LECTURE NOTES 5

ELECTROMAGNETIC WAVES IN VACUUM

THE WAVE EQUATION(S) FOR $\vec{E}$ AND $\vec{B}$

In regions of free space (*i.e.* the vacuum), where no electric charges, no electric currents and no matter of any kind are present, Maxwell's equations (in differential form) are:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

$\left(c^2=1/\varepsilon_o\mu_o\right)$

Set of coupled first-order partial differential equations

We can de-couple Maxwell's equations *e.g.* by applying the curl operator to equations 3) and 4):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$
$$=\vec{\nabla}\left(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$
$$=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$
$$=\boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$
$$=\vec{\nabla}\left(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}\right)-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$
$$=-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$
$$=\boxed{\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

These are three-dimensional de-coupled wave equations for $\vec{E}$ and $\vec{B}$ - note that they have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.

Remember that each of the above equations is explicitly dependent on space and time, *i.e.* $\vec{E}=\vec{E}(\vec{r},t)$ and $\vec{B}=\vec{B}(\vec{r},t)$:

$$\boxed{\nabla^2\vec{E}(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \qquad \boxed{\nabla^2\vec{B}(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

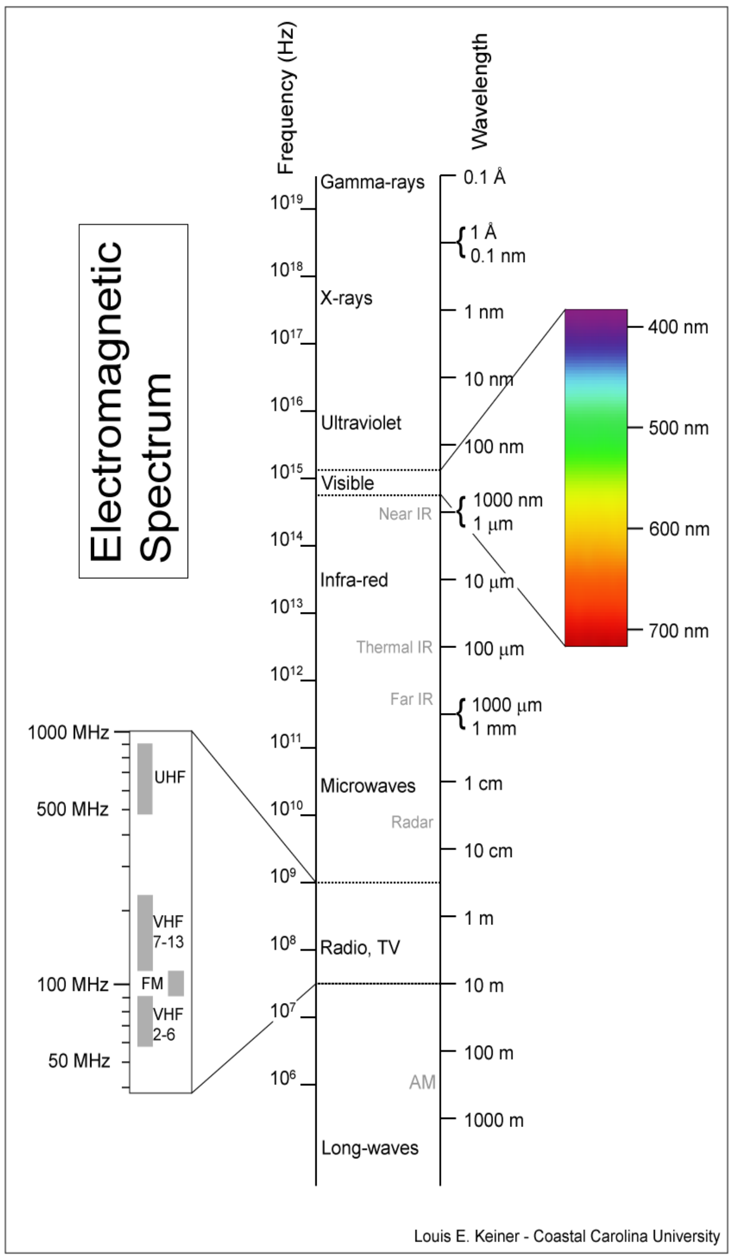
or:

$$\boxed{\nabla^2\vec{E}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0} \qquad \boxed{\nabla^2\vec{B}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0}$$

Thus, Maxwell's equations implies that empty space – the vacuum {which is ***not*** empty, at the ***microscopic*** scale} – supports the propagation of {***macroscopic***} electromagnetic waves, which propagate at the speed of light {in vacuum}: $\boxed{c=1/\sqrt{\varepsilon_o\mu_o}=3\times10^8\text{ m/s}}$.

EM waves have associated with them a frequency *f* and wavelength λ, related to each other via $c = f\lambda$. At the microscopic level, *EM* waves consist of large numbers of {massless} ***real*** photons, each carrying energy $\boxed{E = hf = hc/\lambda}$, linear momentum $|\vec{p}| = h/\lambda = hf/c = E/c$ and angular momentum $\boxed{|\vec{\ell}_z| = 1\hbar}$ where h = Planck's constant = 6.626×10^{-34} Joule-sec and $\boxed{\hbar \equiv h/2\pi}$.

EM waves can have any frequency/any wavelength – the continuum of *EM* waves over the frequency region $0 < f < \infty$ (*c.p.s.* or *Hertz* {*aka Hz*}), or equivalently, over the wavelength region $0 < \lambda < \infty$ (*m*) is known as the electromagnetic spectrum, which has been divided up (for convenience) into eight bands as shown in the figure below (kindly provided by Prof. Louis E. Keiner, of Coastal Carolina University, Conway, SC):

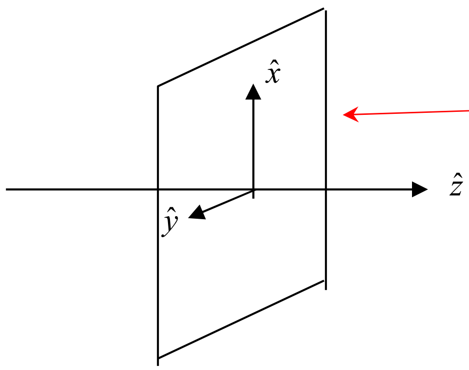


<u>Monochromatic *EM* Plane Waves</u>:

Monochromatic *EM* plane waves propagating in free space/the vacuum are sinusoidal *EM* plane waves consisting of a single frequency f, wavelength $\lambda = c/f$, angular frequency $\omega = 2\pi f$ and wavenumber $k = 2\pi/\lambda$. They propagate with speed $c = f\lambda = \omega/k$.

In the visible region of the *EM* spectrum {~380 nm (violet) $\leq \lambda \leq$ ~ 780 nm (red)}, *EM* light waves (consisting of real photons) of a given frequency / wavelength are perceived by the human eye as having a specific, single <u>color</u>. Hence we call such single-frequency, sinusoidal *EM* waves <u>mono-chromatic</u>.

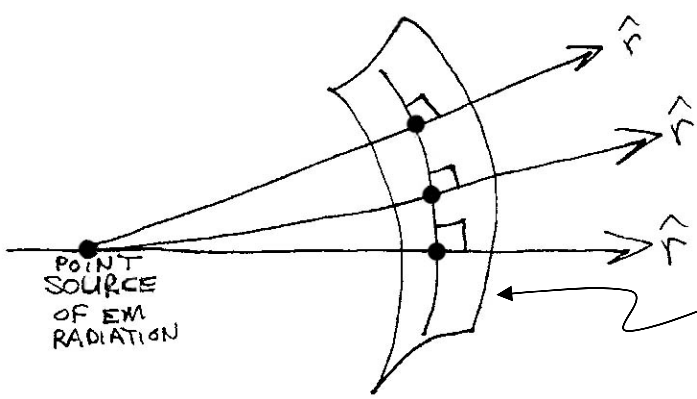
EM waves that propagate *e.g.* in the $+\hat{z}$ direction but which <u>additionally</u> have no explicit x- or y-dependence are known as <u>plane</u> <u>waves</u>, because for a given time, t the <u>wave front(s)</u> of the *EM* wave lie in a <u>plane</u> which is $\perp$ to the $\hat{z}$-axis, as shown in the figure below:



The planar wavefront associated with a plane *EM* wave propagating in the $\hat{k} = +\hat{z}$ direction lies in the *x*-*y* plane.

constant everywhere in (*x*,*y*) on this plane.

Note that there also exist <u>spherical</u> *EM* waves – *e.g.* emitted from a <u>point</u> <u>source</u>, such as an atom, a small antenna or a pinhole aperture – the wavefronts associated with these *EM* waves are spherical, and thus do ***<u>not</u>*** lie in a ***<u>plane</u>*** $\perp$ to the direction of propagation of the *EM* wave:



Portion of a spherical wavefront associated with a spherical wave

n.b. If the point source is ***<u>infinitely</u>*** far away from observer, then a spherical wave → plane wave. In this limit, the radius of curvature $R_C \to \infty$). *i.e.* a spherical surface becomes planar as $R_C \to \infty$.

A criterion for a {good} approximation of spherical wave as a plane wave is: $\boxed{\lambda \ll R_C}$

Monochromatic traveling *EM* plane waves can be represented by complex $\vec{E}$ and $\vec{B}$ fields:

$$\boxed{\tilde{\vec{E}}(z,t)=\tilde{\vec{E}}_o e^{i(kz-\omega t)}} \qquad \boxed{\tilde{\vec{B}}(z,t)=\tilde{\vec{B}}_o e^{i(kz-\omega t)}}$$

Propagating in the $\hat{k}=+\hat{z}$ direction

n.b. complex ***vectors***:

e.g. $\boxed{\tilde{\vec{E}}_o=\tilde{E}_o\hat{x}=\left|\tilde{E}_o\right|e^{i\delta}\hat{x}\equiv E_o e^{i\delta}\hat{x}}$ *e.g.* $\boxed{\tilde{\vec{B}}_o=\tilde{B}_o\hat{y}=\left|\tilde{B}_o\right|e^{i\delta}\hat{y}\equiv B_o e^{i\delta}\hat{y}}$

n.b. The ***real***, ***physical*** instantaneous time-domain *EM* fields are related to their corresponding complex time-domain fields via:

$$\boxed{\vec{E}(\vec{r},t)\equiv\mathrm{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right)}$$
$$\boxed{\vec{B}(\vec{r},t)\equiv\mathrm{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right)}$$

Note that Maxwell's equations for free space impose additional constraints on $\tilde{\vec{E}}_o$ and $\tilde{\vec{B}}_o$.

$\rightarrow$ Not just *any* $\tilde{\vec{E}}_o$ and/or $\tilde{\vec{B}}_o$ is acceptable / allowed !!!

Since: $\vec{\nabla}\cdot\vec{E}=0 = \mathrm{Re}\left(\vec{\nabla}\cdot\tilde{\vec{E}}\right)=0$ and: $\vec{\nabla}\cdot\vec{B}=0 = \mathrm{Re}\left(\vec{\nabla}\cdot\tilde{\vec{B}}\right)=0$

These two relations can ***only*** be satisfied $\forall(\vec{r},t)$ if $\vec{\nabla}\cdot\tilde{\vec{E}}=0\ \ \forall(\vec{r},t)$ and $\vec{\nabla}\cdot\tilde{\vec{B}}=0\ \ \forall(\vec{r},t)$.

In Cartesian coordinates: $\vec{\nabla}=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}$

Thus: $\left(\vec{\nabla}\cdot\tilde{\vec{E}}\right)=0$ and $\left(\vec{\nabla}\cdot\tilde{\vec{B}}\right)=0$ become:

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{E}}_o e^{i(kz-\omega t)}\right)=0 \quad\text{and}\quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we ***do*** allow: $\tilde{\vec{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in }\hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$

$$\tilde{\vec{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in }\hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then:

$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0}$$
$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0}$$

Or: $$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}=0$$

Now: E_{ox}, E_{oy}, E_{oz} = Amplitudes (constants) of the electric field components in *x*, *y*, *z* directions respectively.

B_{ox}, B_{oy}, B_{oz} = Amplitudes (constants) of the magnetic field components in *x*, *y*, *z* directions respectively.

We see that: $\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$ ← has no explicit *x*-dependence

And: $\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$ ← has no explicit *y*-dependence

$\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$ ← has no explicit *x*-dependence

And: $\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$ ← has no explicit *y*-dependence

However: $\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}$

Thus: $\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta}=ikE_{oz}e^{i(kz-\omega t)}e^{i\delta}=0$ ⇐ true ***iff*** $E_{oz}\equiv 0$!!!

$\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta}=ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta}=0$ ⇐ true ***iff*** $B_{oz}\equiv 0$!!!

- Thus, Maxwell's equations additionally tell us/impose the restriction that an electromagnetic plane wave ***cannot*** have ***any*** component of $\vec{E}$ or $\vec{B}$ $\parallel$ to (or anti-$\parallel$ to) the propagation direction (in this case here, the $\hat{k}=+\hat{z}$ -direction)
- Another way of stating this is that an *EM* plane wave ***cannot*** have any ***longitudinal*** components of $\vec{E}$ and $\vec{B}$ (*i.e.* components of $\vec{E}$ and $\vec{B}$ lying along the propagation direction).
- Thus, Maxwell's equations additionally tell us that an *EM* plane wave is a purely ***transverse*** wave (at least while it is propagating in free space) – *i.e.* the components of $\vec{E}$ and $\vec{B}$ must be $\perp$ to propagation direction.
- The ***plane of polarization*** of an *EM* plane wave is ***defined*** (***by convention***) to be ***parallel*** to $\vec{E}$.

Furthermore: Maxwell's equations impose ***yet another*** restriction on the allowed form of $\vec{E}$ and $\vec{B}$ for an *EM* wave:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \mathrm{Re}\left(\nabla \times \tilde{\vec{E}}\right) = \mathrm{Re}\left(-\frac{\partial \tilde{\vec{B}}}{\partial t}\right) \quad \text{and/or:} \quad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mathrm{Re}\left(\nabla \times \tilde{\vec{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial \tilde{\vec{E}}}{\partial t}\right)$$

Can *only* be satisfied $\forall\ (\vec{r},t)$ ***iff***:

$$\nabla \times \tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t} \quad \text{and/or:} \quad \nabla \times \tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial \tilde{\vec{E}}}{\partial t}$$

Thus:

$$\nabla \times \tilde{\vec{E}} = \left(\overbrace{\frac{\partial \tilde{E}_z}{\partial y}}^{=0} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{E}_x}{\partial y}}^{=0}\right)\hat{z} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y} - \overbrace{\frac{\partial \tilde{B}_z}{\partial t}}^{=0}\hat{z}$$

$$\nabla \times \tilde{\vec{B}} = \left(\overbrace{\frac{\partial \tilde{B}_z}{\partial y}}^{=0} - \frac{\partial \tilde{B}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{B}_x}{\partial z} - \overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{B}_x}{\partial y}}^{=0}\right)\hat{z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} + \frac{1}{c^2}\overbrace{\frac{\partial \tilde{E}_z}{\partial t}}^{=0}\hat{z}$$

With:

$$\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \overbrace{\tilde{E}_z}^{=0}\hat{z} = \left(E_{ox}\hat{x} + E_{oy}\hat{y} + \overbrace{E_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \overbrace{\tilde{B}_z}^{=0}\hat{z} = \left(B_{ox}\hat{x} + B_{oy}\hat{y} + \overbrace{B_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

Thus:

$$\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} = \left(E_{ox}\hat{x} + E_{oy}\hat{y}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} = \left(B_{ox}\hat{x} + B_{oy}\hat{y}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\therefore\quad \nabla \times \tilde{\vec{E}} = -\frac{\partial \tilde{E}_y}{\partial z}\hat{x} + \frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y}$$

$$\therefore\quad \nabla \times \tilde{\vec{B}} = -\frac{\partial \tilde{B}_y}{\partial z}\hat{x} + \frac{\partial \tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y}$$

Can ***only*** be satisfied / can ***only*** be true ***iff*** the $\hat{x}$ and $\hat{y}$ relations are ***separately*** / ***independently*** satisfied $\forall\ (\vec{r},t)$!

i.e. $\vec{\nabla}\times\tilde{\vec{E}}$:

$$-\frac{\partial \tilde{E}_y}{\partial z}\hat{x} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} \Rightarrow \frac{\partial \tilde{E}_y}{\partial z} = \frac{\partial \tilde{B}_x}{\partial t} \Rightarrow ikE_{oy} = -i\omega B_{ox} \qquad (1)$$

$$+\frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{E}_x}{\partial z} = -\frac{\partial \tilde{B}_y}{\partial t} \Rightarrow ikE_{ox} = +i\omega B_{oy} \qquad (2)$$

$\vec{\nabla}\times\tilde{\vec{B}}$:

$$-\frac{\partial \tilde{B}_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} \Rightarrow -\frac{\partial \tilde{B}_y}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t} \Rightarrow -ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \qquad (3)$$

$$+\frac{\partial \tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{B}_x}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t} \Rightarrow ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \qquad (4)$$

From (1): $ik\tilde{E}_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right)B_{ox}$ or: $B_{ox} = -\left(\frac{k}{\omega}\right)E_{oy}$

From (2): $ik\tilde{E}_{ox} = +i\omega B_{oy} \Rightarrow E_{ox} = +\left(\frac{\omega}{k}\right)B_{oy}$ or: $B_{oy} = +\left(\frac{k}{\omega}\right)E_{ox}$

From (3): $-ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \Rightarrow B_{oy} = +\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{ox}$

From (4): $ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \Rightarrow B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{oy}$

<u>Now</u>: $c = f\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \left(\frac{\omega}{k}\right)$ and: $\frac{1}{c} = \left(k/\omega\right)$ $\qquad \left(k = 2\pi/\lambda\right)$

$\therefore$ <u>$\vec{\nabla}\times\tilde{\vec{E}}$:</u> (1) $B_{ox} = -\frac{1}{c}E_{oy}$

(2) $B_{oy} = +\frac{1}{c}E_{ox}$

<u>$\vec{\nabla}\times\tilde{\vec{B}}$:</u> (3) $B_{oy} = +\frac{1}{c}E_{ox}$

(4) $B_{ox} = -\frac{1}{c}E_{oy}$

Maxwell's equations also have some ***redundancy*** encrypted into them!

So we really / actually <u>only</u> have <u>*two*</u> independent relations: $B_{ox} = -\frac{1}{c}E_{oy}$ and $B_{oy} = +\frac{1}{c}E_{ox}$

<u>But</u>: $\hat{z}\times\hat{y} = -\hat{x}$ $\qquad$ $\hat{z}\times\hat{x} = +\hat{y}$

<u>Very Useful Table</u>:

$\hat{x}\times\hat{y} = \hat{z}$	$\hat{y}\times\hat{x} = -\hat{z}$
$\hat{y}\times\hat{z} = \hat{x}$	$\hat{z}\times\hat{y} = -\hat{x}$
$\hat{z}\times\hat{x} = \hat{y}$	$\hat{x}\times\hat{z} = -\hat{y}$

$\therefore$ We can write the above two relations succinctly/compactly with ***one*** relation: $\tilde{\vec{B}}_o = \frac{1}{c}\hat{k}\times\tilde{\vec{E}}_o$

Physically, the mathematical relation $\tilde{\vec{B}}_o = \frac{1}{c}\hat{k}\times\tilde{\vec{E}}_o$ tells us that for a monochromatic *EM* plane wave propagating in free space, $\vec{E}$ and $\vec{B}$ are:

a.) ***in phase*** with each other.
b.) ***mutually perpendicular*** to each other **.and.** each is ***perpendicular*** to the ***propagation direction***: $(\vec{E}\perp\vec{B})\perp\hat{k}$ ($\hat{k}=+\hat{z}$ = propagation direction)

The $\vec{E}$ and $\vec{B}$ fields associated with this monochromatic plane *EM* wave are purely transverse { *n.b.* this is as also required by relativity at the microscopic level – for the extreme relativistic particles – the (massless) real photons traveling at the speed of light *c* that make up the macroscopic monochromatic plane *EM* wave.}

The purely ***real/physical amplitudes*** of $\vec{E}$ and $\vec{B}$ are {also} related to each other by: $\boxed{B_o = \frac{1}{c}E_o}$

with $\boxed{B_o = \sqrt{B_{ox}^2 + B_{oy}^2}}$ and $\boxed{E_o = \sqrt{E_{ox}^2 + E_{oy}^2}}$

Griffiths Example 9.2:

A monochromatic (single-frequency) plane *EM* wave that is plane polarized/linearly polarized in the $+\hat{x}$ direction, propagating in the $\hat{k}=+\hat{z}$ direction in free space, has:

$\boxed{\vec{E} = E\,\hat{x}}$ ⇐ definition of linearly polarized *EM* wave, polarized in the $+\hat{x}$ direction.

$$\therefore\quad \boxed{\vec{B} = \tfrac{1}{c}\left(\hat{k}\times\vec{E}\right) = \tfrac{1}{c}\left(\hat{z}\times E\hat{x}\right) = \tfrac{1}{c}E\underbrace{\left(\hat{z}\times\hat{x}\right)}_{=\hat{y}} = \tfrac{1}{c}E\hat{y}}$$

With: $\boxed{\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}}$, $\boxed{B = \frac{1}{c}E}$ and: $\boxed{B_o = \frac{1}{c}E_o}$

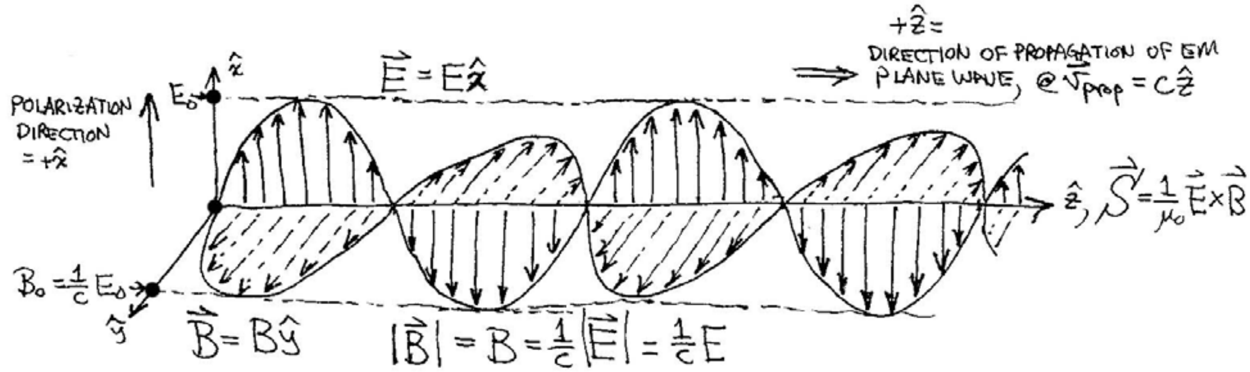
Then:
$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{i(kz-\omega t)}\hat{x} = E_o e^{i(kz-\omega t)}e^{i\delta}\hat{x} = E_o e^{i(kz-\omega t+\delta)}\hat{x}}$$
$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{B}_o e^{i(kz-\omega t)}\hat{y} = B_o e^{i(kz-\omega t)}e^{i\delta}\hat{y} = B_o e^{i(kz-\omega t+\delta)}\hat{y}}$$
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

The ***physical*** instantaneous electric and magnetic fields are given by the following expressions:

$$\boxed{\vec{E}(z,t) = \mathrm{Re}\left(\tilde{\vec{E}}(z,t)\right) = \mathrm{Re}\left\{\overbrace{E_o\cos(kz-\omega t+\delta)\hat{x}}^{real} + i\,\overbrace{E_o\sin(kz-\omega t+\delta)\hat{x}}^{imaginary}\right\}}$$
$$\boxed{\vec{E}(z,t) = E_o\cos(kz-\omega t+\delta)\hat{x}}$$

$$\boxed{\vec{B}(z,t) = \mathrm{Re}\left(\tilde{\vec{B}}(z,t)\right) = \mathrm{Re}\left\{\overbrace{B_o\cos(kz-\omega t+\delta)\hat{y}}^{real} + i\,\overbrace{B_o\sin(kz-\omega t+\delta)\hat{y}}^{imaginary}\right\}}$$
$$\boxed{\vec{B}(z,t) = B_o\cos(kz-\omega t+\delta)\hat{y} = \tfrac{1}{c}E_o\cos(kz-\omega t+\delta)\hat{y}}$$

The ***physical*** instantaneous $\vec{E}$ and $\vec{B}$ fields are in-phase with each other for a linearly polarized *EM* plane wave



Note that: $\left(\vec{E} \perp \vec{B}\right) \perp \hat{z} \Rightarrow \vec{E} \perp \hat{z},\ \vec{B} \perp \hat{z}$ ($\hat{z}$ = direction of propagation of *EM* wave)

Instantaneous Poynting's vector for a linearly polarized *EM* plane wave propagating in free space:

$$\vec{S}(z,t) = \tfrac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \tfrac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}$$

$$\vec{S}(z,t) = \tfrac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}$$

$$\vec{S}(z,t) = \tfrac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\hat{z} \qquad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$\Rightarrow$ *EM* power flows in the direction of propagation of the *EM* plane wave (*here*, $\hat{k} = +\hat{z}$ direction)

Generalization for Propagation of Monochromatic Plane *EM* Waves in an Arbitrary Direction

Obviously, there is nothing special / profound with regard to plane *EM* waves propagating in a specific direction in free space / the vacuum. They can propagate in any direction. We can easily generalize the mathematical description for monochromatic plane *EM* waves traveling in an arbitrary direction as follows:

Introduce the notion / concept of a wave vector (or propagation vector) $\vec{k}$ which points in the direction of propagation, whose magnitude $|\vec{k}| = k$. Then the scalar product $\vec{k}\bullet\vec{r}$ is the appropriate 3-D generalization of kz:

1-D: If: $\vec{k} = k\hat{z}$ with $|\vec{k}| = k$ and: $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$ with $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

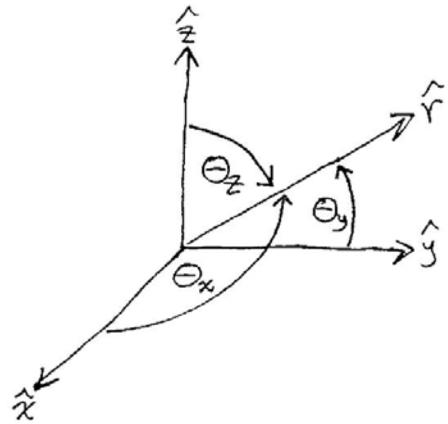
Then: $(\vec{k}\bullet\vec{r}) = k\hat{z}\bullet(x\hat{x}+y\hat{y}+z\hat{z}) = kz$

3-D: If: $\vec{k} = k_x\hat{x}+k_y\hat{y}+k_z\hat{z}$ with $|\vec{k}| = \sqrt{k_x^2+k_y^2+k_z^2}$ and: $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$

Then: $(\vec{k}\bullet\vec{r}) = k_xx+k_yy+k_zz$ with $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

Now: $\left.\begin{array}{l} k_x = k\cos\Theta_x \\ k_y = k\cos\Theta_y \\ k_z = k\cos\Theta_z \end{array}\right\}$ where $\cos\Theta_x,\ \cos\Theta_y,\ \cos\Theta_z$ = direction cosines *w.r.t.* (with respect to) the $\hat{x},\ \hat{y},\ \hat{z}$-axes respectively

Direction Cosines: $\left\{\begin{array}{l} \cos\Theta_x = \hat{k}\cdot\hat{x} = \sin\theta\cos\varphi \\ \cos\Theta_y = \hat{k}\cdot\hat{y} = \sin\theta\sin\varphi \\ \cos\Theta_z = \hat{k}\cdot\hat{z} = \cos\theta \end{array}\right\}$ in spherical-polar coordinates



Note:
$$\sqrt{\cos^2\Theta_x + \cos^2\Theta_y + \cos^2\Theta_z} = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

If *e.g.* $\vec{k} \parallel \vec{r}$ then: $\vec{k}\cdot\vec{r} = kr$. We explicitly demonstrate this in spherical polar coordinates:

For $\vec{k}\parallel\vec{r}$: $\left\{\begin{array}{l} k_x = k\cos\Theta_x = k\sin\theta\cos\varphi \\ k_y = k\cos\Theta_y = k\sin\theta\sin\varphi \\ k_z = k\cos\Theta_z = k\cos\theta \end{array}\right\}$ and: $\left\{\begin{array}{l} x = r\cos\Theta_x = r\sin\theta\cos\varphi \\ y = r\cos\Theta_y = r\sin\theta\sin\varphi \\ z = r\cos\Theta_z = r\cos\theta \end{array}\right\}$

Then:
$$\begin{aligned}(\vec{k}\cdot\vec{r}) &= k_x x + k_y y + k_z z = kx\cos\Theta_x + ky\cos\Theta_y + kz\cos\Theta_z \\ &= kr\cos^2\Theta_x + kr\cos^2\Theta_y + kr\cos^2\Theta_z \\ &= kr\sin^2\theta\cos^2\varphi + kr\sin^2\theta\sin^2\varphi + kr\cos^2\theta \\ &= kr\left\{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta\right\} = kr\left\{\sin^2\theta\left(\cos^2\varphi + \sin^2\varphi\right) + \cos^2\theta\right\} \\ &= kr\left\{\sin^2\theta + \cos^2\theta\right\} = kr\end{aligned}$$

Thus, most generally, we can write the $\tilde{\vec{E}}(\vec{r},t)$ and $\tilde{\vec{B}}(\vec{r},t)$-fields as:

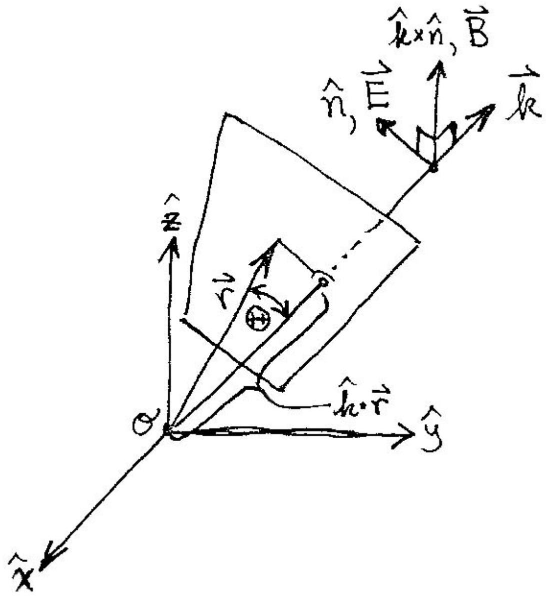
$$\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{n}$$ where: $\hat{n} \equiv$ polarization vector $\hat{n}\perp\hat{k}$

$$\tilde{\vec{B}}(\vec{r},t) = \tilde{B}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n})$$ *i.e.* $\hat{n}\cdot\hat{k} = 0$ because $\tilde{\vec{E}}$ is ***transverse***

$$\tilde{\vec{B}}(\vec{r},t) = \tfrac{1}{c}\hat{k}\times\tilde{\vec{E}}(\vec{r},t) = \tfrac{1}{c}\tilde{E}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \tilde{B}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n})$$

We ***must*** have: $\tilde{\vec{B}}(\vec{r},t)\perp\tilde{\vec{E}}(\vec{r},t)\perp\hat{k}$ *i.e.* $\tilde{\vec{E}}\cdot\tilde{\vec{B}} = 0$ and $\tilde{\vec{E}}\cdot\hat{k} = 0$ and $\tilde{\vec{B}}\cdot\hat{k} = 0$

The Direction of Propagation of a Monochromatic Plane *EM* Wave: $\hat{k}$



The Real/Physical (Instantaneous) *EM* Fields are:

$$\vec{E}(\vec{r},t) = \text{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right) = E_o \cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)\hat{n}$$

where: $\hat{n} \equiv$ ***polarization vector*** $\left(\parallel \vec{E}\right)$

$$\vec{B}(\vec{r},t) = \text{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right) = B_o \cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)\left(\hat{k}\times\hat{n}\right)$$

$\left(B_o = \frac{1}{c}E_o\right)$ in free space

Instantaneous Energy, Linear & Angular Momentum in *EM* Plane Waves (Free Space)

Instantaneous Energy Density Associated with an *EM* Plane Wave (Free Space):

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right) = u_{elect}(\vec{r},t) + u_{mag}(\vec{r},t)$$

where: $u_{elect}(\vec{r},t) = \frac{1}{2}\varepsilon_o E^2(\vec{r},t)$ and $u_{mag}(\vec{r},t) = \frac{1}{2\mu_o}B^2(\vec{r},t) = \frac{1}{2}\varepsilon_o E^2(\vec{r},t)$

But: $B^2 = \frac{1}{c^2}E^2$ and $\frac{1}{c^2} = \varepsilon_o\mu_o$ for *EM* waves propagating in vacuum/free space

Thus: $u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \mu_o}{\mu_o}E^2(\vec{r},t)\right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t)\right) = \varepsilon_o E^2(\vec{r},t)$

Or: $u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)$ $\left(\frac{\text{Joules}}{\text{m}^3}\right)$

n.b. for *EM* plane waves propagating in the vacuum:

$u_{mag}(\vec{r},t) = u_{elect}(\vec{r},t)$ and/or: $u_{mag}(\vec{r},t)/u_{elect}(\vec{r},t) = 1$

Instantaneous Poynting's Vector Associated with an *EM* Plane Wave (Free Space):

$$\boxed{\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=\tfrac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)=\tfrac{1}{\mu_o}\text{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\text{Re}\left\{\tilde{\vec{B}}(z,t)\right\}}\ \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a linearly polarized monochromatic plane *EM* plane wave propagating in the vacuum, *e.g.*:

$$\vec{E}(\vec{r},t)=E_o\cos(kz-\omega t+\delta)\hat{x} \quad \text{and:} \quad \vec{B}(\vec{r},t)=B_o\cos(kz-\omega t+\delta)\hat{y}$$

Then: $\vec{S}(\vec{r},t)=\tfrac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\hat{z}$ but: $B_o=\tfrac{1}{c}E_o$ for *EM* plane waves in vacuum.

Thus: $\vec{S}(\vec{r},t)=\tfrac{1}{\mu_o c}E_o^2\cos^2(kz-\omega t+\delta)\hat{z}$ ← multiply RHS by $1=\left(\frac{c}{c}\right)$

Hence: $\vec{S}(\vec{r},t)=c\left(\dfrac{1}{\mu_o c^2}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z}$ but: $\dfrac{1}{c^2}=\varepsilon_o\mu_o$

Thus: $\vec{S}(\vec{r},t)=c\left(\dfrac{\varepsilon_o\not{\mu_o}}{\not{\mu_o}}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z}=c\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)\hat{z}$

But: $u_{EM}(\vec{r},t)=\varepsilon_oE^2(\vec{r},t)=\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)$

∴ $\boxed{\vec{S}(\vec{r},t)=cu_{EM}(\vec{r},t)\hat{z}}$ *Here*, the propagation ***velocity*** of *EM* field ***energy***: $\boxed{\vec{v}_E=c\hat{z}}$

⇒ Poynting's Vector = Energy Density * (Energy) Propagation Velocity: $\boxed{\vec{S}(\vec{r},t)=u_{EM}(\vec{r},t)\vec{v}_E}$

Instantaneous Linear Momentum Density Associated with an *EM* Plane Wave (Free Space):

$$\boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)}\ \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For linearly polarized monochromatic plane *EM* waves propagating in the vacuum:

$$\vec{\wp}_{EM}=\frac{1}{c^{\not{2}}}\not{c}\,\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)\hat{z}=\frac{1}{c}\underbrace{\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)}_{=u_{EM}}\hat{z}$$

But: $u_{EM}(\vec{r},t)=\varepsilon_oE^2(\vec{r},t)=\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)$

$$\therefore \quad \boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)=\frac{1}{c}u_{EM}(\vec{r},t)\hat{z}}\ \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Angular Momentum Density Associated with an *EM* Plane Wave (Free Space):

$$\boxed{\vec{\ell}_{EM}(\vec{r},t)=\vec{r}\times\vec{\wp}_{EM}(\vec{r},t)} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

But: $$\boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)=\frac{1}{c}u_{EM}(\vec{r},t)\hat{z}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$\therefore$ for an *EM* plane wave propagating in the $+\hat{z}$ direction:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t)=\frac{1}{c^2}\vec{r}\times\vec{S}(\vec{r},t)=\frac{1}{c}u_{EM}(\vec{r},t)(\vec{r}\times\hat{z})} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

n.b. depends on the choice of origin

The *instantaneous* *EM* power flowing into/out of volume *v* with bounding surface *S* enclosing volume *v* (containing *EM* fields in the volume *v*) is:

$$\boxed{P_{EM}(t)=\frac{\partial U_{EM}(t)}{\partial t}=\int_v\frac{\partial u_{EM}(\vec{r},t)}{\partial t}d\tau=-\oint_S\vec{S}(\vec{r},t)\cdot d\vec{a}} \quad \text{(Watts)}$$

n.b. closed surface *S* enclosing volume *v*.

The instantaneous *EM* power crossing an (imaginary) surface (*e.g.* a 2-D plane – a window!) is:

$$\boxed{P_{EM}(t)=-\int_S\vec{S}(\vec{r},t)\cdot d\vec{a}_\perp}$$

The instantaneous total *EM* energy contained in volume *v* is: $\boxed{U_{EM}(t)=\int_v u_{EM}(\vec{r},t)\,d\tau}$ (Joules)

The instantaneous total *EM* linear momentum contained in the volume *v* is:

$$\boxed{\vec{p}_{EM}(t)=\int_v\vec{\wp}_{EM}(\vec{r},t)\,d\tau} \quad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The instantaneous total *EM* angular momentum contained in the volume *v* is:

$$\boxed{\vec{\mathcal{L}}_{EM}(t)=\int_v\vec{\ell}_{EM}(\vec{r},t)\,d\tau} \quad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

3-D Vector Impedance Associated with an *EM* Plane Wave (Free Space):

$\boxed{\vec{Z}(\vec{r},t) \equiv \vec{E}(\vec{r},t) \times \left(1/\vec{H}(\vec{r},t)\right)}$ (Ohms) = Ohm's law for *EM* fields! (*n.b.* a ***vector*** quantity)

Analog of: Ohm's law for AC circuits: $\boxed{Z(t) = V(t)/I(t)}$ (*n.b.* a ***scalar*** quantity)

Complex form of Ohm's Law: $\boxed{\tilde{Z}(t) = \tilde{V}(t)/\tilde{I}(t) = \tilde{V}(t)\cdot \tilde{I}^*(t)/\left|\tilde{I}(t)\right|^2}$

What {precisely} is the mathematical meaning of a "generic" reciprocal vector $1/\vec{A}(\vec{r},t)$???

The ***magnitude*** of the reciprocal vector $\left|1/\vec{A}(\vec{r},t)\right| = 1/\left|\vec{A}(\vec{r},t)\right|$ is ***invariant*** (*i.e.* ***cannot*** change) for arbitrary rotations & translations of the ***coordinate system***. The ***direction*** that the reciprocal vector $1/\vec{A}(\vec{r},t)$ ***points in space*** {at time t} is also ***invariant*** for arbitrary rotations & translations of the coordinate system.

Note further that ***inverse*** unit vectors {such as $1/\hat{x}$, $1/\hat{y}$, $1/\hat{z}$ } are ***meaningless***!

For a purely real "generic" vector $\vec{A}(\vec{r},t) = A_x(\vec{r},t)\hat{x} + A_y(\vec{r},t)\hat{y} + A_z(\vec{r},t)\hat{z}$ {*e.g.* expressed in rectangular/Cartesian coordinates}, the mathematical definition of a purely ***real*** reciprocal vector $1/\vec{A}(\vec{r},t)$, satisfying all of the above requirements is:

$$\boxed{\frac{1}{\vec{A}(\vec{r},t)} \equiv \frac{1}{\left|\vec{A}(\vec{r},t)\right|}\hat{A}(\vec{r},t)} \quad \text{where:} \quad \boxed{\hat{A}(\vec{r},t) \equiv \frac{\vec{A}(\vec{r},t)}{\left|\vec{A}(\vec{r},t)\right|} = \frac{A_x(\vec{r},t)\hat{x} + A_y(\vec{r},t)\hat{y} + A_z(\vec{r},t)\hat{z}}{\left|\vec{A}(\vec{r},t)\right|}}$$

Using rectangular / Cartesian coordinates

Hence, we also see that:

$$\boxed{\frac{1}{\vec{A}(\vec{r},t)} \equiv \frac{\hat{A}(\vec{r},t)}{\left|\vec{A}(\vec{r},t)\right|} = \frac{\vec{A}(\vec{r},t)}{\left|\vec{A}(\vec{r},t)\right|^2} = \frac{A_x(\vec{r},t)\hat{x} + A_y(\vec{r},t)\hat{y} + A_z(\vec{r},t)\hat{z}}{\left|\vec{A}(\vec{r},t)\right|^2}}$$

Using rectangular / Cartesian coordinates

Note: For the more general case of ***complex*** reciprocal vectors $1/\tilde{\vec{A}}(\vec{r},t)$ these relations become:

$$\boxed{\frac{1}{\tilde{\vec{A}}(\vec{r},t)} \equiv \frac{1}{\left|\tilde{\vec{A}}(\vec{r},t)\right|}\tilde{\hat{A}}^*(\vec{r},t)} \quad \text{where:} \quad \boxed{\tilde{\hat{A}}^*(\vec{r},t) \equiv \frac{\tilde{\vec{A}}^*(\vec{r},t)}{\left|\tilde{\vec{A}}(\vec{r},t)\right|} = \frac{\tilde{A}_x^*(\vec{r},t)\hat{x} + \tilde{A}_y^*(\vec{r},t)\hat{y} + \tilde{A}_z^*(\vec{r},t)\hat{z}}{\left|\tilde{\vec{A}}(\vec{r},t)\right|}}$$

Using rectangular / Cartesian coordinates

Hence, we also see that:

$$\boxed{\frac{1}{\tilde{\vec{A}}(\vec{r},t)} \equiv \frac{\tilde{\hat{A}}^*(\vec{r},t)}{\left|\tilde{\vec{A}}(\vec{r},t)\right|} = \frac{\tilde{\vec{A}}^*(\vec{r},t)}{\left|\tilde{\vec{A}}(\vec{r},t)\right|^2} = \frac{\tilde{A}_x^*(\vec{r},t)\hat{x} + \tilde{A}_y^*(\vec{r},t)\hat{y} + \tilde{A}_z^*(\vec{r},t)\hat{z}}{\left|\tilde{\vec{A}}(\vec{r},t)\right|^2}}$$

Using rectangular / Cartesian coordinates

Thus, *e.g.* for a linearly polarized monochromatic *EM* plane wave propagating in the vacuum in the $\hat{k}=+\hat{z}$ direction, with instantaneous/physical purely real time-domain *EM* fields of:

$$\boxed{\vec{E}(\vec{r},t)=E_o\cos(kz-\omega t+\delta)\hat{x}} \quad \text{and:} \quad \boxed{\vec{B}(\vec{r},t)=B_o\cos(kz-\omega t+\delta)\hat{y}} \quad \text{with:} \quad \boxed{B_o=\frac{1}{c}E_o}$$

and: $\boxed{\vec{H}(\vec{r},t)=\tfrac{1}{\mu_o}\vec{B}(\vec{r},t)=\tfrac{1}{\mu_o}B_o\cos(kz-\omega t+\delta)\hat{y}=\tfrac{1}{\mu_o c}E_o\cos(kz-\omega t+\delta)\hat{y}}$

The vector impedance $\vec{Z}(\vec{r},t)$ associated with a monochromatic plane *EM* plane wave propagating in the $\hat{k}=+\hat{z}$ direction in free space is:

$$\vec{Z}(\vec{r},t)\equiv\vec{E}(\vec{r},t)\times\left(1/\vec{H}(\vec{r},t)\right)=\frac{\vec{E}(\vec{r},t)\times\hat{H}(\vec{r},t)}{\left|\vec{H}(\vec{r},t)\right|}=\frac{\overbrace{\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)}^{=S(\vec{r},t)}}{\left|\vec{H}(\vec{r},t)\right|^2}=\frac{\vec{S}(\vec{r},t)}{\left|\vec{H}(\vec{r},t)\right|^2}$$

$$=\frac{c\varepsilon_o \cancel{E_o^2}\cancel{\cos^2(kz-\omega t+\delta)}\hat{z}}{\left(\frac{1}{\mu_o c}\right)^2\cancel{E_o^2}\cancel{\cos^2(kz-\omega t+\delta)}}=\mu_o^2c^3\varepsilon_o\hat{z}=\mu_o c\cdot c^2\cdot(\varepsilon_o\mu_o)\hat{z}$$

$$=\mu_o c\cdot\cancel{\left(\frac{1}{\varepsilon_o\mu_o}\right)}\cdot\cancel{(\varepsilon_o\mu_o)}\hat{z}=\mu_o c\,\hat{z}=\mu_o\frac{1}{\sqrt{\varepsilon_o\mu_o}}\hat{z}=\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\hat{z}\equiv Z_o\,\hat{z}$$

Note the cancellations!!! ***Here***, *Z* has ***no spatial*** and/or ***temporal*** dependence – for a monochromatic *EM* plane wave propagating in free space!

Thus, in free space: $\boxed{\vec{Z}(\vec{r},t)=Z_o\,\hat{z}=\sqrt{\tfrac{\mu_o}{\varepsilon_o}}\,\hat{z}}$ (Ohms)

where: $\boxed{Z_o\equiv\sqrt{\tfrac{\mu_o}{\varepsilon_o}}}$ is known as the {***scalar***!} ***characteristic impedance*** of free space.

The ***vector*** impedance $\vec{Z}(\vec{r},t)$ associated with an *EM* field is a ***physical*** property of the ***medium*** that the ***EM*** field is ***propagating*** – which in ***this*** case {***here***} – is the ***vacuum***.

Microscopically, the quantum numbers of the {*QED*} vacuum – free space {which, at the microscopic level is ***not*** empty!} – must all be associated with ***scalar-type*** quantities – spin = 0, even parity (+) for both space inversion operation *P* and charge conjugation *C*, *i.e.* the quantum numbers of the {*QED*} vacuum are $\boxed{J^{PC}=0^{++}}$.

Note further that ***all*** of the physical ***macroscopic*** (mean-field) parameters of the vacuum ***must*** be ***invariant*** {*i.e.* unchanged} under arbitrary rotations, translations and Lorentz boosts - from one reference frame to any other. This means that ***all*** macroscopic physical parameters of the vacuum intrinsically ***must*** have ***no spatial*** and/or ***temporal*** dependence – they are ***constants***:

$\varepsilon_o=8.85\times10^{-12}\ Farads/m=$ electric permittivity of free space

$\mu_o=4\pi\times10^{-7}\ Henrys/m=$ magnetic permeablity of free space

$c=1/\varepsilon_o\mu_o=3\times10^8\ m/s=$ speed of *EM* waves propagating in in free space

$Z_o=\sqrt{\tfrac{\mu_o}{\varepsilon_o}}=376.82\,\Omega=$ characteristic impedance of *EM* waves propagating in free space

The ***vectorial*** nature of $\vec{Z}(\vec{r},t)$ is simply associated with the direction of propagation of the *EM* wave – here, in ***this*** case {*i.e.* the vacuum} the $\hat{k}=+\hat{z}$ direction, so: $\vec{Z}(\vec{r},t)=Z_o\,\hat{z}=\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\hat{z}$. For *EM* waves propagating in the vacuum, it ***can't*** physically matter which direction they are propagating in – ***any*** direction $\hat{k}$ will give $\vec{Z}(\vec{r},t)=Z_o\hat{k}=\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\hat{k}$!

Note also from the above derivations, that we also have a relation between the vector impedance $\vec{Z}(\vec{r},t)$ and Poynting's vector $\vec{S}(\vec{r},t)$ associated with a propagating *EM* wave:

$$\boxed{\vec{Z}(\vec{r},t)\equiv\vec{E}(\vec{r},t)\times\left(1/\vec{H}(\vec{r},t)\right)=\frac{\vec{E}(\vec{r},t)\times\hat{H}(\vec{r},t)}{\left|\vec{H}(\vec{r},t)\right|}=\frac{\overbrace{\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)}^{=\vec{S}(\vec{r},t)}}{\left|\vec{H}(\vec{r},t)\right|^2}=\frac{\vec{S}(\vec{r},t)}{\left|\vec{H}(\vec{r},t)\right|^2}}$$

For the ***complex*** time-domain representation of *EM* fields – at least those associated with monochromatic (*i.e.* single-frequency) *EM* waves, then in general we have: $\tilde{\vec{E}}(\vec{r},t;\omega)=\tilde{\vec{E}}(\vec{r};\omega)\cdot e^{-i\omega t}$ and: $\tilde{\vec{H}}(\vec{r},t;\omega)=\tilde{\vec{H}}(\vec{r};\omega)\cdot e^{-i\omega t}$, and thus the ***complex*** vector impedance is:

$$\tilde{\vec{Z}}(\vec{r},t;\omega)\equiv\tilde{\vec{E}}(\vec{r},t;\omega)\times\left(1/\tilde{\vec{H}}(\vec{r},t;\omega)\right)=\frac{\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^*(\vec{r},t;\omega)}{\left|\tilde{\vec{H}}(\vec{r},t;\omega)\right|}=\frac{\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^*(\vec{r},t;\omega)}{\left|\tilde{\vec{H}}(\vec{r},t;\omega)\right|^2}\ (Ohms)$$

$$=\frac{\tilde{\vec{E}}(\vec{r};\omega)\cdot\cancel{e^{-i\omega t}}\times\tilde{\vec{H}}^*(\vec{r};\omega)\cdot\cancel{e^{+i\omega t}}}{\tilde{\vec{H}}(\vec{r};\omega)\cdot\cancel{e^{-i\omega t}}\bullet\tilde{\vec{H}}^*(\vec{r};\omega)\cdot\cancel{e^{+i\omega t}}}=\frac{\tilde{\vec{E}}(\vec{r};\omega)\times\tilde{\vec{H}}^*(\vec{r};\omega)}{\tilde{\vec{H}}(\vec{r};\omega)\bullet\tilde{\vec{H}}^*(\vec{r};\omega)}=\frac{\overbrace{\tilde{\vec{E}}(\vec{r};\omega)\times\tilde{\vec{H}}^*(\vec{r};\omega)}^{\equiv\tilde{\vec{S}}(\vec{r};\omega)}}{\left|\tilde{\vec{H}}(\vec{r};\omega)\right|^2}=\frac{\tilde{\vec{S}}(\vec{r};\omega)}{\left|\tilde{\vec{H}};\omega\right|^2}\equiv\tilde{\vec{Z}}(\vec{r};\omega)$$

where $\tilde{\vec{S}}(\vec{r},\omega)$ and $\tilde{\vec{Z}}(\vec{r};\omega)$ are the complex ***frequency-domain*** Poynting's vector and vector impedance, respectively. Note that – at least for monochromatic/single-frequency *EM* waves – that: $\tilde{\vec{Z}}(\vec{r},t;\omega)=\tilde{\vec{Z}}(\vec{r};\omega)$, *i.e.* the complex vector impedance associated with monochromatic *EM* waves has ***no*** time dependence! It is a manifestly ***frequency-domain*** quantity!

For monochromatic *EM* plane waves propagating in free space/the vacuum in the $\hat{k}=+\hat{z}$ direction, the complex vector impedance is a purely real quantity:

$$\tilde{\vec{Z}}(\vec{r},t;\omega)=\tilde{\vec{Z}}(\vec{r};\omega)=Z_o\,\hat{z}=\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\hat{z}=376.82\,\Omega\,\hat{z}$$

Physically, the ***real*** part of the complex vector impedance $\Re e\left\{\tilde{\vec{Z}}(\vec{r};\omega)\right\}$ is associated with ***propagating*** *EM* waves/***propagating*** *EM* wave energy, whereas the ***imaginary*** part of the complex vector impedance $\Im m\left\{\tilde{\vec{Z}}(\vec{r};\omega)\right\}$ is associated with ***non-propagating*** *EM* wave energy – *i.e.* *EM* wave energy that simply ***sloshes back and forth locally***, 2× per cycle of oscillation!

Time-Averaged Quantities Associated with *EM* Waves:

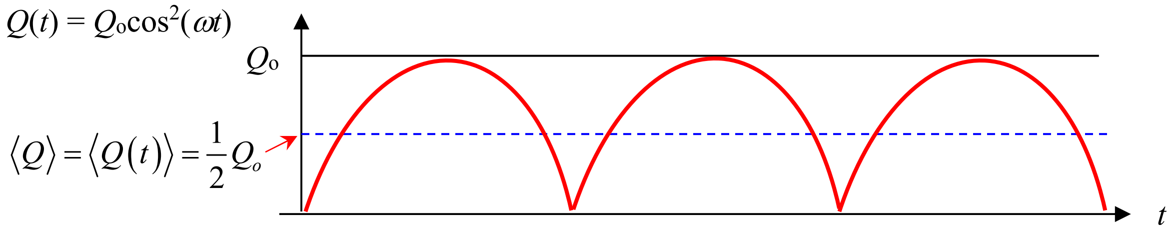
Frequently, we are ***not*** interested in knowing the instantaneous power P(t), energy / energy density, Poynting's vector, linear and angular momentum, *etc*.- *e.g*. simply because experimental measurements of these quantities are very often only ***averages*** over ***many*** extremely fast cycles of oscillation…

(*e.g*. period of oscillation of a ***light*** wave: $\tau_{light} = 1/f_{light} \simeq \dfrac{1}{10^{15}\,\text{cps}} = 10^{-15}\ \text{sec/cycle} = 1$ femto-sec)

$\therefore\ \Rightarrow$ We want/need ***time-averaged*** expressions for each of these quantities (*e.g*. in order to compare directly with experimental data) *e.g*. for monochromatic plane *EM* ***light*** waves:

If we have *e.g*. a "generic" instantaneous physical quantity of the form: $\boxed{Q(t) = Q_o \cos^2(\omega t)}$

The time-average of $Q(t)$ is defined as: $\boxed{\langle Q(t)\rangle \equiv \langle Q\rangle = \dfrac{1}{\tau}\displaystyle\int_{t=0}^{t=\tau} Q(t)\,dt = \dfrac{Q_o}{\tau}\int_{t=0}^{t=\tau}\cos^2(\omega t)\,dt}$



The time average of the $\cos^2(\omega t)$ function:

$$\boxed{\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2} + \frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau - 0) + \left(\frac{\sin 2\omega\tau}{2\omega} - 0\right)\right] = \frac{1}{2\tau}\left[\tau + \frac{\sin 2\omega\tau}{2\omega}\right]}$$

<u>But</u>: $\boxed{\omega\tau = 2\pi f\tau}$ and: $\boxed{f = 1/\tau}$ $\therefore$ $\boxed{\omega\tau = 2\pi(\tau/\tau) = 2\pi}$ $\therefore$ $\boxed{\sin(\omega\tau) = \sin(2\pi) = 0}$

$$\therefore\ \boxed{\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{2\not\tau}\left[\not\tau\right] = \frac{1}{2}}\ \therefore\ \boxed{\langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o}$$

The <u>time-averaged</u> quantities associated with an *EM* plane wave propagating in free space are:

EM Energy Density: $\boxed{u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t)\rangle}$ Total *EM* Energy: $\boxed{U_{EM}(t) \Rightarrow \langle U_{EM}(t)\rangle}$

Poynting's Vector: $\boxed{\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t)\rangle}$ *EM* Power: $\boxed{P_{EM}(t) \Rightarrow \langle P_{EM}(t)\rangle}$

Linear Momentum Density: $\boxed{\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\wp}_{EM}(\vec{r},t)\rangle}$ Linear Momentum: $\boxed{\vec{p}_{EM}(t) \Rightarrow \langle \vec{p}_{EM}(t)\rangle}$

Angular Momentum Density: $\boxed{\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\ell}_{EM}(\vec{r},t)\rangle}$ Angular Momentum: $\boxed{\vec{\mathfrak{L}}_{EM}(t) \Rightarrow \langle \vec{\mathfrak{L}}_{EM}(t)\rangle}$

For a monochromatic *EM* plane wave propagating in free space / vacuum in the $\hat{k} = +\hat{z}$ direction:

Time – averaged quantities for *EM* plane wave propagating in the $+\hat{z}$ direction:

$$\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}c\varepsilon_o E_o^2\hat{z} = c\langle u_{EM}(\vec{r},t)\rangle\hat{z} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\langle \vec{\wp}_{EM}(\vec{r},t)\rangle = \frac{1}{2c}\varepsilon_o E_o^2\hat{z} = \frac{1}{c^2}\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\langle \vec{\ell}_{EM}(\vec{r},t)\rangle = \left(\vec{r}\times\langle \vec{\wp}_{EM}(\vec{r},t)\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\langle \vec{S}(\vec{r},t)\rangle\right) = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle(\hat{r}\times\hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

We define the ***intensity*** *I* associated with an *EM* wave as the ***time average*** of the ***magnitude*** of Poynting's vector:

Intensity of an *EM* wave: $$I(\vec{r}) \equiv \langle S(\vec{r},t)\rangle = \langle |\vec{S}(\vec{r},t)|\rangle = c\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}c\varepsilon_o E_o^2 \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The ***intensity*** of an *EM* wave is also known as the ***irradiance*** of the *EM* wave – it is the so-called ***radiant power incident per unit area*** on a surface.

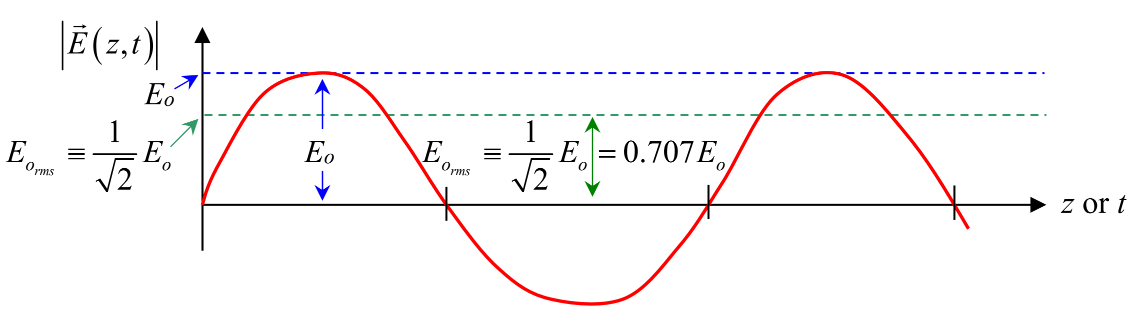
When working with ***time-averaged*** quantities such as $\langle u_{EM}(\vec{r},t)\rangle$, $\langle \vec{S}(\vec{r},t)\rangle$, $\langle \vec{\wp}_{EM}(\vec{r},t)\rangle$, $\langle \vec{\ell}_{EM}(\vec{r},t)\rangle$, *etc*. it is convenient/useful to define the so-called ***root-mean-square*** (≡ *RMS*) values of the $\vec{E}$ and $\vec{B}$ electric and magnetic field ***amplitudes*** (using the mathematical definition of *RMS* from probability and statistics):

For a ***monochromatic*** (*i.e.* ***single*** frequency, ***sinusoidally-varying***) *EM* wave (***only***):

$$\vec{E}_{rms} \equiv \frac{1}{\sqrt{2}}\vec{E} \quad \Rightarrow \quad E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o = 0.707E_o$$

$$\vec{B}_{rms} \equiv \frac{1}{\sqrt{2}}\vec{B} \quad \Rightarrow \quad B_{o_{rms}} \equiv \frac{1}{\sqrt{2}}B_o = 0.707B_o$$

Where: E_o = peak (*i.e.* max) value of the $\vec{E}$-field = amplitude of the $\vec{E}$-field.
B_o = peak (*i.e.* max) value of the $\vec{B}$-field = amplitude of the $\vec{B}$-field.



Thus we see that:

$$\vec{E}_{rms}\bullet\vec{E}_{rms}=\left(\frac{1}{\sqrt{2}}\vec{E}\right)\left(\frac{1}{\sqrt{2}}\vec{E}\right)=\frac{1}{2}\vec{E}\bullet\vec{E} \quad \text{and} \quad \vec{B}_{rms}\bullet\vec{B}_{rms}=\left(\frac{1}{\sqrt{2}}\vec{B}\right)\left(\frac{1}{\sqrt{2}}\vec{B}\right)=\frac{1}{2}\vec{B}\bullet\vec{B}$$

i.e. that:

$$E_{rms}^2=\frac{1}{2}E^2=\frac{1}{2}E_{peak}^2 \Rightarrow E_{o_{rms}}^2=\frac{1}{2}E_o^2 \quad \text{and} \quad B_{rms}^2=\frac{1}{2}B^2=\frac{1}{2}B_{peak}^2 \Rightarrow B_{o_{rms}}^2=\frac{1}{2}B_o^2$$

Then:

For monochromatic *EM* plane waves (***only***):

$$\left\langle u_{EM}^{rms}(t)\right\rangle=\frac{1}{2}\left\langle u_{EM}(t)\right\rangle=\frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}=\frac{1}{4}\varepsilon_o E_o^2=\frac{1}{2}\varepsilon_o E_{o_{rms}}^2 \quad \left(\frac{\text{RMS Joules}}{\text{m}^3}\right)$$

$$\left\langle \vec{S}_{rms}(t)\right\rangle=\frac{1}{2}\left\langle \vec{S}(t)\right\rangle=\frac{1}{2}c\left\langle u_{EM}(t)\right\rangle\hat{z}=c\left\langle u_{EM}^{rms}(t)\right\rangle\hat{z} \quad \left(\frac{\text{RMS Watts}}{\text{m}^2}\right)$$

$$\left\langle \vec{\wp}_{EM}^{rms}(t)\right\rangle=\frac{1}{2c^2}\left\langle \vec{S}(t)\right\rangle=\frac{1}{2c}\left\langle u_{EM}(t)\right\rangle\hat{z}=\frac{1}{c^2}\left\langle \vec{S}_{rms}(t)\right\rangle=\frac{1}{c}\left\langle u_{EM}^{rms}(t)\right\rangle\hat{z} \quad \left(\frac{\text{RMS kg}}{\text{m}^2\text{-sec}}\right)$$

$$\left\langle \vec{\ell}_{EM}^{rms}(t)\right\rangle=\frac{1}{2}\vec{r}\times\left\langle \vec{\wp}_{EM}(t)\right\rangle=\vec{r}\times\left\langle \vec{\wp}_{EM}^{rms}(t)\right\rangle=\frac{1}{c^2}\left(\vec{r}\times\left\langle \vec{S}_{rms}(t)\right\rangle\right)=\frac{1}{c}\left\langle u_{EM}^{rms}(t)\right\rangle\left(\vec{r}\times\hat{z}\right) \quad \left(\frac{\text{RMS kg}}{\text{m-sec}}\right)$$

$$I_{rms}=\left|\left\langle \vec{S}_{rms}(t)\right\rangle\right|=\left\langle\left|\vec{S}_{rms}(t)\right|\right\rangle=\frac{1}{2}I=\frac{1}{2}\left\langle\left|\vec{S}(t)\right|\right\rangle=c\left\langle u_{EM}^{rms}(t)\right\rangle=\frac{1}{2}c\varepsilon_o E_{o_{rms}}^2 \quad \left(\frac{\text{RMS Watts}}{\text{m}^2}\right)$$

Real world example: Here in the U.S., 120 Vac/60 Hz "wall power" refers to the RMS AC voltage! The ***peak*** voltage (*i.e.* the voltage ***amplitude***) is: $V_{peak}=\sqrt{2}V_{rms}=\sqrt{2}\cdot 120=169.7\simeq 170.0$ Volts.

n.b. For *EM* waves ≠ sinusoidal waves, the root-mean-square (RMS) must be defined properly / mathematically – *e.g.* the RMS value of square  or triangle  wave amplitudes (from Fourier analysis these consist of linear combinations of infinite # of harmonics)

$$\text{E}_{rms}^{\sqcap\sqcup\sqcap} \neq \frac{1}{\sqrt{2}}\text{E}^{\sqcap\sqcup\sqcap} \qquad \text{E}_{rms}^{\wedge\wedge} \neq \frac{1}{\sqrt{2}}\text{E}^{\wedge\wedge}$$

(See/refer to probability & statistics reference books!!)

The Relationship(s) Between the Complex Time-Domain Poynting's Vector and the Complex Vector Impedance/Admittance of an *EM* Plane Wave:

Complex Time-Domain Poynting's Vector:

$$\tilde{\vec{S}}(\vec{r},t;\omega)\equiv\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}(\vec{r},t;\omega) \quad \left(Watts/m^2\right)$$

Complex Vector Impedance of an *EM* Plane Wave:

$$\tilde{\vec{Z}}(\vec{r},t;\omega)\equiv\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^{-1}(\vec{r},t;\omega)=\left\{\tilde{\vec{E}}(\vec{r},t;\omega)\times\tilde{\vec{H}}^{*}(\vec{r},t;\omega)\right\}\Big/\left|\tilde{\vec{H}}(\vec{r},t;\omega)\right|^2 \quad (Ohms)$$

Complex Vector Admittance = Reciprocal of Complex Vector Impedance:

$$\tilde{\vec{Y}}(\vec{r},t;\omega)=\tilde{\vec{Z}}^{-1}(\vec{r},t;\omega)=\tilde{\vec{E}}^{-1}(\vec{r},t;\omega)\times\tilde{\vec{H}}(\vec{r},t;\omega)=\left\{\tilde{\vec{E}}^{*}(\vec{r},t;\omega)\times\tilde{\vec{H}}(\vec{r},t;\omega)\right\}\Big/\left|\tilde{\vec{E}}(\vec{r},t;\omega)\right|^2 \quad (Siemens)$$

If we start with Poynting's vector, we show that it is linearly related to vector admittance and/or reciprocal vector impedance {we suppress (here) the argument $(\vec{r},t;\omega)$ for notation clarity}:

$$\tilde{\vec{S}} \equiv \tilde{\vec{E}} \times \tilde{\vec{H}} = \frac{(\tilde{\vec{E}} \bullet \tilde{\vec{E}})}{\tilde{\vec{E}}} \times \tilde{\vec{H}} = (\tilde{\vec{E}} \bullet \tilde{\vec{E}}) \underbrace{\tilde{\vec{E}}^{-1} \times \tilde{\vec{H}}}_{\equiv \tilde{\vec{Y}}} = (\tilde{\vec{E}} \bullet \tilde{\vec{E}}) \tilde{\vec{Y}} = (\tilde{\vec{E}} \bullet \tilde{\vec{E}}) \tilde{\vec{Z}}^{-1} \ \left(Watts/m^2 \right)$$

Note that the units of $\tilde{\vec{E}}\left(Volts/m\right)$, hence: $(\tilde{\vec{E}} \bullet \tilde{\vec{E}})\tilde{\vec{Z}}^{-1}\left(\left(Volts/m\right)^2 / Ohms = Watts/m^2 \text{ !!!}\right)$

We can also obtain the alternate relations:

$$\tilde{\vec{S}} \equiv \tilde{\vec{E}} \times \tilde{\vec{H}} = \tilde{\vec{E}} \times \frac{(\tilde{\vec{H}} \bullet \tilde{\vec{H}})}{\tilde{\vec{H}}} = (\tilde{\vec{H}} \bullet \tilde{\vec{H}}) \underbrace{\tilde{\vec{E}} \times \tilde{\vec{H}}^{-1}}_{\equiv \tilde{\vec{Z}}} = (\tilde{\vec{H}} \bullet \tilde{\vec{H}}) \tilde{\vec{Z}} = (\tilde{\vec{H}} \bullet \tilde{\vec{H}}) \tilde{\vec{Y}}^{-1} \ \left(Watts/m^2 \right)$$

Note that the units of $\tilde{\vec{H}}\left(Amps/m\right)$, hence: $(\tilde{\vec{H}} \bullet \tilde{\vec{H}})\tilde{\vec{Z}} \ \left(\left(Amp/m\right)^2 \cdot Ohms = Watts/m^2 \text{ !!!}\right)$

Note that the above complex relations are the ***vector*** analogs of the complex ***scalar*** power and Ohm's law relations associated with AC circuits {suppressing the arguments $(t;\omega)$ for notational clarity}:

Complex time-domain *AC* power:

$$\tilde{P} \equiv \tilde{V} \cdot \tilde{I} \left(Volts \cdot Amps = Watts \right)$$

Complex Ohm's Law:

$$\tilde{Z} = \tilde{V} / \tilde{I} = (\tilde{V} \cdot \tilde{I}^*) \Big/ \left| \tilde{I} \right|^2 \left(Volts/Amps = Ohms \right)$$

Complex Scalar Admittance = Reciprocal of Complex Scalar Impedance:

$$\tilde{Y} = 1/\tilde{Z} = \tilde{I}/\tilde{V} = \tilde{I} \cdot \tilde{V}^* \Big/ \left| \tilde{V} \right|^2 \left(Amps/Volts = Siemens = Ohms^{-1} \right)$$

Starting with complex time-domain AC power, we show that it is linearly related to scalar admittance and/or reciprocal scalar impedance:

$$\tilde{P} \equiv \tilde{V} \cdot \tilde{I} = \frac{(\tilde{V} \cdot \tilde{V})}{\tilde{V}} \cdot \tilde{I} = (\tilde{V} \cdot \tilde{V}) \frac{\tilde{I}}{\tilde{V}} = (\tilde{V} \cdot \tilde{V}) \tilde{Y} = (\tilde{V} \cdot \tilde{V}) / \tilde{Z} \ \left(Watts \right)$$

Note that the units of $\tilde{V}\left(Volts\right)$, hence: $(\tilde{V} \cdot \tilde{V})/\tilde{Z}\left(Volts^2/Ohms = Watts \text{ !!!}\right)$

We can also obtain the alternate relations:

$$\tilde{P} \equiv \tilde{V} \cdot \tilde{I} = \tilde{V} \cdot \frac{(\tilde{I} \cdot \tilde{I})}{\tilde{I}} = (\tilde{I} \cdot \tilde{I}) \frac{\tilde{V}}{\tilde{I}} = (\tilde{I} \cdot \tilde{I}) \tilde{Z} = (\tilde{I} \cdot \tilde{I}) / \tilde{Y} \ \left(Watts \right)$$

Note that the units of $\tilde{I}\left(Amps\right)$, hence: $(\tilde{I} \cdot \tilde{I})\tilde{Z} \ \left(Amp^2 \cdot Ohms = Watts \text{ !!!}\right)$

Radiation Pressure: $\mathrm{P}_{rad}\left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$

When an *EM* wave impinges (*i.e.* is incident) on a ***perfect absorber*** (*e.g.* a totally ***black*** object with absorbance {*aka* absorption coefficient} $A = 1$, as "seen" at the frequency of the *EM* wave), all of the *EM* energy (by definition) is ***absorbed*** {ultimately winding up as heat…}.

By conservation of energy, linear momentum & angular momentum the object being irradiated by the incident *EM* wave acquires energy, linear momentum & angular momentum from the incident *EM* wave.

The *EM* Radiation Pressure acting on a ***perfect absorber*** for a ***normally incident*** *EM* wave is defined as:

$$\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \frac{\text{Time-Averaged}\,|\text{Force}|}{\perp\text{ Unit Area}} = \frac{\left\langle\left|\vec{F}_{EM}^{net}(t)\right|\right\rangle}{A_\perp} \quad \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

However, the ***time-averaged*** *EM* force is defined as:

$$\left\langle\vec{F}_{EM}^{net}(t)\right\rangle \equiv \frac{d\left\langle\vec{p}_{EM}(t)\right\rangle}{dt} = \frac{\left\langle\Delta\vec{p}_{EM}(t)\right\rangle}{\Delta t} = \text{time rate of change of the time-averaged linear momentum}$$

$\therefore$ the *EM* Radiation Pressure at ***normal incidence*** is:

$$\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \frac{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle}{\Delta t}\frac{1}{A_\perp} \quad \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

In a time interval $\Delta t \gg \tau = 1/f$, the *time-averaged* magnitude of the *EM* linear momentum *transfer* $\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle$ at ***normal incidence*** to a ***perfect absorber*** of *EM* radiation is:

$$\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle = \left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\,\Delta V$$

EM Linear momentum density (↗ $\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle$) Volume of *EM* wave associated with time interval Δt (↗ ΔV)

The volume associated with an *EM* wave propagating in free space over a time interval Δt is:

$\Delta V = A_\perp\cdot(c\Delta t)$ where $c\Delta t$ = distance traveled by the *EM* wave in the time interval Δt.

$$\therefore\quad \mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \frac{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle}{\Delta t}\frac{1}{A_\perp} = \frac{\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\Delta V}{\Delta t}\frac{1}{A_\perp} = \frac{\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle \cancel{A_\perp}\, c\,\cancel{\Delta t}}{\cancel{\Delta t}\;\cancel{A_\perp}} = c\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle$$

Thus, we see that for a monochromatic *EM* plane wave propagating in free space ***normally incident*** on a ***perfect absorber*** (A = 1):

$$\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = c\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle = \frac{1}{2}\varepsilon_o E_o^2 = \left\langle u_{EM}\right\rangle = I/c \quad \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

For a ***perfect reflector*** (*e.g.* a ***perfect*** mirror, with ***reflection coefficient*** $R = 1\{A = 0\}$), note that:

$$\boxed{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle^{\substack{perfect\\reflector}} = 2\times\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle^{\substack{perfect\\absorber}}}$$

Since $\Delta\vec{p}_{EM} \equiv \vec{p}_{EM}^{\,initial} - \vec{p}_{EM}^{\,final}$ and $\vec{p}_{EM}^{\,final} = -\vec{p}_{EM}^{\,initial}$ for an *EM* wave reflecting off of a ***perfect reflector***, then $\Delta\vec{p}_{EM} \equiv \vec{p}_{EM}^{\,initial} - \vec{p}_{EM}^{\,final} = \vec{p}_{EM}^{\,initial} + \vec{p}_{EM}^{\,initial} = 2\,\vec{p}_{EM}^{\,initial}$

i.e. an *EM* wave that reflects off of (*i.e.* "bounces" off of) a ***perfect reflector*** delivers ***twice*** (2×) the momentum kick (*i.e.* impulse) to the ***perfect reflector*** than the same *EM* wave that is absorbed by a ***perfect absorber***! Thus at ***normal incidence***:

$$\therefore\ \boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{reflector}}(R=1)\right\} = 2\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} = 2\left(I/c\right)}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

Note that for a ***partially*** reflecting surface, with reflection coefficient $R < 1$, since $R + A = 1$, the radiation pressure associated with an *EM* wave propagating in free space and reflecting off of a ***partially*** reflecting surface at ***normal incidence*** is given by:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = A\cdot\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} + 2R\cdot\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(R=1)\right\} = (A+2R)\left(I/c\right)}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

Since $A = 1 - R$, we can equivalently re-write this relation as:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = (A+2R)\left(I/c\right) = (1-R+2R)\left(I/c\right) = (1+R)\left(I/c\right)}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

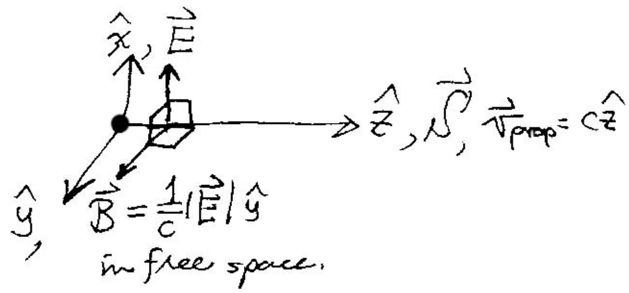
If the *EM* wave is *not* at normal incidence on the absorbing/reflecting surface, but instead makes a finite angle θ with respect to the unit normal of the surface, these relations need to be modified, due to the cosine θ factor $\left\langle\vec{S}\right\rangle\bullet\hat{n} = \left\langle\left|\vec{S}\right|\right\rangle\cos\theta = I\cos\theta$ associated with the flux of *EM* energy/momentum $\left\langle\vec{\wp}_{EM}(t)\right\rangle\bullet\hat{n} = \left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\cos\theta = \varepsilon_o\mu_o\left\langle\left|\vec{S}(t)\right|\right\rangle\cos\theta = \tfrac{1}{c^2}\left\langle\left|\vec{S}(t)\right|\right\rangle\cos\theta = \tfrac{1}{c^2}I\cos\theta$ crossing the surface area $A_\perp$ at a finite angle θ:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} = \left(I/c\right)\cos\theta}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

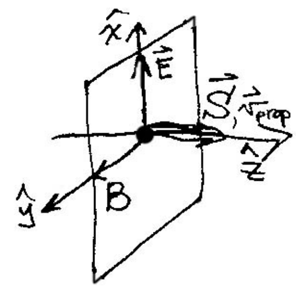
$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{reflector}}(R=1)\right\} = 2\left(I/c\right)\cos\theta}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = (A+2R)\left(I/c\right)\cos\theta = (1+R)\left(I/c\right)\cos\theta}\ \left(\frac{\text{RMS Newtons}}{\text{m}^2}\right)$$

Maxwell's equations (and relativity) for the macroscopic $\vec{E}$ and $\vec{B}$ fields associated with an *EM* plane wave propagating in free space mandate / require that $\vec{E} \perp \vec{B} \perp$ propagation direction (here, $\hat{k} = +\hat{z}$) $\{\vec{v}_{prop} = c\hat{z}\}$, as shown in the figure below:



Compare this microscopic picture to that of a classical / macroscopic *EM* plane wave, polarized in the x-hat direction:

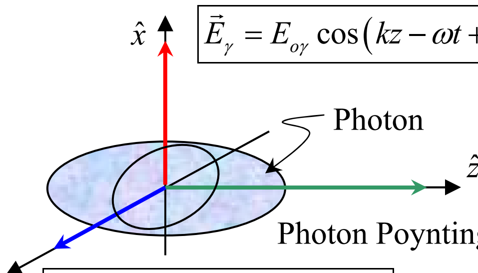


Macroscopic *EM* plane waves propagating in free space are purely transverse waves, *i.e.* $\vec{E} \perp \vec{B}$, and both of the $\vec{E}$ and $\vec{B}$ fields are also $\perp$ to the propagation direction of the *EM* plane wave, *e.g.* $\vec{v}_{prop} = c\hat{z}$. Thus: $\vec{E} \perp \vec{v}_{prop} = c\hat{z}$ and: $\vec{B} \perp \vec{v}_{prop} = c\hat{z}$.

The behavior of the <u>macroscopic</u> $\vec{E}$ and $\vec{B}$ fields associated with *e.g.* a monochromatic *EM* plane wave propagating in free space, at the microscopic scale is simply the sum over (*i.e.* linear superposition of) the $\vec{E}$ and $\vec{B}$-field contributions from {large numbers of} individual <u>real</u> photons making up the *EM* field.

Each real photon has associated with it, its own $\vec{E}$ and $\vec{B}$ field – *e.g.* a <u>linearly</u> <u>polarized</u> real photon, <u>polarized</u> in $+\hat{x}$ direction:

$\boxed{\vec{E}_\gamma = E_{o\gamma}\cos(kz - \omega t + \delta)\,\hat{x}}$ ($+\hat{x}$ = polarization direction)



Real Photon Momentum: $\vec{p}_\gamma = h/\lambda\,\hat{z}$

Photon Poynting's vector: $\boxed{\vec{S}_\gamma = \frac{1}{\mu_o}\left(\vec{E}_\gamma \times \vec{B}_\gamma\right) \parallel +\hat{z}}$

$\boxed{\vec{B}_\gamma = B_{o\gamma}\cos(kz - \omega t + \delta)\,\hat{y}}$

$\boxed{\vec{B}_\gamma = \frac{1}{c}\hat{k} \times \vec{E}_\gamma}$ where the <u>unit</u> <u>wavevector</u> $\boxed{\hat{k} = +\hat{z}}$ {here} and $\boxed{B_{o\gamma} = \frac{1}{c}E_{o\gamma}}$ in vacuum.

Real photon energy: $E_\gamma = hf = p_\gamma c = |\vec{p}_\gamma| c$ (Total Relativistic Energy^2 = $E_\gamma^2 = p_\gamma^2 c^2 + \cancelto{=0}{m_\gamma^2 c^4}$)

Real photon momentum (deBroglie relation): $m_\gamma c^2 \equiv 0$ for <u>real</u> photon

$\boxed{p_\gamma = h/\lambda}$ and $c = f\lambda$ c = speed of light (in vacuum) = 3×10^8 m/sec

---

Question: How many ***real*** visible-light photons per second are emitted *e.g.* from a *EM* power = 10 mW laser? (mW = milli-Watt = 10^{-3} Watt)

Answer: The rate at which visible-light photons from a 10 mW laser depends on the ***color*** (*i.e.* the wavelength λ, frequency f, and/or photon energy E_γ) of the laser beam! $E_\gamma = hf = hc/\lambda$.

When we say a 10 mW power laser, what ***precisely*** does this mean/refer to?

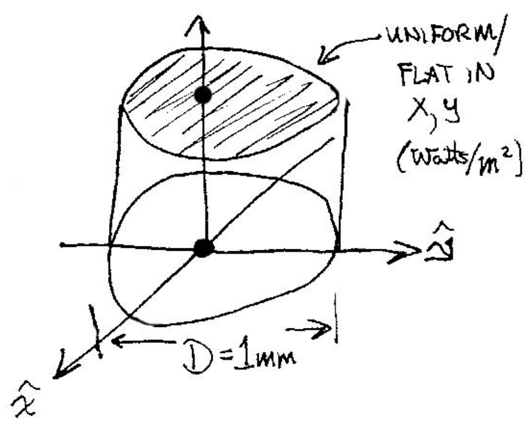
It refers to the ***time-averaged*** *EM* power:

$$\langle P_{laser}(t)\rangle = 10\text{ mW}\,(RMS) = 10\times10^{-3}\,\text{Watts}\,(RMS) = 0.010\text{ Watts}\,(RMS)$$

Let's assume that the laser beam points in the $+\hat{z}$ direction.

Also assume that the diameter of the laser beam is D = 1 mm = 0.001 m (typical).
Further assume (for simplicity's sake): Power flux density = intensity profile $I(x,y)$ is uniform in x and y over the diameter of the laser beam (not true in real life – laser beams have ~ Gaussian intensity profiles in x and y (*i.e.* $I(\rho) = I_o e^{-\rho^2/2\sigma^2}$); note that there also exist *e.g.* diffraction {beam-spreading} effects that should/need to be taken into account…)

$$I(x,y) = \left\langle \left|\vec{S}(x,y,t)\right| \right\rangle$$



In Δt = 1 second, the time-averaged energy associated with the 10 (*RMS*) mW laser beam is:

$$\langle \Delta E_{laser}(t)\rangle = \langle P_{laser}(t)\rangle \Delta t$$

$$\langle \Delta E_{laser}(t)\rangle = 0.010\ (RMS)\,\text{Watts} * 1\ \text{sec}$$

$$\langle \Delta E_{laser}(t)\rangle = 0.010\ \frac{(RMS)\,\text{Joules}}{\text{sec}} * 1\ \text{sec}$$

$$\langle \Delta E_{laser}(t)\rangle = 0.010\,(RMS)\ \text{Joules} = \text{Time-averaged energy of laser beam}$$

The {instantaneous} energy of the laser beam crosses an imaginary planar surface that is ⊥ to the laser beam.

If the laser has red light, *e.g.* λ_{red} = 750 nm (*n.b.* 1 nm = 1 nano-meter = 10^{-9} meters) or if the laser has blue light, *e.g.* λ_{blue} = 400 nm

Since $f = c/\lambda$ the corresponding photon frequencies associated with red and blue laser light are:

$$f_{red}^{\gamma} = \frac{c}{\lambda_{red}^{\gamma}} = \frac{3\times10^{8}\,m/s}{750\times10^{-9}\,m} = 4.0\times10^{14} \text{ cycles/sec (= Hertz, or Hz)}$$

$$f_{blue}^{\gamma} = \frac{c}{\lambda_{blue}^{\gamma}} = \frac{3\times10^{8}\,m/s}{400\times10^{-9}\,m} = 7.5\times10^{14} \text{ cycles/sec (= Hertz, or Hz)}$$

The energy associated with a single, real photon is: $\boxed{E_{\gamma} = hf^{\gamma} = hc/\lambda^{\gamma}}$, where h = Planck's constant: h = 6.626 x 10^{-34} Joule-sec and c = 3 x 10^{8} m/sec (speed of light in vacuum).
Thus, the corresponding photon energies associated with red and blue laser light are:

$$\boxed{E_{\gamma}^{red} = hf_{red}^{\gamma} = hc/\lambda_{red}^{\gamma}} \text{ and: } \boxed{E_{\gamma}^{blue} = hf_{blue}^{\gamma} = hc/\lambda_{blue}^{\gamma}} \quad \text{since } f = c/\lambda$$

$$\boxed{E_{\gamma}^{red} = hf_{red}^{\gamma} = 6.626\times10^{-34}\,\text{Joule/sec} \times 4.0\times10^{14}\,/\text{sec} = 2.6504\times10^{-19}\,\text{Joules}} \text{ (red light)}$$

$$\boxed{E_{\gamma}^{blue} = hf_{blue}^{\gamma} = 6.626\times10^{-34}\,\text{Joule/sec} \times 7.5\times10^{14}\,/\text{sec} = 4.9695\times10^{-19}\,\text{Joules}} \text{ (blue light)}$$

In a time interval of $\Delta t = 1$ sec, the time-averaged energy $\boxed{\langle \Delta E_{laser}(t)\rangle = \langle \Delta N_{\gamma}(t)\rangle E_{\gamma}}$ where $\langle \Delta N_{\gamma}(t)\rangle$ is the {time-averaged} number of photons crossing a ⊥ area in the time interval Δt.

Thus, the number of red (blue) photons emitted from a red (blue) laser in a $\Delta t = 1$ sec time interval is:

$$\text{\# red photons: } \boxed{\langle \Delta N_{\gamma}^{red}(t)\rangle = \frac{\langle \Delta E_{laser}(t)\rangle}{E_{\gamma}^{red}} = \frac{0.010 \text{ Joules}}{2.6504\times10^{-19}\text{ Joules/photon}} = 3.7730\times10^{16}}$$

$$\text{\# blue photons: } \boxed{\langle \Delta N_{\gamma}^{blue}(t)\rangle = \frac{\langle \Delta E_{laser}(t)\rangle}{\text{E}_{\gamma}^{blue}} = \frac{0.010 \text{ Joules}}{4.9695\times10^{-19}\text{ Joules/photon}} = 2.0123\times10^{16}}$$

Thus, the {time-averaged} <u>*rate*</u> of emission of red (blue) photons from a red (blue) laser is:

$$\boxed{\langle R_{\gamma}^{red}(t)\rangle = \frac{\langle \Delta N_{\gamma}^{red}(t)\rangle}{\Delta t} = 3.7730\times10^{16}} \text{ red photons/sec}$$

$$\boxed{\langle R_{\gamma}^{blue}(t)\rangle = \frac{\langle \Delta N_{\gamma}^{blue}(t)\rangle}{\Delta t} = 2.0123\times10^{16}} \text{ blue photons/sec}$$

<u>Note</u>: In a time interval of $\Delta t = 1$ sec, photons (of any color / λ^{γ} / f^{γ} / E_{γ}) will travel a distance of $d = c\Delta t = 3\times10^{8}$ m/s × 1 s = 3×10^{8} meters

If the flux of photons is assumed (for simplicity) to be uniform across the $D = 1$ mm diameter laser beam, then the time-averaged flux of photons (#/m^2/sec) is:

$$\left\langle \mathcal{F}_\gamma^{red}(t)\right\rangle = \frac{\left\langle R_\gamma^{red}(t)\right\rangle}{A_\perp^{laser}} = \frac{3.7730\times10^{16}\left(\gamma/\text{sec}\right)}{\pi\left(\frac{10^{-3}\,\text{m}}{2}\right)^2} = 4.8039\times10^{22}\left(\text{red }\gamma/\text{m}^2/\text{sec}\right)$$

$$\left\langle \mathcal{F}_\gamma^{blue}(t)\right\rangle = \frac{\left\langle R_\gamma^{blue}(t)\right\rangle}{A_\perp^{laser}} = \frac{2.0123\times10^{16}\left(\gamma/\text{sec}\right)}{\pi\left(\frac{10^{-3}\,\text{m}}{2}\right)^2} = 2.562\times10^{22}\left(\text{blue }\gamma/\text{m}^2/\text{sec}\right)$$

If each photon has E_γ Joules of energy, then power associated with red (blue) laser beam:

$$\left\langle P_\gamma^{red}(t)\right\rangle \Big/ A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = E_\gamma^{red}\bullet\left\langle \mathcal{F}_\gamma^{red}(t)\right\rangle = 2.6504\times10^{-19}\,\text{Joules}\times4.8039\times10^{22}\left(\text{red}\,\gamma/\text{m}^2/\text{sec}\right)$$

$$= 1.2732\times10^4\ \text{Watts/m}^2$$

$$\left\langle P_\gamma^{blue}(t)\right\rangle \Big/ A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = E_\gamma^{blue}\bullet\left\langle \mathcal{F}_\gamma^{blue}(t)\right\rangle = 4.9695\times10^{-19}\,\text{Joules}\times2.5621\times10^{22}\left(\text{blue}\,\gamma/\text{m}^2/\text{sec}\right)$$

$$= 1.2732\times10^4\ \text{Watts/m}^2$$

Thus we see that:

$$\left\langle P_\gamma^{red}(t)\right\rangle \Big/ A_\perp^{laser} = \left\langle P_\gamma^{blue}(t)\right\rangle \Big/ A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = 1.2732\times10^4\ \text{Watts/m}^2 \quad \leftarrow \text{10 mW laser}$$

n.b. This is **precisely** ***why*** you ***shouldn't*** look into a laser beam {with your ***one*** remaining eye}!!!

Time-averaged linear momentum density:

$$\left|\left\langle \vec{\wp}_\gamma^{red}(t)\right\rangle\right| = \varepsilon_o\mu_o\left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \frac{1}{c^2}\left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \frac{1}{c}\left\langle u_\gamma^{red}(t)\right\rangle\hat{z} = 1.4147\times10^{-13}\,\text{kg/m}^2\text{-sec}$$

$$\left|\left\langle \vec{\wp}_\gamma^{blue}(t)\right\rangle\right| = \varepsilon_o\mu_o\left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = \frac{1}{c^2}\left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = \frac{1}{c}\left\langle u_\gamma^{blue}(t)\right\rangle\hat{z} = 1.4147\times10^{-13}\,\text{kg/m}^2\text{-sec}$$

Thus: $\left|\left\langle \vec{\wp}_\gamma^{red}\right\rangle\right| = \left|\left\langle \vec{\wp}_\gamma^{blue}\right\rangle\right| = 1.4147\times10^{-13}\,\text{kg/m}^2\text{-sec}$

Momentum density, Poyntings vector, energy density are independent of frequency / wavelength / photon energy

The time-averaged linear momentum contained in $\Delta t = 1$ second's worth of laser beam:

Time averaged linear momentum: $\left\langle\left|\Delta\vec{p}_\gamma(t)\right|\right\rangle$ = momentum density $\left\langle\left|\vec{\wp}_\gamma(t)\right|\right\rangle$ x volume ΔV

Volume $\Delta V = A_\perp^{laser} * (c\Delta t)$ (m^3)

($c\Delta t$:) Distance light travels in Δt sec.

Red light momentum:

$$\boxed{\left\langle\left|\Delta\vec{p}_\gamma^{\,red}(t)\right|\right\rangle=\left\langle\left|\vec{\wp}_\gamma^{\,red}(t)\right|\right\rangle c\Delta tA_\perp=1.4147\times10^{-13}\times3\times10^{8}\times1\times\pi\times\left(\frac{0.001}{2}\right)^2}\left(\frac{\text{kg-m}}{\text{sec}}\right)$$

$$\boxed{=3.3333\times10^{-11}\ \text{kg-m/sec}}$$

Blue light momentum:

$$\boxed{\left\langle\left|\Delta\vec{p}_\gamma^{\,blue}(t)\right|\right\rangle=\left\langle\left|\vec{\wp}_\gamma^{\,blue}(t)\right|\right\rangle c\Delta tA_\perp=1.4147\times10^{-13}\times3\times10^{8}\times1\times\pi\times\left(\frac{0.001}{2}\right)^2}\left(\frac{\text{kg-m}}{\text{sec}}\right)$$

$$\boxed{=3.3333\times10^{-11}\ \text{kg-m/sec}}$$

Thus: $\boxed{\left\langle\left|\Delta\vec{p}_\gamma^{\,red}(t)\right|\right\rangle=\left\langle\left|\Delta\vec{p}_\gamma^{\,blue}(t)\right|\right\rangle=3.3333\times10^{-11}\ \text{kg-m/sec}}$

{"TRICK"}:

For an *EM* plane wave, the time-averaged energy density $\langle u_{EM}(t)\rangle$ = time-averaged momentum density $\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle * c$ (Since photon energy, $E_\gamma = p_\gamma c$). Thus:

$$\boxed{\langle u_\gamma^{red}(t)\rangle=\left\langle\left|\vec{\wp}_\gamma^{\,red}(t)\right|\right\rangle c=1.4147\times10^{-13}\left(\frac{\text{kg}}{\text{m}^2/\text{sec}}\right)\times3\times10^{8}\,(\text{m/s})=4.2441\times10^{-5}\left(\text{Joules/m}^3\right)}$$

$$\boxed{\langle u_\gamma^{blue}(t)\rangle=\left\langle\left|\vec{\wp}_\gamma^{\,blue}(t)\right|\right\rangle c=1.4147\times10^{-13}\left(\frac{\text{kg}}{\text{m}^2/\text{sec}}\right)\times3\times10^{8}\,(\text{m/s})=4.2441\times10^{-5}\left(\text{Joules/m}^3\right)}$$

$$\boxed{\text{Joule}=\frac{\text{kg-m}^2}{\text{s}^2}}\quad\Rightarrow\quad\boxed{\frac{\text{Joule}}{\text{m}^2}=\frac{\text{kg}}{\text{m/s}^2}}$$

The time-averaged energy contained in $\Delta t = 1$ second's worth of laser beam is:
The time-averaged energy $\langle U_\gamma(t)\rangle$ = time-averaged energy density $\langle u_\gamma(t)\rangle *$ volume ΔV

$$\boxed{\Delta V = A_\perp^{laser} * (c\Delta t)}$$

$$\therefore\ \boxed{\langle U_\gamma^{red}(t)\rangle=\langle u_\gamma^{red}(t)\rangle * A_\perp c\Delta t=4.2441\times10^{-5}\left(\frac{\text{Joules}}{\text{m}^3}\right)*\pi\times\left(\frac{0.001}{2}\right)^2*3\times10^{8}*1\left(\text{m}^3\right)}$$

$$\boxed{=0.010\,\text{Joules}=10\,\text{mJ}}$$

$$\boxed{\langle U_\gamma^{blue}(t)\rangle=\langle u_\gamma^{blue}(t)\rangle * A_\perp c\Delta t=4.2441\times10^{-5}\left(\frac{\text{Joules}}{\text{m}^3}\right)*\pi\times\left(\frac{0.001}{2}\right)^2*3\times10^{8}*1\left(\text{m}^3\right)}$$

$$\boxed{=0.010\,\text{Joules}=10\,\text{mJ}}$$

The time-averaged power in the laser beam: $\boxed{\langle P_{laser}^{red}(t)\rangle=\frac{\langle U_{laser}(t)\rangle}{\Delta t}=10\text{mW}=\langle \text{P}_{laser}^{blue}(t)\rangle}$

Time-averaged Power (Watts) = $\dfrac{d\langle U(t)\rangle}{dt}\left(\text{Joules}/_{\text{sec}}\right)$ ← $\Delta t = 1$ sec

Note: P_{laser} (laser power) is measured by the total time-averaged energy $\langle U(t)\rangle$ deposited in (a very accurately) known time interval Δt using an absolutely calibrated photodiode (*e.g.* by NIST).

A typical time interval $\Delta t = 10$ secs $\rightarrow$ $\Delta t \gg \tau$ (oscillation period) $= 1/f$!!

$$\tau_{red} = 1/f_{red} = 2.500\times10^{-15}\text{ sec} = 2.500\text{ femto-sec} = 2.500\text{ fs}$$

$$\tau_{blue} = 1/f_{blue} = 1.333\times10^{-15}\text{ sec} = 1.333\text{ femto-sec} = 1.333\text{ fs}$$

$\rightarrow$ The laser power measured is time-averaged power, *i.e.* $\langle P_{laser}(t)\rangle = \frac{1}{2}P_{laser}^{peak}(t)$

Consider (the time-averaged) energy density associated with this 10 mW laser:

$$\langle u_{EM}(t)\rangle = 4.2441\times10^{-5}\ \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

Now: $\langle u_{EM}(t)\rangle = \langle u_{elect}(t)\rangle + \langle u_{mag}(t)\rangle = \frac{1}{2}\varepsilon_o E_o^2 = \frac{1}{2}u_{EM}^{peak}(t)$

And because: $|\vec{B}(t)| = \frac{1}{c}|\vec{E}(t)|$ for *EM* plane waves propagating in free space / vacuum ($\frac{1}{c^2} = \varepsilon_o\mu_o$)

We showed that: $\langle u_{elect}(t)\rangle = \langle u_{mag}(t)\rangle$

$$\langle u_{elect}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}$$

$$B_o^2 = \frac{1}{c^2}E_o^2 = \varepsilon_o\mu_o E_o^2$$

$$\langle u_{mag}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2\mu_o}B_o^2\right\} = \frac{1}{2}\left\{\frac{\varepsilon_o\not{\mu_o}}{2\not{\mu_o}}E_o^2\right\} = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}$$

Now: E_o = amplitude of the macroscopic electric field: $\vec{E}(z,t) = E_o\cos(kz-\omega t+\delta)\hat{x}$

B_o = amplitude of the macroscopic magnetic field: $\vec{B}(z,t) = B_o\cos(kz-\omega t+\delta)\hat{y}$

Define the RMS (Root-Mean-Square) amplitudes of the $\vec{E}$ and $\vec{B}$ fields:

$$E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o \Rightarrow E_{o_{rms}}^2 = \frac{1}{2}E_o^2$$

$$B_{o_{rms}} \equiv \frac{1}{\sqrt{2}}B_o \Rightarrow B_{o_{rms}}^2 = \frac{1}{2}B_o^2 = \frac{1}{2c^2}E_o^2 \quad \text{in free space / vacuum}$$

Then: $\langle u_{elect}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\} = \frac{1}{2}\varepsilon_o E_{o_{rms}}^2$ (Joules/m^3)

$\langle u_{mag}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2\mu_o}B_o^2\right\} = \frac{1}{2\mu_o}B_{o_{rms}}^2 = \frac{1}{2}\varepsilon_o E_{o_{rms}}^2$ in free space / vacuum

So if: $\langle u_{EM}(t)\rangle = \langle u_{elect}(t)\rangle + \langle u_{mag}(t)\rangle = 2\langle u_{elect}(t)\rangle$ in free space / vacuum

$= 2\varepsilon_o E_{o_{\rm rms}}^2 = 4.2441\times10^{-5}$ Joules/m^3

Then: $E_{o_{\rm rms}}^2 = \dfrac{1}{\varepsilon_o}\langle u_{EM}(t)\rangle$ where $\varepsilon_o = 8.85\times10^{-12}$ Farads/m = electric permittivity of free space

Thus: $E_{o_{\rm rms}}^{laser} = \sqrt{\dfrac{1}{\varepsilon_o}\langle U_{EM}(t)\rangle} = \left[\dfrac{4.2441\times10^{-5}\,\text{Joules/m}}{8.85\times10^{-12}\,\text{Farads/m}}\right]^{1/2}$ (Volts/m)

$E_{o\ \rm rms}^{laser} \simeq 3.0970\times10^{3} = 3097$ RMS Volts/m (*n.b.* ***same*** for red *vs*. blue laser light!)

Then: $E_o^{laser} = E_{peak}^{laser} = \sqrt{2}\ E_{o\ \rm rms}^{laser} \simeq 4380$ Volts/m

Then: $B_{o_{\rm rms}}^{laser} = \dfrac{1}{c}E_{o_{\rm rms}}^{laser} \simeq 10.3232\times10^{-6}\,\text{RMS Tesla}\ \left(=10.3232\times10^{-2}\,\text{RMS Gauss}\right)$

Note: 1 Tesla {SI/MKS units} = 10^4 Gauss {CGS units}

Thus: $B_o^{laser} = \dfrac{1}{c}E_o^{laser} = \sqrt{2}\ B_{o_{\rm rms}}^{laser} \simeq 14.5970\times10^{-6}\ \text{Tesla}\ \left(=14.5970\times10^{-2}\ \text{Gauss}\right)$

Now earlier (above) we calculated the (time-averaged) number of photons present in the {red and blue} laser beams that were emitted in a time interval of $\Delta t = 1$ sec.

\# red photons emitted in $\Delta t = 1$ sec: $\langle \Delta N_\gamma^{red}(t)\rangle = 3.7730\times10^{16}$ red photons

\# blue photons emitted in $\Delta t = 1$ sec: $\langle \Delta N_\gamma^{blue}(t)\rangle = 2.0123\times10^{16}$ blue photons

The volume associated with a D = 1 mm diameter laser beam turned on for $\Delta t = 1$ sec is:

$$\Delta V = A_\perp c\Delta t = \pi\left(\frac{D}{2}\right)^2 c\Delta t = \pi\left(\frac{0.001}{2}\right)^2 \bullet 3\times10^8 \bullet 1 = 235.6194\ \text{m}^3$$

The (time-averaged) number density $\langle n_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta V}$ of {red and blue} photons in the laser beam is:

$$\langle n_\gamma^{red}(t)\rangle = \frac{\langle \Delta N_\gamma^{red}(t)\rangle}{\Delta V} = \frac{3.7730\times10^{16}}{2.3562\times10^{2}} = 1.6009\times10^{14}\ \text{red photons/m}^3$$

$$\langle n_\gamma^{blue}(t)\rangle = \frac{\langle \Delta N_\gamma^{blue}(t)\rangle}{\Delta V} = \frac{2.0123\times10^{16}}{2.3562\times10^{2}} = 8.5405\times10^{13}\ \text{blue photons/m}^3$$

Then the (time-averaged) energy density $\langle u_{EM}(t)\rangle$ of the {red and blue} laser beam is:

Red photon energy: $E_\gamma^{red} = hf_\gamma^{red} = 2.6504\times10^{-19}$ Joules

Blue photon energy: $E_\gamma^{blue} = hf_\gamma^{blue} = 4.9695\times10^{-19}$ Joules

$$\langle u_{EM}^{red}(t)\rangle = \langle n_\gamma^{red}(t)\rangle E_\gamma^{red} = 1.6009\times 10^{14} \bullet 2.6504\times 10^{-19} = 4.2442\times 10^{-5}\ (\text{Joules/m}^3)$$
$$\langle u_{EM}^{blue}(t)\rangle = \langle n_\gamma^{blue}(t)\rangle E_\gamma^{blue} = 8.5405\times 10^{13} \bullet 4.9695\times 10^{-19} = 4.2442\times 10^{-5}\ (\text{Joules/m}^3)$$

The (time-averaged) energy $\langle U_{EM}(t)\rangle = \langle u_{EM}(t)\rangle * \Delta V$ of the {red and blue} laser beams is:

$$\langle U_{EM}^{red}(t)\rangle = \langle u_{EM}^{red}(t)\rangle * \Delta V = 4.2442\times 10^{-5} * 2.3562\times 10^{2} = 0.010 \text{ Joules} = 10 \text{ mJoules}$$
$$\langle U_{EM}^{blue}(t)\rangle = \langle u_{EM}^{blue}(t)\rangle * \Delta V = 4.2442\times 10^{-5} * 2.3562\times 10^{2} = 0.010 \text{ Joules} = 10 \text{ mJoules}$$

Now here is something quite interesting: Given that $E_{o_{rms}} \equiv E_o/\sqrt{2}$ for a monochromatic *EM* plane wave propagating in free space/the vacuum, with time-averaged *EM* energy density:

$$\langle u_{EM}(t)\rangle = \varepsilon_o E_{o_{rms}}^2 = \tfrac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

<u>But</u>: $\langle u_{EM}(t)\rangle = \langle n_\gamma(t)\rangle E_\gamma \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$ with $\langle n_\gamma(t)\rangle$ = photon number density (#/m^3) in laser beam, $E_\gamma = hf_\gamma = hc/\lambda_\gamma$ = energy/photon (Joules)

$\therefore$ $\varepsilon_o E_{o_{rms}}^2 = \langle n_\gamma(t)\rangle E_\gamma$

<u>Or</u>: $E_{o_{rms}}^2 = \dfrac{\langle n_\gamma(t)\rangle}{\varepsilon_o} E_\gamma$

This formula explicitly connects the <u>amplitudes</u> of the <u>macroscopic</u> $\vec{E}$ and $\vec{B}$ fields (since $B_o = E_o/c$) with the <u>microscopic</u> <u>constituents</u> of the $\vec{E}$ and $\vec{B}$ fields (*i.e.* the <u>photons</u>)!!!

n.b. This formula physically says that the <u>number</u> of {real} photons in the *EM* wave (each of photon energy E_γ) is proportional to E_o^2 = the <u>square</u> of the <u>macroscopic</u> electric field <u>amplitude</u>!

We can write this as: $\langle n_\gamma(t)\rangle = \varepsilon_o E_{o_{rms}}^2 / E_\gamma$ and note also that: $E_{o_{rms}} = \sqrt{\dfrac{\langle n_\gamma(t)\rangle}{\varepsilon_o} E_\gamma}$!!!

Thus, we can now see that the {time-averaged} *EM* energy density:

$$\langle u_{EM}(\vec{r},t)\rangle = \varepsilon_o E_{o_{rms}}^2 = \langle n_\gamma(\vec{r},t)\rangle E_\gamma \quad \underline{\text{with}}: \quad \int_v \langle u_{EM}(\vec{r},t)\rangle d\tau = U_{EM}$$

plays a role analogous to that of the <u>probability</u> <u>density</u> in <u>quantum</u> <u>mechanics</u>:

$$\mathcal{P}(\vec{r},t) = \langle \psi(\vec{r},t) | \psi(\vec{r},t)\rangle = |\psi(\vec{r},t)|^2 \quad \underline{\text{with}}: \quad \int_v \mathcal{P}(\vec{r},t)\, d\tau = 1$$

Since: $\langle n_\gamma(\vec{r},t)\rangle = \langle u_{EM}(\vec{r},t)\rangle / E_\gamma = 2\varepsilon_o E_{o_{rms}}^2(\vec{r})/E_\gamma$ and: $\int_v \langle n_\gamma(\vec{r},t)\rangle d\tau = \langle \Delta N_\gamma\rangle$,

Then: $\langle \mathcal{P}_\gamma(\vec{r},t)\rangle \equiv \langle n_\gamma(\vec{r},t)\rangle / \langle \Delta N_\gamma\rangle = \langle \psi_\gamma(\vec{r},t) | \psi_\gamma(\vec{r},t)\rangle = |\psi_\gamma(\vec{r},t)|^2$!!!

Thus, we also see that the macroscopic electric field $\vec{E}(\vec{r},t)$ plays a role analogous to that of the probability density ***<u>amplitude</u>*** $\psi(\vec{r},t)$ in quantum mechanics!!!

The (real) photon number density in the laser beam is: $\boxed{\langle n_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta V} = \dfrac{\langle \Delta N_\gamma(t)\rangle}{A_\perp c\Delta t}}$ $\left(\dfrac{\#}{\text{m}^3}\right)$

Then: $\boxed{\varepsilon_o E_{o_{\text{rms}}}^2 A_\perp c\Delta t = \langle \Delta N_\gamma(t)\rangle E_\gamma}$ or: $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\varepsilon_o A_\perp c\Delta t} E_\gamma}$

But: $\boxed{\langle R_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta t}}$ = the time averaged rate of photons in laser beam (#/sec)

$\therefore$ $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{1}{2\varepsilon_o}\dfrac{\langle R_\gamma(t)\rangle}{A_\perp c} E_\gamma}$ and

$\boxed{\langle \mathscr{F}_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta t}\Big/ A_\perp = \langle R_\gamma(t)\rangle \Big/ A_\perp}$ $\left(\dfrac{\#}{\text{m}^2\text{-s}}\right)$ = flux of photons in the laser beam

$\therefore$ $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{1}{\varepsilon_o c}\langle \mathscr{F}_\gamma(t)\rangle E_\gamma}$ and $\boxed{\langle u_{EM}(t)\rangle = \varepsilon_o E_{o_{\text{rms}}}^2 = \dfrac{1}{c}\langle \mathscr{F}_\gamma(t)\rangle E_\gamma = \langle n_\gamma(\vec{r},t)\rangle E_\gamma}$ $\left(\dfrac{\text{Joules}}{\text{m}^3}\right)$

Thus, we see that the {real} photon flux: $\boxed{\langle \mathscr{F}_\gamma(t)\rangle = c\langle n_\gamma(\vec{r},t)\rangle}$ $\left(\dfrac{\#}{\text{m}^2\text{-s}}\right)$

Thus, the ***intensity*** {*aka* ***irradiance***} of the laser beam is:

$$\boxed{I \equiv \langle \vec{S}(t)\rangle \bullet \hat{k} = c\langle u_{EM}(t)\rangle = \varepsilon_o E_{o_{\text{rms}}}^2 = \langle \mathscr{F}_\gamma(t)\rangle E_\gamma = c\langle n_\gamma(t)\rangle E_\gamma} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The {time-averaged} <***longitudinal*** separation distance> between photons is defined as:

$$\boxed{\langle \Delta d_\parallel(t)\rangle \equiv \frac{c\Delta t}{\langle N_\gamma(t)\rangle}} \quad (\text{m})$$

For $\Delta t = 1$ sec: $\boxed{\langle \Delta d_\parallel^{red}(t)\rangle = \dfrac{3\times 10^8\,\text{m}}{3.773\times 10^{16}\,\gamma's} = 7.85\times 10^{-9}\,\text{m} \sim 8\times 10^{-9}\,\text{m} \simeq 8\text{ nm}}$ (1 nm = 10^{-9} m)

$$\boxed{\langle \Delta d_\parallel^{blue}(t)\rangle = \frac{3\times 10^8\,\text{m}}{2.012\times 10^{16}\,\gamma's} = 1.49\times 10^{-8}\,\text{m} \sim 15\times 10^{-9}\,\text{m} \simeq 15\text{ nm}}$$

Recall that: $\lambda_\gamma^{red} = 750$ nm and $\lambda_\gamma^{blue} = 400$ nm

Thus: $\boxed{\lambda_\gamma \gg \langle \Delta d_\parallel(t)\rangle}$ for either red or blue laser light.

The {time-averaged} <***transverse*** separation distance> between photons is defined as:

$$\boxed{\langle \Delta d_\perp(t)\rangle \equiv \frac{\sqrt{A_\perp}}{\langle N_\gamma(t)\rangle}}$$

Thus:

$$\left\langle \Delta d_{\perp}^{red}(t)\right\rangle = \frac{\sqrt{\pi\left(\frac{0.001}{2}\right)^2}}{3.773\times 10^{16}} = 2.35\times 10^{-20}\,\text{m}$$

$$\left\langle \Delta d_{\perp}^{blue}(t)\right\rangle = \frac{\sqrt{\pi\left(\frac{0.001}{2}\right)^2}}{2.0123\times 10^{16}} = 4.40\times 10^{-20}\,\text{m}$$

We showed above that the time-averaged/mean ***number density*** of photons in the laser beam is:

$$\left\langle n_\gamma(\vec{r},t)\right\rangle = \left\langle u_{EM}(\vec{r},t)\right\rangle / E_\gamma = \tfrac{1}{2}\varepsilon_o E_o^2 / E_\gamma \;\left(photons/m^3\right).$$

The ***instantaneous number density*** of photons in the laser beam is:

$$n_\gamma(\vec{r},t) = u_{EM}(\vec{r},t)/E_\gamma = \varepsilon_o E_o^2 \cos^2(kz-\omega t)/E_\gamma \;\left(photons/m^3\right)$$

We can ***normalize*** the ***instantaneous*** photon number density to obtain an ***instantaneous*** 3-D photon ***probability density***, $\mathcal{P}_\gamma^{3\text{-}D}(z,t)\;\left(1/m^3\right)$. Recall that the laser beam intensity is uniform/constant in the cross-sectional area $A_\perp$ of the laser beam.

The 3-D photon ***probability density***, $\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)$ is:

$$\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t) \equiv n_\gamma(\vec{r},t)/N_\gamma = u_{EM}(\vec{r},t)/\left(N_\gamma\cdot E_\gamma\right) = \varepsilon_o E_o^2\cos^2(kz-\omega t)/\left(N_\gamma\cdot E_\gamma\right)\;\left(1/m^3\right)$$

But note that: $\left(N_\gamma\cdot E_\gamma\right) = \left\langle \Delta U_{EM}\right\rangle \;(Joules)$ in time interval of $\Delta t\; secs$.

Then also note that: $\left\langle \Delta U_{EM}\right\rangle = \left\langle u_{EM}(\vec{r},t)\right\rangle\cdot Vol = \left(\tfrac{1}{2}\varepsilon_o E_o^2\right)\cdot A_\perp\cdot\ell = \left(\left\langle n_\gamma(\vec{r},t)\right\rangle\cdot E_\gamma\right)\cdot A_\perp\cdot\ell$.

Thus: $$\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t) \equiv \frac{n_\gamma(\vec{r},t)}{N_\gamma}\frac{u_{EM}(\vec{r},t)}{\left\langle\Delta U_{EM}\right\rangle} = \frac{\cancel{\varepsilon_o E_o^2}}{\left(\tfrac{1}{2}\cancel{\varepsilon_o E_o^2}\right)\cdot A_\perp\cdot\ell}\cos^2(kz-\omega t) = \frac{2}{A_\perp\cdot\ell}\cos^2(kz-\omega t)\;\left(1/m^3\right)$$

with:

$$\int_v \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)\,d\tau = \frac{2}{A_\perp\cdot\ell}\int_{A_\perp}\int_{z=0}^{z=\ell=c\Delta t}\cos^2(kz-\omega t)\,dz\cdot da = \frac{2\cancel{A_\perp}}{\cancel{A_\perp}\cdot\ell}\int_{z=0}^{z=\ell=c\Delta t}\cos^2(kz-\omega t)\,dz$$
$$= 2\cdot\underbrace{\frac{1}{\ell}\int_{z=0}^{z=\ell=c\Delta t}\cos^2(kz-\omega t)\,dz}_{=\frac{1}{2}} = 1$$

We thus see that the instantaneous normalized ***photon number density*** $\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)\equiv n_\gamma(\vec{r},t)/N_\gamma$ plays a role analogous to that of the ***probability density*** in ***quantum mechanics*** $\mathcal{P}(\vec{r},t) = \left\langle\psi(\vec{r},t)|\psi(\vec{r},t)\right\rangle = \left|\psi(\vec{r},t)\right|^2$, with: $\int_v\mathcal{P}(\vec{r},t)\,d\tau = 1$. Hence, we ***also*** see that the macroscopic electric field ***amplitude*** $\boxed{E_o = \sqrt{\tfrac{1}{\varepsilon_o}\langle n_\gamma(\vec{r},t)\rangle}\,E_\gamma}$ plays a role analogous to the ***probability density amplitude*** in ***quantum mechanics***!

We can calculate the time rate of change of the normalized 3-D photon ***probability density***, $\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)$:

$$\boxed{\frac{\partial \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)}{\partial t} = \frac{2}{A_\perp\cdot\ell}\frac{\partial}{\partial t}\cos^2(kz-\omega t) = +\frac{4\omega}{A_\perp\cdot\ell}\sin(kz-\omega t)\cdot\cos(kz-\omega t)}$$

We can define a normalized 3-D photon ***probability current density*** as: $\boxed{\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t) = c\cdot\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)\hat{z}}$.

We calculate the divergence of the normalized 3-D photon ***probability current density***:

$$\boxed{\begin{aligned}\vec{\nabla}\bullet\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t) &= \vec{\nabla}\bullet\left(c\cdot\mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)\hat{z}\right) = c\cdot\frac{2}{A_\perp\cdot\ell}\vec{\nabla}\bullet\left(\cos^2(kz-\omega t)\hat{z}\right)\\ &= c\cdot\frac{2}{A_\perp\cdot\ell}\left(\frac{\partial}{\partial z}\cos^2(kz-\omega t)\right) = -c\cdot\frac{4k}{A_\perp\cdot\ell}\sin(kz-\omega t)\cdot\cos(kz-\omega t)\end{aligned}}$$

We thus show that the photons in this laser beam obey the ***continuity equation for photons***:

$$\boxed{\frac{\partial \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)}{\partial t} + \vec{\nabla}\bullet\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t) = 0}$$

From above:

$$\boxed{\frac{\partial \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)}{\partial t} = \frac{4\omega}{A_\perp\cdot\ell}\sin(kz-\omega t)\cdot\cos(kz-\omega t)} \text{ and: } \boxed{\vec{\nabla}\bullet\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t) = -c\cdot\frac{4k}{A_\perp\cdot\ell}\sin(kz-\omega t)\cdot\cos(kz-\omega t)}$$

However, in free space, we have: $\boxed{\omega = ck}$.

Hence: $$\boxed{\frac{\partial \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)}{\partial t} = \frac{4ck}{A_\perp\cdot\ell}\sin(kz-\omega t)\cdot\cos(kz-\omega t) = -\vec{\nabla}\bullet\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t)}$$

Thus: $$\boxed{\frac{\partial \mathcal{P}_\gamma^{3\text{-}D}(\vec{r},t)}{\partial t} + \vec{\nabla}\bullet\vec{\mathcal{J}}_\gamma^{3\text{-}D}(\vec{r},t) = 0}$$

i.e. microscopically, photons neither disappear, nor are they created in propagating as this laser beam!

