LECTURE NOTES 6

ELECTROMAGNETIC WAVES IN MATTER

Electromagnetic Wave Propagation in Linear Media

We now consider *EM* wave propagation inside ***linear matter***, but ***only*** in regions where there are NO ***free*** charges $\left(\sigma_{free}=0\right)$ and/or ***free*** currents $\left(\vec{K}_{free}=0\right)$ (*i.e*. the ***linear*** medium is an ***insulator*** / a ***non-conductor***).

For ***this*** situation, Maxwell's equations (in differential form) become:

1) $\boxed{\vec{\nabla}\bullet\vec{D}\left(\vec{r},t\right)=0}$ 2) $\boxed{\vec{\nabla}\bullet\vec{B}\left(\vec{r},t\right)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}\left(\vec{r},t\right)=-\dfrac{\partial\vec{B}\left(\vec{r},t\right)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{H}\left(\vec{r},t\right)=\dfrac{\partial\vec{D}\left(\vec{r},t\right)}{\partial t}}$

The medium in which the *EM* waves propagate is assumed to be linear, homogeneous and isotropic, thus the following relations are valid in this medium:

$$\boxed{\vec{D}\left(\vec{r},t\right)=\varepsilon\vec{E}\left(\vec{r},t\right)} \quad \text{and} \quad \boxed{\vec{H}\left(\vec{r},t\right)=\tfrac{1}{\mu}\vec{B}\left(\vec{r},t\right)}$$

Where:

ε = electric permittivity of the medium.
μ = magnetic permeability of the medium.
$\varepsilon=\varepsilon_o\left(1+\chi_e\right)$, χ_e = electric susceptibility of the medium.
$\mu=\mu_o\left(1+\chi_m\right)$, χ_m = magnetic susceptibility of the medium.
ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.
μ_o = magnetic permeability of free space = $4\pi\times10^{-7}$ Henrys/m.

Thus, Maxwell's equations for the $\vec{E}$ and $\vec{B}$ fields inside this linear, homogeneous and isotropic non-conducting medium become:

1) $\boxed{\vec{\nabla}\bullet\vec{E}\left(\vec{r},t\right)=0}$ 2) $\boxed{\vec{\nabla}\bullet\vec{B}\left(\vec{r},t\right)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}\left(\vec{r},t\right)=-\dfrac{\partial\vec{B}\left(\vec{r},t\right)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}\left(\vec{r},t\right)=\mu\varepsilon\dfrac{\partial\vec{E}\left(\vec{r},t\right)}{\partial t}}$

Note that the above four relations are (almost) identical to those for *EM* waves in free space {*cf* with eqns. 1) - 4) on page 1 of P436 Lect. Notes 5}.

We simply replace the macroscopic *EM* parameters associated with the vacuum $\left\{\varepsilon_o,\mu_o\right\}$ with those associated with the linear, homogeneous and isotropic medium $\left\{\varepsilon,\mu\right\}$.

- In free space/vacuum, the speed of propagation of *EM* waves is:

$\boxed{c = 1/\sqrt{\varepsilon_o \mu_o}} = 3\times 10^8\ \text{m/s}$, the $\vec{E}$ and $\vec{B}$ fields in vacuum obey the wave equation:

$$\boxed{\nabla^2 \vec{E}(\vec{r},t) = \varepsilon_o \mu_o \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}} \quad \boxed{\nabla^2 \vec{B}(\vec{r},t) = \varepsilon_o \mu_o \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}}$$

- In a linear/homogeneous/isotropic medium, the speed of propagation of *EM* waves is: $\boxed{v = 1/\sqrt{\varepsilon\mu}}$ and the $\vec{E}$ and $\vec{B}$ fields in the medium obey the following wave equation:

$$\boxed{\nabla^2 \vec{E}(\vec{r},t) = \varepsilon\mu \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}} \quad \boxed{\nabla^2 \vec{B}(\vec{r},t) = \varepsilon\mu \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}}$$

For linear / homogeneous / isotropic media:

$$\varepsilon = K_e \varepsilon_o = (1+\chi_e)\varepsilon_o \qquad K_e = \frac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity}$$

$$\mu = K_m \mu_0 = (1+\chi_m)\mu_o \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability}$$

$$\therefore \boxed{v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e\varepsilon_o K_m \mu_o}} = \frac{1}{\sqrt{K_e K_m}}\frac{1}{\sqrt{\varepsilon_o\mu_o}} = \frac{1}{\sqrt{K_e K_m}}c} \quad i.e. \quad \boxed{v = \frac{1}{\sqrt{K_e K_m}}c}$$

Now ***if***:

$$\boxed{K_e = \left(\frac{\varepsilon}{\varepsilon_o}\right) = (1+\chi_e) \ge 1} \text{ and } \boxed{K_m = \left(\frac{\mu}{\mu_o}\right) = (1+\chi_m) \ge 1} \quad \text{or if: } \boxed{K_e K_m \ge 1}$$

{true for a wide variety of common/everyday materials – gases, liquids & solids}

Then: $\boxed{\sqrt{K_e K_m} \ge 1}$ thus: $\boxed{\dfrac{1}{\sqrt{K_e K_m}} \le 1} \Rightarrow \boxed{v = \dfrac{1}{\sqrt{K_e K_m}}c \le c}$

Note also that since $\boxed{K_e = \dfrac{\varepsilon}{\varepsilon_o}}$ and $\boxed{K_m = \dfrac{\mu}{\mu_o}}$ are dimensionless quantities, then so is $\boxed{\dfrac{1}{\sqrt{K_e K_m}}}$.

We can now define the index of refraction {*n.b.* a dimensionless quantity} of the linear / homogeneous / isotropic medium as:

$$\boxed{n \equiv \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o\mu_o}}}$$

Thus, for linear / homogeneous / isotropic media: $\boxed{v = c/n\ (\le c)}$ because $\boxed{n \ge 1}$ {*usually*}.

n.b. We will find out {soon!} that ε and μ are in fact ***not*** constants, instead they are {very often} ***frequency-dependent*** quantities, *i.e.* $\boxed{\varepsilon = \varepsilon(\omega)}$ and $\boxed{\mu = \mu(\omega)}$, $\boxed{\omega = 2\pi f}$.

Thus: $$\boxed{K_e = K_e(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_o} = 1 + \chi_e(\omega)} \text{ and } \boxed{K_m = K_m(\omega) = \frac{\mu(\omega)}{\mu_o} = 1 + \chi_m(\omega)}$$

Hence: $$\boxed{n = n(\omega) = \sqrt{K_e(\omega)K_m(\omega)} = \sqrt{\frac{\varepsilon(\omega)\mu(\omega)}{\varepsilon_o\mu_o}} = \sqrt{(1+\chi_e(\omega))(1+\chi_m(\omega))}} \quad \boxed{v(\omega) = \frac{c}{n(\omega)}}$$

For now, we will ignore/neglect any/all frequency-dependent effects, for simplicity, *i.e.*

$$\boxed{v = \frac{c}{n}} = \text{constant} \qquad \boxed{n = \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o\mu_o}} = \sqrt{(1+\chi_e)(1+\chi_m)}} = \text{constant}$$

Now for many (but not all) linear/homogeneous/isotropic materials: $\boxed{\mu = \mu_o(1+\chi_m) \simeq \mu_o}$

(*e.g.* true for many paramagnetic and diamagnetic-type materials) $\Rightarrow$ $\boxed{|\chi_m| \sim \vartheta(10^{-8}) \sim 0}$

Thus: $$\boxed{K_m = \frac{\mu}{\mu_o} = (1+\chi_m) \simeq 1} \Rightarrow \boxed{n \simeq \sqrt{K_e}} \text{ and: } \boxed{v = \frac{c}{n} \simeq \frac{c}{\sqrt{K_e}}}.$$

- The instantaneous *EM* energy density associated with a linear/homogeneous/isotropic material:

$$\boxed{u_{EM}(\vec{r},t) = u_E(\vec{r},t) + u_M(\vec{r},t) = \tfrac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t) + \vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right) = \tfrac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \tfrac{1}{\mu}B^2(\vec{r},t)\right)} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with: $\boxed{\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)}$ and: $\boxed{\vec{H}(\vec{r},t) = \tfrac{1}{\mu}\vec{B}(\vec{r},t)}$. But: $\boxed{\vec{B}(\vec{r},t) = \tfrac{1}{v}\hat{k}\times\vec{E}(\vec{r},t)}$ and: $\boxed{v^2 = \tfrac{1}{\varepsilon\mu}}$

hence: $$\boxed{u_{EM}(\vec{r},t) = \tfrac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \tfrac{1}{\mu v^2}E^2(\vec{r},t)\right) = \tfrac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \tfrac{\varepsilon\mu}{\mu}E^2(\vec{r},t)\right) = \varepsilon E^2(\vec{r},t)}$$

$\Rightarrow \boxed{u_M(\vec{r},t) = u_E(\vec{r},t)}$ *i.e.* $\boxed{u_M(\vec{r},t)/u_E(\vec{r},t) = 1}$ for ***linear*** media.

- The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material:

$$\boxed{\vec{S}(\vec{r},t) = \left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right) = \tfrac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)} \left(\frac{\text{Watts}}{\text{m}^2}\right) \text{ with: } \boxed{\vec{H}(\vec{r},t) = \tfrac{1}{\mu}\vec{B}(\vec{r},t)} \text{ and:}$$

$$\boxed{\vec{B}(\vec{r},t) = \tfrac{1}{v}\hat{k}\times\vec{E}(\vec{r},t)} \Rightarrow \boxed{\vec{S}(\vec{r},t) = \tfrac{1}{\mu v}E^2(\vec{r},t)\hat{k} = \tfrac{v}{\mu v^2}E^2(\vec{r},t)\hat{k} = v\varepsilon E^2(\vec{r},t)\hat{k} = v\cdot u_{EM}(\vec{r},t)\hat{k}}$$

- The vector impedance associated with a linear/homogeneous/isotropic material:

$$\boxed{\vec{Z}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}^{-1}(\vec{r},t) = \left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right)\Big/\left|\vec{H}(\vec{r},t)\right|^2 = \mu v\,\hat{k} = Z_{med}\,\hat{k} = \sqrt{\tfrac{\mu}{\varepsilon}}\hat{k}} \text{ (Ohms)}$$

- For monochromatic (*i.e.* sinusoidal, single frequency) plane *EM* waves propagating in a linear/homogeneous/isotropic medium, $\vec{E}$ and $\vec{B}$ satisfy/obey the wave equation:

$$\boxed{\nabla^2\vec{E}(\vec{r},t) = \varepsilon\mu\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \quad \boxed{\nabla^2\vec{B}(\vec{r},t) = \varepsilon\mu\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

E and *B*-field solutions of wave eqn. for linearly polarized plane *EM* wave with ***polarization*** vector $\hat{n} \perp \hat{k}$ propagating in this linear/homogeneous/isotropic medium are of the form:

$\boxed{\vec{E}(\vec{r},t) = E_o \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{n}}$ and $\boxed{\vec{B}(\vec{r},t) = B_o \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k}\times\hat{n})}$

With: $\boxed{\vec{B}(\vec{r},t) = \frac{1}{v}\hat{k}\times\vec{E}(\vec{r},t)}$, thus: $\boxed{\left|\vec{B}(\vec{r},t)\right| = \frac{1}{v}\left|\vec{E}(\vec{r},t)\right|}$, *i.e.* $\boxed{B_o = \frac{1}{v}E_o}$

And: $\boxed{v = f\lambda = \omega/k}$ with angular frequency $\boxed{\omega = 2\pi f}$ and wavenumber $\boxed{k = 2\pi/\lambda}$.

- The ***intensity*** of an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{I(\vec{r}) \equiv \left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle = v\left\langle u_{EM}(\vec{r},t)\right\rangle = \tfrac{1}{2}v\varepsilon E_o^2(\vec{r}) = \tfrac{1}{2}\left(\tfrac{c}{n}\right)\varepsilon E_o^2(\vec{r}) = \left(\tfrac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Where $\boxed{E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o}$. The **RMS *intensity*** of the *EM* wave is:

$$\boxed{I_{rms}(\vec{r}) \equiv \left\langle\left|\vec{S}_{rms}(\vec{r},t)\right|\right\rangle = v\left\langle u_{EM_{rms}}(\vec{r},t)\right\rangle = \tfrac{1}{2}v\varepsilon E_{o_{rms}}^2(\vec{r}) = \tfrac{1}{2}\left(\tfrac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})} \quad \left(\frac{\text{RMS Watts}}{\text{m}^2}\right)$$

i.e. $\boxed{I_{rms}(\vec{r}) = \frac{1}{2}I(\vec{r})}$, $\boxed{\left\langle\left|\vec{S}_{rms}(\vec{r},t)\right|\right\rangle = \frac{1}{2}\left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle}$, $\boxed{\left\langle u_{EM_{rms}}(\vec{r},t)\right\rangle = \frac{1}{2}\left\langle u_{EM}(\vec{r},t)\right\rangle}$, *etc.*

- The instantaneous linear momentum density associated with an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\mu\vec{S}(\vec{r},t) = \tfrac{1}{v^2}\vec{S}(\vec{r},t) = \varepsilon\cancel{\mu}\tfrac{1}{\cancel{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \tfrac{1}{v^2}v\cdot u_{EM}(\vec{r},t)\hat{k} = \tfrac{1}{v}u_{EM}(\vec{r},t)\hat{k} = \tfrac{1}{v}\varepsilon E^2(\vec{r},t)\hat{k}}$$

- The instantaneous angular momentum density associated with an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r}\times\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

- And of course, an *EM* wave propagating in this medium has, in volume *v*:

Total instantaneous *EM* energy: $\boxed{U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\,d\tau}$ (Joules)

Total instantaneous linear momentum: $\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power in volume *v*: $\boxed{P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = -\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a}}$ (Watts)

n.b. through surface *S* enclosing *v*

Total instantaneous angular momentum: $\boxed{\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$

QUESTION:

What happens when an *EM* wave passes from one linear/homogeneous/isotropic medium into another (*e.g*. vacuum → gas; air → water; water → oil; glass → plastic; *etc*…?

As we saw in the case of mechanical transverse traveling waves propagating on the taught string which had two different mass-per-unit-lengths $(\mu_1 \text{ and } \mu_2)$, we anticipate that *EM* wave ***reflection*** and wave ***transmission*** phenomena will also occur at the interface/boundary between two different linear/homogeneous/isotropic media.

However, in the *EM* wave situation, what actually happens at the boundary/interface between two linear/homogeneous/isotropic media depends on the electro-dynamic versions of the boundary conditions on the $\vec{E}$ and $\vec{B}$-fields at that interface {as we derived last semester in P435 from the integral form of Maxwell's equations}:

BC 1) The NORMAL component of $\vec{D}$ is continuous across the interface @ $z = 0$ (true only when there are ***no*** free surface charges present @ the interface - $\sigma_{free} = 0$):

$$\oint_S \vec{D}(\vec{r},t)\cdot\hat{n}da = 0$$ (Shrink Gaussian pillbox surface S down to $\in$ above/below interface)

$$D_1^{\perp}(\vec{r},t)\Big|_{intf} = D_2^{\perp}(\vec{r},t)\Big|_{intf} \Rightarrow \varepsilon_1 E_1^{\perp}(\vec{r},t)\Big|_{intf} = \varepsilon_2 E_2^{\perp}(\vec{r},t)\Big|_{intf}$$ since: $\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)$

BC 2) The TANGENTIAL component of $\vec{E}$ is {***always***} continuous across interface @ $z = 0$:

$$\int_S \vec{\nabla}\times\vec{E}(\vec{r},t)\cdot\hat{n}da = \int_C \vec{E}(\vec{r},t)\cdot d\vec{\ell} = -\frac{d}{dt}\cancel{\int_S \vec{B}(\vec{r},t)\cdot\hat{n}da} = 0$$

$$E_1^{\parallel}(\vec{r},t)\Big|_{intf} = E_2^{\parallel}(\vec{r},t)\Big|_{intf}$$ (Shrink contour C down to $\in$ above/below interface)

BC 3) The NORMAL component of $\vec{B}$ is {***always***} continuous across the interface @ $z = 0$:

$$\oint_S \vec{B}(\vec{r},t)\cdot\hat{n}da = 0$$ (Shrink Gaussian pillbox surface S down to $\in$ above/below interface)

$$B_1^{\perp}(\vec{r},t)\Big|_{intf} = B_2^{\perp}(\vec{r},t)\Big|_{intf}$$

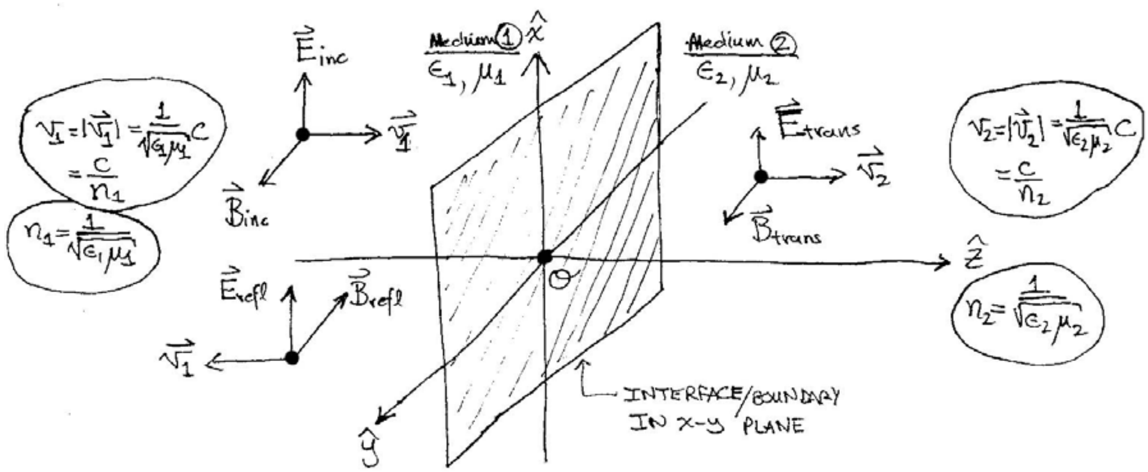
BC 4) The TANGENTIAL component of $\vec{H}$ is continuous across the interface @ $z = 0$ (true only when there are ***no*** free surface currents flowing @ the interface - $\vec{K}_{free} = 0$):

$$\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\cdot\hat{n}da = \int_C \vec{H}(\vec{r},t)\cdot d\vec{\ell} = \mu \cancel{I_{free}^{encl}} + \frac{d}{dt}\cancel{\int_S \vec{D}(\vec{r},t)\cdot\hat{n}da} = 0$$ (Shrink contour C down to $\in$ above/below interface)

$$H_1^{\parallel}(\vec{r},t)\Big|_{intf} = H_2^{\parallel}(\vec{r},t)\Big|_{intf} \Rightarrow \tfrac{1}{\mu_1}B_1^{\parallel}(\vec{r},t)\Big|_{intf} = \tfrac{1}{\mu_2}B_2^{\parallel}(\vec{r},t)\Big|_{intf}$$ since: $\vec{H}(\vec{r},t) = \tfrac{1}{\mu}\vec{B}(\vec{r},t)$

Reflection & Transmission of Linear Polarized Plane *EM* Waves at Normal Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

As shown in the figure below, a boundary between two linear / homogeneous / isotropic media lies in *x*-*y* plane, with a monochromatic plane *EM* wave of frequency ω propagating in the $+\hat{z}$-direction, which is linearly polarized in $+\hat{x}$-direction. Thus this *EM* wave approaches the boundary from the left and is at ***normal incidence*** to the boundary:



Here, we write down the ***complex*** amplitudes for the $\vec{E}$ and $\vec{B}$-fields:

Incident *EM* plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}}$ with: $\boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$

$\boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}}$ since: $\boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}}$

Reflected *EM* plane wave (in medium 1):

Propagates in the $-\hat{z}$-direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\boxed{\hat{n}_{refl} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{x}}$ with: $\boxed{k_{refl} = \left|\vec{k}_{refl}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$

$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{y}}$ since: $\boxed{\hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}}$

Transmitted *EM* plane wave (in medium 2):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\boxed{\hat{n}_{trans} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{x}}$ with: $\boxed{k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2}$

$\boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{y}}$ since: $\boxed{\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}}$

Note that {***here***, in ***this*** situation} the $\vec{E}$-field / ***polarization*** vectors are all oriented in the ***same*** direction, *i.e.* $\boxed{\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}}$ or equivalently: $\boxed{\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)}$.

At the interface / boundary between the two linear / homogeneous / isotropic media, *i.e.* at $z = 0$ {in the *x*-*y* plane} the boundary conditions 1) – 4) ***must*** be satisfied for the total $\vec{E}$ and $\vec{B}$-fields immediately present on either side of the interface between the two media:

BC 1) Normal $\vec{D}$ continuous @ $z = 0$: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}}(z=0,t) = \varepsilon_2 E^{\perp}_{2_{Tot}}(z=0,t)}$
(*n.b.* $\perp$ refers to the *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous @ $z = 0$: $\boxed{E^{\parallel}_{1_{Tot}}(z=0,t) = E^{\parallel}_{2_{Tot}}(z=0,t)}$
(*n.b.* || refers to the *x*-*y* boundary, *i.e.* in the *x*-*y* plane)

BC 3) Normal $\vec{B}$ continuous @ $z = 0$: $\boxed{B^{\perp}_{1_{Tot}}(z=0,t) = B^{\perp}_{2_{Tot}}(z=0,t)}$
($\perp$ to *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 4) Tangential $\vec{H}$ continuous @ $z = 0$: $\boxed{\tfrac{1}{\mu_1} B^{\parallel}_{1_{Tot}}(z=0,t) = \tfrac{1}{\mu_2} B^{\parallel}_{2_{Tot}}(z=0,t)}$
(|| to *x*-*y* boundary, *i.e.* in *x*-*y* plane)

For plane *EM* waves at ***normal*** incidence on the boundary at $z = 0$ lying in the *x*-*y* plane, note that ***no*** components of $\vec{E}$ or $\vec{B}$ (incident, reflected or transmitted waves) are allowed to be along the $\pm\hat{z}$ propagation direction(s) because of the $\vec{E}$ and $\vec{B}$-field ***transversality*** requirement(s) on the propagation of *EM* waves {arising from the constraints imposed by Maxwell's equations}.

Thus, because of this, we see that BC 1) and BC 3) impose ***no*** restrictions {***here***} on such *EM* waves since: { $E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0$; $E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0$ } and: { $B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0$; $B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0$ } @ $z = 0$.

$\Rightarrow$ The only restrictions on plane *EM* waves propagating with normal incidence on the boundary at $z = 0$ {lying in the *x*-*y* plane} are imposed by BC 2) and BC 4).

$\therefore$ <u>In medium 1) (*i.e.* $z \le 0$)</u>:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}(z,t) = \tilde{\vec{E}}_{inc}(z,t) + \tilde{\vec{E}}_{refl}(z,t)} \text{ and: }$$
$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}(z,t) = \frac{1}{\mu_1}\tilde{\vec{B}}_{inc}(z,t) + \frac{1}{\mu_1}\tilde{\vec{B}}_{refl}(z,t)}$$

<u>In medium 2) (*i.e.* $z \ge 0$)</u>:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{2_{Tot}}(z,t) = \tilde{\vec{E}}_{trans}(z,t)} \text{ and: }$$
$$\boxed{\frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}(z,t) = \frac{1}{\mu_2}\tilde{\vec{B}}_{trans}(z,t)}$$

These relations also hold/are valid ***on*** the boundary, *i.e.* @ $z = 0$.

Then BC 2) (Tangential $\vec{E}$ is continuous on the boundary @ $z = 0$) requires that:

$$\boxed{\tilde{\vec{E}}_{1_{Tot}}^{\parallel}\Big|_{z=0} = \tilde{\vec{E}}_{2_{Tot}}^{\parallel}\Big|_{z=0}} \text{ or: } \boxed{\tilde{\vec{E}}_{inc}(z=0,t) + \tilde{\vec{E}}_{refl}(z=0,t) = \tilde{\vec{E}}_{trans}(z=0,t)}.$$

Then BC 4) (Tangential $\vec{H}$ is continuous on the boundary @ $z = 0$) requires that:

$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}_{1_{Tot}}^{\parallel}\Big|_{z=0} = \frac{1}{\mu_2}\tilde{\vec{B}}_{2_{Tot}}^{\parallel}\Big|_{z=0}} \text{ or: } \boxed{\frac{1}{\mu_1}\tilde{\vec{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}_{refl}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}_{trans}(z=0,t)}$$

Inserting the explicit expressions for the complex $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ fields

$$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{y}}$$

into the above boundary condition relations, these equations become:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

We see from the common $e^{-i\omega t}$ factors that these relations are satisfied for ***any*** time t provided that:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Assuming that {μ_1 and μ_2} and {v_1 and v_2} are known / given for the two media, we have ***two*** equations {from BC 2) and BC 4)} and ***three*** unknowns { $\tilde{E}_{o_{inc}}$, $\tilde{E}_{o_{refl}}$, $\tilde{E}_{o_{trans}}$ }

→ Solve above equations simultaneously for { $\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}$ } in terms of / scaled to $\tilde{E}_{o_{inc}}$.

First (for convenience) let us define: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$. Note: $Z_1 = \mu_1 v_1$, $Z_2 = \mu_2 v_2$ so: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{Z_1}{Z_2}}$!!

Then BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$) relation becomes: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\,\tilde{E}_{o_{trans}}}$

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\,\tilde{E}_{o_{trans}}}$ with $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{Z_1}{Z_2}}$

Add BC 2) and BC 4) relations: $2\tilde{E}_{o_{inc}} = (1+\beta)\,\tilde{E}_{o_{trans}}$ $\Rightarrow$ $\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}$ (2+4)

Subtract (BC 2) – BC 4)) relations: $2\tilde{E}_{o_{refl}} = (1-\beta)\tilde{E}_{o_{trans}}$ $\Rightarrow$ $\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{2}\right)\tilde{E}_{o_{trans}}$ (2–4)

Insert the result of eqn. (2+4) into eqn. (2–4): $\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{\not{2}}\right)\left(\dfrac{\not{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\dfrac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}$

$\therefore$ $\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}$ and $\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}$, or: $\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{1-\beta}{1+\beta}\right)$ and: $\dfrac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{2}{1+\beta}\right)$

Now: $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{Z_1}{Z_2}$ and: $v_1 = \dfrac{c}{n_1}$, $v_2 = \dfrac{c}{n_2}$ where: $n_1 = \sqrt{\dfrac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}$ and: $n_2 = \sqrt{\dfrac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}$

$$\therefore\quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}$$

Now if the two media are ***both*** paramagnetic and/or diamagnetic, such that $\left|\chi_{m_{1,2}}\right| \ll 1$

i.e. $\mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o$ and: $\mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o$

{very common for many (but not all) non-conducting linear/homogeneous/isotropic media}

Then: $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\dfrac{v_1}{v_2}\right) = \left(\dfrac{n_2}{n_1}\right)$ for $\mu_1 \approx \mu_2 \approx \mu_o$ or $\left|\chi_{m_{1,2}}\right| \ll 1$

Then: $\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{1-\beta}{1+\beta}\right) \simeq \left(\dfrac{1-(v_1/v_2)}{1+(v_1/v_2)}\right) = \left(\dfrac{v_2-v_1}{v_2+v_1}\right)$

$\dfrac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{2}{1+\beta}\right) \simeq \left(\dfrac{2}{1+(v_1/v_2)}\right) = \left(\dfrac{2v_2}{v_2+v_1}\right)$

n.b. these relations are ***identical*** to the those we obtained for traveling transverse waves on a taught string with $\mu_1 = m_1/L_1$ and $\mu_2 = m_2/L_2$ with a {massless} knot at $z = 0$ {see *p.* 16, P436 Lect. Notes 4}!!!

We can alternatively express these relations in terms of the indices of refraction n_1 & n_2:

$\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} \simeq \left(\dfrac{n_1-n_2}{n_1+n_2}\right)$ and $\dfrac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} \simeq \left(\dfrac{2n_1}{n_1+n_2}\right)$

Now since:

$\tilde{E}_{o_{inc}} = E_{o_{inc}}e^{i\delta}$ — *n.b.* plane waves have ***constant*** phase on (x,y) wavefront @ $z = 0$

$\tilde{E}_{o_{refl}} = E_{o_{refl}}e^{i\delta}$ — δ = phase angle (in radians) defined at the zero of time, $t = 0$

$\tilde{E}_{o_{trans}} = E_{o_{trans}}e^{i\delta}$ — phase factor $e^{i\delta}$ ***common*** to all 3 electric field amplitudes @ $z = 0$

Thus, for the above ***ratios*** of electric field amplitudes @ $z = 0$, these relations become:

Monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media

$$\frac{E_{o_{refl}}}{E_{o_{inc}}} = \left(\frac{1-\beta}{1+\beta}\right) \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) \quad \text{for } \mu_1 \simeq \mu_2 \simeq \mu_o$$

$$\frac{E_{o_{trans}}}{E_{o_{inc}}} = \left(\frac{2}{1+\beta}\right) \simeq \left(\frac{2v_2}{v_1 + v_1}\right) = \left(\frac{2n_1}{n_1 + n_2}\right) \quad \text{for } \mu_1 \simeq \mu_2 \simeq \mu_o$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \frac{Z_1}{Z_2}$$

For a monochromatic plane *EM* wave at ***normal*** incidence on a boundary @ $z = 0$ between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$, note the following points:

- If $v_2 > v_1$ (*i.e.* $Z_2 > Z_1$, or: $n_2 < n_1$) {*e.g.* medium 1) = glass $\Rightarrow$ medium 2) = air}:

$$\frac{E_{o_{refl}}}{E_{o_{inc}}} = \left(\frac{1-\beta}{1+\beta}\right) = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \simeq \left(\frac{n_1 - n_2}{n_1 + n_2}\right) \Rightarrow$$

$E_{o_{refl}}$ is precisely in-phase with $E_{o_{inc}}$ because $(v_2 - v_1) > 0$.

- If $v_2 < v_1$ (*i.e.* $Z_2 < Z_1$, or: $n_2 > n_1$) {*e.g.* medium 1) = air $\Rightarrow$ medium 2) = glass}:

$$\frac{E_{o_{refl}}}{E_{o_{inc}}} = \left(\frac{1-\beta}{1+\beta}\right) = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \simeq \left(\frac{n_1 - n_2}{n_1 + n_2}\right) \Rightarrow$$

$E_{o_{refl}}$ is 180° out-of-phase with $E_{o_{inc}}$ because $(v_2 - v_1) < 0$.

i.e.
$$\frac{E_{o_{refl}}}{E_{o_{inc}}} = -\left|\frac{1-\beta}{1+\beta}\right| = -\left|\frac{Z_2 - Z_1}{Z_2 + Z_1}\right| \simeq -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| \simeq -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| \Rightarrow$$

The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (*i.e.* $n_2 > n_1$) !!!

- $E_{o_{trans}}$ is ***always*** in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $(n_1 \,\&\, n_2)$ because:

$$\frac{E_{o_{trans}}}{E_{o_{inc}}} = \left(\frac{2}{1+\beta}\right) = \left(\frac{2Z_2}{Z_1 + Z_2}\right) \simeq \left(\frac{2v_2}{v_1 + v_1}\right) \simeq \left(\frac{2n_1}{n_1 + n_2}\right)$$

What ***fraction*** of the incident *EM* wave ***energy*** is ***reflected*** back from the interface @ $z = 0$?

What ***fraction*** of the incident *EM* wave ***energy*** is ***transmitted*** through the interface @ $z = 0$?

In a given linear/homogeneous/isotropic medium with: $v = \sqrt{\dfrac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = \dfrac{c}{n}$

The time-averaged energy density in the *EM* wave is: $\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r})$ $\left(\dfrac{\text{Joules}}{\text{m}^3}\right)$

The time-averaged Poynting's vector is: $\langle \vec{\tilde{S}}(\vec{r},t)\rangle = v\langle u_{EM}(\vec{r},t)\rangle \hat{k}$ $\left(\dfrac{\text{Watts}}{\text{m}^2}\right)$

The time-averaged *EM* field power flowing through/crossing an imaginary/conceptual surface S *e.g.* at/on the boundary/interface @ $z = 0$ is: $\boxed{\text{P}_{EM}(z=0) = \int_S \langle \tilde{\vec{S}}(\vec{r},t)\rangle \cdot \hat{n}da \ \ (\text{Watts})}$

where $\hat{n}$ is the unit ***normal*** of the imaginary/conceptual surface S, $\hat{n}$ is oriented on the surface S in the direction that *EM* energy is flowing such that $\langle \tilde{\vec{S}}(\vec{r},t)\rangle \cdot \hat{n}$ is ***always*** a positive quantity.

Thus ***here***, $\hat{n} = +\hat{z}$ direction. We can define the ***intensity*** (*aka* ***irradiance***) of the *EM* wave as:

$$\boxed{I(z=0) \equiv \langle \tilde{\vec{S}}(z=0,t)\rangle \cdot \hat{n}} \ \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The 3 Poynting's vectors associated with ***this*** problem have: $\vec{S}_{inc} \parallel (+\hat{z})$, $\vec{S}_{refl} \parallel (-\hat{z})$ and: $\vec{S}_{trans} \parallel (+\hat{z})$.
Then, for ***this*** problem {***here***}, each of the 3 intensities is of the form:

$$\boxed{I(z=0) \equiv \langle \tilde{\vec{S}}(z=0,t)\rangle \cdot \hat{n} = v\langle u_{EM}(z=0,t)\rangle = v\left(\tfrac{1}{2}\varepsilon E_o^2\right) = \tfrac{1}{2}\varepsilon v E_o^2 = \varepsilon v E_{o_{rms}}^2} \ \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$ @ $z = 0$:

$$\boxed{\frac{E_{o_{refl}}}{E_{o_{inc}}} = \left(\frac{1-\beta}{1+\beta}\right) = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \simeq \left(\frac{n_1 - n_2}{n_1 + n_2}\right)} \quad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \frac{Z_1}{Z_2}}$$

$$\boxed{\frac{E_{o_{trans}}}{E_{o_{inc}}} = \left(\frac{2}{1+\beta}\right) = \left(\frac{2Z_2}{Z_1 + Z_2}\right) \simeq \left(\frac{2v_2}{v_1 + v_1}\right) \simeq \left(\frac{2n_1}{n_1 + n_2}\right)}$$

Then square each of these relations:

$$\boxed{\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \quad \text{and:}$$

$$\boxed{\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 = \left(\frac{2Z_2}{Z_1 + Z_2}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Define the ***reflection*** coefficient as the ratio of reflected to incident ***intensities*** @ $z = 0$:

$$\boxed{R \equiv \frac{I_{refl}(0)}{I_{inc}(0)} = \frac{\langle \vec{S}_{refl}(0,t)\rangle \cdot (-\hat{z})}{\langle \vec{S}_{inc}(\vec{r},t)\rangle \cdot \hat{z}} = \frac{v_1 \langle u_{EM}^{refl}(0,t)\rangle}{v_1 \langle u_{EM}^{inc}(0,t)\rangle} = \frac{\tfrac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2}{\tfrac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2} = \frac{E_{o_{refl}}^2}{E_{o_{inc}}^2}}$$

Define the ***transmission*** coefficient as the ratio of transmitted to incident ***intensities*** @ $z = 0$:

$$\boxed{T \equiv \frac{I_{trans}(0)}{I_{inc}(0)} = \frac{\langle \vec{S}_{trans}(0,t)\rangle \cdot \hat{z}}{\langle \vec{S}_{inc}(0,t)\rangle \cdot \hat{z}} = \frac{v_2 \langle u_{EM}^{trans}(0,t)\rangle}{v_1 \langle u_{EM}^{inc}(0,t)\rangle} = \frac{\tfrac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2}{\tfrac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2}{\varepsilon_1 v_1 E_{o_{inc}}^2}}$$

For a linearly-polarized monochromatic plane *EM* wave at normal incidence on a boundary @ $z = 0$ between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$:

Reflection coefficient @ $z = 0$:
$$R \equiv \frac{I_{refl}(0)}{I_{inc}(0)} = \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2$$

Transmission coefficient @ $z = 0$:
$$T \equiv \frac{I_{trans}(0)}{I_{inc}(0)} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2$$

But:
$$\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$
and:
$$\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 = \left(\frac{2Z_2}{Z_1 + Z_2}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Thus, the reflection and transmission coefficients for *EM* plane wave at normal incidence are:

$$R \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad \beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$T \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2Z_2}{Z_1 + Z_2}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Now:
$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2} = \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}\frac{v_1}{v_2}\frac{\varepsilon_2}{\varepsilon_1} = \frac{v_2^2}{v_1^2}\frac{v_1}{v_2}\frac{\varepsilon_2}{\varepsilon_1} = \frac{v_2}{v_1}\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(= \frac{Z_1}{Z_2}\right)$$

for $\mu_1 \approx \mu_2 \approx \mu_o$

∴
$$T = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2 + v_1)^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

Thus:

$$R + T = \frac{(1-\beta)^2}{(1+\beta)^2} + \frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2 + 4\beta}{(1+\beta)^2} = \frac{1 - 2\beta + \beta^2 + 4\beta}{(1+\beta)^2} = \frac{1 + 2\beta + \beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1$$

∴ $R + T = 1$ ⇒ *EM* energy ***is*** conserved at the interface/boundary between two linear / homogeneous / isotropic media in this process !!!

EXAMPLE:

A monochromatic plane *EM* wave is incident on an air-glass interface (@ $z = 0$) at normal incidence:

Indices of refraction for air and glass (*n.b.* both are non-magnetic materials) $\begin{pmatrix} n_1 = n_{air} \simeq 1.0 \\ n_2 = n_{glass} \simeq 1.5 \end{pmatrix}$

Reflection coefficient: $R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1.0-1.5}{1.0+1.5}\right)^2 = \left(\frac{-0.5}{2.5}\right)^2 = \left(-\frac{1}{5}\right)^2 = \frac{1}{25} = 0.04 = 4\%$

Transmission coefficient: $T = \frac{4n_1n_2}{(n_1+n_2)^2} = \frac{4\cdot 1.0\cdot 1.5}{(1.0+1.5)^2} = \frac{6.0}{(2.5)^2} = \frac{6.0}{6.25} = 0.96 = 96\%$

$R + T = 0.04 + 0.96 = 1.00$

QUESTION: Is *EM* linear momentum conserved in this process?

The time-averaged linear momentum ***densities*** associated with the 3 *EM* waves are:

$$\left\langle \vec{\wp}_{EM}^{inc}(\vec{r},t)\right\rangle = +\frac{1}{v_1}\left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle\hat{z} = +\frac{1}{v_1}\left(\tfrac{1}{2}\varepsilon_1 E_{o_{inc}}^2(\vec{r})\right)\hat{z}$$

$$\left\langle \vec{\wp}_{EM}^{refl}(\vec{r},t)\right\rangle = -\frac{1}{v_1}\left\langle u_{EM}^{refl}(\vec{r},t)\right\rangle\hat{z} = -\frac{1}{v_1}\left(\tfrac{1}{2}\varepsilon_1 E_{o_{refl}}^2(\vec{r})\right)\hat{z}$$

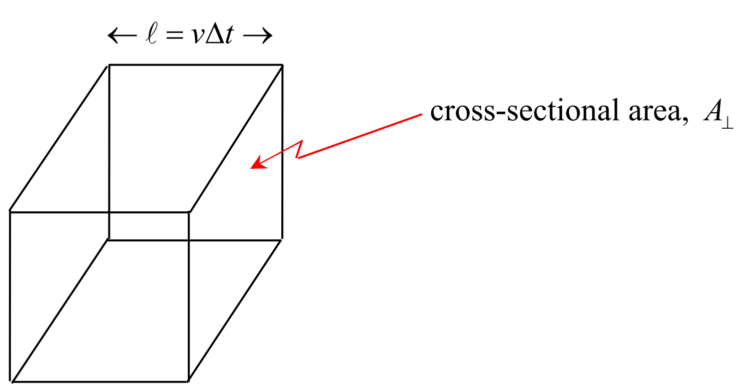
$$\left\langle \vec{\wp}_{EM}^{trans}(\vec{r},t)\right\rangle = +\frac{1}{v_2}\left\langle u_{EM}^{trans}(\vec{r},t)\right\rangle\hat{z} = +\frac{1}{v_2}\left(\tfrac{1}{2}\varepsilon_2 E_{o_{trans}}^2(\vec{r})\right)\hat{z}$$

In order that *EM* linear momentum be conserved at the interface, we must have the time-averaged ***initial*** *EM* linear momentum at the interface = the time-averaged ***final*** *EM* linear momentum at the interface, *i.e.* $\left\langle \vec{p}_{EM}^{initial}(\vec{r},t)\right\rangle\big|_{z=0} = \left\langle \vec{p}_{EM}^{final}(\vec{r},t)\right\rangle\big|_{z=0}$.

{*n.b.* we (again) use time-averages here, in order to make direct comparisons with experimental measurements of these quantities}.

Now: $\left\langle \vec{p}(\vec{r},t)\right\rangle = \int_v \left\langle \vec{\wp}(\vec{r},t)\right\rangle d\tau = \left\langle \vec{\wp}(\vec{r},t)\right\rangle$ * Volume, ΔV where the volume associated with the *EM* wave over the time interval Δt is $\Delta V = \ell \cdot A_\perp = v\Delta t \cdot A_\perp$

Incident plane *EM* wave ⇒



Thus: $\left\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\right\rangle = \left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle \Delta V_{inc} = \left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle v_1 \Delta t \mathrm{A}_{\perp}$

$\left\langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\right\rangle = \left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle \Delta V_{refl} = \left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle v_1 \Delta t \mathrm{A}_{\perp}$

$\left\langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\right\rangle = \left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle \Delta V_{trans} = \left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle v_2 \Delta t \mathrm{A}_{\perp}$

Then: $\left\langle \vec{p}_{EM}^{\,inital}(\vec{r},t)\right\rangle\Big|_{z=0} = \left\langle \vec{p}_{EM}^{\,final}(\vec{r},t)\right\rangle\Big|_{z=0} \Rightarrow \left\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\right\rangle\Big|_{z=0} = \left\langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\right\rangle\Big|_{z=0} + \left\langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\right\rangle\Big|_{z=0}$

Thus: $\left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle \Delta V_{inc}\Big|_{z=0} = \left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle \Delta V_{refl}\Big|_{z=0} + \left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle \Delta V_{trans}\Big|_{z=0}$

or: $\left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle v_1 \Delta t \cdot \mathrm{A}_{\perp}\Big|_{z=0} = \left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle v_1 \Delta t \cdot \mathrm{A}_{\perp}\Big|_{z=0} + \left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle v_2 \Delta t \cdot \mathrm{A}_{\perp}\Big|_{z=0}$

i.e: $v_1 \left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle\Big|_{z=0} = v_1 \left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle\Big|_{z=0} + v_2 \left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle\Big|_{z=0}$

But: $\left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\right\rangle = +\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{inc}}^2(\vec{r})\right)\hat{z}$

$\left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\right\rangle = -\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{refl}}^2(\vec{r})\right)\hat{z}$

$\left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\right\rangle = +\frac{1}{v_2}\left(\frac{1}{2}\varepsilon_2 E_{o_{trans}}^2(\vec{r})\right)\hat{z}$

Thus: $\frac{\cancel{v_1}}{\cancel{v_1}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_1 E_{o_{inc}}^2\right)\Bigg|_{z=0} = -\frac{\cancel{v_1}}{\cancel{v_1}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_1 E_{o_{refl}}^2\right)\Bigg|_{z=0} + \frac{\cancel{v_2}}{\cancel{v_2}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_2 E_{o_{trans}}^2\right)\Bigg|_{z=0}$ @ $z=0$

or: $\varepsilon_1\left(E_{o_{inc}}^2 + E_{o_{refl}}^2\right)\Big|_{z=0} = \varepsilon_2 E_{o_{trans}}^2\Big|_{z=0}$ @ $z=0$

Divide this relation on both sides by $E_{o_{inc}}^2$: $1+\underbrace{\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2}_{\equiv R} = \frac{\varepsilon_2}{\varepsilon_1}\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{v_1}{v_2}\right)\underbrace{\left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2}_{\equiv T}$

Thus: $1+R = \left(\frac{v_1}{v_2}\right)T$ But: $R+T=1$ or: $R=1-T$

$\therefore$ $1+(1-T) = \left(\frac{v_1}{v_2}\right)T$ or: $2-T = \left(\frac{v_1}{v_2}\right)T$ or: $2 = \left[1+\left(\frac{v_1}{v_2}\right)\right]T$

Thus: $T = \frac{2}{\left[1+(v_1/v_2)\right]} = \frac{2v_2}{(v_1+v_2)}$

But: $T = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2+v_1)^2} \neq \frac{2v_2}{(v_1+v_2)}$ {from above} !!! where: $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}$

$\Rightarrow$ Linear momentum carried by *EM* wave is ***NOT*** conserved in/at the interface between two linear / homogeneous / isotropic media !!! Why???? How???

The physical reason for this is because {again} we're ***not*** "counting ***all*** of the beans" ***here***…

The *EM* waves that are present in each of the linear / homogeneous / isotropic media (*i.e.* the *EM* waves that exist in medium 1 and medium 2) ***polarize*** the atoms/molecules in that medium and create an ***additional co-traveling*** momentum in that medium – which results from the {mechanical} momentum of the electrons associated with the atomic/molecular ***induced electric dipole moments*** that arise in response to the ***induced polarization*** associated with the incident/reflected/transmitted traveling *EM* waves! Please see/read P436 Lect. Notes 7.5….

Thus, ***overall*** linear momentum ***is*** conserved when the *EM* wave ***and*** its co-traveling electron / atom / molecule induced electric dipole mechanical momentum associated with the medium is included.

In medium 1: $\boxed{\langle \vec{p}_{Tot}^{\,inc}(\vec{r},t)\rangle = \langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\rangle + \langle \vec{p}_{e^-dipole}^{\,inc}(\vec{r},t)\rangle}$

In medium 1: $\boxed{\langle \vec{p}_{Tot}^{\,refl}(\vec{r},t)\rangle = \langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\rangle + \langle \vec{p}_{e^-dipole}^{\,refl}(\vec{r},t)\rangle}$

In medium 2: $\boxed{\langle \vec{p}_{Tot}^{\,trans}(\vec{r},t)\rangle = \langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\rangle + \langle \vec{p}_{e^-dipole}^{\,trans}(\vec{r},t)\rangle}$

Hence for ***total/overall*** momentum conservation, we ***must*** have: $\langle \vec{p}_{Tot}^{\,inital}(\vec{r},t)\rangle\big|_{z=0} = \langle \vec{p}_{Tot}^{\,final}(\vec{r},t)\rangle\big|_{z=0}$ *i.e.* we ***must*** have @ $z = 0$ at/on the interface:

$$\boxed{\begin{aligned}\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{e^-dipole}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} &= \langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{e^-dipole}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} \\ &\quad + \langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{e^-dipole}^{\,trans}(\vec{r},t)\rangle\big|_{z=0}\end{aligned}}$$

Or:

$$\boxed{\begin{aligned}&\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} - \langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} - \langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\rangle\big|_{z=0} \\ &= -\langle \vec{p}_{e^-dipole}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{e^-dipole}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{e^-dipole}^{\,trans}(\vec{r},t)\rangle\big|_{z=0} \neq 0\end{aligned}}$$

It is curious that the time-averaged *EM* field ***energy*** (alone) ***is*** conserved, whereas the time-averaged *EM* field ***linear momentum*** is ***not*** conserved at the interface of two L/H/I media. Microscopically, note that a photon's energy $E_\gamma = hf$ is ***unchanged*** in such a medium, whereas a photon's ***linear momentum*** $p_\gamma = h/\lambda_\gamma$ ***is*** changed. Since ***macroscopic*** *EM* field ***linear momentum*** is ***not*** conserved at the interface of two L/H/I media, neither will *EM* field ***angular momentum*** / *EM* field ***angular momentum density*** be conserved, since: $\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t)}$.

For further details on this subject, see/read:

1.) J.D. Jackson, *Classical Electrodynamics*, *p*. 262, 3rd Ed. Wiley, NY
2.) R.E. Peierls, *Proc. Roy. Soc. London* **347**, *p*. 475 (1976).
3.) R.E. Peierls, *Proc. Roy. Soc. London* **355**, *p*. 141 (1971).
4.) R. Loudon, L. Allen and D.F. Nelson, *Phys. Rev. E***55**, *p*. 1071 (1997).

Arbitrary/Generalized Polarization States of a Plane *EM* Wave; Elliptical, Circular and Linear Polarization

As we saw in the previous discussion, a monochromatic, linearly-polarized plane *EM* wave *e.g.* propagating in the $+\hat{z}$ direction in medium 1, which is also at normal incidence to a boundary between two linear / homogenous / isotropic media {located as before at $z = 0$ in the x-y plane} has the following mathematical forms {for linear polarization in the $+\hat{x}$ direction} for the complex $\vec{E}$ and $\vec{B}$ fields:

Incident monochromatic, linearly-polarized *EM* plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with linear polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}^{LP}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}} \quad \text{with: } \boxed{k_{inc} = |\vec{k}_{inc}| = k_1 = |\vec{k}_1| = 2\pi/\lambda_1 = \omega/v_1} \quad \text{and: } \boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$$

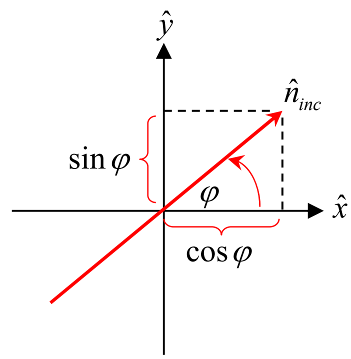
$$\boxed{\tilde{\vec{B}}^{LP}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}^{LP}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z}\times\hat{x} = +\hat{y}}$$

In general, this monochromatic, linearly-polarized *EM* plane wave incident on the boundary between two linear / homogenous / isotropic media can be polarized in *any* direction in the x-y plane. More generally then, we can write the polarization vector $\hat{n}_{inc}$ as:

$$\boxed{\hat{n}_{inc} = \cos\varphi\hat{x} + \sin\varphi\hat{y}} \quad \text{where} \quad \boxed{0 \le \varphi < 2\pi}$$

$\varphi = 0^o$: $\Rightarrow$ LP in $+\hat{x}$-direction

$\varphi = 90^o$: $\Rightarrow$ LP in $+\hat{y}$-direction



Thus, more generally, we can write the complex $\vec{E}$ and $\vec{B}$ fields for the incident monochromatic, but arbitrarily linearly-polarized *EM* plane wave (in medium 1) as:

Incident monochromatic, arbitrarily linearly-polarized *EM* plane wave (in medium 1):

$+\hat{z}$ propagation direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), arbitrary linear polarization $\boxed{\hat{n}_{inc} = \cos\varphi\hat{x} + \sin\varphi\hat{y}}$

$$\boxed{\tilde{\vec{E}}^{LP}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{n}_{inc} = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\left[\cos\varphi\hat{x} + \sin\varphi\hat{y}\right]}$$

with: $\boxed{k_{inc} = |\vec{k}_{inc}| = k_1 = |\vec{k}_1| = 2\pi/\lambda_1 = \omega/v_1}$ and: $\boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$

$$\boxed{\tilde{\vec{B}}^{LP}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}^{LP}_{inc}(z,t) = \frac{1}{v_1}\tilde{\vec{E}}_{o_{inc}} e^{i(k_1 z - \omega t)}\left(\hat{k}_{inc} \times \hat{n}_{inc}\right)}$$

But: $\boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z}\times\left[\cos\varphi\hat{x} + \sin\varphi\hat{y}\right] = \cos\varphi\left(\hat{z}\times\hat{x}\right) + \sin\varphi\left(\hat{z}\times\hat{y}\right) = +\cos\varphi\hat{y} - \sin\varphi\hat{x}}$

Very Useful Table:

$\hat{x}\times\hat{y} = +\hat{z}$	$\hat{y}\times\hat{x} = -\hat{z}$
$\hat{y}\times\hat{z} = +\hat{x}$	$\hat{z}\times\hat{y} = -\hat{x}$
$\hat{z}\times\hat{x} = +\hat{y}$	$\hat{x}\times\hat{z} = -\hat{y}$

Thus, the complex $\vec{B}$-field can be equivalently written as:

$$\tilde{\vec{B}}_{inc}^{LP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{LP}(z,t)=\frac{1}{v_1}\tilde{E}_{o_{inc}}e^{i(k_1z-\omega t)}\left(\hat{k}_{inc}\times\hat{n}_{inc}\right)=\frac{1}{v_1}\tilde{E}_{o_{inc}}e^{i(k_1z-\omega t)}\left[\cos\varphi\hat{y}-\sin\varphi\hat{x}\right]$$

As always, the ***physical*** $\vec{E}$ and $\vec{B}$ fields associated with this *EM* wave are of the form:

$$\begin{aligned}\vec{E}_{inc}^{LP}(z,t)&=\text{Re}\left\{\tilde{\vec{E}}_{inc}^{LP}(z,t)\right\}=\text{Re}\left\{\tilde{E}_{o_{inc}}e^{i(k_1z-\omega t)}\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\right\},\quad\text{but}:\ \tilde{E}_{o_{inc}}=E_{o_{inc}}e^{i\delta}\\&=\text{Re}\left\{E_{o_{inc}}e^{i\delta}e^{i(k_1z-\omega t)}\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\right\}=\text{Re}\left\{E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\right\}\\&=E_{o_{inc}}\ \text{Re}\left\{e^{i(k_1z-\omega t+\delta)}\right\}\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\\&=E_{o_{inc}}\ \text{Re}\left\{\cos(k_1z-\omega t+\delta)+i\sin(k_1z-\omega t+\delta)\right\}\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\\&=E_{o_{inc}}\cos(k_1z-\omega t+\delta)\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]\end{aligned}$$

$$\begin{aligned}\vec{B}_{inc}^{LP}(z,t)&=\text{Re}\left\{\tilde{\vec{B}}_{inc}^{LP}(z,t)\right\}=\frac{1}{v_1}\hat{k}_{inc}\times\vec{E}_{inc}^{LP}(z,t)=\frac{1}{v_1}E_{o_{inc}}\cos(k_1z-\omega t+\delta)\left(\hat{k}_{inc}\times\hat{n}_{inc}\right)\\&=\frac{1}{v_1}E_{o_{inc}}\cos(k_1z-\omega t+\delta)\left[\cos\varphi\hat{y}-\sin\varphi\hat{x}\right]\end{aligned}$$

Now, for a circularly-polarized monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence, the *physical* $\vec{E}$ and $\vec{B}$ fields can be written mathematically as follows:

$$\vec{E}_{inc}^{CP}(z,t)=E_{o_{inc}}\left[\cos(k_1z-\omega t+\delta)\hat{x}\pm\sin(k_1z-\omega t+\delta)\hat{y}\right]\quad\text{with}\quad\hat{k}_{inc}=+\hat{k}_1=+\hat{z}$$

$$\begin{aligned}\vec{B}_{inc}^{CP}(z,t)&=\tfrac{1}{v_1}\hat{k}_{inc}\times\vec{E}_{inc}^{CP}(z,t)=\tfrac{1}{v_1}E_{o_{inc}}\left\{\hat{z}\times\left[\cos(k_1z-\omega t+\delta)\hat{x}\pm\sin(k_1z-\omega t+\delta)\hat{y}\right]\right\}\\&=\tfrac{1}{v_1}E_{o_{inc}}\left[\cos(k_1z-\omega t+\delta)(\hat{z}\times\hat{x})\pm\sin(k_1z-\omega t+\delta)(\hat{z}\times\hat{y})\right]\\&=\tfrac{1}{v_1}E_{o_{inc}}\left[\cos(k_1z-\omega t+\delta)\hat{y}\mp\sin(k_1z-\omega t+\delta)\hat{x}\right]\end{aligned}$$

Note that the $\pm$ signs between the 90° out-of-phase $\hat{x}$ and $\hat{y}$ components for $\vec{E}$ (and the corresponding $\mp$ signs for $\vec{B}$) denote the ***handedness*** of the circularly polarized *EM* wave – *i.e.* whether it is right- or left-circularly polarized!

A right- (left-) circularly-polarized monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence, the ***physical*** $\vec{E}$ and $\vec{B}$ fields can be written mathematically as follows:

RCP *EM* Wave:

$$\vec{E}_{inc}^{RCP}(z,t) = E_{o_{inc}}\left[\cos\left(k_1 z - \omega t + \delta\right)\hat{x} + \sin\left(k_1 z - \omega t + \delta\right)\hat{y}\right]$$

$$\vec{B}_{inc}^{RCP}(z,t) = \frac{1}{v_1}E_{o_{inc}}\left[\cos\left(k_1 z - \omega t + \delta\right)\hat{y} - \sin\left(k_1 z - \omega t + \delta\right)\hat{x}\right]$$

LCP *EM* Wave:

$$\vec{E}_{inc}^{LCP}(z,t) = E_{o_{inc}}\left[\cos\left(k_1 z - \omega t + \delta\right)\hat{x} - \sin\left(k_1 z - \omega t + \delta\right)\hat{y}\right]$$

$$\vec{B}_{inc}^{RCP}(z,t) = \frac{1}{v_1}E_{o_{inc}}\left[\cos\left(k_1 z - \omega t + \delta\right)\hat{y} + \sin\left(k_1 z - \omega t + \delta\right)\hat{x}\right]$$

Note that at $(z,t) = (0,0)$ these *EM* fields at that point/at that time are:

RCP *EM* Wave:

$$\vec{E}_{inc}^{RCP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} + \sin\delta\hat{y}\right]$$

$$\vec{B}_{inc}^{RCP}(0,0) = \frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y} - \sin\delta\hat{x}\right]$$

LCP *EM* Wave:

$$\vec{E}_{inc}^{LCP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} - \sin\delta\hat{y}\right]$$

$$\vec{B}_{inc}^{LCP}(0,0) = \frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y} + \sin\delta\hat{x}\right]$$

Or more generally for circularly-polarized *EM* waves (right- or left-handed):

CP *EM* Wave:

$$\vec{E}_{inc}^{CP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} \pm \sin\delta\hat{y}\right] \quad (+ = \text{RCP}, - = \text{LCP})$$

$$\vec{B}_{inc}^{CP}(0,0) = \frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y} \mp \sin\delta\hat{x}\right] \quad (- = \text{RCP}, + = \text{LCP})$$

If we compare these formulae to their equivalents for arbitrarily linearly-polarized *EM* waves, with $\boxed{\hat{n}_{LP} = \hat{n}_{inc} \equiv \cos\varphi\hat{x} + \sin\varphi\hat{y}}$:

LP *EM* Wave:

$$\vec{E}_{inc}^{LP}(0,0) = E_{o_{inc}}\cos\delta\left[\cos\varphi\hat{x} + \sin\varphi\hat{y}\right] = E_{o_{inc}}\cos\delta\,\hat{n}_{LP} = E_{o_{inc}}\cos\delta\,\hat{n}_{inc}$$

$$\vec{B}_{inc}^{LP}(0,0) = \frac{1}{v_1}E_{o_{inc}}\cos\delta\left[\cos\varphi\hat{y} - \sin\varphi\hat{x}\right] = \frac{1}{v_1}E_{o_{inc}}\cos\delta\left(\hat{k}\times\hat{n}_{inc}\right)$$

Then we see that we can {analogously} define right- and left-circular transverse polarization unit vectors (*i.e.* lying in the *x*-*y* plane, $\perp$ to the direction of propagation {here, in the $+\hat{z}$ direction}):

RCP *EM* Wave: $\boxed{\hat{n}_{RCP} = \hat{n}_{+} \equiv \cos\delta\hat{x} + \sin\delta\hat{y}}$

LCP *EM* Wave: $\boxed{\hat{n}_{LCP} = \hat{n}_{-} \equiv \cos\delta\hat{x} - \sin\delta\hat{y}}$

Thus, we can write the physical instantaneous $\vec{E}$ and $\vec{B}$ fields at $(z,t)=(0,0)$ associated with a right- (left-) circularly-polarized monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence as follows, for $\boxed{\hat{n}_{RCP}=\hat{n}_+\equiv\cos\delta\hat{x}+\sin\delta\hat{y}}$ and $\boxed{\hat{n}_{LCP}=\hat{n}_-\equiv\cos\delta\hat{x}-\sin\delta\hat{y}}$:

RCP *EM* Wave:

$$\vec{E}_{inc}^{RCP}(0,0)=E_{o_{inc}}[\cos\delta\hat{x}+\sin\delta\hat{y}]=E_{o_{inc}}\hat{n}_{RCP}=E_{o_{inc}}\hat{n}_+$$

$$\vec{B}_{inc}^{RCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}[\cos\delta\hat{y}-\sin\delta\hat{x}]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{RCP}\right)=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_+\right)$$

LCP *EM* Wave:

$$\vec{E}_{inc}^{LCP}(0,0)=E_{o_{inc}}[\cos\delta\hat{x}-\sin\delta\hat{y}]=E_{o_{inc}}\hat{n}_{LCP}=E_{o_{inc}}\hat{n}_-$$

$$\vec{B}_{inc}^{LCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}[\cos\delta\hat{y}+\sin\delta\hat{x}]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{LCP}\right)=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_-\right)$$

Or more generally for ***circularly-polarized*** *EM* waves (right- or left-handed):

CP *EM* Wave:

$$\vec{E}_{inc}^{RCP}(0,0)=E_{o_{inc}}[\cos\delta\hat{x}\pm\sin\delta\hat{y}]=E_{o_{inc}}\hat{n}_\pm$$

$$\vec{B}_{inc}^{RCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}[\cos\delta\hat{y}\mp\sin\delta\hat{x}]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_\pm\right)$$

Defining right- and left- ***complex*** circular-polarization unit vectors, respectively as:

$$\boxed{\hat{\epsilon}_{RCP}=\hat{\epsilon}_-\equiv\tfrac{1}{\sqrt{2}}[\hat{x}-i\hat{y}]}\quad\text{and:}\quad\boxed{\hat{\epsilon}_{LCP}=\hat{\epsilon}_+\equiv\tfrac{1}{\sqrt{2}}[\hat{x}+i\hat{y}]}:$$

The corresponding ***complex*** CP (RCP or LCP) *EM* waves are of the following forms

RCP *EM* Wave:

$$\tilde{\vec{E}}_{inc}^{RCP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}[\hat{x}-i\hat{y}]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{RCP}=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_-$$

$$\tilde{\vec{B}}_{inc}^{RCP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{RCP}(z,t)$$

LCP *EM* Wave:

$$\tilde{\vec{E}}_{inc}^{LCP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}[\hat{x}+i\hat{y}]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{LCP}=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_+$$

$$\tilde{\vec{B}}_{inc}^{LCP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{LCP}(z,t)$$

CP *EM* Wave:

$$\tilde{\vec{E}}_{inc}^{CP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}[\hat{x}\mp i\hat{y}]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_\mp$$

$$\tilde{\vec{B}}_{inc}^{CP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{CP}(z,t)$$

(*n.b.* $-$ = RCP, $+$ = LCP here !!!)

At a ***fixed*** point in space (*e.g.* $z = 0$), an observer looking into an ***oncoming/incident*** LCP *EM* plane wave sees the electric field vector $\vec{E}_{inc}^{LCP}(z=0,t)$ spinning/rotating ***counter-clockwise*** (CCW) at angular frequency ω for a LCP *EM* wave as ***time*** progresses.

Similarly, at a ***fixed*** point in space (*e.g.* $z = 0$), an observer looking into an ***oncoming/incident*** RCP *EM* plane wave sees the electric field vector $\vec{E}_{inc}^{RCP}(z=0,t)$ spinning/rotating ***clockwise*** (CW) in a circle at angular frequency ω for RCP light as ***time*** progresses.

Note that both ***linearly-polarized*** and ***circularly-polarized*** *EM* plane waves are limiting/special cases of the more general class of ***elliptically-polarized*** *EM* plane waves.

For a ***generally-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field have the ***same*** phase, *i.e.* $\boxed{\tilde{E}_{ox} = E_{ox}e^{i\delta}}$ and $\boxed{\tilde{E}_{oy} = E_{oy}e^{i\delta}}$, this is a ***linearly-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction: $\boxed{\tilde{\vec{E}}^{LP}(z,t) = \left[E_{ox}\hat{x} + E_{oy}\hat{y}\right]e^{i(k_1 z - \omega t + \delta)}}$.

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field have the ***same*** amplitude and the ***same*** phase, *i.e.* $\boxed{\tilde{E}_{ox} = E_o e^{i\delta}}$ and $\boxed{\tilde{E}_{oy} = E_o e^{i\delta}}$, this is a monochromatic *EM* plane wave ***linearly-polarized*** at +45° (*w.r.t.* the $\hat{x}$-axis) propagating in the $+\hat{z}$ direction: $\boxed{\tilde{\vec{E}}^{LP}(z,t) = E_o\left[\hat{x} + \hat{y}\right]e^{i(k_1 z - \omega t + \delta)}}$.

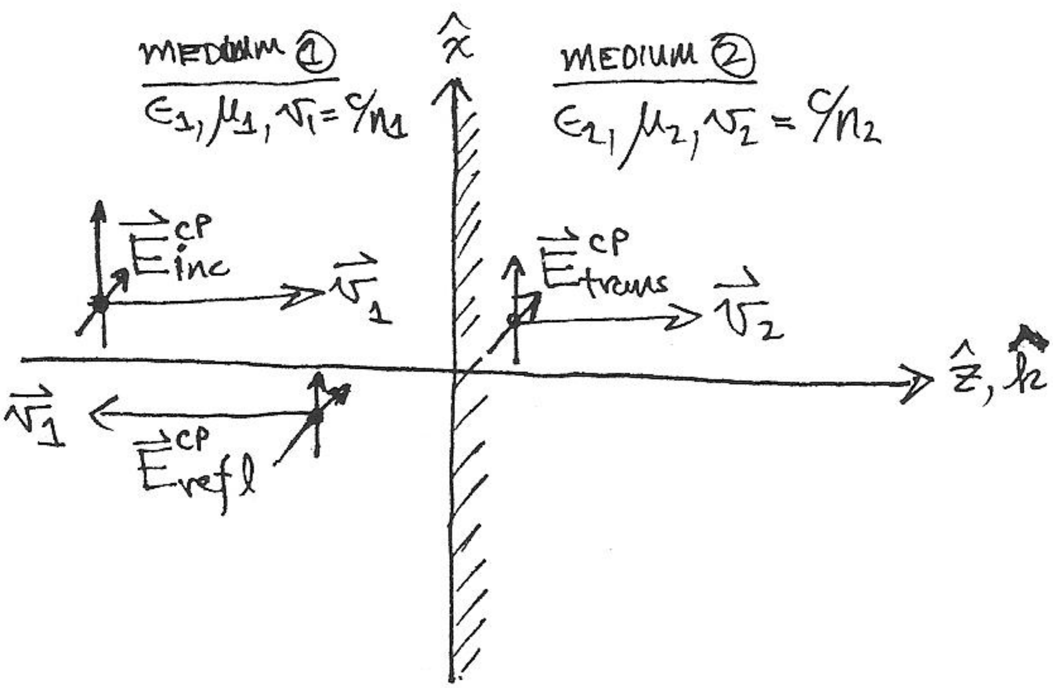
Other special cases of linear polarization, such as LP in the $\hat{x}$-only, or the $\hat{y}$-only direction, or *e.g.* $\varphi = -45°$ (*w.r.t.* the $\hat{x}$-axis) can also be easily worked out.

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$ of the ***generally-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction have ***different*** phases, *i.e.* $\boxed{\tilde{E}_{ox} = E_{ox}e^{i\delta_x}}$ and $\boxed{\tilde{E}_{oy} = E_{oy}e^{i\delta_y}}$, this *EM* wave is ***elliptically-polarized***.

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$ of the ***generally-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction have the ***same*** amplitudes {*i.e.* $\boxed{E_{ox} = E_{oy} = E_o}$} but their phases ***differ*** by $\boxed{\delta_x - \delta_y = \pm 90^o = \pm\pi/2}$ radians, *i.e.* $\boxed{\tilde{E}_{ox} = E_o e^{i\delta_x}}$ and $\boxed{\tilde{E}_{oy} = E_o e^{i\delta_y} = E_o e^{i(\delta_x \mp \pi/2)} = E_o e^{i\delta_x}e^{\mp i\pi/2} = \mp iE_o e^{i\delta_x} = \mp i\tilde{E}_{ox}}$ {since: $\boxed{e^{\mp i\pi/2} = \cos(\pi/2) \mp i\sin(\pi/2) = \mp i}$}, then: $\boxed{\left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right] = E_o\left[\hat{x} \mp i\hat{y}\right] = \sqrt{2}E_o\,\hat{\epsilon}_{\mp}}$ this monochromatic *EM* plane wave is ***circularly-polarized***.

Reflection & Transmission of Circularly Polarized Plane *EM* Waves at Normal Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

A circularly-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction is normally incident on a boundary {in the *x*-*y* plane} between two linear, homogeneous and isotropic media as shown in the figure below:



The complex amplitudes for the CP $\vec{E}$ and $\vec{B}$ fields are summarized below:

Incident CP monochromatic plane *EM* wave:

$$\tilde{\vec{E}}^{CP}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} [\hat{x} \mp i\hat{y}] = E_{o_{inc}} e^{i(k_1 z - \omega t + \delta)} [\hat{x} \mp i\hat{y}] \quad n.b. \quad \hat{k}_{inc} = \hat{k}_1 = +\hat{z}$$

$$\tilde{\vec{B}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1} \hat{k}_{inc} \times \tilde{\vec{E}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} [\hat{y} \pm i\hat{x}] = \tfrac{1}{v_1} E_{o_{inc}} e^{i(k_1 z - \omega t + \delta)} [\hat{y} \pm i\hat{x}]$$

Reflected CP monochromatic plane *EM* wave:

$$\tilde{\vec{E}}^{CP}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(k_1 z - \omega t)} [\hat{x} \mp i\hat{y}] = E_{o_{refl}} e^{i(k_1 z - \omega t + \delta)} [\hat{x} \mp i\hat{y}] \quad n.b. \quad \hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$$

$$\tilde{\vec{B}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1} \hat{k}_{refl} \times \tilde{\vec{E}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1} \tilde{E}_{o_{refl}} e^{i(k_1 z - \omega t)} [-\hat{y} \mp i\hat{x}] = -\tfrac{1}{v_1} E_{o_{refl}} e^{i(k_1 z - \omega t + \delta)} [\hat{y} \pm i\hat{x}]$$

Transmitted CP monochromatic plane *EM* wave:

$$\tilde{\vec{E}}^{CP}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} [\hat{x} \mp i\hat{y}] = E_{o_{trans}} e^{i(k_2 z - \omega t + \delta)} [\hat{x} \mp i\hat{y}] \quad n.b. \quad \hat{k}_{trans} = \hat{k}_2 = +\hat{z}$$

$$\tilde{\vec{B}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2} \hat{k}_{trans} \times \tilde{\vec{E}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2} \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} [\hat{y} \pm i\hat{x}] = \tfrac{1}{v_2} E_{o_{trans}} e^{i(k_2 z - \omega t + \delta)} [\hat{y} \pm i\hat{x}]$$

The boundary conditions on the CP $\vec{E}$ and $\vec{B}$ fields @ $z = 0$ in the x-y plane are summarized below:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$

(*n.b.* $\perp$ refers to the x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$

(*n.b.* || refers to the x-y boundary, *i.e.* in the x-y plane)

BC 3) Normal $\vec{B}$ continuous: $\boxed{B^{\perp}_{1_{Tot}} = B^{\perp}_{2_{Tot}}}$ ($\perp$ to x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 4) Tangential $\vec{H}$ continuous: $\boxed{\frac{1}{\mu_1} B^{\parallel}_{1_{Tot}} = \frac{1}{\mu_2} B^{\parallel}_{2_{Tot}}}$ (|| to x-y boundary, *i.e.* in x-y plane)

Thus, at $z = 0$:

Again, because the ***transversality*** requirements (from Maxwell's equations) of the $\vec{E}$ and $\vec{B}$ fields, we see that BC 1) and BC 3) impose ***no*** restrictions {here} on such CP *EM* waves since: { $E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0$; $E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0$ } and { $B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0$; $B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0$ }

$\Rightarrow$ Again, the ***only*** restrictions on plane *EM* waves propagating with normal incidence on the boundary at $z = 0$ {lying in the x-y plane} are imposed by BC 2) and BC 4).

$\therefore$ In medium 1) (*i.e.* $z \le 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}(z,t) = \tilde{\vec{E}}^{CP}_{inc}(z,t) + \tilde{\vec{E}}^{CP}_{refl}(z,t)} \text{ and:}$$

$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}(z,t) = \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{inc}(z,t) + \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{refl}(z,t)}$$

In medium 2) (*i.e.* $z \ge 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{2_{Tot}}(z,t) = \tilde{\vec{E}}^{CP}_{trans}(z,t)} \text{ and:}$$

$$\boxed{\frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}(z,t) = \frac{1}{\mu_2}\tilde{\vec{B}}^{CP}_{trans}(z,t)}$$

Then BC 2) (Tangential $\vec{E}$ is continuous @ $z = 0$) requires that:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \tilde{\vec{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \text{ or: } \boxed{\tilde{\vec{E}}^{CP}_{inc}(z=0,t) + \tilde{\vec{E}}^{CP}_{refl}(z=0,t) = \tilde{\vec{E}}^{CP}_{trans}(z=0,t)}.$$

Then BC 4) (Tangential $\vec{H}$ is continuous @ $z = 0$) requires that:

$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \text{ or: } \boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{refl}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}^{CP}_{trans}(z=0,t)}$$

Inserting the explicit expressions for the complex $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ fields

$$\tilde{\vec{E}}_{inc}^{CP}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\left[\hat{x} \mp i\hat{y}\right] \qquad \tilde{\vec{B}}_{inc}^{CP}(z,t) = \tfrac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{CP}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\left[\hat{y} \pm i\hat{x}\right]$$

$$\tilde{\vec{E}}_{refl}^{CP}(z,t) = \tilde{E}_{o_{refl}} e^{i(k_1 z-\omega t)}\left[\hat{x} \mp i\hat{y}\right] \qquad \tilde{\vec{B}}_{refl}^{CP}(z,t) = \tfrac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}^{CP}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(k_1 z-\omega t)}\left[-\hat{y} \mp i\hat{x}\right]$$

$$\tilde{\vec{E}}_{trans}^{CP}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}\left[\hat{x} \mp i\hat{y}\right] \qquad \tilde{\vec{B}}_{trans}^{CP}(z,t) = \tfrac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}^{CP}(z,t) = \tfrac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}\left[\hat{y} \pm i\hat{x}\right]$$

into the above boundary condition relations, these equations become:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} - \dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \dfrac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations, we have at $z = 0$ {*n.b.* ***everywhere*** in the *x-y* plane @ $z = 0$, independent of/valid for ***any*** time *t*}:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \dfrac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}$

Note that these last two relations for ***circularly-polarized*** *EM* plane waves are ***identical*** to those we obtained for the ***linearly-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction is normally incident on a boundary {@ $z = 0$ in the *x-y* plane} between two linear, homogeneous and isotropic media.

$\Rightarrow$ The BC constraints on the $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ are decoupled from their polarization states!

Thus, we obtain ***precisely*** the same reflection and transmission coefficients for the ***circularly-polarized*** *EM* plane wave as we did for the ***linearly-polarized*** monochromatic *EM* plane wave propagating in the $+\hat{z}$ direction, normally incident on a boundary {@ $z = 0$ in the *x-y* plane} between two linear, homogeneous and isotropic media:

Reflection coefficient: $$R \equiv \frac{I_{refl}(0)}{I_{inc}(0)} = \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \frac{(1-\beta)^2}{(1+\beta)^2} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad (\mu_1 \simeq \mu_2 \simeq \mu_o)$$

Transmission coefficient: $$T \equiv \frac{I_{trans}(0)}{I_{inc}(0)} = \beta\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4 v_2 v_1}{(v_2 + v_1)^2} = \frac{4 n_1 n_2}{(n_1 + n_2)^2} \qquad (\mu_1 \simeq \mu_2 \simeq \mu_o)$$

$R + T = 1$ and: $$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{Z_1}{Z_2}$$