wave undergoes a 180° ***phase shift*** with respect to the ***incident*** wave at the interface/boundary at $z = 0$ in the x-y plane. A ***perfect*** conductor screens out ***all*** *EM* waves from propagating in its interior.

For the case of a ***good*** conductor, the conductivity σ_C is finite-large, but not infinite. The reflection coefficient R for monochromatic plane *EM* waves at normal incidence on a ***good*** conductor is ***not*** unity, but ***very*** close to it. {This is why ***good*** conductors make ***good*** mirrors!}

For a ***good*** conductor: $R \equiv \left(\dfrac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = \left|\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*$

Where: $\tilde{\beta} \equiv \left(\dfrac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2$ and: $\tilde{k}_2 = k_2 + i\kappa_2$. For a ***good*** conductor: $k_2 \simeq \kappa_2 \simeq \sqrt{\dfrac{\mu_2\omega\sigma_C}{2}}$

Thus: $\tilde{\beta} = \left(\dfrac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2 = \left(\dfrac{\mu_1 v_1}{\mu_2\omega}\right)\sqrt{\dfrac{\mu_2\omega\sigma_C}{2}}(1+i) = \mu_1 v_1\sqrt{\dfrac{\sigma_C}{2\mu_2\omega}}(1+i)$

Define: $\gamma \equiv \mu_1 v_1\sqrt{\dfrac{\sigma_C}{2\mu_2\omega}}$ Then: $\tilde{\beta} = \gamma(1+i)$

Thus, the reflection coefficient R for monochromatic plane *EM* waves at normal incidence on a ***good*** conductor is {*n.b.* ***frequency-dependent***!}:

$$R = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \left[\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}\right]$$

with: $\gamma \equiv \mu_1 v_1\sqrt{\dfrac{\sigma_C}{2\mu_2\omega}} = \gamma(\omega)$

Obviously, only a {very} ***small*** fraction of the normally-incident monochromatic plane *EM* wave is ***transmitted*** into the ***good*** conductor, since $R \lesssim 1$ and $\boxed{T = 1 - R}$, *i.e.*:

$$\boxed{T = 1 - R = 1 - \left[\frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}\right] \quad (\ll 1)} \quad \text{with:} \quad \boxed{\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2 \omega}} = \gamma(\omega)}$$

Note that the ***transmitted*** wave is ***exponentially*** attenuated in the *z*-direction; the $\vec{E}$ and $\vec{B}$ fields in the ***good*** conductor fall to 1/*e* of their initial $\{z = 0\}$ values (at/on the interface) after the monochromatic plane *EM* wave propagates a distance of one skin depth in *z* into the conductor:

$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \simeq \sqrt{\frac{2}{\mu_2 \omega \sigma_C}}}$$

Note also that the ***energy*** associated with the ***transmitted*** monochromatic plane *EM* wave is ultimately dissipated in the conducting medium as ***heat***.

In {bulk} metals, since the transmitted wave is {rapidly} absorbed/attenuated in the metal, experimentally we are only able to study/measure the reflection coefficient *R*. A full/detailed mathematical description of the physics of reflection from the surface of a metal conductor as a function of angle of incidence *i.e.* $R(\omega, \theta_{inc})$ and also requires the use of a complex dispersion relation $\tilde{k}(\omega) = \omega/\tilde{v}(\omega) = (\omega/c)\tilde{n}(\omega)$ with complex $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$ ***and*** a complex propagation speed $\tilde{v}(\omega) = v(\omega) + i\text{v}(\omega) = c/\tilde{n}(\omega)$ with accompanying complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$, and is hence commensurately more mathematically complicated….

So-called ***ellipsometry*** measurements of the *EM* radiation reflected from the surface of the metal as a function of angle of incidence yields information on the real and imaginary parts of the complex index of refraction of the metal $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$, and thus the real and imaginary parts of the complex dielectric constant and/or the complex electric susceptibility of the metal, since $\tilde{n}(\omega) = \sqrt{\tilde{\varepsilon}(\omega)/\varepsilon_o} = \sqrt{1 + \tilde{\chi}_e(\omega)}$ or $\tilde{n}^2(\omega) = \tilde{\varepsilon}(\omega)/\varepsilon_o = 1 + \tilde{\chi}_e(\omega)$.

If interested in learning more about this, *e.g.* please see/read Physics 436 Lect. Notes 8, and *e.g.* please see/read ***Optics***, M.V. Klein, p. 588-592, Wiley, 1970 {P436 reference book on reserve in the Physics library}. Please also tsee/read he UIUC P402 Optics/Light Lab Ellipsometry Lab Handout C4 and especially the references at the end. Available at: **http://online.physics.uiuc.edu/courses/phys402/exp/C4/C4.pdf**

We will discuss the ***dispersive*** nature of dielectric, ***non***-conducting materials in the next lecture…

But first, we need to remind the reader of the full Maxwell's equations in matter…

Full Maxwell Equations in Matter:

The electromagnetic state of matter at a given observation point $\vec{r}$ at a given time *t* is described by four ***macroscopic*** quantities:

1.) The volume density of free charge: $\boxed{\rho_{free}(\vec{r},t)}$ $\Leftarrow$ *aka* {free} charge density

2.) The volume density of electric dipole moments: $\boxed{\vec{\mathrm{P}}(\vec{r},t)}$ $\Leftarrow$ *aka* electric polarization

3.) The volume density of magnetic dipole moments: $\boxed{\vec{\mathrm{M}}(\vec{r},t)}$ $\Leftarrow$ *aka* magnetization

4.) The free electric current/unit area: $\boxed{\vec{J}_{free}(\vec{r},t)}$ $\Leftarrow$ *aka* {free} current density

All four of these quantities are macroscopically averaged - *i.e.* the microscopic fluctuations due to atomic/molecular makeup of matter have been smoothed out.

The four above quantities are related to the macroscopic $\vec{E}$ and $\vec{B}$ fields by the four Maxwell equations for matter (see Physics 435 Lect. Notes 24, *p.* 14):

1) Gauss' Law: $\boxed{\vec{\nabla}\bullet\vec{E} = \frac{\rho_{Tot}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\left(\rho_{free}+\rho_{bound}\right)}$, where: $\boxed{\rho_{bound} = -\vec{\nabla}\bullet\vec{\mathrm{P}}}$

Auxiliary relation: $\boxed{\vec{D} = \varepsilon_o\vec{E}+\vec{\mathrm{P}}}$ & constitutive relation: $\boxed{\vec{D} = \varepsilon\vec{E}}$

Electric polarization $\boxed{\vec{\mathrm{P}} = (\varepsilon-\varepsilon_o)\vec{E} = \varepsilon_0\chi_e\vec{E}}$, electric susceptibility: $\boxed{\chi_e = \left(\frac{\varepsilon}{\varepsilon_o}-1\right)}$

$$\boxed{\vec{\nabla}\bullet\vec{D} = \varepsilon_o\vec{\nabla}\bullet\vec{E}+\vec{\nabla}\bullet\vec{\mathrm{P}} = \rho_{free}}$$

2) No magnetic charges/monopoles: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$

Auxiliary relation: $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B}-\vec{\mathrm{M}}}$ $\Rightarrow$ $\boxed{\vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{\mathrm{M}}}$ & constitutive relation: $\boxed{\vec{B} = \mu\vec{H}}$

3) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t}-\mu_0\frac{\partial\vec{\mathrm{M}}}{\partial t}}$

Magnetization: $\boxed{\vec{\mathrm{M}} = \left(\frac{\mu}{\mu_o}-1\right)\vec{H} = \chi_m\vec{H}}$, magnetic susceptibility: $\boxed{\chi_m = \left(\frac{\mu}{\mu_o}-1\right)}$

4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J}_{ToT}+\mu_o\vec{J}_D}$ with: $\boxed{\vec{J}_D = \varepsilon_o\frac{\partial\vec{E}}{\partial t}}$

Total current density: $\boxed{\vec{J}_{ToT} = \vec{J}_{free}+\vec{J}^{mag}_{bound}+\vec{J}^{P}_{bound}}$ $\boxed{\vec{J}^{mag}_{bound} = \vec{\nabla}\times\vec{\mathrm{M}}}$ $\boxed{\vec{J}^{P}_{bound} = \frac{\partial\vec{\mathrm{P}}}{\partial t}}$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_o J_{free}+\mu_o\vec{\nabla}\times\vec{\mathrm{M}}+\mu_o\frac{\partial\vec{\mathrm{P}}}{\partial t}+\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}}$$

$$\boxed{\vec{\nabla}\times\vec{H} = \mu_o\vec{J}_{free}+\mu_o\frac{\partial\vec{D}}{\partial t}}$$

We also have Ohm's Law: $\boxed{\vec{J} = \sigma_C \vec{E}}$ and the 3 continuity equation(s): $\boxed{\vec{\nabla} \bullet \vec{J}_\alpha = -\frac{\partial \rho_\alpha}{\partial t}}$ associated with subscript index α for $\alpha =$ free, bound and total electric charge conservation.

For many/most (but <u>not</u> all!!!) physics problems, *e.g.* in optics/condensed matter physics, materials of interest are frequently ***non-magnetic*** (or ***negligibly*** magnetic) and have ***no*** free charge densities present, *i.e.* $\rho_{free} = 0$. If $\mu \simeq \mu_o$, then: $\vec{M} = 0$ and thus: $\vec{H} = \vec{B}/\mu_o$ in such non-magnetic materials.

Then Maxwell's equations in matter, for $\rho_{free} = 0$ and $\vec{M} = 0$ reduce to:

1) <u>Gauss' Law</u>: $\boxed{\vec{\nabla} \bullet \vec{D} = 0}$ or: $\boxed{\vec{\nabla} \bullet \vec{E} = -\frac{1}{\varepsilon_o} \vec{\nabla} \bullet \vec{P} = \frac{\rho_{bound}}{\varepsilon_o}}$

2) <u>No magnetic charges</u>: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$

3) <u>Faraday's Law</u>: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

4) <u>Ampere's Law</u>: $\boxed{\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \mu_o \frac{\partial \vec{P}}{\partial t} + \mu_o \vec{J}_{free}}$

We also have Ohm's Law $\boxed{\vec{J}_{free} = \sigma_c \vec{E}}$ and the Continuity eqn. $\boxed{\vec{\nabla} \bullet \vec{J}_{free} = 0}$ {here}.

Then applying the curl operator to Faraday's Law:

$$\boxed{\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_o \frac{\partial^2 \vec{P}}{\partial t^2} - \mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \vec{\nabla}\left(\vec{\nabla} \bullet \vec{E}\right) - \nabla^2 \vec{E} = \frac{1}{\varepsilon_o} \vec{\nabla} \rho_{bound} - \nabla^2 \vec{E}}$$

We obtain the ***inhomogeneous*** wave equation:

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_o} \vec{\nabla} \rho_{bound} + \mu_o \frac{\partial^2 \vec{P}}{\partial t^2} + \mu_o \frac{\partial \vec{J}_{free}}{\partial t}}_{\text{source terms}}}$$ {and a similar/analogous one for $\vec{B}$}

For nonconducting/poorly-conducting media, *i.e.* insulators/dielectrics, the first two terms on the RHS of the above equation are important – *e.g.* they explain many optical effects such as dispersion (wavelength/frequency-dependence of the index of refraction), absorption, double – refraction/bi-refringence, optical activity, ...

Note that the $\vec{\nabla} \rho_{bound} = -\vec{\nabla}\left(\vec{\nabla} \bullet \vec{P}\right)$ term is often zero, *e.g.* if the electric polarization $\vec{P}$ is uniform:

where: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$ and: $\vec{\nabla} \bullet \vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} = 0$ if $\vec{P}$ is uniform

or, if the polarization $\vec{P} \propto \vec{E}$, where *e.g.* $\vec{E}\left(\vec{r}, t\right) = E_o \cos\left(kz - \omega t + \delta\right)\hat{x}$, then: $\vec{\nabla} \bullet \vec{P} = 0$ also.

For ***good*** conductors (*e.g.* metals), the ***conduction*** ***term*** $\boxed{\mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \mu_o \sigma_C \frac{\partial \vec{E}}{\partial t}}$ is the most important, because it explains the ***opacity*** of metals (*e.g.* in the visible light region) and also explains the ***high*** ***reflectance*** of metals.

All source terms on the RHS of the above inhomogeneous wave equation are of importance for ***semi-conductors*** – however a ***proper/more*** ***complete*** physics description of *EM* wave propagation in semiconductors also requires the addition of quantum theory for rigorous treatment…