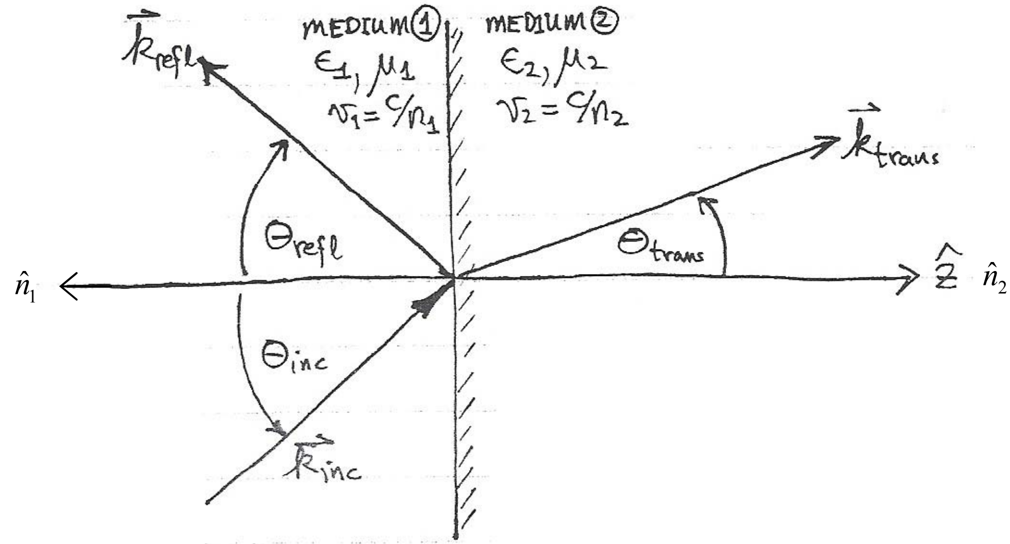
LECTURE NOTES 6.5

Reflection & Transmission of Monochromatic Plane *EM* Waves at Oblique Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

A monochromatic *EM* plane wave is incident at an ***oblique*** angle θ_{inc} on a boundary between two linear/homogeneous/isotropic media. A portion of this *EM* wave is reflected at angle θ_{refl}, a portion of this *EM* wave is transmitted, at angle θ_{trans}. The three angles are defined with respect to the unit ***normals*** to the interface $\hat{n}_1, \hat{n}_2$, as shown in the figure below:



The incident *EM* wave is: $\boxed{\tilde{\vec{E}}_{inc}(\vec{r},t) = \tilde{\vec{E}}_{o_{inc}} e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}}$ and: $\boxed{\tilde{\vec{B}}_{inc}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(\vec{r},t)}$

The reflected *EM* wave is: $\boxed{\tilde{\vec{E}}_{refl}(\vec{r},t) = \tilde{\vec{E}}_{o_{refl}} e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)}}$ and: $\boxed{\tilde{\vec{B}}_{refl}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}(\vec{r},t)}$

The transmitted *EM* wave is: $\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}}$ and: $\boxed{\tilde{\vec{B}}_{trans}(\vec{r},t) = \frac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}(\vec{r},t)}$

Note that all three *EM* waves have the same *frequency*, $f = \omega/2\pi$

This is due to the fact that at the microscopic level, the energy of real photon does *not* change in a medium, *i.e.* $E_\gamma^{vac} = E_\gamma^{med} = E_\gamma$, and since $E_\gamma = hf_\gamma$ for real photons, then $hf_\gamma^{vac} = hf_\gamma^{med} = hf_\gamma$. Thus, the frequency of a real photon does ***not*** change in a medium, *i.e.* $f_\gamma^{vac} = f_\gamma^{med} = f_\gamma$ {*n.b.* An experimental fact: *colors* of objects do *not* change when placed & viewed *e.g.* underwater}.

However, the momentum of a real photon *does* change in a medium! This is because the momentum of the real photon in a medium depends on index of refraction of that medium n_{med} via the relation $p_\gamma^{med} = n_{med} p_\gamma^{vac}$ where $n_{med} = c/v_{med}$. Thus the photon momentum depends {inversely} on the speed of propagation in the medium!

From the DeBroglie relation between momentum and wavelength of the real photon $p_\gamma = h/\lambda_\gamma$ we see that $p_\gamma^{med} = n_{med}\, p_\gamma^{vac} = n_{med}\left(h/\lambda_\gamma^{vac}\right) = h\left(n_{med}/\lambda_\gamma^{vac}\right) = h/\lambda_\gamma^{med}$ and hence $\lambda_\gamma^{med} = \lambda_\gamma^{vac}/n_{med}$.

Thus, for ***macroscopic*** *EM* waves propagating in the two linear/homogeneous/isotropic media (1) and (2), we have $f_1 = f_2 = f$, and since $\omega = 2\pi f$ then $\omega_1 = \omega_2 = \omega$.

But since: $\omega = kv$ then: $\omega_1 = \omega_2 = \omega \;\Rightarrow\; k_1 v_1 = k_2 v_2$ thus: $\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$

Now: $\boxed{k_{inc} = \left|\vec{k}_{inc}\right| = 2\pi/\lambda_1}$; $\boxed{k_{refl} = \left|\vec{k}_{refl}\right| = 2\pi/\lambda_1}$; $\boxed{k_{trans} = \left|\vec{k}_{trans}\right| = 2\pi/\lambda_2}$

And: $\boxed{\omega = \omega_1 = \omega_2 = 2\pi\left(v_1/\lambda_1\right) = 2\pi\left(v_2/\lambda_2\right)}$

Then: $\boxed{\omega = 2\pi \underset{\uparrow\, f_{inc}}{f} = 2\pi \underset{\uparrow\, f_{refl}}{f_1} = 2\pi \underset{\uparrow\, f_{trans}}{f_2}} \;\Rightarrow\; \boxed{f_1 = f_2 = f_{inc} = f_{refl} = f_{trans}}$

Then: $\boxed{\lambda_1 = \lambda_o/n_1}$ $\boxed{\lambda_2 = \lambda_o/n_2}$ where: $\boxed{\lambda_o = \text{ vacuum wavelength } = c/f}$

And: $\boxed{v_1 = c/n_1}$ $\boxed{v_2 = c/n_2}$

Thus: $\boxed{k_1 = n_1 k_o}$ $\boxed{k_2 = n_2 k_o}$ where: $\boxed{k_o = \text{ vacuum wavenumber } = 2\pi/\lambda_o = \omega/c}$

From: $\boxed{\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2}$

We see that: $\boxed{k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right) k_{trans} = \left(\frac{v_2}{v_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_{trans} = \left(\frac{n_1}{n_2}\right) k_2}$ Since $\boxed{v_i = c/n_i \quad i = 1,2}$

The ***total*** (*i.e.* combined) *EM* fields in medium 1):

$$\boxed{\tilde{\vec{E}}_{Tot_1}(\vec{r},t) = \tilde{\vec{E}}_{inc}(\vec{r},t) + \tilde{\vec{E}}_{refl}(\vec{r},t)} \quad \text{and:} \quad \boxed{\tilde{\vec{B}}_{Tot_1}(\vec{r},t) = \tilde{\vec{B}}_{inc}(\vec{r},t) + \tilde{\vec{B}}_{refl}(\vec{r},t)}$$

must be matched (*i.e.* joined smoothly) to the total *EM* fields in medium 2):

$$\boxed{\tilde{\vec{E}}_{Tot_2}(\vec{r},t) = \tilde{\vec{E}}_{trans}(\vec{r},t)} \quad \text{and:} \quad \boxed{\tilde{\vec{B}}_{Tot_2}(\vec{r},t) = \tilde{\vec{B}}_{trans}(\vec{r},t)}$$

using the boundary conditions BC1) → BC4) at $z = 0$ (in the *x*-*y* plane).

At $z = 0$, these four boundary conditions generically are of the form:

$$\boxed{(\sim\sim)\, e^{i\left(\vec{k}_{inc}\cdot\vec{r} - \omega t\right)} + (\sim\sim)\; e^{i\left(\vec{k}_{refl}\cdot\vec{r} - \omega t\right)} = (\sim\sim)\; e^{i\left(\vec{k}_{trans}\cdot\vec{r} - \omega t\right)}}$$

These boundary conditions ***must*** hold for ***all*** (x,y) on the interface at $z = 0$, and also ***must*** hold for ***arbitrary***/***any***/***all times***, t. The above relation is ***already*** satisfied for arbitrary time, t, since the factor $e^{-i\omega t}$ is common to ***all*** terms.

Thus, the following generic relation ***must*** hold for ***any*/*all*** (x,y) on the interface at $z=0$:

$$\boxed{(\sim\sim)\,e^{i\left(\vec{k}_{inc}\cdot\vec{r}\right)}+(\sim\sim)\,e^{i\left(\vec{k}_{refl}\cdot\vec{r}\right)}=(\sim\sim)\,e^{i\left(\vec{k}_{trans}\cdot\vec{r}\right)}}$$

$\Rightarrow$ For $z=0$ (*i.e.* ***on*** the interface in the x-y plane) we ***must*** have: $\vec{k}_{inc}\bullet\vec{r}=\vec{k}_{refl}\bullet\vec{r}=\vec{k}_{trans}\bullet\vec{r}$

More explicitly: $k_{inc_x}x+k_{inc_y}y+\underbrace{\cancel{k_{inc_z}z}}_{z=0}=k_{refl_x}x+k_{refl_y}y+\underbrace{\cancel{k_{refl_z}z}}_{z=0}=k_{trans_x}x+k_{trans_y}y+\underbrace{\cancel{k_{trans_z}z}}_{z=0}$

or: $k_{inc_x}x+k_{inc_y}y=k_{refl_x}x+k_{refl_y}y=k_{trans_x}x+k_{trans_y}y$ @ $z=0$ in the x-y plane.

The above relation can ***only*** hold for ***arbitrary*** $(x,y,z=0)$ ***iff*** (= if and only if):

$$k_{inc_x}x=k_{refl_x}x=k_{trans_x}x\quad\Rightarrow\quad k_{inc_x}=k_{refl_x}=k_{trans_x}$$

and: $k_{inc_y}y=k_{refl_y}y=k_{trans_y}y\quad\Rightarrow\quad k_{inc_y}=k_{refl_y}=k_{trans_y}$

Since this problem has ***rotational invariance*** (*i.e.* rotational ***symmetry***) about the $\hat{z}$-axis, (see above pix on *p*. 1), without any loss of generality we can *e.g.* choose $\vec{k}_{inc}$ to lie entirely within the x-z plane, as shown in the figure below…

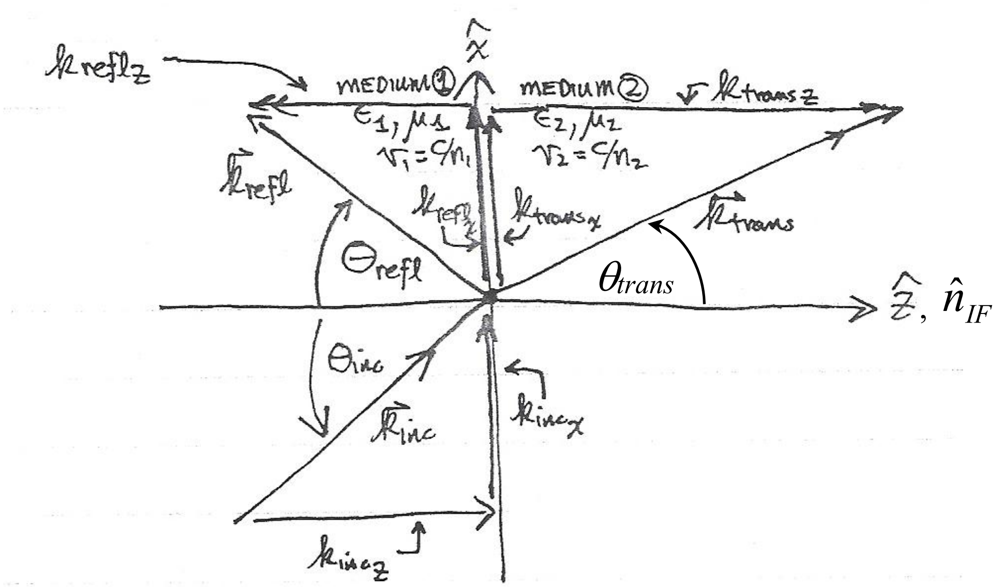
Then: $k_{inc_y}=k_{refl_y}=k_{trans_y}=0$ and thus: $k_{inc_x}=k_{refl_x}=k_{trans_x}$.

i.e. the ***transverse*** components of $\vec{k}_{inc},\vec{k}_{refl},\vec{k}_{trans}$ are all equal and point in the {same} $+\hat{x}$ direction.

The First Law of Geometrical Optics (All wavevectors *k* lie in a common plane):

The above result tells us that the three wave vectors $\vec{k}_{inc},\vec{k}_{refl}$ and $\vec{k}_{trans}$ ALL LIE IN A PLANE known as the **plane of incidence** (***here***, the x-z plane) ***and*** that: $k_{inc_x}=k_{refl_x}=k_{trans_x}$ as shown in the figure below. Note that the plane of incidence also includes the ***unit normal*** to the interface {***here***}, $\hat{n}_{IF}=+\hat{z}$-axis.

The *x*-*z* Plane of Incidence:



The Second Law of Geometrical Optics (Law of Reflection):

From the above figure, we see that:

$$\boxed{k_{inc_x} = k_{inc}\sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl}\sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans}\sin\theta_{trans}}$$

But: $\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$

$\Rightarrow$ Angle of Incidence = Angle of Reflection $\boxed{\theta_{inc} = \theta_{refl}}$ Law of Reflection!

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle, θ_{trans} we see that: $\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}}$

In medium 1): $\boxed{k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o}$

where $\boxed{k_o = \text{ vacuum wave number } = 2\pi/\lambda_o}$ and $\boxed{\lambda_o = \text{ vacuum wave length}}$

In medium 2): $\boxed{k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o}$

Thus: $\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}} \Rightarrow \boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}}$

But since: $\boxed{k_{inc} = k_1 = n_1 k_o}$ and $\boxed{k_{trans} = k_2 = n_2 k_o}$

Then: $\boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}} \Rightarrow \boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}}$ Law of Refraction (Snell's Law)

Which can also be written as: $\boxed{\dfrac{\sin\theta_{trans}}{\sin\theta_{inc}} = \dfrac{n_1}{n_2}}$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1) we can also write Snell's Law as:

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \quad \text{or:} \quad \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}}$$

(incident) (transmitted)

Because of the above ***three laws of geometrical optics***, we see that:

$\boxed{\vec{k}_{inc}\cdot\vec{r}\,\big|_{z=0} = \vec{k}_{refl}\cdot\vec{r}\,\big|_{z=0} = \vec{k}_{trans}\cdot\vec{r}\,\big|_{z=0}}$ everywhere at/on the interface @ $z = 0$ in the x-y plane.

Thus we see that: $\boxed{e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}\big|_{z=0} = e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)}\big|_{z=0} = e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}\big|_{z=0}}$ everywhere at/on the interface at $z = 0$ in the x-y plane, this relation is also valid/holds for any/all time(s) t, since ω is the same in either medium (1 or 2).

Thus, the boundary conditions BC 1) → BC 4) for a monochromatic plane *EM* wave incident at/ on an interface at an oblique angle θ_{inc} between two linear/homogeneous/isotropic media become:

BC 1): Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges):

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}}} \quad \left\{\text{using } \vec{D} = \varepsilon\vec{E}\right\}$$

BC 2): Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_{x,y}}} + \tilde{E}_{o_{refl_{x,y}}}\right) = \tilde{E}_{o_{trans_{x,y}}}}$$

BC 3): Normal (*i.e.* *z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 4): Tangential (*i.e.* *x*-, *y*-) components of $\vec{H}$ continuous at $z = 0$ (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}}$$

Note that in each of the above, we also have the relation $\boxed{\tilde{\vec{B}}_o = \frac{1}{v}\hat{k}\times\tilde{\vec{E}}_o}$

For a *EM* plane wave incident on a boundary between two linear / homogeneous / isotropic media at an ***oblique*** angle of incidence, there are ***three*** possible **polarization** cases to consider:

Case I): $\tilde{\vec{E}}_{inc} \perp$ plane of incidence – known as **Transverse Electric (*TE*) Polarization**

{ $\tilde{\vec{B}}_{inc} \parallel$ plane of incidence}

Case II): $\tilde{\vec{E}}_{inc} \parallel$ plane of incidence – known as **Transverse Magnetic (*TM*) Polarization**

{ $\tilde{\vec{B}}_{inc} \perp$ plane of incidence}

Case III): ***The most general case***: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the plane of incidence.

{$\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the plane of incidence}

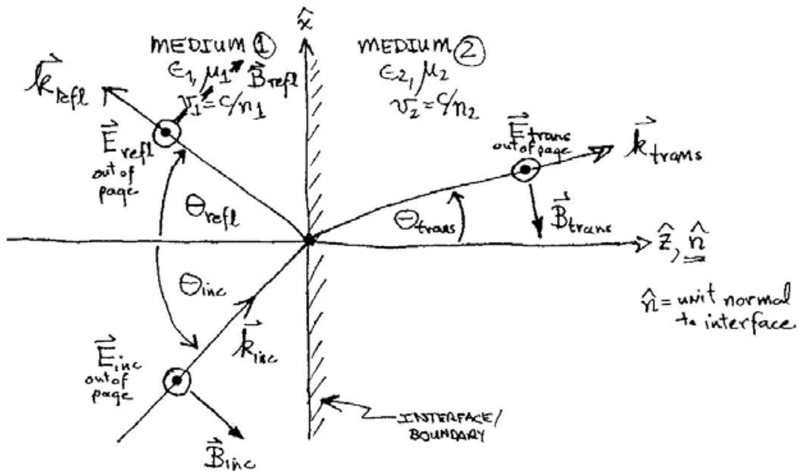
i.e. Case III is a linear ***vector*** combination of Cases I) and II) above!

- LP vector: $\boxed{\hat{n}_{inc}^{LP} = \cos\varphi\hat{x} + \sin\varphi\hat{y} = \cos\varphi\,\hat{\in}_{\perp} + \sin\varphi\,\hat{\in}_{\parallel}}$
- CP vector: $\boxed{\hat{n}_{inc}^{CP} = \hat{x} \mp i\hat{y}}$, also the EP (elliptical polarization case.

$\Rightarrow$ Simply decompose the polarization components for the general-case incident *EM* plane wave into its $\hat{x} = \hat{\in}_{\perp}$ and $\hat{y} = \hat{\in}_{\parallel}$ vector components – *i.e.* the *E*-field components perpendicular to and parallel to the plane of incidence (*TE* polarization and *TM* polarization respectively).
Solve these separately, then combine results vectorially…

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE*) Polarization

A monochromatic plane *EM* wave is incident {from the left} on a boundary located at $z = 0$ in the *x*-*y* plane between two linear / homogeneous / isotropic media at an oblique angle of incidence. The ***polarization*** of the incident *EM* wave (*i.e.* the ***orientation*** of $\vec{E}_{inc}$ is ***transverse*** (*i.e.* $\perp$) to the plane of incidence {= the *x*-*z* plane containing the three wavevectors $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ ***and*** the unit normal to the boundary/interface, $\hat{n} = +\hat{z}$ }), as shown in the figure below:



Note that all three $\vec{E}$-field vectors are $\parallel \hat{y}$ {*i.e.* point out of the page} and thus all three $\vec{E}$-field vectors are $\parallel$ to the boundary/interface at $z = 0$, which lies in the *x*-*y* plane.

Since the three $\vec{B}$-field vectors are related to their respective $\vec{E}$-field vectors by the right-hand rule cross-product relation $\vec{B} = \frac{1}{v}\hat{k} \times \vec{E}$ then we see that all three $\vec{B}$-field vectors lie in the *x*-*z* plane {the plane of incidence}, as shown in the figure above.

The four boundary conditions on the {complex} $\vec{E}$- and $\vec{B}$-fields on the boundary at $z = 0$ are:

BC 1) Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges)

$$\boxed{\varepsilon_1\left(\cancelto{=0}{\tilde{E}_{o_{inc_z}}} + \cancelto{=0}{\tilde{E}_{o_{refl_z}}}\right) = \varepsilon_2\, \cancelto{=0}{\tilde{E}_{o_{trans_z}}}} \;\Rightarrow\; \boxed{0 + 0 = 0} \quad \text{\{see/refer to above figure\}}$$

BC 2) Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_y}} + \tilde{E}_{o_{refl_y}}\right) = \tilde{E}_{o_{trans_y}}} \;\Rightarrow\; \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}} \quad \{n.b.\text{ All } E_x\text{'s} = 0 \text{ for } TE \text{ Polarization}\}$$

BC 3) Normal (*i.e.* *z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 3) {continued}: *n.b.* Since only the <u>*z*-components</u> of $\vec{B}$'s on either side of interface are involved here, and all unit wavevectors $\hat{k}_{inc}$, $\hat{k}_{refl}$ and $\hat{k}_{trans}$ lie in the plane of incidence (*x*-*y* plane) and all $\vec{E}$-field vectors are || to the $+\hat{y}$ direction for *TE* polarization, then because of the cross-product nature of $\tilde{\vec{B}} = \frac{1}{v}\hat{k}\times\tilde{\vec{E}}$, we only need the <u>*x*-components</u> of the unit wavevectors, *i.e.*:

$$\left.\begin{array}{l}
\boxed{\hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z}} \\
\boxed{\hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z}} \\
\boxed{\hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z}}
\end{array}\right\} \boxed{\text{See/refer to above figure}}$$

$$\boxed{\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z}} = \boxed{\frac{1}{v_1}\left(\hat{k}_{inc_x}\times\tilde{E}_{o_{inc_y}}\hat{y} + \hat{k}_{refl_x}\times\tilde{E}_{o_{refl_y}}\hat{y}\right) = \frac{1}{v_2}\left(\hat{k}_{trans_x}\times\tilde{E}_{o_{trans_y}}\hat{y}\right)} \quad \boxed{\{\hat{x}\times\hat{y} = +\hat{z}\}}$$

$$= \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc}\{\hat{x}\times\hat{y}\} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\{\hat{x}\times\hat{y}\}\right) = \frac{1}{v_2}\left(\tilde{E}_{o_{trans}}\sin\theta_{trans}\{\hat{x}\times\hat{y}\}\right)}$$

$$= \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}}$$

BC 4) Tangential (*i.e.* *x*-, *y*-) components of $\vec{H}$ continuous at *z* = 0 (no free surface currents):

n.b. Same reasoning as in BC3 above, but here we only need the <u>*z*-components</u> of the unit wavevectors, *i.e.*:

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x} + \tilde{B}_{o_{refl_x}}\hat{x}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}} \quad \{n.b.\ \text{All } B_y^{'s} = 0 \text{ for } TE \text{ Polarization – see above pix}\}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\hat{k}_{inc_z}\times\tilde{E}_{o_{inc_y}}\hat{y} + \hat{k}_{refl_z}\times\tilde{E}_{o_{refl_y}}\hat{y}\right) = \frac{1}{\mu_2 v_2}\left(\hat{k}_{trans_z}\times\tilde{E}_{o_{trans_y}}\hat{y}\right)} \quad \boxed{\{\hat{z}\times\hat{y} = -\hat{x}\}}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}\cos\theta_{inc}\{\hat{z}\times\hat{y}\} + \tilde{E}_{o_{refl}}\cos\theta_{refl}\{-\hat{z}\times\hat{y}\}\right) = \frac{1}{\mu_2 v_2}\left(\tilde{E}_{o_{trans}}\cos\theta_{trans}\{\hat{z}\times\hat{y}\}\right)}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}(-\cos\theta_{inc}) + \tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}(-\cos\theta_{trans})\hat{x}}$$

<u>Thus, we obtain</u>: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$ (from BC 2))

Using the Law of Reflection $\theta_{inc} = \theta_{refl}$ on the BC 3) result: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Using Snell's Law / Law of Refraction:

$$\boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

<u>or</u>: $\boxed{v_2\sin\theta_{inc} = v_1\sin\theta_{trans}}$ <u>or</u>: $\boxed{\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

$$\therefore \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \tilde{E}_{o_{trans}}}$$ *i.e.* BC 3) gives the same info as BC 1) !

From the BC 4) result: $$\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

Thus, {again} from BC 1) → BC 4) we have ***only*** **two** independent relations, but ***three*** unknowns for the case of transverse electric (*TE*) polarization:

1) $$\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$$

2) $$\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

Now: $\boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \frac{Z_1}{Z_2}}$ and we also define: $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ {*n.b.* Both α and $\beta > 0$ and ***real***}

Then eqn. 2) above becomes: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\,\tilde{E}_{o_{trans}}}$ and eqn. 1) is: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Add equations 1) + 2) to get: $\boxed{2\tilde{E}_{o_{inc}} = (1+\alpha\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ eqn. (1+2)

Subtract eqn's 2) − 1) to get: $\boxed{2\tilde{E}_{o_{refl}} = (1-\alpha\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\tilde{E}_{o_{trans}}}$ eqn. (2−1)

Plug eqn. (2+1) into eqn. (2−1) to obtain: $\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$

Thus: $\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ and $\boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ or: $\boxed{\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}$ and $\boxed{\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)}$

n.b. since ***both*** α and $\beta > 0$ and purely ***real*** quantities then: $\left(\frac{2}{1+\alpha\beta}\right) > 0$ and hence the transmitted wave is ***always*** in-phase with the incident wave for *TE* polarization.

The ratios of electric field amplitudes @ $z = 0$ for transverse electric (*TE*) polarization are thus:

The Fresnel Equations for $\vec{E}$ || to Interface @ *z* = 0

$\vec{E}\perp$ to Plane of Incidence = Transverse Electric (*TE*) Polarization

$$\boxed{\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}\ \text{and:}\ \boxed{\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)}\ \text{with:}\ \boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}\ \text{and:}\ \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

Now because the incident monochromatic plane *EM* wave strikes the interface (lying in the *x-y* plane) at an ***oblique*** angle θ_{inc}, the time-averaged EM wave ***intensity*** is $I(z=0) \equiv \left\langle \vec{\tilde{S}}(z=0,t) \right\rangle \cdot \hat{n}$ $\left(Watts/m^2\right)$ where $\hat{n}$ is the unit normal of the interface, oriented in the direction of energy flow.

Because the ***incident*** *EM* wave is propagating in a linear / homogeneous / isotropic medium, we have the relation: $\left\langle \vec{\tilde{S}}_{inc}(z=0,t) \right\rangle = v_1 \left\langle u_{EM}^{inc}(z=0,t) \right\rangle \hat{k}_{inc}$. Thus, the time-averaged ***incident*** intensity (*aka* ***irradiance***) for an ***oblique*** angle of incidence is:

$$\boxed{I_{inc}(z=0) \equiv \left\langle \vec{\tilde{S}}_{inc}(z=0,t) \right\rangle \cdot \hat{z} = v_1 \left\langle u_{EM}^{inc}(z=0,t) \right\rangle \hat{k}_{inc} \cdot \hat{z} = v_1 \left\langle u_{EM}^{inc}(z=0,t) \right\rangle \cos\theta_{inc}}$$

<u>For *TE* polarization the incident, reflected and transmitted intensities are</u>:

$$I_{inc}^{TE}(z=0) \equiv \left\langle \vec{\tilde{S}}_{inc}(z=0,t) \right\rangle \cdot \hat{z} = v_1 \left\langle u_{EM}^{inc}(z=0,t) \right\rangle \hat{k}_{inc} \cdot \hat{z} = \tfrac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}$$

$$I_{refl}^{TE}(z=0) \equiv \left\langle \vec{\tilde{S}}_{refl}(z=0,t) \right\rangle \cdot (-\hat{z}) = v_1 \left\langle u_{EM}^{refl}(z=0,t) \right\rangle \hat{k}_{inc} \cdot (-\hat{z}) = \tfrac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos\theta_{inc}$$

n.b. used law of reflection: $\theta_{refl} = \theta_{inc}$

$$I_{trans}^{TE}(z=0) \equiv \left\langle \vec{\tilde{S}}_{trans}(z=0,t) \right\rangle \cdot \hat{z} = v_2 \left\langle u_{EM}^{trans}(z=0,t) \right\rangle k_{trans} \cdot \hat{z} = \tfrac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}$$

Thus the reflection and transmission coefficients for transverse electric (*TE*) polarization (with all $\vec{E}$-field vectors oriented $\perp$ to the plane of incidence) @ $z = 0$ are:

$$\boxed{R_{TE} \equiv \frac{I_{refl}^{TE}(z=0)}{I_{inc}^{TE}(z=0)} = \frac{\tfrac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos\overbrace{\theta_{inc}}^{=\theta_{refl}}}{\tfrac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

$$\boxed{T_{TE} \equiv \frac{I_{trans}^{TE}(z=0)}{I_{inc}^{TE}(z=0)} = \frac{\tfrac{1}{2}\varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}}{\tfrac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

<u>But</u>: $\boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right) = \frac{Z_1}{Z_2}}$ <u>and</u>: $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ $\therefore$ $\boxed{T_{TE} = \alpha\beta \left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$

<u>And from above (*p*. 8)</u>: $\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}$ and $\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\frac{2}{1+\alpha\beta}\right)}$

<u>Thus</u>: $\boxed{R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$ <u>and</u>: $\boxed{T_{TE} = \alpha\beta \left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}}$

Explicit Check: Does $R_{TE}+T_{TE}=1$? (*i.e.* is *EM* wave ***energy*** conserved?)

$$\frac{(1-\alpha\beta)^2}{(1+\alpha\beta)^2}+\frac{4\alpha\beta}{(1+\alpha\beta)^2}=\frac{1-2\alpha\beta+\alpha^2\beta^2+4\alpha\beta}{(1+\alpha\beta)^2}=\frac{1+2\alpha\beta+\alpha^2\beta^2}{(1+\alpha\beta)^2}=\frac{\cancel{(1+\alpha\beta)^2}}{\cancel{(1+\alpha\beta)^2}}=1 \quad \textbf{\textit{Yes}}\ !!!$$

Note that at ***normal*** incidence: $\boxed{\theta_{inc}=0} \Rightarrow \boxed{\theta_{refl}=0}$ and $\boxed{\theta_{trans}=0}$ {See/refer to above figure}

Then: $\boxed{\alpha\equiv\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)=\frac{\cos 0}{\cos 0}=1} \Rightarrow \boxed{\alpha=1}$

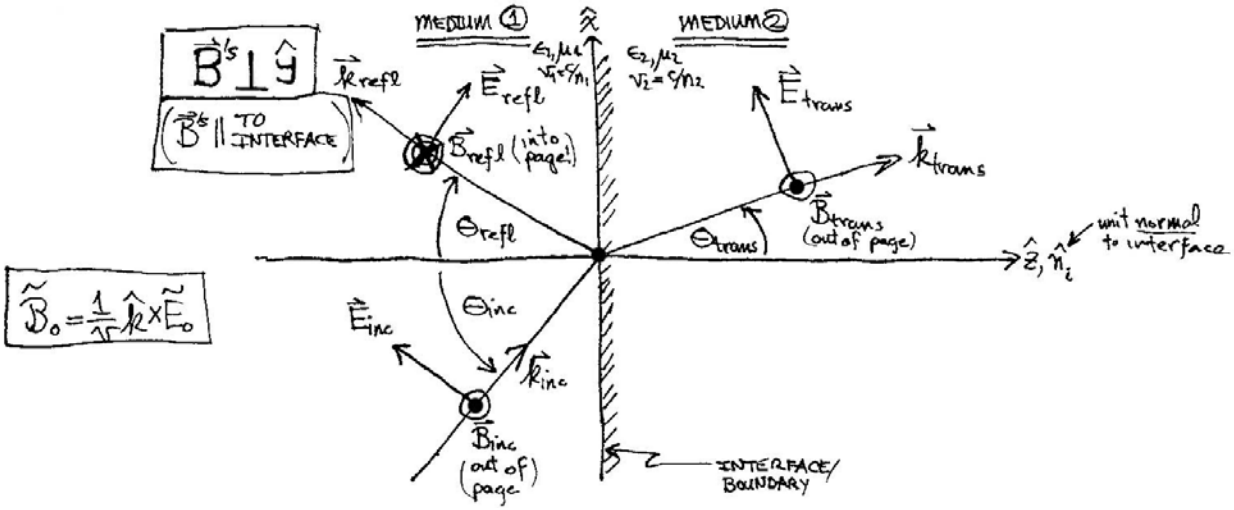
Thus, at ***normal*** incidence: $\boxed{R_{TE}\big|_{\theta_{inc}=0}=\left(\frac{1-\beta}{1+\beta}\right)^2}$ and: $\boxed{T_{TE}\big|_{\theta_{inc}=0}=\frac{4\beta}{(1+\beta)^2}}$

Note that these results for $R_{TE}\big|_{\theta_{inc}=0}$ and $T_{TE}\big|_{\theta_{inc}=0}$ are the same/identical to those we obtained previously for a monochromatic plane *EM* wave at ***normal*** incidence on interface!!!

In the special/limiting-case situation of normal incidence, where $\boxed{\theta_{inc}=\theta_{refl}=\theta_{trans}=0}$, the plane of incidence ***collapses*** into a ***line*** (the $\hat{z}$ axis), the problem then has ***rotational invariance*** about the $\hat{z}$ axis, and thus for normal incidence the polarization direction associated with the spatial orientation of $\vec{E}_{inc}$ no longer has any physical consequence(s).

Case II): Electric Field Vectors Parallel to the Plane of Incidence: Transverse Magnetic (*TM*) Polarization

A monochromatic plane *EM* wave is incident {from the left} on a boundary located at $z = 0$ in the *x-y* plane between two linear / homogeneous / isotropic media at an oblique angle of incidence. The ***polarization*** of the incident *EM* wave (*i.e.* the ***orientation*** of $\vec{E}_{inc}$ is now ***parallel*** (*i.e.* $\|$) to the ***plane of incidence*** {= the *x-z* plane containing the three wavevectors $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ ***and*** the unit normal to the boundary/interface, $\hat{n} = +\hat{z}$ }), as shown in the figure below:



For *TM* polarization, all three $\vec{E}$-field vectors lie in the plane of incidence.

Since the three $\vec{B}$-field vectors are related to their respective $\vec{E}$-field vectors by the right-hand rule cross-product relation $\vec{B} = \frac{1}{v}\hat{k} \times \vec{E}$ then we see that all three $\vec{B}$-field vectors are $\| \hat{y}$ {*i.e.* either point out of or into the page} and thus are $\perp$ to the plane of incidence {hence the origin of the name ***transverse magnetic polarization***}; hence note that all three $\vec{B}$-field vectors are $\|$ to the boundary/interface at $z = 0$, which lies in the *x-y* plane as shown in the figure above.

The four boundary conditions on the {complex} $\vec{E}$- and $\vec{B}$-fields on the boundary at $z = 0$ are:

BC 1) Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges)

$$\boxed{\varepsilon_1 \left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}} \right) = \varepsilon_2 \tilde{E}_{o_{trans_z}}}$$

$$\boxed{\varepsilon_1 \left(-\tilde{E}_{o_{inc}} \sin\theta_{inc} + \tilde{E}_{o_{refl}} \sin\theta_{refl} \right) = \varepsilon_2 \left(-\tilde{E}_{o_{trans}} \sin\theta_{trans} \right)}$$ {*n.b.* see/refer to above figure}

BC 2) Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_x}} + \tilde{E}_{o_{refl_x}} \right) = \tilde{E}_{o_{trans_x}}}$$

$$\boxed{\left(\tilde{E}_{o_{inc}} \cos\theta_{inc} + \tilde{E}_{o_{refl}} \cos\theta_{refl} \right) = \tilde{E}_{o_{trans}} \cos\theta_{trans}}$$ {*n.b.* see/refer to above figure}

BC 3) Normal (*i.e. z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\overset{=0}{\cancel{\tilde{B}_{o_{inc_z}}}} + \overset{=0}{\cancel{\tilde{B}_{o_{refl_z}}}}\right) = \overset{=0}{\cancel{\tilde{B}_{o_{trans_z}}}}} \Rightarrow \boxed{0+0=0} \quad \{n.b. \text{ see/refer to above figure}\}$$

BC 4) Tangential (*i.e. x*-, *y*-) components of $\vec{H}$ continuous at $z = 0$ (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \quad \{n.b. \text{ All } B_x^{\,'s} = 0 \text{ for } TM \text{ Polarization}\}$$

$$\therefore \quad \boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}}\hat{y} + \tilde{B}_{o_{refl_y}}\hat{y}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\hat{y}\right)} \quad n.b. \text{ Can use full cross-product(s) } \tilde{\vec{B}} = \tfrac{1}{v}\hat{k}\times\tilde{\vec{E}} \text{ here!}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\hat{k}_{inc}\times\tilde{\vec{E}}_{o_{inc}} + \hat{k}_{refl}\times\tilde{\vec{E}}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\left(\hat{k}_{trans}\times\tilde{\vec{E}}_{o_{trans}}\right)}$$

Use right-hand rule for all cross-products

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}\hat{y} - \tilde{E}_{o_{refl}}\hat{y}\right) = \frac{1}{\mu_2 v_2}\left(\tilde{E}_{o_{trans}}\hat{y}\right)} \quad \{n.b. \text{ see/refer to above figure}\}$$

$$\therefore \quad \boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \Rightarrow \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$$

From BC 1) at $z = 0$:

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} - \tilde{E}_{o_{refl}}\sin\theta_{refl}\right) = \varepsilon_2\left(\tilde{E}_{o_{trans}}\sin\theta_{trans}\right)}$$

Redundant info – both BC's give same relation

<u>But:</u> $\boxed{\theta_{inc} = \theta_{refl}}$ (Law of Reflection) <u>and</u>: $\boxed{n_1 = \frac{c}{v_1}}$, $\boxed{n_2 = \frac{c}{v_2}}$

<u>And:</u> $\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \Rightarrow \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{\sin\theta_{trans}}{\sin\theta_{inc}} = \frac{n_1}{n_2}(\text{Snell's Law}) = \frac{v_2}{v_1}}$

$$\therefore \quad \boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2}\right)\tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\tilde{E}_{o_{trans}} = \beta\,\tilde{E}_{o_{trans}}}$$

From BC 4) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\tilde{E}_{o_{trans}} = \beta\tilde{E}_{o_{trans}}} \quad \text{where: } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$$

From BC 2) at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc}}\cos\theta_{inc} + \tilde{E}_{o_{refl}}\cos\theta_{refl}\right) = \tilde{E}_{o_{trans}}\cos\theta_{trans}} \quad \text{but: } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\therefore \quad \boxed{\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha\tilde{E}_{o_{trans}}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\tilde{E}_{o_{trans}}} \text{ and } \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha\tilde{E}_{o_{trans}}} \text{ with } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)} \text{ and } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

Solving these two above equations simultaneously, we obtain:

$$\boxed{2\tilde{E}_{o_{inc}} = (\alpha+\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}}$$

and: $$\boxed{2\tilde{E}_{o_{refl}} = (\alpha-\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}}}$$

$$\Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}}$$

The real / physical electric field amplitudes for transverse magnetic (*TM*) polarization are thus:

The Fresnel Equations for $\vec{B} \parallel$ to Interface

$\vec{B} \perp$ to Plane of Incidence = Transverse Magnetic (*TM*) Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)} \text{ and } \boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{2}{\alpha+\beta}\right)} \text{ with } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}} \text{ and } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right) = \frac{Z_1}{Z_2}}$$

The Fresnel relations for *TM* polarization are ***not*** identical to the Fresnel relations for *TE* polarization:

$$\boxed{\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \text{ and } \boxed{\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{2}{1+\alpha\beta}\right)}$$

We define the incident, reflected & transmitted intensities at oblique incidence on the interface @ $z = 0$ for the *TM* case exactly as we did for the *TE* case:

$$\boxed{I^{TM}_{inc}(z=0) = \left\langle \vec{\tilde{S}}^{TM}_{inc}(z=0,t)\right\rangle \cdot \hat{z} = \tfrac{1}{2}\varepsilon_1 v_1 \left(E^{TM}_{o_{inc}}\right)^2 \cos\theta_{inc}}$$

$$\boxed{I^{TM}_{refl}(z=0) = \left\langle \vec{S}^{TM}_{refl}(z=0,t)\right\rangle \cdot (-\hat{z}) = \tfrac{1}{2}\varepsilon_1 v_1 \left(E^{TM}_{o_{refl}}\right)^2 \cos\theta_{inc} \quad \left(\theta_{ref} = \theta_{inc}\right)}$$

$$\boxed{I^{TM}_{trans}(z=0) = \left\langle \vec{S}^{TM}_{trans}(z=0t)\right\rangle \cdot \hat{z} = \tfrac{1}{2}\varepsilon_2 v_2 \left(E^{TM}_{o_{trans}}\right)^2 \cos\theta_{trans}}$$

The reflection and transmission coefficients for transverse magnetic (*TM*) polarization (with all $\vec{B}$ -field vectors oriented $\perp$ to the plane of incidence @ $z = 0$) are:

$$\boxed{R_{TM} \equiv \frac{I^{TM}_{refl}(z=0)}{I^{TM}_{inc}(z=0)} = \left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2}$$

$$\boxed{T_{TM} \equiv \frac{I^{TM}_{trans}(z=0)}{I^{TM}_{inc}(z=0)} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right)^2 = \alpha\beta\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}}$$

i.e. $\boxed{R_{TM} = \left(\dfrac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2}$ and: $\boxed{T_{TM} = \alpha\beta\left(\dfrac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \dfrac{4\alpha\beta}{(\alpha+\beta)^2}}$

Again, note that the reflection and transmission coefficients for transverse magnetic (*TM*) polarization are ***not*** identical/the same as those for the transverse electric case:

$\boxed{R_{TE} = \left(\dfrac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$ and: $\boxed{T_{TE} = \alpha\beta\left(\dfrac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \dfrac{4\alpha\beta}{(1+\alpha\beta)^2}}$

Explicit Check: Does $R_{TM} + T_{TM} = 1$? (*i.e.* is *EM* wave ***energy*** conserved?)

$$R_{TM} + T_{TM} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 + \frac{4\alpha\beta}{(\alpha+\beta)^2} = \frac{\alpha^2 - 2\alpha\beta + \beta^2 + 4\alpha\beta}{(\alpha+\beta)^2} = \frac{\alpha^2 + 2\alpha\beta + \beta^2}{(\alpha+\beta)^2} = \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2} = 1 \text{ Yes !!!}$$

Note again at ***normal*** incidence: $\theta_{inc} = 0 \;\Rightarrow\; \theta_{refl} = 0$ and $\theta_{trans} = 0$ {See/refer to above figure}

Then: $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) = \dfrac{\cos 0}{\cos 0} = 1} \Rightarrow \boxed{\alpha = 1}$

Thus at ***normal*** incidence: $\boxed{R_{TM}\big|_{\theta_{inc}=0} = \left(\dfrac{1-\beta}{1+\beta}\right)^2}$ and $\boxed{T_{TM}\big|_{\theta_{inc}=0} = \dfrac{4\beta}{(1+\beta)^2}}$

These ***are*** identical to those for the *TE* case at normal incidence, as expected – due to rotational invariance / symmetry about the $\hat{z}$ axis:

At ***normal*** incidence: $\boxed{R_{TE}\big|_{\theta_{inc}=0} = \left(\dfrac{1-\beta}{1+\beta}\right)^2}$ and $\boxed{T_{TE}\big|_{\theta_{inc}=0} = \dfrac{4\beta}{(1+\beta)^2}}$

The Fresnel Relations

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)=\frac{2}{(1+\alpha\beta)}}$$

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}=\frac{Z_1}{Z_2}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)=\frac{2}{(\alpha+\beta)}}$$

$$\boxed{v_1={c}/{n_1}={1}/{\sqrt{\varepsilon_1\mu_1}}}$$

$$\boxed{v_2={c}/{n_2}={1}/{\sqrt{\varepsilon_2\mu_2}}}$$

Reflection and Transmission Coefficients *R* & *T*
R + *T* = 1

TE Polarization

$$\boxed{R_{TE}\equiv\frac{I_{refl}^{TE}}{I_{inc}^{TE}}=\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$$

$$\boxed{T_{TE}\equiv\left(\frac{I_{trans}^{TE}}{I_{inc}^{TE}}\right)=\alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2=\frac{4\alpha\beta}{(1+\alpha\beta)^2}}$$

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}=\frac{Z_1}{Z_2}}$$

TM Polarization

$$\boxed{R_{TM}\equiv\frac{I_{refl}^{TM}}{I_{inc}^{TM}}=\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2}$$

$$\boxed{T_{TM}\equiv\left(\frac{I_{trans}^{TM}}{I_{inc}^{TM}}\right)=\alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2=\frac{4\alpha\beta}{(\alpha+\beta)^2}}$$

$$\boxed{v_1={c}/{n_1}={1}/{\sqrt{\varepsilon_1\mu_1}}}$$

$$\boxed{v_2={c}/{n_2}={1}/{\sqrt{\varepsilon_2\mu_2}}}$$

Note that since $E_{o_{1,2}}^2=\left\langle n_{\gamma_{1,2}}(t)\right\rangle E_\gamma/\varepsilon_{1,2}$, the reflection coefficient/reflectance *R* can thus be seen as the statistical/ensemble average ***probability*** that at the microscopic scale, individual photons will be reflected at the interface: $R=\left(E_{o_{refl}}/E_{o_{inc}}\right)^2=\left\langle n_{\gamma_{refl}}(t)\right\rangle/\left\langle n_{\gamma_{inc}}(t)\right\rangle=P_{refl}$, and since $R+T=1$ then $T=1-R=1-P_{refl}=P_{trans}$, since we must have $P_{refl}+P_{trans}=1$!!!

Now we want to explore / investigate the physics associated with the Fresnel relations and the reflection and transmission coefficients – comparing results for *TE vs. TM* polarization for the cases of ***external*** reflection ($n_1 < n_2$) and ***internal*** reflection $n_1 > n_2$)

Just as β can be written several different but equivalent ways (see above), so can the Fresnel relations, as well as the expressions for *R* & *T* using various relations including Snell's Law.

Starting with the Fresnel relations as given above, explicitly writing these out alternate versions:

Fresnel Relations

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)=\frac{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}-\left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}+\left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)=\frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}+\left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)=\frac{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc}-\left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc}+\left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)=\frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc}+\left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

If we now neglect / ignore the magnetic properties of the two media – *e.g.* if paramagnetic / diamagnetic such that $|\chi_m|\ll 1$ then $\mu_1\simeq\mu_2\simeq\mu_o$ the Fresnel relations then become:

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)\simeq\frac{n_1\cos\theta_{inc}-n_2\cos\theta_{trans}}{n_1\cos\theta_{inc}+n_2\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)\simeq\frac{2n_1\cos\theta_{inc}}{n_1\cos\theta_{inc}+n_2\cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)\simeq\frac{-n_2\cos\theta_{inc}+n_1\cos\theta_{trans}}{n_2\cos\theta_{inc}+n_1\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)\simeq\frac{2n_1\cos\theta_{inc}}{n_2\cos\theta_{inc}+n_1\cos\theta_{trans}}$$

Using Snell's Law $n_1\sin\theta_1=n_2\sin\theta_2 \;\Rightarrow\; n_{inc}\sin\theta_{inc}=n_{trans}\sin\theta_{trans}$ and various trigonometric identities, the above Fresnel relations can also equivalently be written as:

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)\simeq-\frac{\sin(\theta_{inc}-\theta_{trans})}{\sin(\theta_{inc}+\theta_{trans})}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)\simeq\frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc}+\theta_{trans})}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)\simeq-\frac{\tan(\theta_{inc}-\theta_{trans})}{\tan(\theta_{inc}+\theta_{trans})}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{trans}}^{TM}}\right)\simeq\frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc}+\theta_{trans})\cos(\theta_{inc}-\theta_{trans})}$$

n.b. the signs correlate to the *TE* & *TM* $\vec{E}$-field vectors as shown in the above figures!

We now use Snell's Law $n_{inc}\sin\theta_{inc} = n_{trans}\sin\theta_{trans}$ to eliminate θ_{trans}:

TE Polarization

$$\boxed{\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

$$\boxed{\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{2\cos\theta_{inc}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

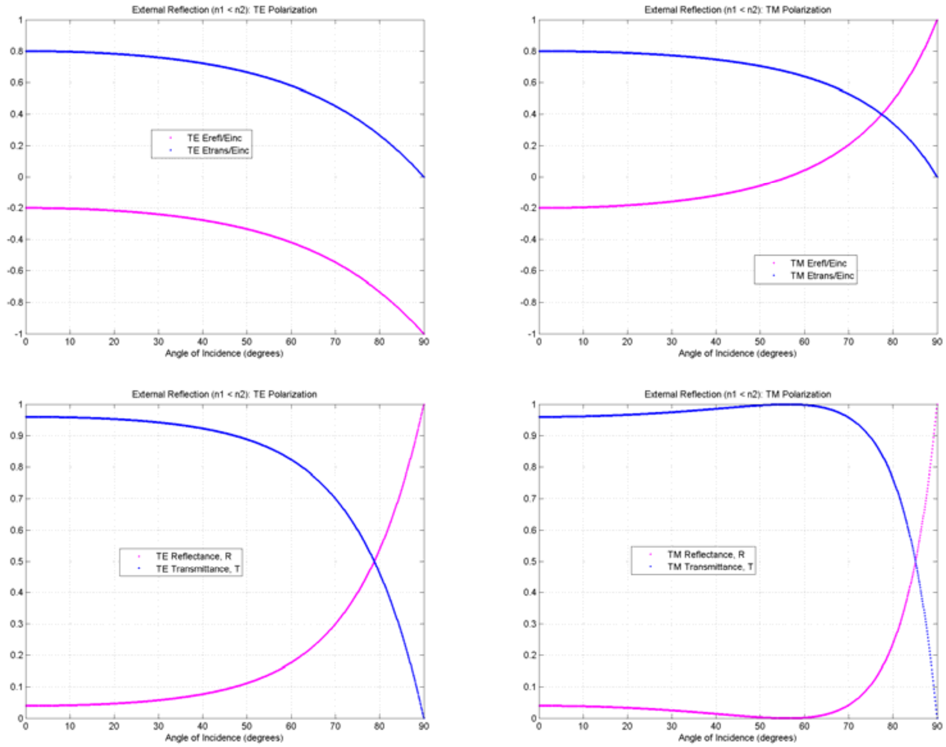
TM Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

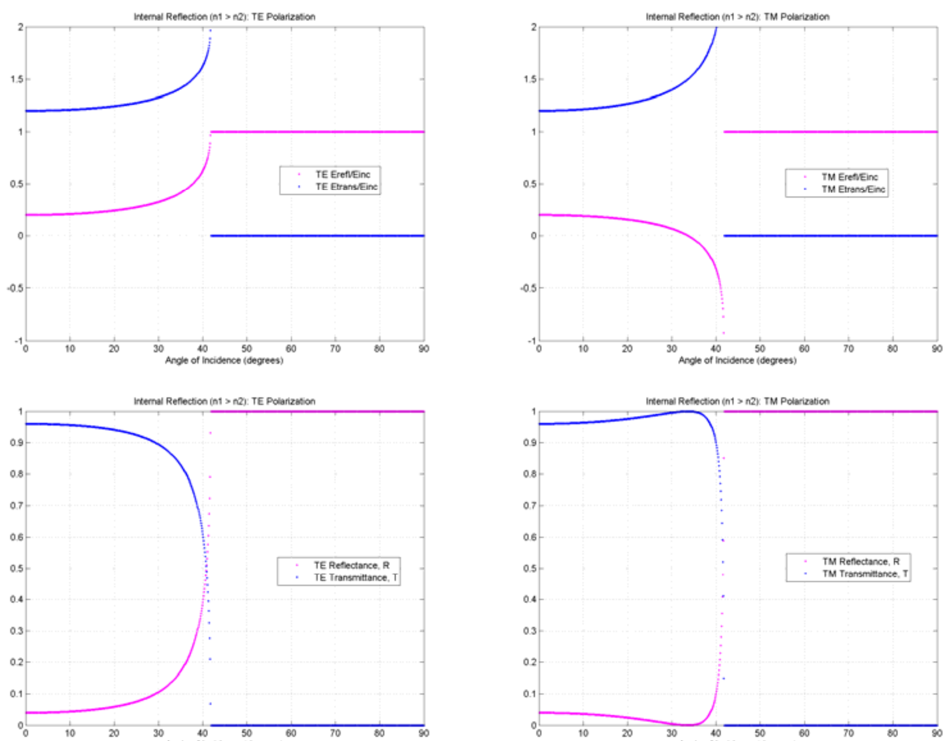
$$\boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{2\left(\frac{n_2}{n_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

The functional dependence of $\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)$, $\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)$, the reflection coefficient $R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2$ and the transmission coefficient $T = \alpha\beta\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \frac{\sqrt{(n_2/n_1)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc}}\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2$ as a function of the angle of incidence θ_{inc} for ***external*** reflection ($n_1 < n_2$) and ***internal*** reflection ($n_1 > n_2$) for *TE* & *TM* polarization are shown in the figures below:

External Reflection ($n_1 = 1.0 < n_2 = 1.5$):



Internal Reflection ($n_1 = 1.5 > n_2 = 1.0$):



Comment 1):

When $\left(E_{refl}/E_{inc}\right)<0$, $E_{o_{refl}}$ is 180° out-of-phase with $E_{o_{inc}}$ since the ***numerators*** of the **original** Fresnel relations for *TE* & *TM* polarization are $(1-\alpha\beta)$ and $(\alpha-\beta)$ respectively.

Comment 2):

For *TM* Polarization (***only***), there exists an angle of ***incidence*** where $\left(E_{refl}/E_{inc}\right)=0$, *i.e.* ***no reflected*** wave occurs at this incident angle for *TM* polarization! This ***incidence*** angle is known as **Brewster's angle** θ_B (also known as the ***polarizing angle*** θ_P - because *e.g.* an incident wave that is a ***linear combination*** of *TE* ***and*** *TM* polarizations will have a **reflected** wave which is 100% pure-*TE* polarized for an ***incidence*** angle $\theta_{inc}=\theta_B=\theta_P$!!). * *n.b.* **Brewster's angle** θ_B exists for ***both*** **external** ($n_1 < n_2$) & **internal** reflection ($n_1 > n_2$) for *TM* polarization (***only***). *

Brewster's Angle θ_B / the Polarizing Angle θ_P for Transverse Magnetic (*TM*) Polarization

From the numerator of $\left(E_{o_{refl}}^{TM}/E_{o_{inc}}^{TM}\right)=\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)$ of the {originally-derived} expression for *TM* polarization, this numerator = 0 at Brewster's angle θ_B (*aka* the polarizing angle θ_P), which occurs when $(\alpha-\beta)=0$, *i.e.* when $\alpha=\beta$.

But: $\boxed{\alpha\equiv\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}}$ and $\boxed{\beta\equiv\left(\dfrac{\mu_1 v_1}{\mu_2 v_2}\right)=\dfrac{\mu_1 n_2}{\mu_2 n_1}\simeq\dfrac{n_2}{n_1}}$ for $\boxed{\mu_1\simeq\mu_2\simeq\mu_o}$

Now: $\cos\theta_{trans}=\sqrt{1-\sin^2\theta_{trans}}$ and Snell's Law: $n_1\sin\theta_{inc}=n_2\sin\theta_{trans}\;\Rightarrow\;\sin\theta_{trans}=\left(\dfrac{n_1}{n_2}\right)\sin\theta_{inc}$

$\therefore$ at Brewster's angle $\theta_{inc}=\theta_B$ = polarizing angle θ_P, where $\alpha=\beta$, this relation becomes:

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}=\beta\equiv\frac{\mu_1 n_2}{\mu_2 n_1}\simeq\frac{n_2}{n_1}}\text{ for }\boxed{\mu_1\simeq\mu_2\simeq\mu_o}\;\Rightarrow\;\boxed{\alpha=\frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}}}{\cos\theta_{inc}}\simeq\left(\frac{n_2}{n_1}\right)=\beta}$$

or: $$\boxed{1-\frac{1}{\beta^2}\sin^2\theta_{inc}=\beta^2\cos^2\theta_{inc}=\beta^2\left(1-\sin^2\theta_{inc}\right)}\leftarrow\text{Solve for }\sin^2\theta_{inc}$$

$$\boxed{1-\beta^2=\left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc}}\;\Rightarrow\;\boxed{\sin^2\theta_{inc}=\frac{1-\beta^2}{1/\beta^2-\beta^2}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}}$$

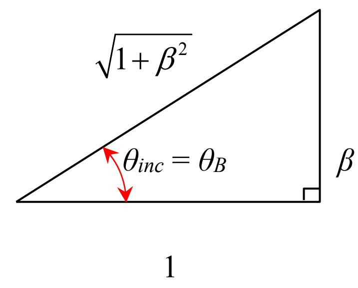
But: $$\boxed{1-\beta^4=\left(1-\beta^2\right)\left(1+\beta^2\right)}$$

$$\therefore\quad\boxed{\sin^2\theta_{inc}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)}=\frac{\beta^2}{1+\beta^2}}\;\Rightarrow\;\boxed{\sin\theta_{inc}=\frac{\beta}{\sqrt{1+\beta^2}}}$$

Geometrically: $\boxed{\sin\theta_{inc} = \dfrac{\beta}{\sqrt{1+\beta^2}}} = \boxed{\dfrac{\text{opp. side}}{\text{hypotenuse}}}$

$\boxed{\cos\theta_{inc} = \dfrac{1}{\sqrt{1+\beta^2}}} = \boxed{\dfrac{\text{adjacent}}{\text{hypotenuse}}}$

$\boxed{\tan\theta_{inc} = \beta} = \boxed{\dfrac{\text{opp. side}}{\text{adjacent}}} \simeq \left(\dfrac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where ***no*** ***reflected*** wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \dfrac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \dfrac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

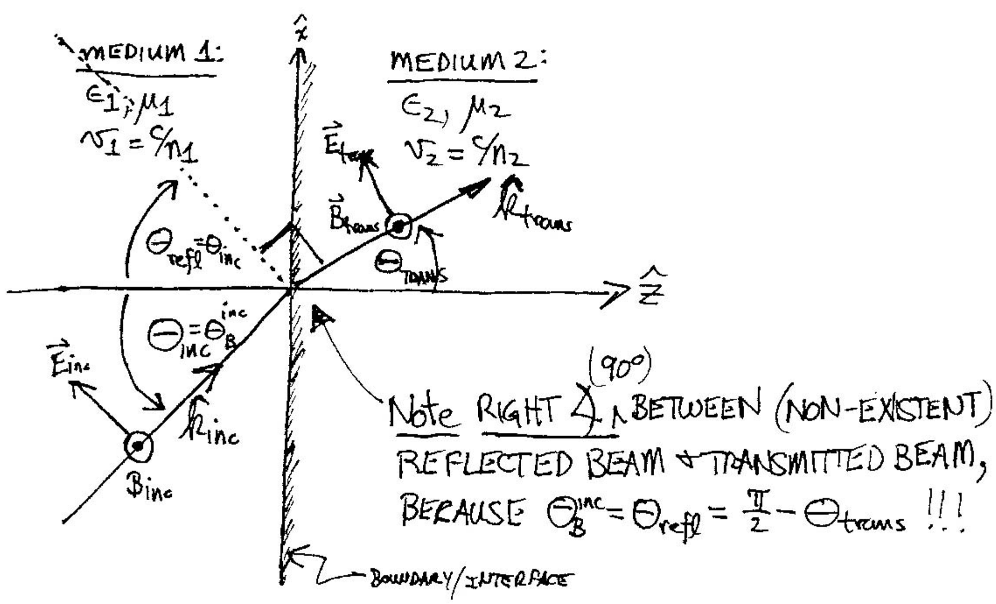
So what's so interesting about this???

Well: $\boxed{\cos\theta_B^{inc} = \sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta_B^{inc} - \overset{=0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} \sin\theta_B^{inc} = \sin\theta_{trans}}$ *i.e.* $\boxed{\sin\left(\dfrac{\pi}{2} - \theta_B^{inc}\right) = \sin\theta_{trans}}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

Thus: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_{\mathrm{P}}^{inc}$ and θ_{trans} are ***complimentary*** angles !!!

TM Polarized EM Wave Incident at Brewster's Angle θ_B^{inc}:

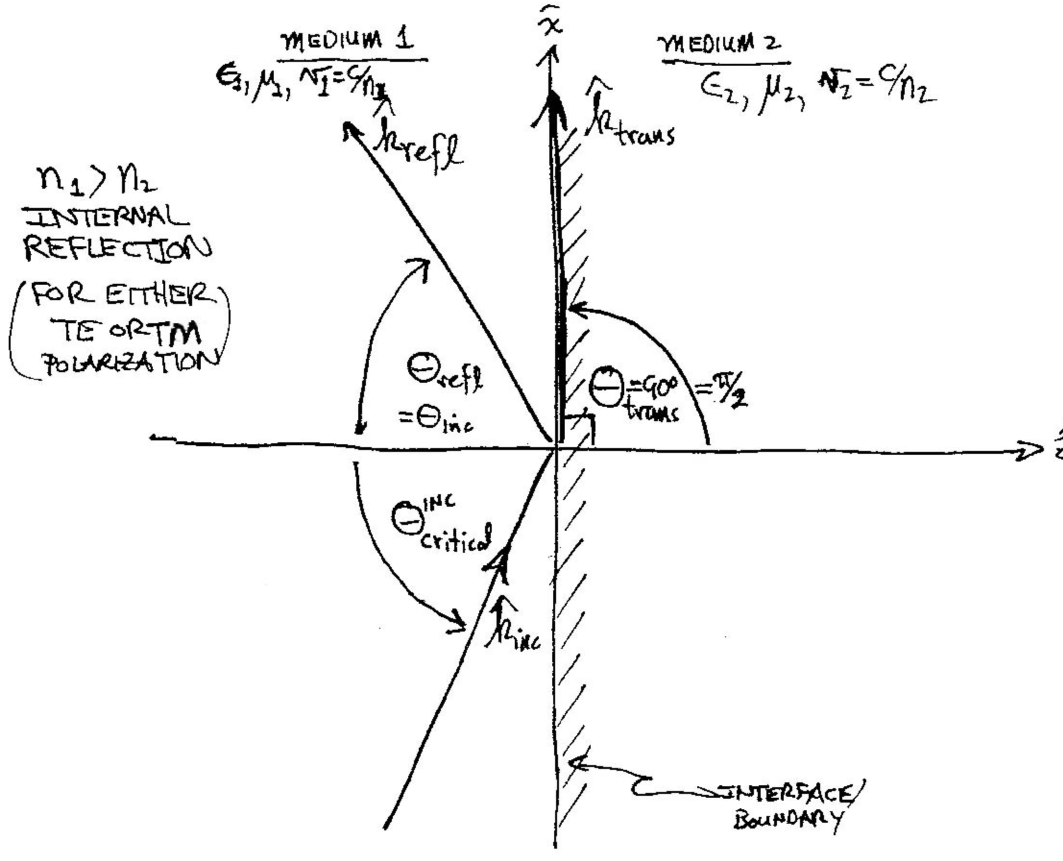


Thus, *e.g.* if an ***unpolarized*** *EM* wave (*i.e.* one which contains all polarizations/random polarizations) or an *EM* wave which is a linear combination of *TE* and *TM* polarization is incident on the interface between two linear/homogeneous/isotropic media ***at*** Brewster's angle $\theta_B^{inc} \equiv \theta_P^{inc}$, the ***reflected*** beam will be 100% pure *TE* polarization!!
Hence, this is why Brewster's angle θ_B is also known as the polarizing angle θ_P.

Comment 3):

For ***internal*** reflection ($n_1 > n_2$) there exists a critical angle of incidence $\theta_{critical}^{inc}$ past which ***no*** transmitted beam exists for either *TE* or *TM* polarization. The critical angle does ***not*** depend on polarization – it is actually dictated / defined by Snell's Law:

$\boxed{n_1 \sin\theta_{critical}^{inc} = n_2 \sin\theta_{trans}^{\max} = n_2 \sin\left(\frac{\pi}{2}\right) = n_2}$ or: $\boxed{\sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right)}$ or: $\boxed{\theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$

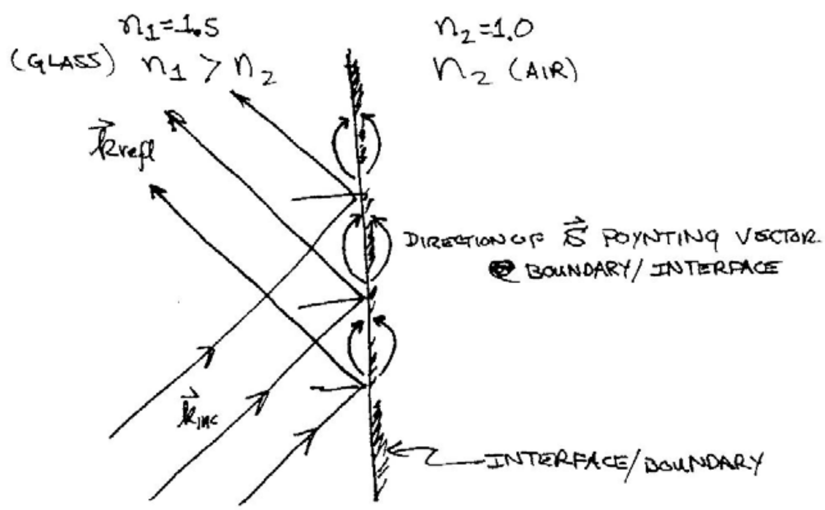


For $\theta_{inc} \geq \theta_{critical}^{inc}$, ***no*** transmitted beam exists → incident beam is **totally internally reflected**.
For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually **exponentially** damped – it becomes a so-called

Evanescent Wave: $\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} \underbrace{e^{-\alpha z}}_{\text{Exponential damping in } z} \underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right) - \omega t\right)}}_{\text{Oscillatory along interface in } x\text{-direction}}}$ $\boxed{\alpha = k_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}}$

Exponential damping in *z* Oscillatory along ***interface*** in *x*-direction

For Total Internal Reflection $\theta_{inc} \geq \theta_{critical}^{inc}$, $n_1 > n_2$:



For total internal reflection ($\theta_{inc} \geq \theta_{critical}^{inc}$, $n_1 > n_2$), the ***reflected*** wave is actually displaced laterally, along the interface (in the direction of the evanescent wave), relative to the prediction from geometrical optics. The lateral displacement is known as the Goos-Hänchen effect [F. Goos and H. Hänchen, Ann. Phys. (Leipzig) (6) **1**, 333-346 (1947)] and is different for *TE vs. TM* polarization:

$$\boxed{D_{TE} = \frac{\lambda_1}{\pi}\frac{\sin\theta_{inc}}{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}} \quad \text{where: } \lambda_1 = \lambda_o/n_1 \quad \text{and: } \lambda_o = c/f = \text{vacuum wavelength}$$

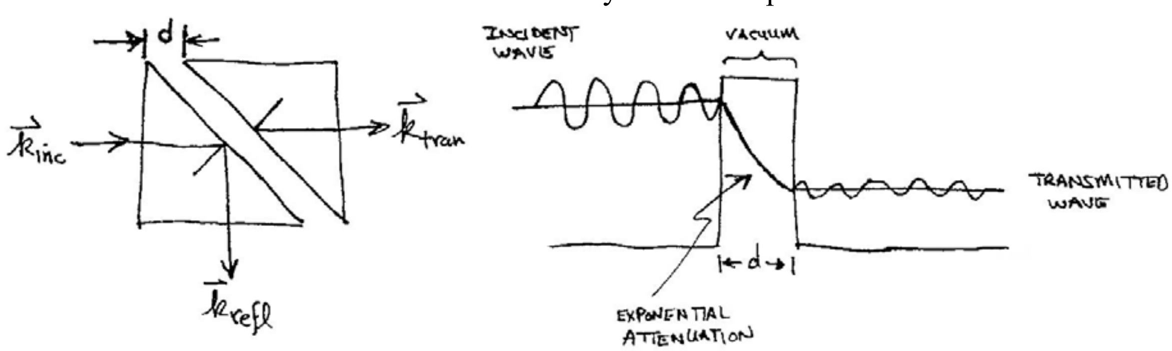
$$\boxed{D_{TM} = \frac{\lambda_1}{\pi}\frac{\sin\theta_{inc}}{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}\frac{\left(\frac{n_2}{n_1}\right)^2}{\left[\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2\cos^2\theta_{inc}\right]} = D_{TE}\cdot\frac{\left(\frac{n_2}{n_1}\right)^2}{\left[\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2\cos^2\theta_{inc}\right]}}$$

Experiment to demonstrate that the transmitted *EM* wave for $\theta_{inc} \geq \theta_{critical}^{inc}$ ***is*** exponentially damped:

Use two 45° prisms, separated by a small distance d as shown in the figure below – (*e.g.* use glass prisms {for light}; can use $\underbrace{\text{paraffin prisms \{for microwaves\}}}$!!)

UIUC Physics 401 Experiment # 34 !!!

See *e.g.* D.D. Coon, Am. J. Phys. **34**, 240 (1966)



⇒ Microscopically, this experiment ***is*** an example of quantum mechanical barrier penetration / quantum mechanical tunneling phenomenon (using ***real*** photons) !!!

The above lateral displacement(s) for *TE vs. TM* polarization are also correlated with ***phase shifts*** that occur in the ***reflected*** wave when $\theta_{inc} \geq \theta_{critical}^{inc}$ for total internal reflection ($n_1 > n_2$):

Using the (last) version of Fresnel Equations (*p*. 17 of these lecture notes):

TE Polarization / ***TM Polarization***

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}} \quad \boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

When $\theta_{inc} \geq \theta_{critical}^{inc}$, Snell's Law is: $\sin\theta_{critical}^{inc} = \left(n_2/n_1\right)$ $\{$since $\sin\theta_{trans} = \sin 90^o = 1\}$

The above *E*-field amplitude ratios become ***complex*** for ***internal*** reflection, because for $\left(\frac{n_2}{n_1}\right) < 1$,

when: $\sin^2\theta_{inc} > \left(\frac{n_2}{n_1}\right) < 1$, then: $\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}$ becomes ***imaginary***. Thus, for $\theta_{inc} \geq \theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$ for $n_1 > n_2$ (***internal*** reflection), we can re-write the above $\vec{E}$-field ratios as:

TE Polarization / ***TM Polarization***

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\cos\theta_{inc} - i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}} \quad \boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}}$$

It is easy to verify that these ratios lie on the unit circle in the complex plane – simply multiply them by their complex conjugates to show $AA^* = 1$, as they must for total internal reflection.

These formulae imply a ***phase change*** of the ***reflected*** wave (***relative*** to the ***incident*** wave) that depends on the angle of incidence $\theta_{inc} \geq \theta_{critical}^{inc} = \sin^{-1}\left(n_2/n_1\right)$ for total internal reflection.

We set: $\boxed{-\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right) = e^{-i\delta} = \frac{ae^{-i\alpha}}{ae^{+i\alpha}}} \Rightarrow \boxed{\delta = 2\alpha}$ and $\boxed{\tan\left(\delta/2\right) = \tan\left(\alpha\right)}$

Where δ = ***phase change*** (in radians) of the ***reflected*** wave ***relative*** to the ***incident*** wave.

Thus, we see that (from the numerators of the above formulae) that:

$$\boxed{\tan\left(\frac{\delta_{TE}}{2}\right) = \frac{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_{inc}}} \quad \text{and:} \quad \boxed{\tan\left(\frac{\delta_{TM}}{2}\right) = \frac{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc}}}$$

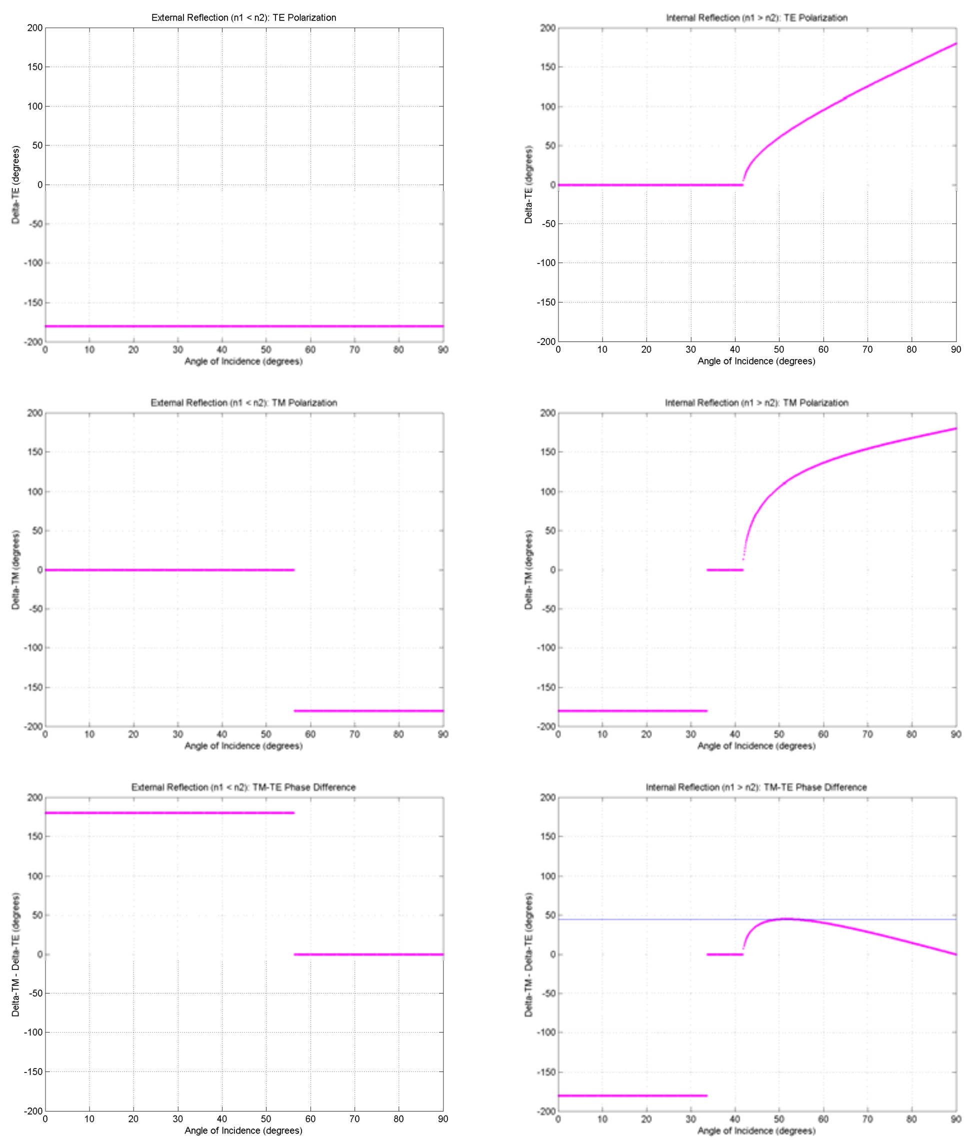
The ***relative phase difference*** between total internally-reflected *TM* *vs*. *TE* polarized waves $\Delta \equiv \delta_{TM} - \delta_{TE}$ can also be calculated:

$$\boxed{\tan\left(\frac{\Delta}{2}\right) = \tan\left(\frac{\delta_{TM} - \delta_{TE}}{2}\right) = \frac{\cos\theta_{inc}\sqrt{\sin^2\theta_{inc} - (n_2/n_1)^2}}{\sin^2\theta_{inc}}}$$

Phase shifts of the ***reflected*** wave ***relative*** to the ***incident*** wave for external, internal reflection and for *TE*, *TM* polarization are shown in the following graphs:

Phase Shifts Upon Reflection:

External Reflection ($n_1 = 1.0 < n_2 = 1.5$): **Internal Reflection ($n_1 = 1.5 > n_2 = 1.0$):**



Note that a phase shift of -180° is equivalent to a phase shift of $+180^{\circ}$.

<u>An Example of the {Clever} Use of Internal Reflection Phase Shifts - The Fresnel Rhomb</u>:

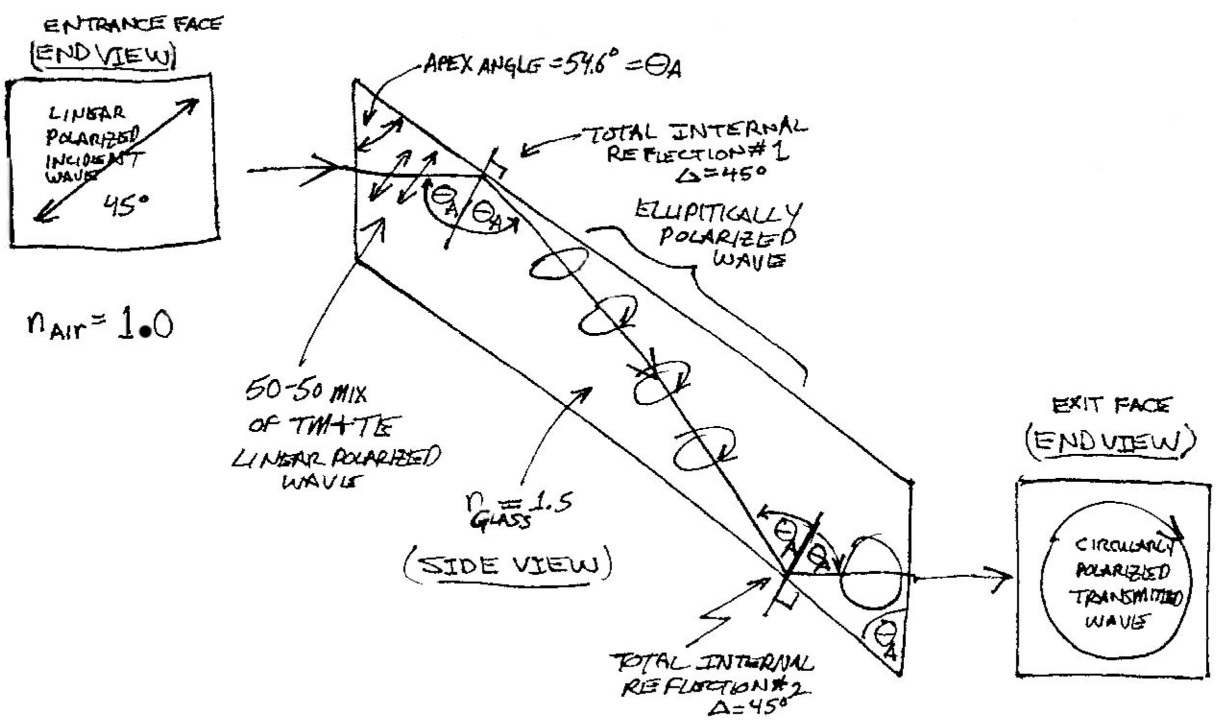
From last graph of the internal reflection phase shifts (above), we see that the ***<u>relative difference</u>*** in *TM vs. TE* phase shifts for total internal reflection at a glass-air interface ($n_1 = 1.5$ {glass}, $n_2 = 1.0$ {air}) is $\Delta \equiv \delta_{TM} - \delta_{TE} = \pi/4 = 45^o$ when $\theta_{inc} = 54.6^o$

Fresnel used this *TM vs. TE* relative phase-shift fact associated with total internal reflection and developed / designed a glass rhomb-shaped prism that converted ***<u>linearly-polarized</u>*** light to ***<u>circularly-polarized</u>*** light, as shown in the figure below.

He used light incident on the glass rhomb-shaped prism with polarization angle at 45° with respect to face-edge of the glass rhomb (thus the incident light was a 50-50 mix of *TE* and *TM* polarization). Note that the transmitted wave actually undergoes ***<u>two</u>*** total internal reflections before emerging from rhomb at the exit face, with a –45° relative phase *TM-TE* phase shift occurring at *<u>each</u>* total internal reflection. Thus, the first total internal reflection converts a linearly polarized wave into an elliptically polarized wave, the second total internal reflection converts the elliptically polarized wave into a circularly polarized wave!!!

The total phase shift (for 2 internal reflections): $\Delta_{tot} = 2\Delta = 2\left(\delta_{TM} - \delta_{TE}\right) = \pi/2 = 90^o$

(for rhomb apex angle $\theta_A = 54.6°$, $n_{air} = 1.0$ and $n_{glass} = 1.5$)



<u>NOTE</u>: By time-reversal invariance of the *EM* interaction, we can also can see from the above that Fresnel's rhomb can also be used to convert circularly-polarized incident light into linearly polarized light !!!

The Fresnel relations for *TE* / *TM* polarization for internal / external reflection are in fact useful for any type of polarized *EM* wave – linear polarization with $\hat{n} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$, elliptic or circular polarization – these are all ***vectorial*** linear combinations of the two orthogonal polarization cases - *TE* and *TM* polarization. For each case, the results associated with the *TE* and *TM* components of that *EM* wave, but then have to be combined vectorially.

Finally, for the limiting case of ***normal*** incidence (where the plane of incidence collapses) the reflection coefficient *R* (valid for both *TE* and *TM* polarization) at $\theta_{inc} = 0$ is:

$$\boxed{R = \left(\frac{1-\left(\frac{n_2}{n_1}\right)^2}{1+\left(\frac{n_2}{n_1}\right)}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \approx 4\% \;\; \text{for} \left\{\begin{matrix} n_1 = 1.0 \text{ (air)} \\ n_2 = 1.5 \text{ (glass)}\end{matrix}\right\}}$$

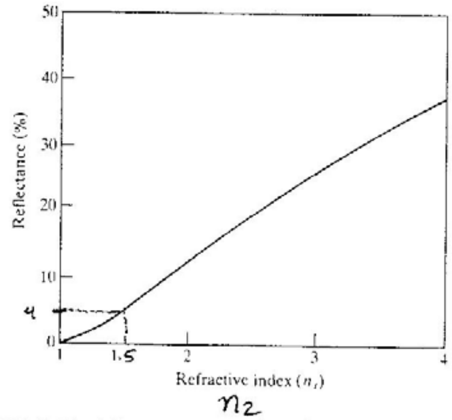


FIGURE 4.49 Reflectance at normal incidence in air (n_i = 1.0) at a single interface. $n_1 = n_{AIR} = 1.0$

Can There be a Brewster's Angle θ_B^{inc} for Transverse Electric (*TE*) Polarization Reflection / Refraction at an Interface?

The Fresnel Equations:

TE Polarization | **TM Polarization**

$$\boxed{r_\perp \equiv \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \qquad \boxed{r_\parallel \equiv \left(\frac{\mathrm{E}_{0_{refl}}^{TE}}{\mathrm{E}_{0_{inc}}^{TE}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)}$$

with: $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ and: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2} = \frac{Z_1}{Z_2}}$

$$\boxed{v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}} \quad \boxed{v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}} \quad \boxed{n_1 = \sqrt{\varepsilon_1\mu_1/\varepsilon_0\mu_0}} \quad \boxed{n_2 = \sqrt{\varepsilon_2\mu_2/\varepsilon_0\mu_0}}$$

We saw P436 lecture notes above (*p*. 19-21) that for *TM* polarized *EM* Waves (where $\vec{B} \perp$ plane of incidence {*i.e.* $\vec{B} \parallel$ to plane of the interface}, with unit normal to the plane of incidence defined as $\hat{n}_{inc} \equiv \hat{k}_{inc} \times \hat{k}_{refl}$) that when $\theta_{inc} = \theta_B^{inc}$ = Brewster's angle (*aka* θ_P^{inc} = polarizing angle), that $E_{o_{refl}}^{TM} = 0$ because the ***numerator*** of $r_\parallel$, , $(\alpha-\beta) = 0$ *i.e.* $\alpha = \beta$ when $\theta_{inc} = \theta_B^{inc} = \theta_P^{inc}$
Thus, for an incident *TM* polarized monochromatic plane *EM* wave, when:

$$\boxed{\alpha=\beta\Big|_{\theta_{inc}=\theta_B^{inc}}} \Rightarrow \boxed{\left(\frac{\cos\theta_{trans}^{TM}}{\cos\theta_{inc}^{TM}}\right)_{TM}=\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}=\beta}$$

<u>or</u>: $$\boxed{\beta=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\mu_1}{\mu_2}\sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}=\sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}=\frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$$

For <u>non-magnetic</u> media: $|\chi_m|\ll 1$ *i.e.* $\mu_1\simeq\mu_2\simeq\mu_0$ <u>then</u>: $\boxed{\left(\frac{\cos\theta_{trans}^{TM}}{\cos\theta_{inc}^{TM}}\right)\simeq\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}=\frac{n_2}{n_1}}$

We also derived the Brewster angle relation for *TM* polarization: $\boxed{\tan\theta_B^{inc}\equiv\tan\theta_P^{inc}\simeq\frac{n_2}{n_1}}$

For the case of *TE* polarization, we see that: $E_{o_{refl}}^{TE}=0$ when the numerator of $r_\perp$, $(1-\alpha\beta)=0$ *i.e.* when: $\alpha\beta=1$ or: $\beta=1/\alpha$. What does this mean physically??

For *TE* Polarization: $\beta=1/\alpha \Rightarrow \boxed{\sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}=1\Big/\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)=\left(\frac{\cos\theta_{inc}}{\cos\theta_{trans}}\right)}$

For <u>non-magnetic</u> media where: $|\chi_m|\ll 1$ *i.e.* $\mu_1\simeq\mu_2\simeq\mu_o$ <u>then</u>: $\boxed{\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\simeq\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}=\frac{n_2}{n_1}}$

<u>Thus</u>: $$\boxed{\left(\frac{n_2}{n_1}\right)^2=\frac{\cos^2\theta_{inc}}{\cos^2\theta_{trans}}=\frac{\cos^2\theta_{inc}}{1-\sin^2\theta_{trans}}=\frac{1-\sin^2\theta_{inc}}{1-\sin^2\theta_{trans}}}$$

<u>From Snell's Law</u>: $\boxed{n_1\sin\theta_1=n_2\sin\theta_2}$ or: $\boxed{n_1\sin\theta_{inc}=n_2\sin\theta_{trans}}$

$\boxed{\sin\theta_{trans}=\left(\frac{n_1}{n_2}\right)\sin\theta_{inc}}$ or: $\boxed{\sin^2\theta_{trans}=\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}}$

<u>Then</u>: $\boxed{\left(\frac{n_2}{n_1}\right)^2=\frac{\left(1-\sin^2\theta_{inc}\right)}{\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}\right)}}$ <u>or</u>: $\boxed{\left(\frac{n_2}{n_1}\right)^2-\cancel{\sin^2\theta_{inc}}=1-\cancel{\sin^2\theta_{inc}}}$

$\Rightarrow \boxed{\left(\frac{n_2}{n_1}\right)^2=1}$ or: $\boxed{n_1=n_2}$ $\Rightarrow$ can get $\theta_B^{inc}=\theta_P^{inc}$ for *TE* Polarization ***<u>only</u>*** when $n_1=n_2$

However, when $n_1=n_2$, this corresponds to ***<u>no</u>*** interface boundary, at least for <u>non-magnetic</u> material(s), where $|\chi_m|\ll 1$ and $\mu_1\simeq\mu_2\simeq\mu_0$.

Is there a possibility of a Brewster's angle for incident *TE* polarization for ***<u>magnetic</u>*** materials???

For incident *TE* polarization, we still need to satisfy the condition $\beta = 1/\alpha$.

i.e. $\boxed{\sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{\cos\theta_{inc}^B}{\cos\theta_{trans}}}$ <u>or</u>: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) = \frac{\cos^2\theta_{inc}^B}{\cos^2\theta_{trans}} = \frac{1-\sin^2\theta_{inc}^B}{1-\sin^2\theta_{trans}} = \frac{1-\sin^2\theta_{trans}^B}{1-\left(\frac{n_1}{n_2}\right)\sin^2\theta_{inc}^B}}$

<u>but</u>: $\boxed{\left(\frac{n_1}{n_2}\right)^2 = \left(\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\right)}$ <u>thus</u>: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) = \frac{1-\sin^2\theta_{inc}^B}{1-\left(\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\right)\sin^2\theta_{inc}^B}}$

<u>thus</u>: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) - \left(\frac{\cancel{\varepsilon_2}\mu_1}{\cancel{\varepsilon_1}\mu_2}\right)\left(\frac{\cancel{\varepsilon_1}\mu_1}{\cancel{\varepsilon_2}\mu_2}\right)\sin^2\theta_{inc}^B = 1-\sin^2\theta_{inc}^B}$

$\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) - \left(\frac{\mu_1}{\mu_2}\right)^2 \sin^2\theta_{inc}^B = \left(1-\sin^2\theta_{inc}^B\right)}$ multiply both sides of this eqn. by $\left(\frac{\mu_2}{\mu_1}\right)$

$\boxed{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_1}{\mu_2}\right)\sin^2\theta_{inc}^B = \left(\frac{\mu_2}{\mu_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)\sin^2\theta_{inc}^B}$

$\boxed{\left[\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)\right] = \left[\left(\frac{\mu_1}{\mu_2}\right) - \left(\frac{\mu_2}{\mu_1}\right)\right]\sin^2\theta_{inc}^B}$

$\Rightarrow \boxed{\sin^2\theta_{inc}^B = \left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right) - \left(\frac{\mu_2}{\mu_1}\right)}\right]}$

<u>Note</u>: $\boxed{0 \le \sin^2\theta_{inc}^B \le 1}$ with $\boxed{\theta_{inc}^B = 0^o}$ and $\boxed{\theta_{inc}^B = 90^o}$

<u>Define</u>: $\boxed{\sin\theta_{inc}^B = \sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right) - \left(\frac{\mu_2}{\mu_1}\right)}} \equiv \sqrt{A}}$ *i.e.* $\boxed{A \equiv \left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right) - \left(\frac{\mu_2}{\mu_1}\right)}\right]}$

<u>Brewster's angle for *TE* polarization</u>: $\boxed{\theta_{\substack{inc\\TE}}^B = \sin^{-1}\sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right) - \left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right) - \left(\frac{\mu_2}{\mu_1}\right)}} = \sin^{-1}\sqrt{A}}$

Let us assume that ε_1 and ε_2 are fixed {*i.e.* electric properties of medium 1) and 2) are fixed} but that we can engineer/design/manipulate the magnetic properties of medium 1) and 2) in such a way as to obtain a ratio $(\mu_1/\mu_2) \neq 1$ to give $0 \le A \le 1$!!!

Then if $\theta_{\substack{inc\\TE}}^{B} = \sin^{-1}\sqrt{A}$ can be achieved, it might also be possible to engineer the magnetic properties (μ_1/μ_2) such that $A < 0$ – *i.e.* θ_{inc}^{B} becomes imaginary!!!

Note also that in the above formula that $(\mu_1/\mu_2) = 1$ does ***not*** mean $\sin\theta_{inc} = \infty$ because the original formula for $(\mu_1/\mu_2) = 1$ was:

$$\boxed{\left(\frac{n_2}{n_1}\right)^2 \left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc}\right) = \left(1 - \sin^2\theta_{inc}\right)}$$

which is perfectly mathematically fine/OK for $(n_2/n_1) = 1$.