LECTURE NOTES 4

A Mini-Review of "Generic" Wave Phenomena:

Waves in 1-Dimension

What is a 1-dimensional (1-D) wave?

- A (classical) 1-D *traveling* wave is a quasi-coherent collective phenomenon – that of a "disturbance" associated with a localized excess of energy (above ambient thermal / background energy) in a macroscopic, continuous medium, which propagates (*i.e.* translates in 1-D space) as time progresses.

  In a dissipationless (*i.e.* lossless), non-dispersive medium, the shape / profile /envelope (*i.e.* crests and troughs) of the wave propagates with constant velocity.

- In a dispersive medium, a traveling wave consisting of a linear combination of several / many different frequencies, the various frequency components of the wave will each propagate with different speed, thus the overall shape of the wave will change with time in a dispersive medium.

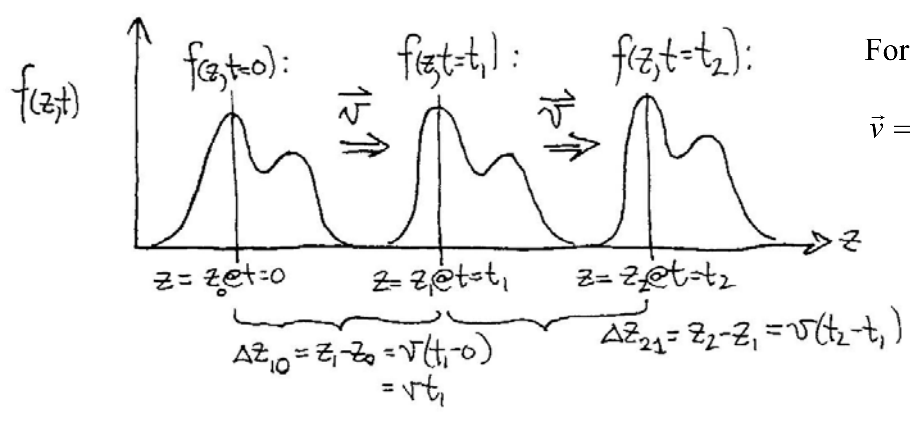
- In a dissipative (but non-dispersive) medium the wave amplitude(s) will decrease with time (or equivalently propagation distance), Often, real media are not only dissipative, but also dispersive, thus dissipation in a medium may also be frequency dependent.

- Classical media can be both dispersive and dissipative – one or both or neither.

- A (classical) 1-D *standing* wave = a linear superposition of two counter-propagating traveling waves (*e.g.* a standing wave on a stringed instrument.)

- Standing waves do not propagate in space, although they can / do evolve in time (due to dispersion, dissipation and other processes).

Let us consider a 1-dimensional transverse *traveling* wave, *e.g.* on a taught / tight string:



For $|\vec{v}|$ = constant

$$\vec{v} = v\hat{z} = \frac{\Delta z}{\Delta t}\hat{z} = \left(\frac{z_2 - z_1}{t_2 - t_1}\right)\hat{z}$$

For non-dissipative, non-dispersive media, the transverse shape { = transverse displacement from equilibrium shape of string} of a traveling wave $f(z,t)$ is invariant under space translations and time translations.

Mathematically, this means that:

$$\boxed{f\left(z,t_2\right)=f\left(z-v\left(t_2-t_1\right),t_1\right)=\underbrace{f\left(z-v\left(t_2-0\right),0\right)}_{=f\left(z-vt_2,0\right)}=f\left(z-vt_2,0\right)}$$

$$\boxed{f\left(z,t_1\right)=\underbrace{f\left(z-v\left(t_1-0\right),0\right)}_{=f\left(z-vt_1,0\right)}=f\left(z-vt_1,0\right)}$$

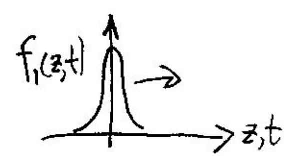
In general, $f\left(z,t\right)$ for a 1-D traveling wave at position z and time t = a <u>special</u> type of function $g(z-vt)$. The function $f\left(z,t\right)$ that mathematically describes the 1-D wave motion / wave propagation is <u>*not*</u> arbitrary / will-nilly – it is a very special / very specific <u>causal</u> relationship of the location(s) of the 1-D wave in <u>*both*</u> space and time: $f\left(z,t\right)$ is restricted to the <u>causal</u> subset of functions $g(z-vt)$, *i.e.* classical traveling waves <u>obey</u> causality.

This means that the <u>argument</u> of the causal g-functions, $(z-vt)$ = constant, independent of space (z) and time (t) , *i.e.* the argument $(z - vt)$ = constant $\forall$ (for all) allowed (z, t) .

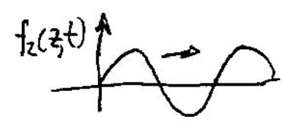
The following are some examples of mathematically acceptable / causal functions describing dissipationless, dipersionless classical traveling wave in 1-D (*n.b.* here, A and b = constants, *e.g.* independent of frequency):

For 1-D traveling wave propagation: $(z-vt)$ = constant

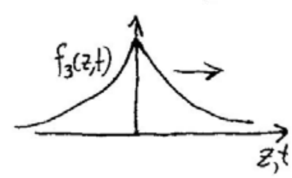
$$f_1\left(z,t\right)=Ae^{-b(z-vt)^2}$$ (Gaussian wave)



$$f_2\left(z,t\right)=A\underbrace{\sin}_{\text{or cos}}\left(b\left(z-vt\right)\right)$$ (Sine/Cosine Wave)



$$f_3\left(z,t\right)=\frac{A}{b\left(z-vt\right)^2+1}$$ ("Cusp" Wave)



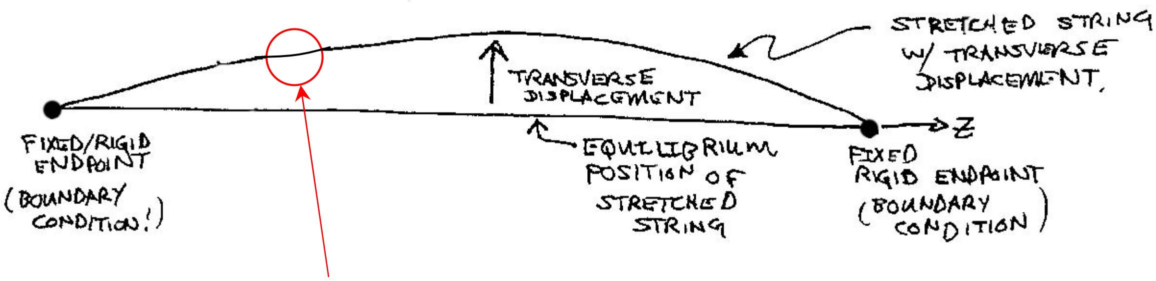
Some examples of mathematically <u>unacceptable</u> / <u>a-causal</u> 1-D "wave" functions:

$$f_4\left(z,t\right)=Ae^{-b(z^2+vt)}$$ (*n.b.* here again, A and b = constants)

$$f_5\left(z,t\right)=A\sin\left(bz\right)\cos\left(bvt\right)^3$$

Example: 1-D transverse mechanical traveling waves on a string obey Newton's 2nd Law: $\boxed{\vec{F} = m\vec{a}}$

If a stretched string is transversely displaced from its equilibrium position, as shown in the figure below, the transverse *displacement* (in SI units: meters) of the string from its equilibrium position at a point z along the string at a given instant in time t is mathematically described by the function $f(z,t)$.



Let us investigate / analyze the forces acting on small/infinitesimal segment of the string:

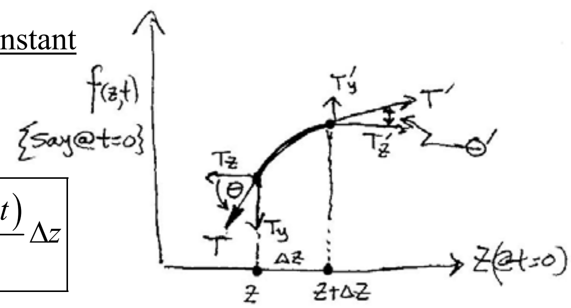
As can be seen from the figure below, at any given instant in time, t the *net* transverse force $\Delta F_y(z,t)$ acting on an infinitesimal string segment (of length Δz) between z and ($z+\Delta z$) on a string with tension T (Newtons) is: $\boxed{\Delta F_y(z,t) = T_y'(z+\Delta z,t) - T_y(z,t) = T\sin\theta' - T\sin\theta}$

For small transverse displacements: String tension T = constant

$\boxed{\sin\theta \approx \theta \approx \tan\theta}$

$\therefore \boxed{\Delta F_y(z,t) \cong T(\tan\theta' - \tan\theta)}$ *n.b.* $\tan\theta$ = slope

$$\boxed{\Delta F_y(z,t) = T\left(\frac{\partial f(z,t)}{\partial z}\bigg|_{z+\Delta z} - \frac{\partial f(z,t)}{\partial z}\bigg|_{z}\right) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$$



Thus: $\boxed{\Delta F_y(z,t) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$ for small transverse displacements of the string from its equilibrium (*i.e.* zero excess energy) configuration.

If the mass per unit length of the string is: $\boxed{\mu = m_{string}/L}$ (kg/m) (where the total string mass = m_{string} and the total length of the string = L) then Newton's 2nd Law: $\boxed{\Delta F_y(z,t) = ma_y(z,t)}$ where $a_y(z,t)$ = transverse acceleration (in the $\hat{y}$-direction) at the point z at time t is

$$\boxed{a_y(z,t) = \frac{\partial^2 f(z,t)}{\partial t^2}}$$

The string segment of infinitesimal length Δz has mass $\boxed{m = \mu\Delta z}$ $\left(=\left[m_{string}/L\right]\Delta z\right)$

$$\therefore \boxed{\Delta F_y(z,t) = ma_y(z,t) \simeq \mu\Delta z\frac{\partial^2 f(z,t)}{\partial t^2}}$$

But: $\boxed{\Delta F_y(z,t) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$ from the transverse force imbalance relation (above)

$\therefore$ for small displacements: $\boxed{T\dfrac{\partial^2 f(z,t)}{\partial z^2}\cancel{\Delta z} = \mu\cancel{\Delta z}\dfrac{\partial^2 f(z,t)}{\partial t^2}}$ or: $\boxed{\dfrac{\partial^2 f(z,t)}{\partial z^2} = \left(\dfrac{\mu}{T}\right)\dfrac{\partial^2 f(z,t)}{\partial t^2}}$

Note that, from dimensional analysis: $\boxed{\dfrac{T}{\mu} = \dfrac{\text{Force, Newtons}}{\text{mass}/\text{unit length}} = \dfrac{\cancel{\text{kg}}\text{-m/s}^2}{\cancel{\text{kg}}/\text{m}} = \dfrac{\text{m}^2}{\text{s}^2} = \left(\dfrac{\text{m}}{\text{s}}\right)^2}$

From conservation of energy associated with traveling waves propagating on a taught string, it can be shown that $\boxed{v = \sqrt{T/\mu}}$ = longitudinal speed of propagation of transverse waves on a string. For dispersionless media, note that v = constant ≠ fcn(frequency, f).

Thus we arrive at the 1-D wave equation for transverse traveling waves propagating *e.g.* on a taught/stretched string:

$$\boxed{\frac{\partial^2 f(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f(z,t)}{\partial t^2}} \quad \text{with:} \quad \boxed{v = \sqrt{T/\mu}}$$

We can re-arrange the wave equation into its more traditional form: $\boxed{\dfrac{\partial^2 f(z,t)}{\partial z^2} - \dfrac{1}{v^2}\dfrac{\partial^2 f(z,t)}{\partial t^2} = 0}$

Thus, we see that the wave equation is a 2nd order linear and homogeneous differential equation.

Solutions of wave equation are all functions $f(z,t)$ of the form where the longitudinal position z and time t are causally connected to each other by $(z - vt)$ = constant, *i.e.* all of the functions $g(z - vt)$ = constant.

The requirement / restriction that $f(z,t) = g(z - vt)$ explicitly means that: $u \equiv (z - vt) =$ argument of the g-functions, *i.e.* that:

$$\boxed{\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}} \quad \text{because} \quad \boxed{\frac{\partial u}{\partial z} = \frac{\partial(z - vt)}{\partial z} = 1}$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}} \quad \text{because} \quad \boxed{\frac{\partial u}{\partial t} = \frac{\partial(z - vt)}{\partial t} = -v}$$

And thus:

1.) $\boxed{\dfrac{\partial^2 f}{\partial z^2} = \dfrac{\partial}{\partial z}\left(\dfrac{dg}{du}\right) = \dfrac{d^2 g}{du^2}\dfrac{\partial u}{\partial z} = \dfrac{d^2 g}{du^2}}$ 2.) $\boxed{\dfrac{\partial^2 f}{\partial t^2} = -v\dfrac{\partial}{\partial t}\left(\dfrac{dg}{du}\right) = -v\dfrac{d^2 g}{du^2}\dfrac{\partial u}{\partial t} = +v^2\dfrac{d^2 g}{du^2}}$

$\therefore$ $\boxed{{}^{1)}\dfrac{d^2 g}{du^2} = {}^{2)}\dfrac{d^2 g}{du^2}} \Rightarrow \boxed{\dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}}$ 1-D Wave Equation

Thus, $g(u)$ can be any differentiable function satisfying $\boxed{u = (z - vt)}$

Note that since the wave equation involves the square of the longitudinal propagation speed v, then another acceptable form of a solution is: $f(z,t) = g(z + vt)$

Thus there are two "generic" possible acceptable solutions:

a.) $f(z,t) = g(z - vt)$ where $(z - vt)$ = constant, thus if t increases $\rightarrow z$ also increases
b.) $f(z,t) = g(z + vt)$ where $(z + vt)$ = constant, thus if t increases $\rightarrow z$ decreases

Physically this means that:

a.) $f(z,t) = g(z - vt)$ represents a 1-D wave propagating in the $+\hat{z}$ direction
b.) $f(z,t) = g(z + vt)$ represents a 1-D wave propagating in the $-\hat{z}$ direction

The Linear Wave Equation & the Linear Superposition Principle

Provided that the initial (simplifying) assumption that the displacement from equilibrium is small, such that $\sin\theta \approx \theta \approx \tan\theta$ is valid, then the principle of linear superposition tells us that:

$$\boxed{f_{TOT}(z,t) = \sum_{i=1}^{n} f_i(z,t) = f_1(z,t) + f_2(z,t) + f_3(z,t) + \ldots}$$

is also a solution of the linear wave equation. Note that in general:

$$\boxed{f_{TOT}(z,t) = \underbrace{\sum_{i=1}^{n} g_i(z - vt)} + \underbrace{\sum_{j=1}^{m} h_j(z + vt)}}$$

Traveling waves propagating in the $+\hat{z}$ direction (*i.e.* to the right)

Traveling waves propagating in the $-\hat{z}$ direction (*i.e.* to the left)

Most generally, *i.e.* $f_{TOT}(z,t) =$ linear superposition of left & right-moving / propagating waves.

Standing Waves:

A standing wave (one which is *stationary* in space) is formed by superposing two identical traveling waves, except that one is a is left-going traveling wave and the other is right-going:

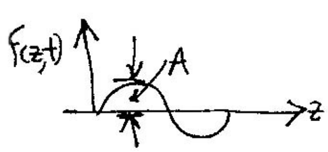
$$\boxed{f_{TOT}(z,t) = g(z - vt) + g(z + vt)} \text{ where } e.g.$$

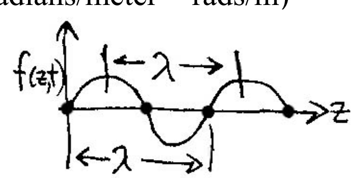
$$\boxed{g(z - vt) = A\sin(k[z - vt]) = A\sin(kz - \omega t)} \text{ and: } \boxed{g(z + vt) = A\sin(k[z + vt]) = A\sin(kz + \omega t)}$$

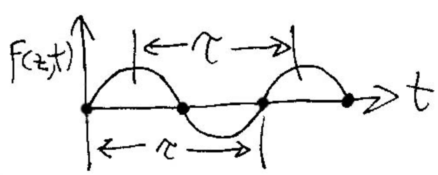
Definition of nomencature used in wave propagation:

A = amplitude (= absolute value of maximum displacement from equilibrium) (m)
$v = f\lambda$ = longitudinal speed of propagation of wave (m/s)
f = frequency of vibration of wave (cycles per sec = *c.p.s.* = *Hz* (*Hertz*))
τ = period of wave $= 1/f$ (seconds, per cycle of oscillation)
λ = wavelength (m) = spatial oscillation distance
$\omega \equiv 2\pi f$ = "angular" frequency (radians/sec = rad/sec)
$k \equiv 2\pi/\lambda$ = wavenumber (radians/meter = rads/m)

$$\boxed{v = f\lambda = \frac{2\pi f}{2\pi/\lambda} = \omega/k}$$







Returning to the discussion of standing waves as a linear superposition of a left-going and a right-going traveling wave:

$$\boxed{f_{TOT}(z,t) = \overbrace{g(z-vt)}^{right} + \overbrace{g(z+vt)}^{left}}$$

<u>where</u>: $\boxed{g(z-vt) = A\sin(kz-\omega t)}$ and: $\boxed{g(z+vt) = A\sin(kz+\omega t)}$

<u>then</u>: $\boxed{f_{TOT}(z,t) = A\sin(kz-\omega t) + A\sin(kz+\omega t)}$

<u>now</u>: $\boxed{\sin(a\pm b) = \sin a\cos b \pm \cos a \sin b}$

$\therefore$ $\boxed{f_{TOT}(z,t) = A\left\{\sin kz\cos\omega t - \cancel{\cos kz \sin\omega t} + \sin kz \cos\omega t + \cancel{\cos kz\sin\omega t}\right\} \equiv 2A\sin kz\cos\omega t}$

<u>Thus</u>: $\boxed{f_{TOT}(z,t) = A'\sin kz\cos\omega t = 2A\sin kz\cos\omega t}$ *i.e.* define $\boxed{A' \equiv 2A}$

= 1-D standing wave (*i.e.* does <u>not</u> propagate/move in longitudinal $\pm\hat{z}$ -direction)

<u>Explicit check</u>: Does $f_{TOT}(z,t) = A'\sin kz\cos\omega t$ obey the wave equation?

i.e. does $\dfrac{\partial^2 f_{TOT}(z,t)}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f_{TOT}(z,t)}{\partial t^2}$??

$$\frac{\partial f(z,t)}{\partial z} = kA'\cos kz\cos\omega t \qquad \frac{\partial f(z,t)}{\partial t} = -\omega A'\sin kz\sin\omega t$$

$$\frac{\partial^2 f(z,t)}{\partial z^2} = -k^2 A'\sin kz\cos\omega t \qquad \frac{\partial^2 f(z,t)}{\partial t^2} = -\omega^2 A'\sin kz\cos\omega t$$

$$-k^2 \cancel{A'}\ \cancel{\sin kz\ \cos\omega t} = \frac{1}{v^2}\left(-\omega^2 \cancel{A'}\ \cancel{\sin kz\ \cos\omega t}\right)$$

$$-k^2 = -\omega^2/v^2 \quad \Rightarrow \quad v^2 = \omega^2/k^2 = \left(\omega/k\right)^2$$

$$\therefore \quad v = \omega/k \qquad \text{Yes!}$$

<u>The Sinusoidal Traveling Wave</u>:

The most familiar wave: $f(z,t) = A\cos\left(k(z-vt)+\delta\right)$

Amplitude

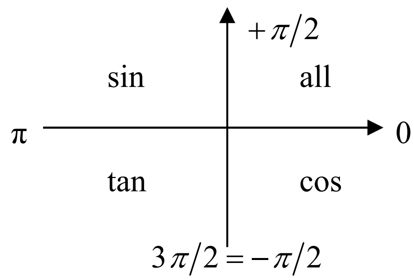
wave number $k = 2\pi/\lambda$

speed of propagation

phase (radians) = constant (usually phase is defined between 0 and 2π)

<u>Note</u>: $\cos\left(k(z-vt) \pm \overbrace{\pi/2}^{=\delta}\right)$ but: $\cos(a\pm b) = \cos a\cos b \mp \sin a \sin b$

$$= \cos\left(k(z-vt)\right)\cos(\pm\pi/2) \mp \sin\left(k(z-vt)\right)\sin(\pi/2)$$



<u>But</u>: $\sin(+\pi/2) = +1$ and $\sin(-\pi/2) = -1$

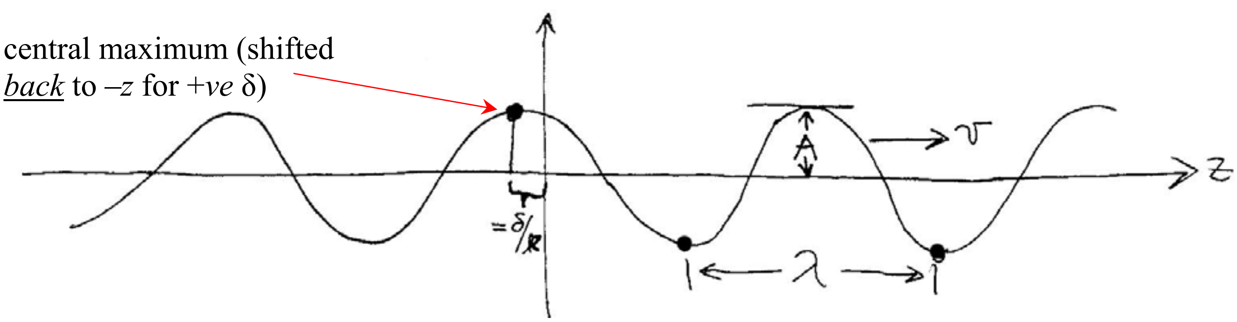
$\therefore \boxed{\cos\left(k\left(z - vt\right) \pm \pi/2\right) = \mp \sin\left(k\left(z - vt\right)\right)}$

<u>Note also</u>: $f_{TOT}(z, t) = g(z - vt) + g(z + vt)$ ← Two waves in phase ($\delta = 0$) with each other

$f_{TOT}(z, t) = g(z - vt) - g(z + vt)$ ← 180° (= π radians) out-of-phase (δ = 180° = π radians)

Thus the function $f(z, t) = A\cos(k(z - vt) + \delta)$ at $t = 0$ appears as shown in the following figure:

$f(z, t = 0)$ = snapshot of wave at $t = 0$:



$$\delta/k = \frac{\delta}{\left(2\pi/\lambda\right)} = \left(\frac{\delta}{2\pi}\right)\lambda \qquad \left(\frac{\delta}{2\pi}\right) = \text{fractional phase}$$

Note the various alternate/equivalent mathematical forms describing the same traveling wave:

$$\begin{aligned}
f(z, t) &= A\cos(k(z-vt) + \delta) & \quad v &= \lambda f = \omega/k \quad \text{(m/s)}\\
&= A\cos\left(\left(kz - kvt\right) + \delta\right) & \omega &= 2\pi f \quad \text{(rads/s)}\\
&= A\cos\left(\left(kz - \omega t\right) + \delta\right) & k &= 2\pi/\lambda \quad \text{(1/m)}\\
&= A\cos\left(\left(\frac{2\pi z}{\lambda} + 2\pi f t\right) + \delta\right) & \omega &= kv\\
&= A\cos\left(2\pi\left(\frac{z}{\lambda} - ft\right) + \delta\right) & &\\
&= A\cos\left[2\pi\left\{\left(\frac{z}{\lambda} - ft\right) + \frac{\delta}{2\pi}\right\}\right] & &
\end{aligned}$$

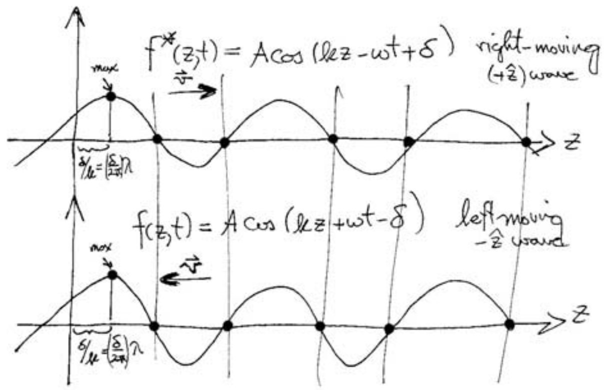
Note that because cos (x) is an even function of x, *i.e.* cos ($-x$) = cos (x)

Then: $f(z,t) = A\cos(kz+\omega t-\delta)$ = left-moving wave ($-\hat{z}$ direction)

$$= A\cos(-kz-\omega t+\delta)$$

But: $f^*(z,t) = A\cos(kz-\omega t+\delta)$ = right-moving wave ($+\hat{z}$ direction)

→ Switching the sign of k produces a wave with the same amplitude, phase, frequency and wavelength, but one which is traveling in the opposite direction.



Complex Notation:

Euler's Formula:

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ $\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

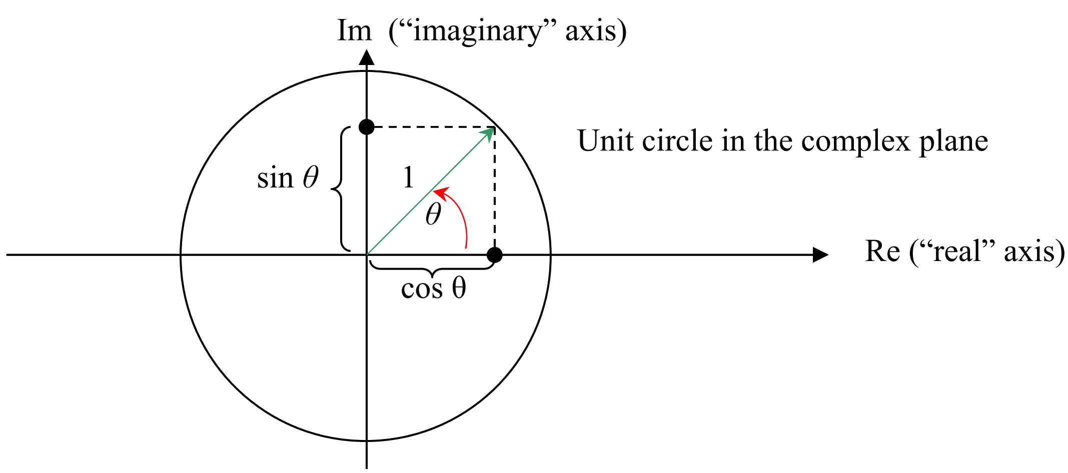
$\boxed{\cos\theta = \tfrac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)}$ $\boxed{\sin\theta = \tfrac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)}$

$\boxed{i \equiv \sqrt{-1}}$ $\boxed{i^* = -i = -\sqrt{-1}}$ $\boxed{i^* i = i i^* = +1}$

The magnitude of $e^{i\theta}$ is defined as $\left|e^{i\theta}\right|$:

$$\boxed{\left|e^{i\theta}\right| \equiv \sqrt{e^{i\theta}e^{-i\theta}} = 1 = \sqrt{(\cos\theta+i\sin\theta)*(\cos\theta-i\sin\theta)} = \sqrt{\cos^2\theta+\sin^2\theta}}$$

Projections of a complex unit vector $e^{i\theta} = \cos\theta + i\sin\theta$ in the complex plane:



Real/physical amplitudes: $\boxed{\cos\omega t = \tfrac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)}$ and/or: $\boxed{\cos kz = \tfrac{1}{2}\left(e^{ikz}+e^{-ikz}\right)}$ Think about this!

We will use the tilde symbol (~) over/above a physical variable to denote its <u>complex</u> nature:

Complex #: $\boxed{\tilde{z} = x + iy}$ $\boxed{\mathrm{Re}(\tilde{z}) = x}$ $\boxed{\mathrm{Im}(\tilde{z}) = y}$

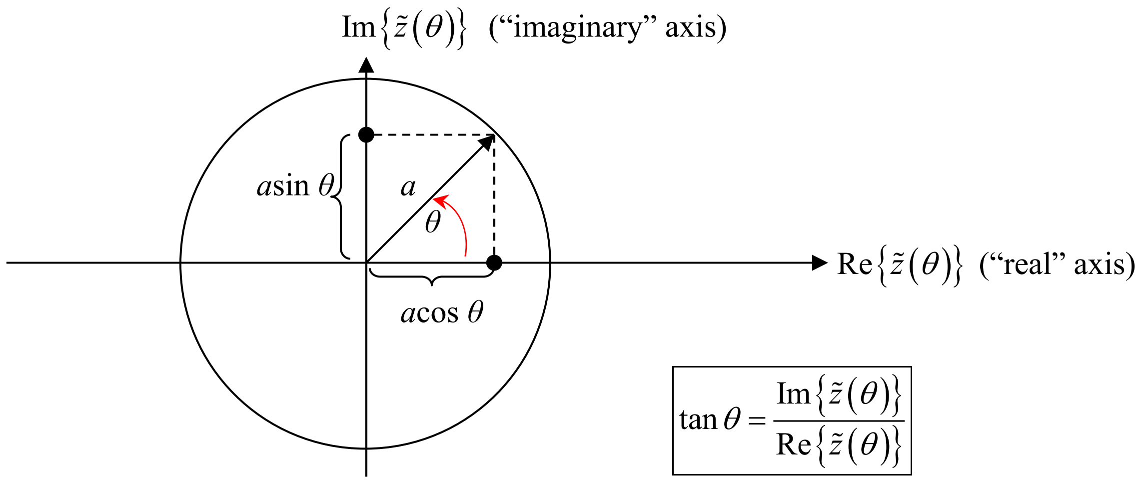
Complex conjugate ($i \to i^* = -i$): $\boxed{\tilde{z}^* = (x+iy)^* = x - iy}$ $\boxed{\mathrm{Re}(\tilde{z}^*) = x}$ $\boxed{\mathrm{Im}(\tilde{z}^*) = -y}$

<u>Suppose</u>: $\boxed{\tilde{z}(\theta) = ae^{i\theta} = \text{complex \#}}$ where $a = \text{real constant}$

$$\boxed{= a(\cos\theta + i\sin\theta) = a\cos\theta + ia\sin\theta}$$

The <u>magnitude</u> of $|\tilde{z}(\theta)|$: $\boxed{|\tilde{z}(\theta)| = a}$ $\boxed{|\tilde{z}(\theta)| = \sqrt{|\mathrm{Re}\{\tilde{z}(\theta)\}|^2 + |\mathrm{Im}\{\tilde{z}(\theta)\}|^2}}$

$\boxed{\mathrm{Re}\{\tilde{z}(\theta)\} = a\cos\theta}$ and $\boxed{\mathrm{Im}\{\tilde{z}(\theta)\} = a\sin\theta}$



$$\boxed{\tan\theta = \frac{\mathrm{Im}\{\tilde{z}(\theta)\}}{\mathrm{Re}\{\tilde{z}(\theta)\}}}$$

$$\boxed{|\tilde{z}(\theta)| = |ae^{i\theta}| = a|e^{i\theta}| = a = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a\sqrt{\cos^2\theta + \sin^2\theta} = a}$$

For a purely <u>real</u> wave function $f(z,t)$: $\boxed{f(z,t) = A\cos(kz - \omega t + \delta)}$ we can equivalently write this using complex notation as: $\boxed{f(z,t) = A\cos(kz - \omega t + \delta) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]}$

For a <u>complex</u> wave function $\tilde{f}(z,t)$: $\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} = \tilde{A}\cos(kz-\omega t) + i\tilde{A}\sin(kz-\omega t)}$

with <u>complex</u> amplitude $\tilde{A}$: $\boxed{\tilde{A} \equiv Ae^{i\delta}}$

A = <u>Real</u> number

<u>Then</u>: $\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} = Ae^{i\delta}e^{i(kz-\omega t)} = Ae^{i[(kz-\omega t)+\delta]} = A\{\cos(kz-\omega t+\delta) + i\sin(kz-\omega t+\delta)\}}$

Griffiths Example 9.1 - Linear Superposition of Two Sinusoidal Waves:

Suppose that we have a situation where two real sinusoidal traveling waves $f_1(z,t) = A_1\cos(kz-\omega t+\delta_1)$ and $f_2(z,t) = A_2\cos(kz-\omega t+\delta_2)$ are simultaneously present at the same point *z* that have the same frequency *f* (and thus same wavelength λ, angular frequency ω, and wavenumber *k*) but have different amplitudes A_1, A_2 and (absolute) phases δ_1, δ_2 {defined relative to a common chosen origin of time, $t = 0$}.

We *can* simply add the two waves together: $\boxed{f_3(z,t) = f_1(z,t)+f_2(z,t)}$ however, this approach will involve some rather tedious algebra and use of trigonometric identities to obtain $f_3(z,t)$.

A much easier method is to carry this out using complex notation:

$$\boxed{f_3(z,t) = f_1(z,t)+f_2(z,t) = \mathrm{Re}\left\{\tilde{f}_1(z,t)\right\}+\mathrm{Re}\left\{\tilde{f}_2(z,t)\right\} = \mathrm{Re}\left\{\tilde{f}_1(z,t)+\tilde{f}_2(z,t)\right\} = \mathrm{Re}\left\{\tilde{f}_3(z,t)\right\}}$$

with: $\boxed{\tilde{f}_3(z,t) = \tilde{f}_1(z,t)+\tilde{f}_2(z,t)}$ $\boxed{\tilde{f}_1(z,t) \equiv \tilde{A}_1 e^{i(kz-\omega t)}}$ $\boxed{\tilde{f}_2(z,t) \equiv \tilde{A}_2 e^{i(kz-\omega t)}}$ $\boxed{\tilde{f}_3(z,t) \equiv \tilde{A}_3 e^{i(kz-\omega t)}}$

$$\boxed{\tilde{f}_3(z,t) \equiv \tilde{A}_3 e^{i(kz-\omega t)} = \tilde{f}_1(z,t)+\tilde{f}_2(z,t) = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)}}$$

thus: $\boxed{\tilde{A}_3 e^{i(kz-\omega t)} = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)}}$ $\Rightarrow$ $\boxed{\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2}$ or: $\boxed{A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}}$

writing this last relation out in its explicit complex form:

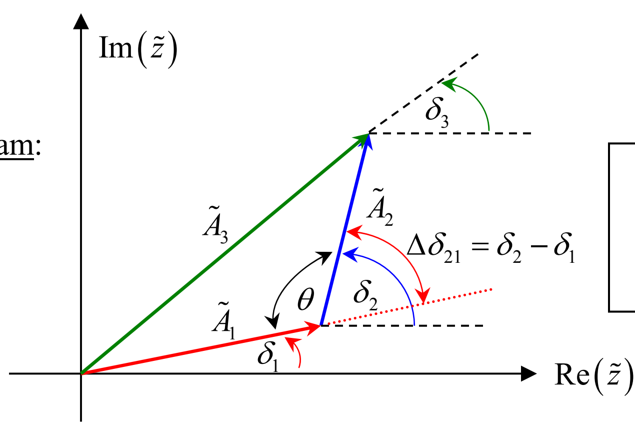
$$\boxed{A_3\cos\delta_3 + iA_3\sin\delta_3 = A_1\cos\delta_1 + iA_1\sin\delta_1 + A_2\cos\delta_2 + iA_2\sin\delta_2}$$

Thus we see that:

$$\boxed{\mathrm{Re}\left(\tilde{A}_3\right) = A_3\cos\delta_3 = A_1\cos\delta_1 + A_2\cos\delta_2} \qquad \boxed{\mathrm{Im}\left(\tilde{A}_3\right) = A_3\sin\delta_3 = A_1\sin\delta_1 + A_2\sin\delta_2}$$

We can either use the so-called *Phasor Diagram* in the complex plane to obtain $\mathrm{Re}\left(\tilde{A}_3\right)$ and $\mathrm{Im}\left(\tilde{A}_3\right)$,or wade through the tedious trigonometry and algebra.

Phasor Diagram:



n.b. phases are always defined relative to the Real axis

$$\theta = \pi - \Delta\delta_{21} = \pi - (\delta_2 = \delta_1)$$

The use of the phasor diagram does not allow us to evade the use of algebra and trigonometry…

What we are essentially doing here is nothing more than adding two 2-dimensional vectors together, *i.e.* $\boxed{\vec{C}=\vec{A}+\vec{B}}$ where $\boxed{\vec{A}\equiv a_x\hat{x}+a_y\hat{y}=a\cos\theta_1\hat{x}+a\sin\theta_1\hat{y}}$ $\boxed{\vec{B}\equiv b_x\hat{x}+b_y\hat{y}=b\cos\theta_2\hat{x}+b\sin\theta_2\hat{y}}$ and $\boxed{\vec{C}\equiv c_x\hat{x}+c_y\hat{y}=c\cos\theta_3\hat{x}+c\sin\theta_3\hat{y}}$

Then: $\boxed{c_x=c\cos\theta_3=a_x+b_x=a\cos\theta_1+b\cos\theta_2}$ and $\boxed{c_y=c\sin\theta_3=a_y+b_y=a\sin\theta_1+b\sin\theta_2}$.

The magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are $\boxed{\left|\vec{A}\right|=a=\sqrt{a_x^2+a_y^2}}$, $\boxed{\left|\vec{B}\right|=b=\sqrt{b_x^2+b_y^2}}$ and $\boxed{\left|\vec{C}\right|=c=\sqrt{c_x^2+c_y^2}}$

The phase angles are: $\boxed{\delta_1=\tan^{-1}\left(a_y/a_x\right)}$, $\boxed{\delta_2=\tan^{-1}\left(b_y/b_x\right)}$ and $\boxed{\delta_3=\tan^{-1}\left(c_y/c_x\right)}$.

Thus, for the addition of two complex amplitudes, we see that:

$$\begin{aligned}\left|\tilde{A}_3\right|&=\sqrt{\tilde{A}_3\bullet\tilde{A}_3^*}=\sqrt{\left(\tilde{A}_1+\tilde{A}_2\right)\bullet\left(\tilde{A}_1+\tilde{A}_2\right)^*}=\sqrt{\left(A_1e^{i\delta_1}+A_2e^{i\delta_2}\right)\left(A_1e^{-i\delta_1}+A_2e^{-i\delta_2}\right)}\\&=\sqrt{A_1^2+A_2^2+A_1A_2\left(e^{i\delta_1}e^{-i\delta_2}+e^{-i\delta_1}e^{i\delta_2}\right)}=\sqrt{A_1^2+A_2^2+A_1A_2\left[2\cos\left(\delta_2-\delta_1\right)\right]}\\&=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\delta_{12}}\qquad\text{where }\ \Delta\delta_{21}\equiv\left(\delta_2-\delta_1\right)\end{aligned}$$

Then: $\boxed{\left|\tilde{A}_3\right|=A_3=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\delta_{12}}}$ where: $\boxed{\Delta\delta_{21}\equiv\left(\delta_2-\delta_1\right)}$

n.b. this is simply the law of cosines!!!

$\boxed{\left|\tilde{A}_3\right|=A_3=\sqrt{A_1^2+A_2^2-2A_1A_2\cos\theta}}$ where: $\boxed{\theta=\pi-\Delta\delta_{21}=\pi-\left(\delta_2=\delta_1\right)}$

The {absolute} phase angle δ_3 can be obtained from:

$$\boxed{\tan\delta_3=\frac{A_3\sin\delta_3}{A_3\cos\delta_3}=\frac{\sin\delta_3}{\cos\delta_3}=\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}}\ i.e.\ \boxed{\delta_3=\tan^{-1}\left\{\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}\right\}}\ \text{radians}$$

Why Use Complex Notation?

$$\boxed{\tilde{f}\left(z,t\right)=\tilde{A}e^{i(kz-\omega t)}=Ae^{i(kz-\omega t+\delta)}}$$

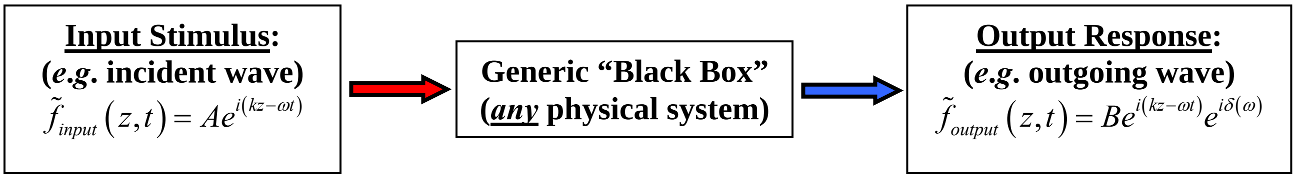
Whenever we have two or more waves, if $\exists$ (there exists) a definite phase relation (defined at some specific origin of time $t=t_o$) between them (*i.e.* the two or more waves are coherent) then the waves will ***interfere*** with each other at a given point in space, z.

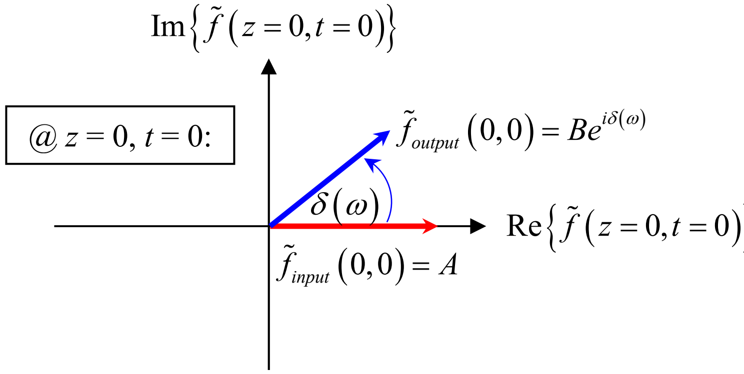
Interference phenomena occurs at the ***amplitude*** level – *i.e.* wave amplitudes interfere for waves that are coherent / have a (well defined) definite phase-relation.

Using complex notation, phase information can be explicitly {and easily} obtained.

There is ***physical*** meaning for real vs. imaginary components of a complex quantity.

Note also that the use of complex notation allows us to explicitly describe properly / mathematically the phase-shifts that can / do occur in the response of a system (a "black box") to an input stimulus / input signal:





Classical Systems – Interference of Wave Amplitudes

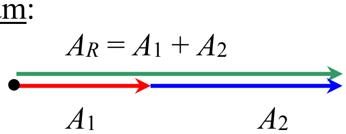
Consider two traveling waves interfering with each other in a non-dispersive medium with different frequencies and amplitudes. For a non-dispersive medium, this means that $\boxed{v = f_1\lambda_1 = f_2\lambda_2 = \omega_1/k_1 = \omega_2/k_2}$ with angular frequencies of $\boxed{\omega_1 = 2\pi f_1}$, $\boxed{\omega_2 = 2\pi\lambda_2}$ and wavenumbers $\boxed{k_1 = 2\pi/\lambda_1}$, $\boxed{k_2 = 2\pi/\lambda_2}$.

Then: $\boxed{\tilde{f}_{TOT}(z,t) = \tilde{f}_1(z,t) + \tilde{f}_2(z,t) = A_1 e^{i(k_1 z - \omega_1 t + \delta_1)} + A_2 e^{i(k_2 z - \omega_2 t + \delta_2)}}$

Easy cases of phase relations between the two waves:

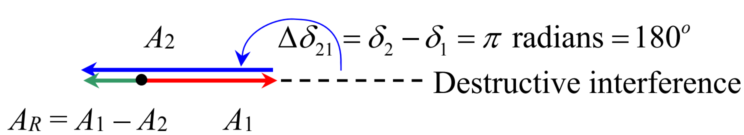
1.) $\delta_1 = \delta_2$ (in phase). Then $\Delta\delta_{21} = \delta_2 - \delta_1 = 0$ radians $= 0^\circ$

Phasor diagram:

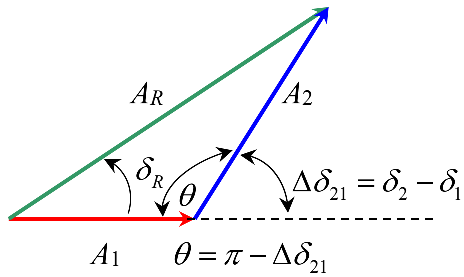


Constructive interference

2.) $\delta_2 = \delta_1 + \pi$ (180° out of phase). Then: $\Delta\delta_{21} = \delta_2 - \delta_1 = \pi$ radians $= 180^o$



3.) <u>The General Case</u>:



From the diagram, we see that the magnitude of the resultant (*i.e.* net or total) amplitude A_R is:

$$\boxed{\begin{aligned} A_R = \left|\tilde{f}_{TOT}(z,t)\right| &= \sqrt{A_1^2 + A_2^2 - 2A_1A_2\cos(\theta)} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\Delta\delta_{21})} \\ &= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_2 - \delta_1)} \end{aligned}}$$

n.b. this is simply the law of cosines!!!

$$\boxed{A_R = \sqrt{\tilde{f}_{TOT}(z,t) * \underbrace{\tilde{f}^{*}_{TOT}(z,t)}}}$$

Complex conjugate of $\tilde{f}_{TOT}(z,t)$

<u>Thus if</u>: $\boxed{\tilde{f}_1(z,t) = A_1 e^{i(kz-\omega t+\delta_1)}}$ and $\boxed{\tilde{f}_2(z,t) = A_2 e^{i(kz-\omega t+\delta_2)}}$

<u>Then</u>: $\boxed{\tilde{f}_{TOT}(z,t) = \tilde{f}_1(z,t) + \tilde{f}_2(z,t) = A_R e^{i(kz-\omega t+\delta_R)}}$

<u>Where</u>: $\boxed{A_R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_2 - \delta_1)} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}}$

n.b. Cosine is an even function of its argument, thus $\boxed{\Delta\delta_{21} = \Delta\delta}$

Then if the two waves have <u>equal</u> amplitudes, *i.e.* $\boxed{A_1 = A_2 = A}$:

1.) If $\delta_1 = \delta_2$ (in phase with each other), then $\Delta\delta = \delta_2 - \delta_1 = 0$ and hence:

$\cos(\delta_2 - \delta_1) = 1 \Rightarrow$ {total} <u>constructive</u> interference.

<u>Resultant Amplitude</u>: $\boxed{A_R = \sqrt{A^2 + A^2 + 2A^2\cos 0} = \sqrt{4A^2} = 2A}$

2.) If $\delta_2 = \delta_1 \pm \pi \Rightarrow \Delta\delta = \delta_2 - \delta_1 = \pm\pi = \pm 180^o$ out of phase with each other and hence:

$\cos(\delta_2 - \delta_1) = \cos(\pm\pi) = -1 \Rightarrow$ {total} <u>destructive</u> interference.

<u>Resultant Amplitude</u>: $\boxed{A_R = \sqrt{A^2 + A^2 + 2A^2\cos\pi} = \sqrt{A^2 + A^2 - 2A^2} = 0}$

Note that classical wave interference effects <u>can/do</u> occur at the amplitude level even if $f_1 \neq f_2$ *e.g.* Sound waves on strings, in air, …

Electronic signals	$f_1 \approx f_2 \rightarrow$ <u>beats</u> phenomena (this is a form of interference)
EM waves	$f_1 \gg f_2 \rightarrow$ <u>modulation</u> phenomena (<u>also</u> a form of interference)
etc.	(or vice versa)

Note also that amplitude interference effects occur in the world of quantum mechanics – *i.e.* matter waves – but is somewhat more complicated – *e.g.* by line width effects and/or uncertainty principle effects. Only identical particles with the exact same quantum numbers (external and internal) can interfere with each other…

Fourier's Theorem – Fourier Transforms

Any arbitrary wave {whose derivatives exist / are well-defined everywhere} can be expressed mathematically as a linear superposition of harmonic (*i.e.* sine/cosine type) waves:

Space Domain (z) ⇔ Wavenumber Domain (k):

Fourier transform of waveform viewed in wavenumber space (k) to waveform in real space (z):

$$\boxed{\tilde{f}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)e^{i(kz-\omega t)}dk} \qquad \text{where: } k=\omega/v$$

Fourier transform of waveform viewed in real space (z) to waveform in wavenumber space (k):

$$\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z,t)e^{-i(kz-\omega t)}dz}$$

The complex wavenumber amplitude $\tilde{A}(k)$ can also be obtained from initial conditions @ $t=0$:

$$\tilde{f}(z,t=0)=\underline{\;xxx\;} \qquad \text{and} \qquad \dot{\tilde{f}}(z,t=0)\equiv\frac{\partial\tilde{f}(z,t=0)}{\partial t}=\underline{\;yyy\;}$$

and use of the inverse Fourier transform: $\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z,t)e^{-i(kz-\omega t)}dz}$

Obtaining: $$\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{f}(z,0)+\frac{i}{\omega}\dot{\tilde{f}}(z,0)\right]e^{-ikz}dz}$$

See Griffiths Problem 9.32

We say that z and k are ***Fourier conjugate variable***s of each other.

Fourier uncertainty principle for (z,k): $\boxed{\Delta z\cdot\Delta k\geq 1}$

σ_z = 1-sigma localization uncertainty in (real) space

σ_k = 1-sigma localization uncertainty in wavenumber space

$\boxed{\sigma_z\cdot\sigma_k\geq 1}$ Equality ($\sigma_z\cdot\sigma_k=1$) achieved ***only*** for Gaussian waveforms, *i.e.* $\frac{1}{\sqrt{2\pi}\cdot\sigma_z}e^{-\alpha z^2/2\sigma_z^2}$

For rigorous mathematical proof of Fourier uncertainty principle, see *e.g.* H.Weyl's book: "Theory of Groups and Quantum Mechanics", Dover, NY (1950), Appendix A.

Time Domain (t) ⇔ {Angular} Frequency Domain (ω):

Fourier transform of waveform viewed in frequency space ($\omega = 2\pi f$) to waveform in time (t):

$$\boxed{\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(\omega) e^{i(kz-\omega t)} d\omega} \qquad \text{where: } k = \omega/v$$

Fourier transform of waveform viewed in time (t) to waveform in frequency space (ω):

$$\boxed{\tilde{A}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(z,t) e^{-i(kz-\omega t)} dt}$$

We say that t and ω are ***Fourier conjugate variable***s of each other.

Fourier uncertainty principle for (t, ω): $\boxed{\Delta\omega \cdot \Delta t \ge 1}$

σ_t = 1-sigma localization uncertainty in time

σ_ω = 1-sigma localization uncertainty in {angular} frequency space

$\boxed{\sigma_t \cdot \sigma_\omega \ge 1}$ Equality ($\sigma_t \cdot \sigma_\omega = 1$) achieved ***only*** for Gaussian waveforms, *i.e.* $\frac{1}{\sqrt{2\pi}\cdot\sigma_t} e^{-\alpha t^2/2\sigma_t^2}$

Wave Intensity, I:

Wave intensities are proportional to $\left(\tilde{A}\tilde{A}^*\right)$. Thus, the total intensity $I \;\alpha \left(\tilde{f}_{TOT}(z,t) \bullet \tilde{f}^*_{TOT}(z,t)\right)$

In above previous example, the individual normalized intensities are: $\boxed{I_1 = A_1^2}$ and $\boxed{I_2 = A_2^2}$.

The corresponding resultant normalized intensity is: $\boxed{I_R = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos(\delta_2 - \delta_1)}$

Then for the special case of $A_1 = A_2 = A$ or equivalently $I_1 = I_2 = I_o$, Then *e.g.* for ($z = 0$, $t = 0$):

1.) $\delta_2 = \delta_1$ then: $\Delta\delta = \delta_2 - \delta_1 = 0$, $\cos(\delta_2 - \delta_1) = +1$: $I_R = 4\, I_o$ constructive interference.

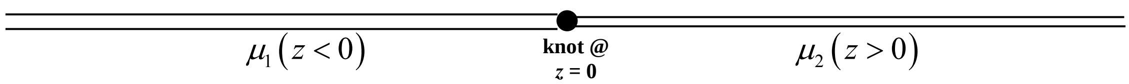
2.) $\delta_2 = \delta_1 \pm \pi$ then: $\Delta\delta = \delta_2 - \delta_1 = \pm\pi$, $\cos(\delta_2 - \delta_1) = -1$: $I_R = 0$ destructive interference.

Let's consider our vibrating string problem again:

Boundary Conditions (End Conditions), Wave Reflection and Wave Transmission

Mechanical wave behavior / wave motion *e.g.* on an ideal string (perfectly compliant – no stiffness) as a function of time and space depends critically on the end conditions / boundary conditions – *i.e.* on how rigidly (or not) the string is attached at its ends.

Or, *e.g.* could have two dispersionless ideal strings tied together in a knot (say @ $z = 0$), but one string has mass per unit length $\mu_1(z<0)$, the second string has mass per unit length $\mu_2(z>0)$. Both strings are stretched – have common tension T – rigidly attached at LHS and RHS ends @ $z = \pm\infty$ (*i.e.* string is infinitely long).



For this latter situation, the longitudinal speed of propagation of waves on a dispersionless ideal string is $v = \sqrt{T/\mu}$ where the string tension T = same in both strings {otherwise $F \neq 0$ in equilibrium – this can't happen, because if $F \neq 0$, then Newton's 2nd Law $F = ma \rightarrow$ *something* accelerates $\rightarrow$ therefore *must* have $F = 0$ in equilibrium}.

Thus, for the 1st string with $\mu_1(z<0)$: $\boxed{v_1 = \sqrt{T/\mu_1}\ \ (z<0)}$

and for the 2nd string with $\mu_2(z>0)$: $\boxed{v_2 = \sqrt{T/\mu_2}\ (z>0)}$

But: $\boxed{\dfrac{v_1}{v_2} = \dfrac{f_1\lambda_1}{f_2\lambda_2} = \dfrac{f\lambda_1}{f\lambda_2} = \dfrac{\lambda_1}{\lambda_2}}$

The frequencies of oscillation associated with a single ideal vibrating string composed of the 2 different strings types of tied together @ $z = 0$ are the same - *i.e.* $f_1 = f_2 = f$, hence their angular frequencies are the same: $\omega_1 = 2\pi f_1 = \omega_2 = 2\pi f_2 = \omega$.

$$\therefore \boxed{\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1/2\pi}{\lambda_2/2\pi} = \frac{2\pi/\lambda_2}{2\pi/\lambda_1} = \frac{k_2}{k_1}}$$

Suppose that an incident harmonic traveling wave propagates in the $+\hat{z}$ direction, from the LHS portion ($z < 0$) of string 1, *i.e.* to the left of the knot @ $z = 0$: $\boxed{\tilde{f}_{inc}(z,t) = \tilde{A}_{inc}e^{i(k_1 z - \omega t)}}$ ($z < 0$).

This harmonic wave is incident on the knot/discontinuity @ $z = 0$.

Because of the mismatch/discontinuity in materials of the ideal string to the left ($z < 0$) and to the right ($z > 0$) of $z = 0$, a portion of incident wave is reflected backwards from the knot / discontinuity, and propagates in $-\hat{z}$ direction along string 1: $\boxed{\tilde{f}_{refl}(z,t) = \tilde{A}_{refl}e^{i(-k_1 z - \omega t)}}$ ($z < 0$).

A portion of the incident wave is transmitted past/through knot @ $z = 0$ and propagates in the $+\hat{z}$ direction along string 2: $\boxed{\tilde{f}_{trans}(z,t) = \tilde{A}_{trans}e^{i(k_2 z - \omega t)}}$ ($z > 0$).

Assuming that both strings are ideal (*i.e.* they are dissipationless), then energy and linear momentum are both conserved in this scattering process at the discontinuity / knot @ $z = 0$.

For simplicity's sake, if the incident wave is an infinitely long sinusoidal wave, the net disturbance = net/total displacement amplitude, using the superposition principle is:

$$\boxed{\tilde{f}_{TOT}^{LHS}(z,t) = \tilde{A}_{inc}e^{i(k_1 z-\omega t)} + \tilde{A}_{refl}e^{i(-k_1 z-\omega t)}} \qquad (z<0)$$
$$\boxed{\tilde{f}_{TOT}^{RHS}(z,t) = \tilde{A}_{trans}e^{i(k_2 z-\omega t)}} \qquad (z>0)$$

However @ $z = 0$: $\tilde{f}_{TOT}(z,t)$ <u>must</u> be continuous {recall that physically, $\tilde{f}_{TOT}(z,t)$ corresponds to the transverse displacement of the string at the space-time point (z,t) }.
Mathematically, this translates to a ***<u>Dirichlet</u>***-type boundary condition @ $z = 0$:
$\boxed{\tilde{f}_{TOT}^{LHS}(0,t) = \tilde{f}_{TOT}^{RHS}(0,t)}$ = transverse displacement = same value on both sides of "point" knot.

If the knot physically has zero (or negligible) mass, then the ***<u>slopes</u>*** of $\tilde{f}_{TOT}(z=0,t)$ must also be the same on both sides of the "point" knot.
Mathematically, this translates to a ***<u>Neumann</u>***-type boundary condition @ $z = 0$:
$\boxed{\left.\frac{\partial \tilde{f}_{TOT}^{LHS}(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial \tilde{f}_{TOT}^{RHS}(z,t)}{\partial z}\right|_{z=0}}$ = transverse slope = same value on both sides of "point" knot.

BC1 @ $z = 0$: The complex value of total amplitude @ $z = 0$: $\boxed{\tilde{f}_{TOT}^{LHS}(z=0,t) = \tilde{f}_{TOT}^{RHS}(z=0,t)}$

BC2 @ $z = 0$: The complex value of amplitude slopes @ $z = 0$: $\boxed{\left.\frac{\partial \tilde{f}_{TOT}^{LHS}(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial \tilde{f}_{TOT}^{RHS}(z,t)}{\partial z}\right|_{z=0}}$

Physically, continuity of the slope implies that there are ***<u>no</u>*** additional forces operative at the knot ($z = 0$).

From BC1 @ $z = 0$: $\boxed{\tilde{A}_{inc} + \tilde{A}_{refl} = \tilde{A}_{trans}}$ *n.b.* We have ***<u>two</u>*** equations…

From BC2 @ $z = 0$: $\boxed{k_1\left(\tilde{A}_{inc} - \tilde{A}_{refl}\right) = k_2\tilde{A}_{trans}}$ and ***<u>three</u>*** unknowns $\left(\tilde{A}_{inc}, \tilde{A}_{refl}, \tilde{A}_{trans}\right)$!

Wavenumbers k_1 and k_2 are assumed to be known (*e.g.* $T = 100\ N$, $f = 100\ Hz$ specified), and $\{\mu_1, \mu_2\}$ specified.

→ Can express $\tilde{A}_{refl} = \left|\tilde{A}_{refl}\right| \cdot e^{i\delta_R} \equiv A_{refl} \cdot e^{i\delta_R}$, $\tilde{A}_{trans} = \left|\tilde{A}_{trans}\right| \cdot e^{i\delta_T} \equiv A_{trns} \cdot e^{i\delta_T}$ in terms of $\tilde{A}_{inc} = \left|\tilde{A}_{inc}\right| \cdot e^{i\delta_I} = A_{inc} \cdot e^{i\delta_I}$ – Solve BC1 and BC2 simultaneously – obtain the following relations:

$$\boxed{\frac{\tilde{f}_{refl}(z=0,t)}{\tilde{f}_{inc}(z=0,t)} = \frac{\tilde{A}_{refl}\cancel{e^{i(-\omega t)}}}{\tilde{A}_{inc}\cancel{e^{i(-\omega t)}}} = \frac{\tilde{A}_{refl}}{\tilde{A}_{inc}} = \frac{A_{refl}\cdot e^{i\delta_R}}{A_{inc}\cdot e^{i\delta_I}} = \left(\frac{k_1-k_2}{k_1+k_2}\right) = \left(\frac{v_2-v_1}{v_1+v_2}\right)} \Leftarrow \text{using } \boxed{\left(\frac{k_2}{k_1}\right) = \left(\frac{v_1}{v_2}\right)}$$

$$\boxed{\frac{\tilde{f}_{trns}(z=0,t)}{\tilde{f}_{inc}(z=0,t)} = \frac{\tilde{A}_{trns}\cancel{e^{i(-\omega t)}}}{\tilde{A}_{inc}\cancel{e^{i(-\omega t)}}} = \frac{\tilde{A}_{trns}}{\tilde{A}_{inc}} = \frac{A_{trns}\cdot e^{i\delta_T}}{A_{inc}\cdot e^{i\delta_I}} = \left(\frac{2k_1}{k_1+k_2}\right) = \left(\frac{2v_2}{v_1+v_2}\right)} \Leftarrow \text{using } \boxed{\left(\frac{k_2}{k_1}\right) = \left(\frac{v_1}{v_2}\right)}$$

Define the so-called ***characteristic*** ***impedance*** Z (in acoustic Ohms) associated with ***longitudinal*** traveling waves propagating on an ***ideal*** string: $\boxed{Z = \rho v\left(kg/m^2\text{-}s \equiv \Omega_{ac}\right)}$ where:

$v\left(m/s\right)$ = ***longitudinal*** speed of propagation of waves on string

$\rho \equiv M/V = M/\left(L\cdot A_\perp\right) = \mu/A_\perp\left(kg/m^3\right)$ = volume mass density of the string

$\mu \equiv M/L\left(kg/m\right)$ = mass per unit length of the string

$A_\perp = \pi r_{string}^2\left(m^2\right)$ = cross-sectional area of the string

For ***this*** problem, if the 2 string segments ($z < 0$) and ($z > 0$) are made up of the ***same*** material, then: $\rho_1\left(z<0\right) = \rho_2\left(z>0\right) \equiv \rho$, and: $\mu_1 = \rho A_{\perp_1}$, $\mu_2 = \rho A_{\perp_2}$ and: $v_1 = \sqrt{T/\mu_1}$, $v_2 = \sqrt{T/\mu_2}$,
and: $Z_1 = \rho v_1 = \rho\sqrt{T/\mu_1} = \sqrt{\mu_1 T}/A_{\perp_1}$, $Z_2 = \rho v_2 = \rho\sqrt{T/\mu_2} = \sqrt{\mu_2 T}/A_{\perp_2}$.

Then the above relations can ***also*** be equivalently written as:

$$\boxed{\frac{\tilde{f}_{refl}\left(z=0,t\right)}{\tilde{f}_{inc}\left(z=0,t\right)} = \frac{\tilde{A}_{refl}}{\tilde{A}_{inc}} = \frac{A_{refl}\cdot e^{i\delta_R}}{A_{inc}\cdot e^{i\delta_I}} = \left(\frac{k_1-k_2}{k_1+k_2}\right) = \left(\frac{v_2-v_1}{v_1+v_2}\right) = \left(\frac{Z_2-Z_1}{Z_1+Z_2}\right)}$$

$$\boxed{\frac{\tilde{f}_{trns}\left(z=0,t\right)}{\tilde{f}_{inc}\left(z=0,t\right)} = \frac{\tilde{A}_{trns}}{\tilde{A}_{inc}} = \frac{A_{trns}\cdot e^{i\delta_T}}{A_{inc}\cdot e^{i\delta_I}} = \left(\frac{2k_1}{k_1+k_2}\right) = \left(\frac{2v_2}{v_1+v_2}\right) = \left(\frac{2Z_2}{Z_1+Z_2}\right)}$$

The Z-terms on the RHS of the above relations make it very clear that an ***impedance mis-match*** to the ***longitudinal flow*** of ***wave energy*** exists @ $z = 0$ if $\mu_1\left(z<0\right) \neq \mu_2\left(z>0\right)$.

We define ***reflection*** and ***transmission*** ***coefficients***: (*n.b.* from conservation of ***energy***…)

$R \equiv I_{refl}/I_{inc}$ = ratio of {***flux*** of reflected energy/ ***flux*** of incident energy}

$T \equiv I_{trns}/I_{inc}$ = ratio of {***flux*** of transmitted energy/ ***flux*** of incident energy}

The linear wave energy density in a 1-D vibrating string is: $\boxed{u_{string} = \tfrac{1}{2}\mu\omega^2\left|\tilde{A}\right|^2\left(Joules/m\right)}$.

The flux of wave energy in a 1-D vibrating string is: $\boxed{I_{string} = v\cdot u_{string} = \tfrac{1}{2}\mu v\omega^2\left|\tilde{A}\right|^2\left(Joules/s = Watts\right)}$.

Thus, we define the ***reflection*** and ***transmission*** ***coefficients***:

$$\boxed{R \equiv \frac{I_{refl}}{I_{inc}} = \frac{\tfrac{1}{2}\mu_1 v_1\omega^2\left|\tilde{A}_{refl}\right|^2}{\tfrac{1}{2}\mu_1 v_1\omega^2\left|\tilde{A}_{inc}\right|^2} = \frac{\left|\tilde{A}_{refl}\right|^2}{\left|\tilde{A}_{inc}\right|^2} = \left(\frac{v_2-v_1}{v_1+v_2}\right)^2}$$

Use: $\mu = T/v^2$ in transmittance, T below:

$$\boxed{T \equiv \frac{I_{trns}}{I_{inc}} = \frac{\tfrac{1}{2}\mu_2 v_2\omega^2\left|\tilde{A}_{trns}\right|^2}{\tfrac{1}{2}\mu_1 v_1\omega^2\left|\tilde{A}_{inc}\right|^2} = \frac{\left(T/v_2^2\right)v_2\left|\tilde{A}_{trns}\right|^2}{\left(T/v_1^2\right)v_1\left|\tilde{A}_{inc}\right|^2} = \frac{v_1\left|\tilde{A}_{trns}\right|^2}{v_2\left|\tilde{A}_{inc}\right|^2} = \left(\frac{v_1}{v_2}\right)\frac{\left|\tilde{A}_{trns}\right|^2}{\left|\tilde{A}_{inc}\right|^2} = \left(\frac{v_1}{v_2}\right)\left(\frac{2v_2}{v_1+v_2}\right)^2}$$

Then: $$\boxed{R+T = \frac{I_{refl}+I_{trns}}{I_{inc}} = \left(\frac{v_2-v_1}{v_1+v_2}\right)^2 + \left(\frac{v_1}{v_2}\right)\left(\frac{2v_2}{v_1+v_2}\right)^2 = \frac{v_1^2 - 2v_1\cdot v_2 + v_2^2 + 4v_1\cdot v_2}{\left(v_1+v_2\right)^2} = \frac{\left(v_1+v_2\right)^2}{\left(v_1+v_2\right)^2} = 1} \;\surd$$

We can also define the complex ***reflectance*** $\tilde{\mathcal{R}}$ and the complex ***transmittance*** $\tilde{\mathcal{T}}$ associated with an incident, right-moving wave impinging on the knot/discontinuity at $z = 0$:

Define: $R \equiv \left|\tilde{\mathcal{R}}\right|^2$, $T \equiv \left|\tilde{\mathcal{T}}\right|^2$, thus: $R + T = \left|\tilde{\mathcal{R}}\right|^2 + \left|\tilde{\mathcal{T}}\right|^2 = 1$. Hence we see that {here}:

$$\boxed{\tilde{\mathcal{R}} \equiv \left(\frac{\tilde{f}_{refl}(z=0,t)}{\tilde{f}_{inc}(z=0,t)}\right) = \left(\frac{\tilde{A}_{refl}\cancel{e^{i(-\omega t)}}}{\tilde{A}_{inc}\cancel{e^{i(-\omega t)}}}\right) = \left(\frac{\tilde{A}_{refl}}{\tilde{A}_{inc}}\right) = \left(\frac{A_{refl}\cdot e^{i\delta_R}}{A_{inc}\cdot e^{i\delta_I}}\right) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)}$$

$$\boxed{\tilde{\mathcal{T}} \equiv \sqrt{\frac{v_1}{v_2}}\left(\frac{\tilde{f}_{trns}(z=0,t)}{\tilde{f}_{inc}(z=0,t)}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{\tilde{A}_{trns}\cancel{e^{i(-\omega t)}}}{\tilde{A}_{inc}\cancel{e^{i(-\omega t)}}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{\tilde{A}_{trns}}{\tilde{A}_{inc}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}\cdot e^{i\delta_T}}{A_{inc}\cdot e^{i\delta_I}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{2v_2}{v_1 + v_2}\right)}$$

Note that the RHS of both of the above relations are ***purely real*** quantities.

If string 2 (on the RHS, $z > 0$) is ***lighter*** than string 1 (on the LHS, $z < 0$), *i.e.* $\mu_2 < \mu_1 \Rightarrow v_2 > v_1$ since: $\boxed{v_2 = \sqrt{T/\mu_2}\,(z>0)}$, $\boxed{v_1 = \sqrt{T/\mu_1}\,(z<0)}$ then: $\boxed{(v_2 - v_1) > 0}$

→ All three wave amplitudes ***must*** have the ***same*** phase angle, *i.e.* $\boxed{\delta_I = \delta_R = \delta_T}$.

Thus, for the case where $\mu_2 < \mu_1$ (or $v_2 > v_1$): $\boxed{\delta_I = \delta_R = \delta_T}$ All phases the same, hence:

$$\boxed{\tilde{\mathcal{R}} = \left(\frac{A_{refl}\cdot \cancel{e^{i\delta_R}}}{A_{inc}\cdot \cancel{e^{i\delta_I}}}\right) = \left(\frac{A_{refl}}{A_{inc}}\right) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)}$$

$$\boxed{\tilde{\mathcal{T}} = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}\cdot \cancel{e^{i\delta_T}}}{A_{inc}\cdot \cancel{e^{i\delta_I}}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}}{A_{inc}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{2v_2}{v_1 + v_2}\right)}$$

If string 2 (on the RHS, $z > 0$) is ***heavier*** than string 1 (on the LHS, $z < 0$), *i.e.* $\mu_2 > \mu_1 \Rightarrow v_2 < v_1$:

then: $\boxed{(v_2 - v_1) < 0}$ → the reflected wave is 180° (= π radians) out-of-phase relative to the incident wave *i.e.* $\cos(-k_1 z - \omega t + \delta_I - \pi) = -\cos(-k_1 z - \omega t + \delta_I)$.

The ***polarity*** of the ***reflected*** wave is ***flipped relative*** to the ***incident*** wave, *i.e.* $\boxed{\delta_I = \delta_R \pm \pi = \delta_T}$!

Note that: $\boxed{e^{\pm i\pi} = \cos\pi \pm i\sin\pi = -1}$

Thus, for the case where $\mu_2 > \mu_1$ (or $v_2 < v_1$): $\boxed{\delta_I = \delta_R \pm \pi = \delta_T}$ and hence:

$$\boxed{\tilde{\mathcal{R}} = \left(\frac{A_{refl}\cdot \cancel{e^{i\delta_R}}\, e^{i\pi}}{A_{inc}\cdot \cancel{e^{i\delta_I}}}\right) = \left(\frac{-A_{refl}}{A_{inc}}\right) = -\left(\frac{v_2 - v_1}{v_1 + v_2}\right) = +\left(\frac{v_1 - v_2}{v_1 + v_2}\right)}$$

$$\boxed{\tilde{\mathcal{T}} = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}\cdot \cancel{e^{i\delta_T}}}{A_{inc}\cdot \cancel{e^{i\delta_I}}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}}{A_{inc}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{2v_2}{v_1 + v_2}\right)}$$

If *e.g.* string 2 (on the RHS, $z > 0$) is ***infinitely*** massive, *i.e.* $\boxed{\mu_2 = \infty} \Rightarrow \boxed{v_2 = \sqrt{T/\mu_2} = 0}$ $(z > 0)$.
Then we see that ***no*** wave is transmitted, the incident wave is ***totally*** reflected, and with a 180° phase shift in polarity, relative to the incident wave:

$$\boxed{\tilde{\mathcal{R}} = \left(\frac{-A_{refl}}{A_{inc}}\right) = +\left(\frac{v_1 - 0}{v_1 + 0}\right) = 1}$$

$$\boxed{\tilde{\mathcal{T}} = \sqrt{\frac{v_1}{v_2}}\left(\frac{A_{trns}}{A_{inc}}\right) = \sqrt{\frac{v_1}{v_2}}\left(\frac{2v_2}{v_1 + v_2}\right) = \left(\frac{2\sqrt{v_1 \cdot v_2}}{v_1 + v_2}\right) = \left(\frac{2\sqrt{v_1 \cdot 0}}{v_1 + 0}\right) = 0}$$

Wave Polarization (on next page…)

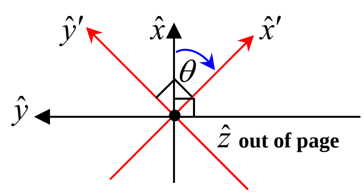
Wave Polarization

Depending (largely) on the type of wave and the nature of the medium that the waves are propagating in/on, the waves can have another degree of freedom known as ***polarization***.

Waves propagating in $\pm\hat{z}$ direction with a small transverse displacement amplitude (*e.g.* on a string) are known as *transverse* waves because the *displacement* of string (relative to its *equilibrium* position) is *transverse* (perpendicular) to the direction of propagation – (*e.g.* $\vec{v}_{prop} = \pm\hat{z}$).

Thus, a transverse displacement wave $\boxed{\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}}$ can be oriented *e.g.* in $\pm\hat{x}$ and/or $\pm\hat{y}$ directions for a traveling transverse displacement wave *e.g.* propagating along the $+\hat{z}$ direction.

→ Transverse traveling waves have *two* polarization states, either the $\pm\hat{x}$ or the $\pm\hat{y}$ direction, or equivalently 2 *orthogonal* {*i.e.* mutually-perpendicular} linear combinations of the $\pm\hat{x}$ and $\pm\hat{y}$ basis states for waves propagating in the $\pm\hat{z}$ direction:



Propagation of *e.g.* longitudinal sound waves in solid or non-solid media, *e.g.* normal gases, liquids and solids also has longitudinal polarization – because longitudinal sound waves have longitudinal displacements of atoms/molecules – *i.e.* along / against (*i.e.* parallel/anti-parallel to) the direction of propagation of the longitudinal wave, *e.g.* in the $\pm\hat{z}$ direction.

Here, the longitudinal displacement amplitude is (also) of the form: $\boxed{\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}}$

→ Longitudinal traveling waves have only *one* polarization state, *e.g.* the $\hat{z}$ direction.

Both longitudinal and transverse waves obey the same wave equation: $\boxed{\dfrac{\partial^2 \tilde{f}(z,t)}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 \tilde{f}(z,t)}{\partial t^2}}$

Longitudinal Waves: *e.g.* sound waves/acoustic waves – liquids, gases and solids and *e.g.* large amplitudes in strings (compression waves)

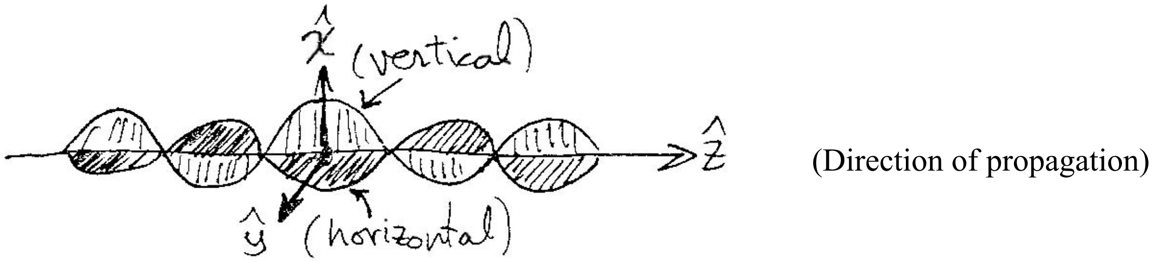
Transverse Waves: *e.g.* Small and large amplitudes in strings, long solid rods, solid bars, *etc.* (shear waves). *EM* waves are transverse waves.

∃ Two orthogonal polarization states for transverse waves – we represent the transverse amplitude as a vector quantity, indicating its polarization state:

If the transverse displacement of a traveling wave *e.g.* on a stretched string is in:

a.) the vertical plane ($\hat{x}$) – "vertical" polarization (up & down): $\boxed{\vec{\tilde{f}}_x(z,t) = \tilde{A}e^{i(kz-wt)}\hat{x}}$

b.) the horizontal plane ($\hat{y}$) – "horizontal" polarization (sideways): $\boxed{\vec{\tilde{f}}_y(z,t) = \tilde{A}e^{i(kz-wt)}\hat{y}}$

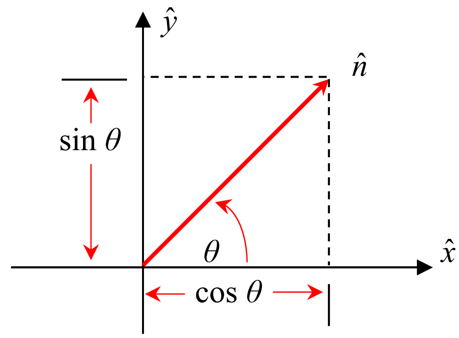


The polarization unit vector $\hat{n}$ lying in the transverse plane defines the plane of polarization: (*i.e.* plane of transverse string displacements {here})

Note that: $\boxed{\hat{n}\bullet\hat{z}\equiv 0}$

Define the polarization angle θ w.r.t. $\hat{x}$ axis:

$$\boxed{\hat{n}=\cos\theta\hat{x}+\sin\theta\hat{y}}$$



Linearly Polarized Transverse Waves

A linearly polarized transverse wave propagating in the $+\hat{z}$ direction has $\hat{x}$ and $\hat{y}$ components that are either precisely in-phase with each other (*i.e.* have 0° ***relative*** phase), or are precisely ± 180° out-of-phase with each other.

Examples of Linearly Polarized Transverse Waves:

$$\boxed{\vec{f}(z,t)=A\cos\theta\cos(kz-\omega t)\hat{x}+A\sin\theta\cos(kz-\omega t)\hat{y}}$$
$$\boxed{\vec{f}(z,t)=A\cos\theta\cos(kz-\omega t)\hat{x}-A\sin\theta\cos(kz-\omega t)\hat{y}}$$

Polarization angle θ defined w.r.t. $\hat{x}$ -axis

Or:

$$\boxed{\vec{f}(z,t)=A\cos\theta\cos\left[(kz-\omega t)+\delta\right]\hat{x}+A\sin\theta\cos\left[(kz-\omega t)+\delta\right]\hat{y}}$$
$$\boxed{\vec{f}(z,t)=A\cos\theta\cos\left[(kz-\omega t)+\delta\right]\hat{x}-A\sin\theta\cos\left[(kz-\omega t)+\delta\right]\hat{y}}$$

Or:

$$\boxed{\tilde{\vec{f}}(z,t)=\tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x}+\tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}}$$
$$\boxed{\tilde{\vec{f}}(z,t)=\tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x}-\tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}}$$

Or:

$$\boxed{\tilde{\vec{f}}(z,t)=\tilde{A}\cos\theta e^{i\left[(kz-\omega t)+\delta\right]}\hat{x}+\tilde{A}\sin\theta e^{i\left[(kz-\omega t)+\delta\right]}\hat{y}}$$
$$\boxed{\tilde{\vec{f}}(z,t)=\tilde{A}\cos\theta e^{i\left[(kz-\omega t)+\delta\right]}\hat{x}-\tilde{A}\sin\theta e^{i\left[(kz-\omega t)+\delta\right]}\hat{y}}$$

Note that: $\boxed{e^{i\delta}=e^{\pm i\pi}=\cos(\pm\pi)+i\sin(\pm\pi)=\cos(\pi)=-1}$ (= phase shift of ± 180°)

Circularly Polarized Transverse Plane Waves:

A circularly polarized transverse plane wave propagating *e.g.* in the $+\hat{z}$ direction has equal amplitude, but $\pm 90^o = \pm\frac{\pi}{2}$ out-of-phase $\hat{x}$ and $\hat{y}$ components (*e.g.* $\delta_x = 0^o$, $\delta_y = \pm 90^o = \pm\frac{\pi}{2}$):

Left Circularly Polarized (LCP) Transverse Plane Wave:

$$\boxed{\vec{f}_x(z,t) = A\cos(kz-\omega t)\hat{x}} \quad \boxed{\vec{f}_y(z,t) = A\cos\left[(kz-\omega t)+90^o\right]\hat{y} = -A\sin(kz-\omega t)\hat{y}}$$

$$\boxed{\vec{f}_{LCP}(z,t) = \vec{f}_x(z,t)+\vec{f}_y(z,t) = A\cos(kz-\omega t)\hat{x} - A\sin(kz-\omega t)\hat{y}}$$

Note that: $\boxed{f_x^2 + f_y^2 = A^2}$ Hence: $\boxed{\vec{f}(z,t) = \vec{f}_x(z,t)+\vec{f}_y(z,t)}$ lies on a circle of radius A.

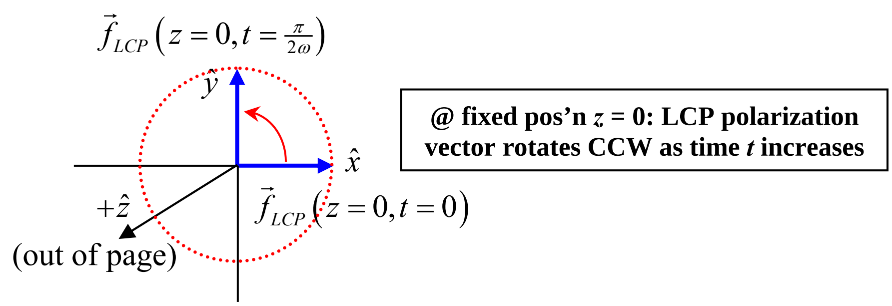
At a fixed ***position*** in ***space*** (*e.g.* $z = 0$):

$$\boxed{\vec{f}_{LCP}(z=0,t) = A\cos(-\omega t)\hat{x} - A\sin(-\omega t)\hat{y} = A\cos(\omega t)\hat{x} + A\sin(\omega t)\hat{y}}$$

At time $t = 0$: $\boxed{\vec{f}_{LCP}(z=0,t=0) = A\cos(0)\hat{x} - A\sin(0)\hat{y} = A\hat{x}}$

At time $t = \frac{\pi}{2\omega}$: $\boxed{\vec{f}_{LCP}(z=0,t=\frac{\pi}{2\omega}) = A\cos(-90^o)\hat{x} - A\sin(-90^o)\hat{y} = A\hat{y}}$ *i.e.* $\left(\omega t = \frac{\pi}{2}\right)$

At a fixed ***position*** in ***space*** (*e.g.* $z = 0$), looking directly into the on-coming LCP wave:
The LCP polarization vector rotates counter-clockwise CCW in the *x-y* plane as time increases for a LCP transverse plane wave propagating in the $+\hat{z}$ direction:



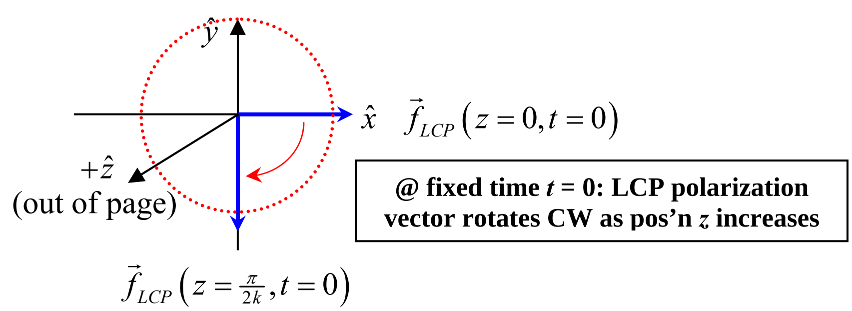
At a fixed ***time*** (*e.g.* $t = 0$):

$$\boxed{\vec{f}_{LCP}(z,t=0) = A\cos(kz)\hat{x} - A\sin(kz)\hat{y}}$$

At pos'n $z = 0$: $\boxed{\vec{f}_{LCP}(z=0,t=0) = A\cos(0)\hat{x} - A\sin(0)\hat{y} = A\hat{x}}$

At pos'n $z = \frac{\pi}{2k}$: $\boxed{\vec{f}_{LCP}(z=\frac{\pi}{2k},t=0) = A\cos(90^o)\hat{x} - A\sin(90^o)\hat{y} = -A\hat{y}}$ *i.e.* $\left(kz = \frac{\pi}{2}\right)$

At a fixed ***time*** (*e.g.* $t = 0$), looking directly into the on-coming LCP wave:
The LCP polarization vector rotates clockwise CW in the *x-y* plane as *z* increases for a LCP transverse plane wave propagating in the $+\hat{z}$ direction:



Right Circularly Polarized (RCP) Transverse Plane Wave:

$$\boxed{\vec{f}_x(z,t) = A\cos(kz-\omega t)\,\hat{x}} \quad \boxed{\vec{f}_y(z,t) = A\cos\left[(kz-\omega t)-90^o\right]\hat{y} = +A\sin(kz-\omega t)\,\hat{y}}$$

$$\boxed{\vec{f}_{RCP}(z,t) = \vec{f}_x(z,t)+\vec{f}_y(z,t) = A\cos(kz-\omega t)\,\hat{x} + A\sin(kz-\omega t)\,\hat{y}}$$

Again $\boxed{f_x^2+f_y^2 = A}$ and hence $\boxed{\vec{f}(z,t) = \vec{f}_x(z,t)+\vec{f}_y(z,t)}$ lies on a circle of radius A.

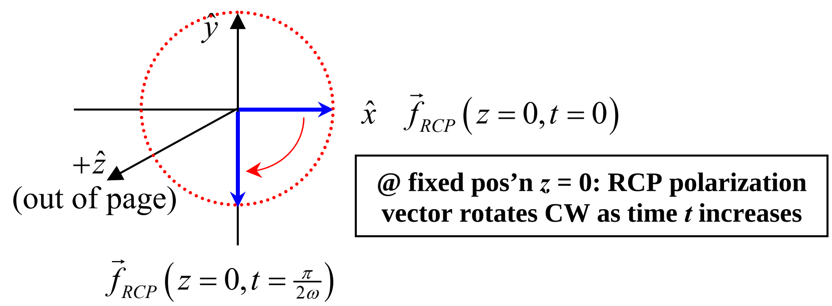
At a fixed ***position*** in ***space*** (*e.g.* $z = 0$):

$$\boxed{\vec{f}_{RCP}(z=0,t) = A\cos(-\omega t)\,\hat{x} + A\sin(-\omega t)\,\hat{y} = A\cos(\omega t)\,\hat{x} - A\sin(\omega t)\,\hat{y}}$$

At time $t = 0$: $\boxed{\vec{f}_{RCP}(z=0,t=0) = A\cos(0)\,\hat{x} + A\sin(0)\,\hat{y} = A\hat{x}}$

At time $t = \frac{\pi}{2\omega}$: $\boxed{\vec{f}_{RCP}(z=0,t=\tfrac{\pi}{2\omega}) = A\cos(-90^o)\,\hat{x} + A\sin(-90^o)\,\hat{y} = -A\hat{y}}$ *i.e.* $(\omega t = \frac{\pi}{2})$

At a fixed ***position*** in ***space*** (*e.g.* $z = 0$), looking directly into the on-coming RCP wave:
The RCP polarization vector rotates clockwise CW in the x-y plane as time increases for a RCP transverse plane wave propagating in the $+\hat{z}$ direction:



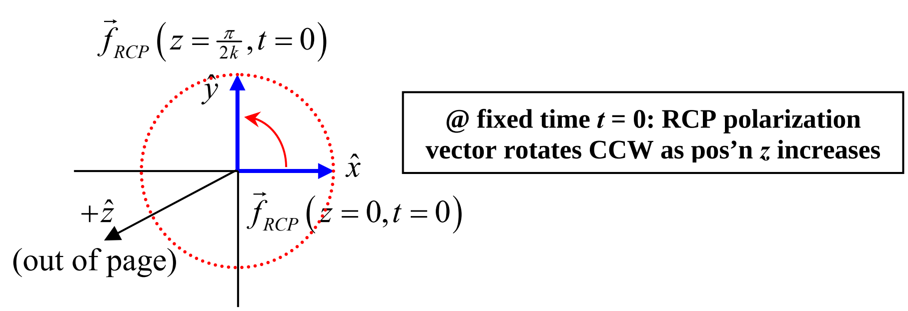
At a fixed ***time*** (*e.g.* $t = 0$):

$$\boxed{\vec{f}_{RCP}(z,t=0) = A\cos(kz)\,\hat{x} + A\sin(kz)\,\hat{y}}$$

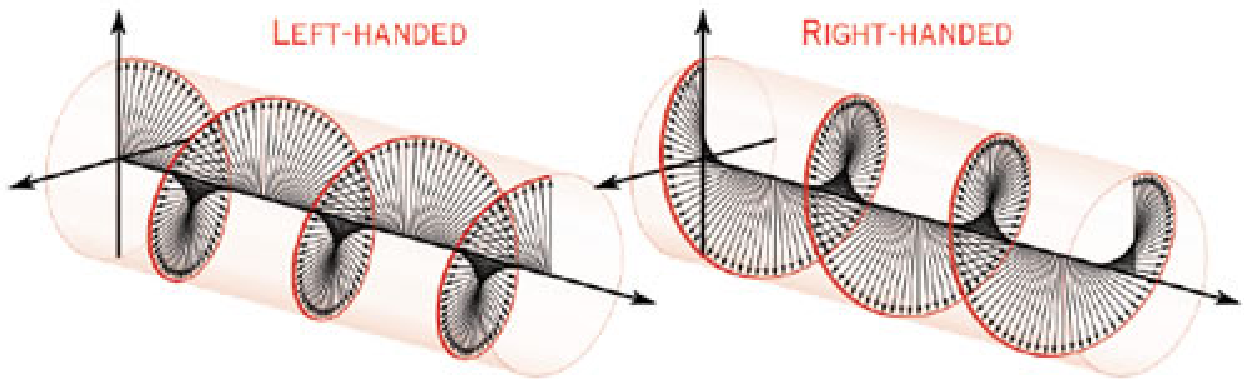
At pos'n $z = 0$: $\boxed{\vec{f}_{RCP}(z=0,t=0) = A\cos(0)\,\hat{x} + A\sin(0)\,\hat{y} = A\hat{x}}$

At pos'n $z = \frac{\pi}{2k}$: $\boxed{\vec{f}_{RCP}(z=\tfrac{\pi}{2k},t=0) = A\cos(90^o)\,\hat{x} + A\sin(90^o)\,\hat{y} = A\hat{y}}$ *i.e.* $(kz = \frac{\pi}{2})$

At a fixed ***time*** (*e.g.* $t = 0$), looking directly into the on-coming RCP wave:
The RCP polarization vector rotates counter-clockwise CCW in the x-y plane as z increases for a RCP transverse plane wave propagating in the $+\hat{z}$ direction:



3-D pictures of *LCP vs. RCP* transverse traveling plane *EM* waves propagating in ***space*** (*e.g.* the $+\hat{z}$ direction) at a ***fixed*** time (*e.g.* $t = 0$) are shown in the figure below:



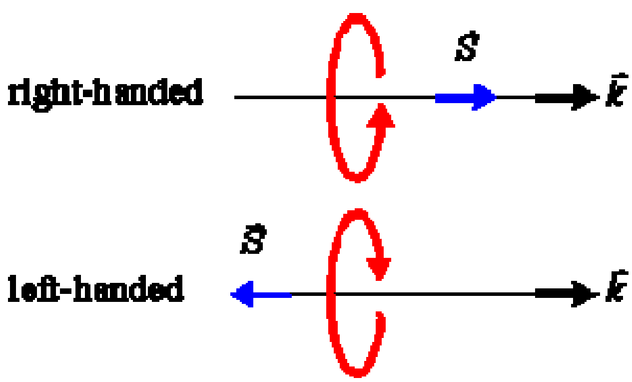
For *LCP vs. RCP* transverse traveling plane *EM* waves propagating in ***space*** (*e.g.* the $+\hat{z}$ direction) at a ***fixed*** time (*e.g.* $t = 0$):

For the *LCP* wave, using one's ***left*** hand, curling up one's fingers in the direction of rotation of the LCP vector with increasing *z*, your ***left*** thumb points in the direction of propagation.

For the *RCP* wave, using one's ***right*** hand, curling up one's fingers in the direction of rotation of the *RCP* vector with increasing *z*, your ***right*** thumb points in the direction of propagation.

***Microscopically*:**

- De-excitation of atoms/atomic transitions – copious source(s) of ***real*** *LCP*/*RCP* photons – due to QM selection rules – angular momentum change by $\pm 1\hbar$. A ***real*** *RCP* (*LCP*) photon has $\vec{s}_z^{\,RCP} = +1\hbar\hat{k}$ $\left(\vec{s}_z^{\,LCP} = -1\hbar\hat{k}\right)$ respectively. $\hat{k}$ = propagation direction of photon:



- ***Real*** photons radiated from *e.g.* an electric dipole antenna are ***linearly polarized***.
- Free electrons (@ high energy) *e.g.* passing through a horizontal/vertical magnetic undulator structure (or *FEL* – Free Electron Laser) produce ***linearly-polarized*** (LP) photons $\vec{s}_z^{\,LP} = 0\hbar\hat{k}$.

A *LP* photon is a ***superposition*** of equal amts of *RCP* & *LCP* states: $|LP\rangle_\gamma = \frac{1}{\sqrt{2}}\left(|RCP\rangle_\gamma \pm |LCP\rangle_\gamma\right)$

Can also produce ***elliptically polarized*** (*EP*) photons – use an elliptical magnetic undulator!

An *EP* photon is also a ***superposition*** of *RCP* & *LCP* states: $|EP\rangle_\gamma = \left(\cos\phi|RCP\rangle_\gamma \pm \sin\phi|LCP\rangle_\gamma\right)$

Now connect this back to the ***macroscopic*** classical *EM* wave – as per ***mean field theory***.

A classical *EM* wave ***can*** be 100% RCP, 100% LCP, 100% LP, but this does ***not*** {uniquely} mean that it is 100% comprised – at the microscopic level – ***solely*** of RCP, LCP, LP, photons respectively!

e.g. Can form a ***macroscopic*** linearly polarized (LP) *EM* wave by ***suitably*** superposing two beams – one consisting of 100% pure RCP photons, the other - 100% pure LCP photons.

e.g. Can form a ***macroscopic*** RCP (or LCP) *EM* wave by ***suitably*** superposing two beams – one consisting of 100% pure LP photons, the other – orthogonally-polarized 100% pure LP photons.

Can make arbitrary mix/arbitrary linear combination(s) of distinct/pure photon states to make up a ***macroscopic*** classical *EM* wave.

If the photon polarization content associated with a macroscopic EM wave is important to know, but don't ***a-priori*** know this, then one must explicitly carry out various polarization-intensity measurements in order to uniquely determine the actual photon polarization content!

If interested in comparing/understanding the similarities/commonalities of *EM* wave phenomena *vs*. acoustic wave phenomena, information on the propagation of acoustical waves, acoustic wave phenomena in general, solving the wave equation in 1-, 2- & 3-dimensions, … is available online (PDF format) from Professor Errede's lecture notes for the UIUC Physics 406 Acoustical Physics of Music/Musical Instruments course, available on the web at the following URL: https://courses.physics.illinois.edu/phys406/

