Physics Phun with Motional Effects and the Magnetic Vector Potential $\vec{A}$, *Eh*!!!

E&M Physics in Different Reference Frames

We have seen/learned, using Faraday's Law: $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$ and $\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$

That: $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{r},t) = -\vec{\nabla}\times\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$

Or: $\vec{\nabla}\times\vec{E}(\vec{r},t) = \vec{\nabla}\times\left(-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right)$ Or: $\vec{\nabla}\times\left(\vec{E}(\vec{r},t) + \frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right) = 0$

But: $\vec{\nabla}\times\left(-\vec{\nabla}V(\vec{r},t)\right) = 0$ ← always

Hence: $\vec{E}(\vec{r},t) + \frac{\partial\vec{A}(\vec{r},t)}{\partial t} = -\vec{\nabla}V(\vec{r},t)$

Or: $\vec{E}(\vec{r},t) = -\vec{\nabla}V(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$

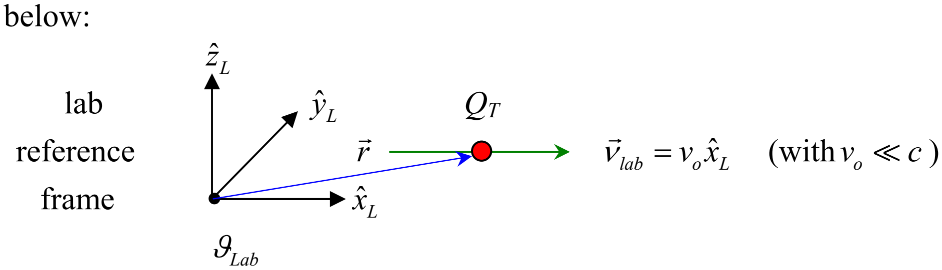
In some E&M physics situations, e.g. either $V(\vec{r},t)$ = constant or is = 0 altogether, and hence $-\vec{\nabla}V(\vec{r},t) = 0$ (i.e. vanishes), then only $\vec{E}(\vec{r},t) = -\partial\vec{A}(\vec{r},t)/\partial t$ remains.

Consider an inertial (i.e. non-accelerating) rest frame with a test charge Q_T in it.
If $V(\vec{r},t)$ = constant or is = 0 altogether, then the test charge will experience a force:

$$\vec{F}_R(\vec{r},t) = Q_T\vec{E}_R(\vec{r},t) = -Q_T\frac{\partial\vec{A}_R(\vec{r},t)}{\partial t} \Rightarrow \vec{E}_R(\vec{r},t) = -\frac{\partial\vec{A}_R(\vec{r},t)}{\partial t}$$

If a time-varying magnetic vector potential $\vec{A}(\vec{r},t)$ with $\partial\vec{A}(\vec{r},t)/\partial t \neq 0$ exists in this reference frame, (the *rest* frame of the test charge, Q_T) the test charge will "see" an electric field as a consequence of the time-varying A-field, and hence will experience a net force acting on it!

We now wish to consider a situation where the test charge Q_T is moving with constant velocity $\vec{v}_{lab} = v_o\hat{x}_L$ $\left(|\vec{v}_{lab}| \ll c\right)$ in the lab reference frame, where a *static* (i.e. time-independent) magnetic field is present {produced by a permanent magnet, or a coil of wire carrying a steady current I}, but has *spatial* variations from place-to-place, i.e. $\vec{B}_{lab} = \vec{B}_{lab}(\vec{r}) \neq fcn(t)$ as shown in the figure below:



Keeping in mind that $-\vec{\nabla} V(\vec{r},t) = 0$ here (i.e. $\exists$ no electrostatic charges/fields are present anywhere – purely for simplicity!) the moving test charge Q_T will experience a Lorentz force $\vec{F}(\vec{r}) = Q_T\left(\vec{v}_{lab} \times \vec{B}_{lab}(\vec{r})\right)$. Since the field/observation point is at the location of the moving test charge in the lab frame, we see that the position vector $\boxed{\vec{r} = \vec{r}(t) = \vec{r}_o + \vec{v}_{lab}t = \vec{r}_o + v_o\hat{x}_L}$ is explicitly time-dependent, thus:

$$\boxed{\vec{F}_{lab}(\vec{r}(t)) = Q_T\left[\vec{v}_{lab}(\vec{r}) \times \vec{B}_{lab}(\vec{r}(t))\right]} \text{ with } \boxed{\vec{v}_{lab} = v_o\hat{x}_L}$$

Since $|\vec{v}_{lab}| \ll c$ we do not need to consider relativistic effects; hence Galilean transformations from the lab frame to/from the rest frame of the test charge Q_T suffice for our purposes, here.

In the rest frame of the test charge Q_T, the test charge "sees" a time-varying electric field $\boxed{\vec{E}_R(\vec{r}(t)) = \vec{v}_{lab} \times \vec{B}_{lab}(\vec{r}(t))}$ due to spatial variations in the static $\vec{B}_{lab}(\vec{r}(t))$ encountered by the test charge moving in the lab frame, with corresponding force:

$$\boxed{\vec{F}_R(\vec{r}(t)) = Q_T\vec{E}_R(\vec{r}(t)) = Q_T\left[\vec{v}_{lab}(\vec{r}) \times \vec{B}_{lab}(\vec{r}(t))\right] = \vec{F}_{lab}(\vec{r}(t))} \text{ with } \boxed{\vec{v}_{lab} = v_o\hat{x}_L}$$

Note that if $\vec{B}_{lab}(\vec{r}(t)) = B_o\hat{z}_L =$ constant/uniform magnetic field, then in the rest frame of the test charge $\vec{E}_R(\vec{r}(t)) = \vec{v}_{lab} \times \vec{B}_{lab}(\vec{r}(t)) = v_o\hat{x}_L \times B_o\hat{z}_L = -v_oB_o\hat{y}_L =$ constant $\neq fcn(t)$ with corresponding force $\vec{F}_R(\vec{r}(t)) = Q_T\vec{E}_R(\vec{r}(t)) = -Q_Tv_oB_o\hat{y}_L$ and this would be the end of the story, i.e. just a very simple, static-only problem…

We're interested in the situation where the static, time-independent $\vec{B}_{lab}(\vec{r}) \neq$ constant/uniform magnetic field, one which has spatial variations – i.e. it has spatial gradients. We're also wanting to understand this problem from the perspective of using the magnetic vector potential, via its relation to the magnetic field, i.e. $\vec{B}_{lab}(\vec{r}) = \vec{\nabla} \times \vec{A}_{lab}(\vec{r})$. Since $\vec{B}_{lab}(\vec{r})$ is static/time-independent in the lab frame, then we see that $\vec{A}_{lab}(\vec{r})$ is also static/time-independent in the lab frame.

In the rest frame of the test charge Q_T, as the particle moves in the lab frame, passing through spatial variations in the static magnetic field $\vec{B}_{lab}(\vec{r}(t))$, viewed from the test charge's perspective, the electric field will have explicit time-dependence because of this: $\boxed{\vec{E}_R(\vec{r}(t)) = \vec{v}_{lab} \times \vec{B}_{lab}(\vec{r}(t))}$ but we can also view this as explicit variations in the time-dependence of the magnetic vector potential, viewed in the rest frame of the test charge: $\boxed{\vec{E}_R(\vec{r},t) = -\partial\vec{A}_R(\vec{r},t)/\partial t}$. Thus we see that $\boxed{\vec{E}_R(\vec{r}(t)) = \vec{v}_{lab} \times \vec{B}_{lab}(\vec{r}(t)) = -\partial\vec{A}_R(\vec{r},t)/\partial t}$ i.e. that $\boxed{\vec{v}_{lab} \times \vec{B}_{lab}(\vec{r}(t)) = -\partial\vec{A}_R(\vec{r},t)/\partial t}$ and that $\boxed{\vec{F}_R(\vec{r}(t)) = Q_T\vec{E}_R(\vec{r}(t)) = Q_T\left[\vec{v}_{lab}(\vec{r}) \times \vec{B}_{lab}(\vec{r}(t))\right] = -Q_T\,\partial\vec{A}_R(\vec{r},t)/\partial t}$

But: $\boxed{\vec{B}_{lab}\left(\vec{r}\left(t\right)\right)=\vec{\nabla}\times\vec{A}_{lab}\left(\vec{r}\left(t\right)\right)}$, thus $\boxed{\vec{v}_{lab}\times\vec{B}_{lab}\left(\vec{r}\left(t\right)\right)=\vec{v}_{lab}\times\vec{\nabla}\times\vec{A}_{lab}\left(\vec{r}\left(t\right)\right)=-\partial\vec{A}_{R}\left(\vec{r},t\right)/\partial t}$

And $\boxed{\vec{E}_{R}\left(\vec{r}\left(t\right)\right)=\vec{v}_{lab}\times\vec{B}_{lab}\left(\vec{r}\left(t\right)\right)=\vec{v}_{lab}\times\vec{\nabla}\times\vec{A}_{lab}\left(\vec{r}\left(t\right)\right)=-\partial\vec{A}_{R}\left(\vec{r},t\right)/\partial t}$ and

$$\boxed{\vec{F}_{R}\left(\vec{r}\left(t\right)\right)=Q_{T}\vec{E}_{R}\left(\vec{r}\left(t\right)\right)=Q_{T}\left[\vec{v}_{lab}\left(\vec{r}\right)\times\vec{B}_{lab}\left(\vec{r}\left(t\right)\right)\right]=Q_{T}\left[\vec{v}_{lab}\left(\vec{r}\right)\times\vec{\nabla}\times\vec{A}_{lab}\left(\vec{r}\left(t\right)\right)\right]=-Q_{T}\,\partial\vec{A}_{R}\left(\vec{r},t\right)/\partial t}$$

Now let us use the vector identity (Griffiths product rule #4):

$$\vec{\nabla}\left(\vec{A}_{lab}\cdot\vec{v}_{lab}\right)=\vec{A}_{lab}\times\left(\vec{\nabla}\times\vec{v}_{lab}\right)+\underbrace{\vec{v}_{lab}\times\left(\vec{\nabla}\times\vec{A}_{lab}\right)}+\left(\vec{A}_{lab}\cdot\vec{\nabla}\right)\vec{v}_{lab}+\left(\vec{v}_{lab}\vec{\nabla}\right)\vec{A}_{lab}$$

Or: $$\boxed{\vec{v}_{lab}\times\left(\vec{\nabla}\times\vec{A}_{lab}\right)=\nabla\left(\vec{A}_{lab}\cdot\vec{v}_{lab}\right)-\vec{A}_{lab}\times\left(\vec{\nabla}\times\vec{v}_{lab}\right)-\left(\vec{A}_{lab}\cdot\vec{\nabla}\right)\vec{v}_{lab}-\left(\vec{v}_{lab}\cdot\vec{\nabla}\right)\vec{A}_{lab}}$$

Then, suppressing the $\left(\vec{r}\left(t\right)\right)$ arguments and using the suffix $L=lab$:

$$\boxed{\vec{v}_{L}\times\vec{B}_{L}=\vec{v}_{L}\times\left(\vec{\nabla}\times\vec{A}_{L}\right)=\vec{\nabla}\underbrace{\left(\vec{A}_{L}\cdot\vec{v}_{L}\right)}_{(1)}-\vec{A}_{L}\times\underbrace{\left(\vec{\nabla}\times\vec{v}_{L}\right)}_{(2)}-\underbrace{\left(\vec{A}_{L}\cdot\vec{\nabla}\right)\vec{v}_{L}}_{(3)}-\underbrace{\left(\vec{v}_{L}\cdot\vec{\nabla}\right)\vec{A}_{L}}_{(4)}}$$

Thus, we see that there are <u>four</u> (!!) different possible mechanisms whereby the moving test charge (in the <u>lab</u> frame) can experience a Lorentz type force due to motional effects in the presence of an $\vec{A}$-field, or the gradient of an $\vec{A}$-field:

$$\boxed{\vec{F}_{L}=\vec{F}_{L}^{(1)}+\vec{F}_{L}^{(2)}+\vec{F}_{L}^{(3)}+\vec{F}_{L}^{(4)}=Q_{T}\vec{v}_{L}\times\vec{B}_{L}=Q_{T}\vec{v}_{L}\times\left(\vec{\nabla}\times\vec{A}_{L}\right)}$$

$$\boxed{\vec{F}_{L}=\underbrace{Q_{T}\vec{\nabla}\left(\vec{A}_{L}\cdot\vec{v}_{L}\right)}_{(1)}-\underbrace{Q_{T}\vec{A}_{L}\times\left(\vec{\nabla}\times\vec{v}_{L}\right)}_{(2)}-\underbrace{Q_{T}\left(\vec{A}_{L}\cdot\vec{\nabla}\right)\vec{v}_{L}}_{(3)}-\underbrace{Q_{T}\left(\vec{v}_{L}\cdot\vec{\nabla}\right)\vec{A}_{L}}_{(4)}}$$

- The <u>first</u> Lorentz force term, $\boxed{\vec{F}_{L}^{(1)}=Q_{T}\left\{\vec{\nabla}\left(\vec{A}_{L}\cdot\vec{v}_{L}\right)\right\}=Q_{T}\vec{\nabla}\left(A_{L}v_{L}\cos\Theta\right)}$ where Θ = opening angle between $\vec{A}_{L}$ and $\vec{v}_{L}$ $\left(0\le\Theta\le\pi\right)$. Note that the nature of $\vec{A}_{L}\cdot\vec{v}_{L}=Av_{L}\cos\Theta$ must be such that a <u>non-zero</u> <u>spatial</u> <u>gradient</u> of $\left(\vec{A}_{L}\cdot\vec{v}_{L}\right)$ must exist, in order for $\vec{F}_{L}^{(1)}\neq 0$.
- The <u>second</u> Lorentz force term, $\vec{F}_{L}^{(2)}=-Q_{T}\left\{\vec{A}_{L}\times\left(\vec{\nabla}\times\vec{v}_{L}\right)\right\}$ must be such that the velocity field, $\vec{v}_{L}\left(\vec{r}\right)$ has <u>non-zero</u> <u>curl</u> (e.g. a rotational vector field) and $\left(\vec{\nabla}\times\vec{v}_{L}\right)$ must also be non-parallel to $\vec{A}_{L}$, in order for $\vec{F}_{L}^{(2)}\neq 0$.
- The <u>third</u> Lorentz force term, $\vec{F}_{L}^{(3)}=-Q_{T}\left(\vec{A}_{L}\cdot\vec{\nabla}\right)\vec{v}_{L}$ must be such that the velocity field, $\vec{v}_{L}\left(\vec{r}\right)$ has <u>non-zero</u> <u>spatial</u> <u>gradient</u>, and $\vec{A}_{L}$ must have a component <u>parallel</u> to the gradient of $\vec{v}_{L}$, in order for $\vec{F}_{L}^{(3)}\neq 0$.

- The fourth Lorentz force term, $\vec{F}_L^{(4)} = -Q_T\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L$ must be such that the $\vec{A}$-field, $\vec{A}_L\left(\vec{r}\right)$ has has non-zero spatial gradient, and $\vec{v}_L$ must have a component parallel to the gradient of $\vec{A}_L$, in order for $\vec{F}_L^{(4)} \neq 0$.

The (total) Lorentz force acting on the test charge moving in the lab frame is $\vec{F}_L = \sum_{i=1}^{4} \vec{F}_L^{(i)}$ is the force acting on the test charge. The $\vec{B}_L$-field, $\vec{A}_L$-field and $\vec{v}_L$ are quantities all defined in the lab frame, and hence all $\vec{\nabla}\times\vec{A}_L$, $\vec{\nabla}\vec{A}_L$, $\vec{\nabla}\times\vec{v}_L$, $\vec{\nabla}\vec{v}_L$ etc. quantities are also defined in the lab frame.

As stated earlier, here the $\vec{B}_L$-field, and hence the $\vec{A}_L$-field, are static (i.e. time-independent) quantities in the lab frame (merely to keep it simple). An observer in the lab frame will ascribe the force $\vec{F}_L$ acting on the test charge Q_T moving in the lab frame with velocity $\vec{v}_L\left(\vec{r}\left(t\right)\right)$ as being due to:

$$\boxed{\begin{aligned}\vec{F}_L &= \sum_{i=1}^{4} \vec{F}_L^{(i)} = Q_T\left(\vec{v}_L \times \vec{B}_L\right) = Q_t\left\{\vec{v}_L \times \left(\vec{\nabla}\times\vec{A}_L\right)\right\} \\ &= Q_T\left\{\underbrace{\vec{\nabla}\left(\vec{A}_L \bullet \vec{v}_L\right)}_{(1)} - \underbrace{\vec{A}_L \times \left(\vec{\nabla}\bullet\vec{v}_L\right)}_{(2)} - \underbrace{\left(\vec{A}_L \bullet \vec{\nabla}\right)\vec{v}_L}_{(3)} - \underbrace{\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L}_{(4)}\right\}\end{aligned}}$$

Whereas an observer in the rest frame of the charged particle will ascribe the (same) force (for $v_L \ll c$), $\vec{F}_R$ acting on the test charge as being due to a time-varying $\vec{A}_R$-field,

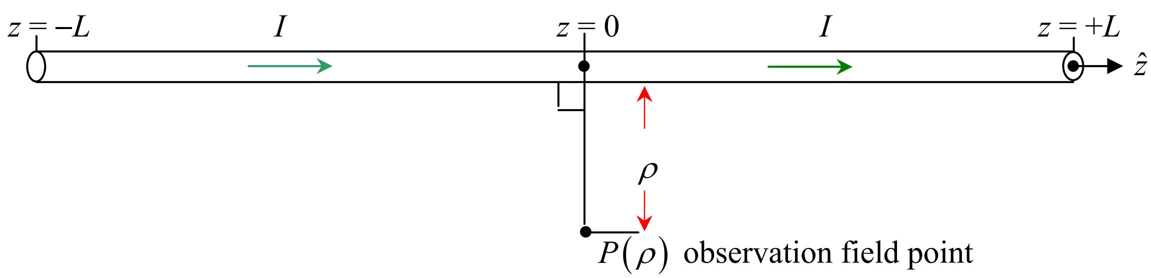
$$\boxed{\vec{F}_R\left(t\right) = Q_T\vec{E}_R\left(t\right) = -Q_T\frac{\partial\vec{A}_R\left(t\right)}{\partial t}}$$

Since $\vec{F}_L = \vec{F}_R$ (i.e. the force acting on the test charge is the same in the lab as in the rest frame),

we see that: $$\boxed{-\left.\frac{\partial\vec{A}_R\left(t\right)}{\partial t}\right|_{\text{rest frame}} \rightleftharpoons \left\{\underbrace{\vec{\nabla}\left(\vec{A}_L \bullet \vec{v}_L\right) - \vec{A}_L \times \left(\vec{\nabla}\bullet\vec{v}_L\right) - \left(\vec{A}_L \bullet \vec{\nabla}\right)\vec{v}_L - \left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L}_{\text{lab frame quantities}}\right\}}$$

In the rest frame of the charged particle, moving with velocity $\vec{v}_L$ in the lab, the $\vec{A}_R$-field, as seen in the rest frame varies in time because the charged particle is passing through/going over e.g. spatial gradients in the lab $\vec{A}_L$-field (which is static in the lab frame).

Let's now explore the physics of this further with a specific example/application/use of this four-term formula for $\vec{F}_L$. Let's consider a long straight wire of length $2L$ carrying a steady (line) current, I in the $+\hat{z}$ direction (see P435 lecture notes 15, p. 7-8) where we found (in cylindrical coordinates):



$$\boxed{\vec{A}_{\substack{long\\wire}}(\rho) = \left(\frac{\mu_0}{2\pi}\right) I \ln\left[\left(\frac{L}{\rho}\right)\left\{1+\sqrt{1+\left(\rho/L\right)^2}\right\}\right]\hat{z}} \leftarrow \vec{A}_{\substack{long\\wire}} \parallel \vec{I} = I\hat{z}$$

for a long wire of length L carrying a steady current I.

Then: $$\boxed{\vec{B}_{\substack{long\\wire}}(\rho) = \vec{\nabla}\times\vec{A}_{\substack{long\\wire}}(\rho) = -\frac{\partial A_z(\rho)}{\partial\rho}\hat{\varphi}}$$

$$\boxed{\vec{B}_{\substack{long\\wire}}(\rho) = \left(\frac{\mu_0}{2\mu}\right) I \left(\frac{1}{\rho}\right)\left\{1-\left(\frac{\rho}{L}\right)^2 \frac{1}{\sqrt{1+\left(\rho/L\right)^2}\left(1+\sqrt{1+\left(\rho/L\right)^2}\right)}\right\}\hat{\varphi}}$$

If $L \gg \rho$, then these expressions simplify to:

$$\left.\begin{aligned} \vec{A}_{\substack{long\\wire}}(\rho) &\simeq \left(\frac{\mu_0}{2\pi}\right) I \ln\left(\frac{2L}{\rho}\right)\hat{z} \\ \vec{B}_{\substack{long\\wire}}(\rho) &\simeq \left(\frac{\mu_0}{2\pi}\right)\left(\frac{I}{\rho}\hat{\varphi}\right) \end{aligned}\right\} \text{for } L \gg \rho$$

n.b. This is the exact same $\vec{B}$-field result we obtained using Ampere's Law for the ∞- long straight wire carrying steady current $\vec{I} = I\hat{z}$ {see P435 lecture notes 14, pp4-5}.

<u>Again</u>, this is the magnetic vector potential $\vec{A}_L$ and corresponding $\vec{B}_L$-field, $\vec{B}_L = \vec{\nabla}\times\vec{A}_L$ in the <u>lab frame</u>, where the long steady current current-carrying wire is <u>stationary/not moving/at rest</u>.

Now let us consider a test charge, Q_T moving with <u>constant/uniform velocity</u>, $\vec{v}_L = v_o\hat{x}$ $\left(\perp \vec{A}_{\substack{long\\wire}}\right)$ in the lab frame, again with $v_o \ll c$.

Then because $\vec{A}_L \perp \vec{v}_L$, the 1st term in the 4-term lab-frame force equation vanishes (here), because $\left(\vec{A}_L \bullet \vec{v}_L\right) = 0$, hence $\boxed{\vec{F}_L^{(1)} = Q_T \vec{\nabla}\left(\vec{A}_L \bullet \vec{v}_L\right) = 0}$.

Then because $\vec{v}_L = v_o \hat{x}$ = constant/uniform velocity, the 2nd term in the lab-frame force equation also vanishes (here), because $\vec{\nabla} \times \vec{v}_L = \vec{\nabla} \times v_o \hat{x} = 0$. Hence $\boxed{\vec{F}_L^{(2)} = -Q_T \vec{A}_L \times \left(\vec{\nabla} \times \vec{v}_L\right) = 0}$.

Similarly, because $\vec{v}_L = v_o \hat{x}$ = constant/uniform velocity, the 3rd term in the lab-frame force equation also vanishes (here), because $\vec{\nabla} \times \vec{v}_L = \vec{\nabla} \times v_o \hat{x} = 0$. Hence $\boxed{\vec{F}_L^{(3)} = -Q_T \left(\vec{A}_L \bullet \vec{\nabla}\right) \vec{v}_L = 0}$.

Thus (here), only the 4th term in the 4-term lab-frame force equation is non-zero, i.e.

$$\boxed{\vec{F} = \vec{F}_L^{(4)} = -Q_T \left(\vec{v}_L \bullet \vec{\nabla}\right) \vec{A}_L}.$$

Now, if we use (for simplicity's sake) the $L \gg \rho$ approximation $\boxed{\vec{A}_{\substack{long \\ wire}}(\rho) \simeq \left(\frac{\mu_o}{2\rho}\right) I \ln\left(\frac{2L}{\rho}\right) \hat{z}}$

Then: $\boxed{\vec{\nabla}\vec{A}_L = \left\{\frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{z}\right\}\left(\frac{\mu_0}{2\pi}\right) I \ln\left(\frac{2L}{\rho}\right)\hat{z}}$ $\Leftarrow$ n.b. $\vec{A}_L(\rho)$ has no explicit φ or z dependence

$$= \frac{\partial}{\partial \rho}\hat{\rho}\left[\left(\frac{\mu_o}{2\pi}\right) I \ln\left(\frac{2L}{\rho}\right)\hat{z}\right]$$

$$= \left\{\left(\frac{\mu_o}{2\pi}\right) I \frac{\partial}{\partial \rho}\left[\ln\left(\frac{2L}{\rho}\right)\right]\hat{\rho}\right\}\hat{z}$$

Now: $$\frac{\partial}{\partial \rho}\left[\ln\left(\frac{2L}{\rho}\right)\right] = \left(\frac{\rho}{2L}\right)\frac{\partial}{\partial \rho}\left\{\frac{2L}{\rho}\right\} = \left(\frac{\rho}{\cancel{2L}}\right)\cancel{2L}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right) = \rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right)$$

$$= \rho\frac{\partial}{\partial \rho}\left(\rho^{-1}\right) = -\rho\left(\frac{1}{\rho^2}\right) = -\frac{1}{\rho}$$

$\therefore$ $\boxed{\vec{\nabla}\vec{A} = -\left\{\left(\frac{\mu_o}{2\pi}\right)\frac{I}{\rho}\hat{\rho}\right\}\hat{z}}$ $\Leftarrow$ n.b. $\hat{\rho}$ is associated with the gradient, $\vec{\nabla}$ and $\hat{z}$ is associated with $\vec{A}$. IMPORTANT!!!

Then:
$$\vec{F}_L = \vec{F}_L^{(4)} = -Q_T\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L = -Q_T\left(\vec{v}_o \hat{x} \bullet \vec{\nabla}\right)\left\{\left(\frac{\mu_o}{2\pi}\right) I \ln\left(\frac{2L}{\rho}\right)\hat{z}\right\} = +Q_T\left(v_o \hat{x} \bullet \left(\frac{\mu_o}{2\pi}\right)\frac{I}{\rho}\hat{\rho}\right)\hat{z}$$
$$= +Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\hat{x} \bullet \hat{\rho}\right)\hat{z}$$ $\Leftarrow$ n.b. must 1st take dot product $\left(\vec{v}_L \bullet \vec{\nabla}\right)$ here!!!

Now: $\hat{\rho} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$ (in cylindrical coordinates)

$$\therefore \boxed{\vec{F}_L = \vec{F}_L^{(4)} = +Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\hat{x}\bullet[\cos\varphi\hat{x} + \sin\varphi\hat{y}]\right)\hat{z} = +Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\cos\varphi\left(\overbrace{\hat{x}\bullet\hat{x}}^{=1}\right) + \sin\varphi\underbrace{\left(\hat{x}\bullet\hat{y}\right)}_{=0}\right)\hat{z}}$$

$$\therefore \boxed{\vec{F}_L = \vec{F}_L^{(4)} = +Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\cos\varphi\hat{z}}$$ $\Leftarrow$ n.b. the lab-frame force acting on moving test charge, that has constant/uniform velocity $\vec{v}_L = v_0\hat{x}$ ($\perp \vec{I} = I\hat{z}$) in proximity $(\rho \ll L)$ of long wire carrying uniform/steady current $\vec{I} = I\hat{z}$ is in the $\hat{z}$ -direction, parallel to direction of current flow!

$\Rightarrow$ IS THIS RESULT ACTUALLY CORRECT ??? $\Leftarrow$

As an explicit check that this result is indeed correct, we calculate the Lorentz force (in the lab reference frame) acting on the test charge, Q_T from $\vec{F}_L = Q_T\left(\vec{v}_L \times \vec{B}_{\substack{long\\wire}}\right)$ with test charge moving in the lab frame with constant/uniform velocity, $\vec{v}_L = v_o\hat{x}$ $(v_o \ll c)$:

$$\vec{B}_{\substack{long\\wire}}(\rho) \simeq \left(\frac{\mu_o}{2p}\right)\frac{I}{\rho}\hat{\varphi}, \text{ for } \rho \ll L$$

$$\vec{F}_L = Q_T\left(\vec{v}_L \times \vec{B}_{\substack{long\\wire}}(\rho)\right) = Q_T\left(v_0\hat{x}\times\left(\frac{\mu_o}{2\pi}\right)\frac{I}{\rho}\hat{\varphi}\right) = Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\hat{x}\times\hat{\varphi}\right)$$

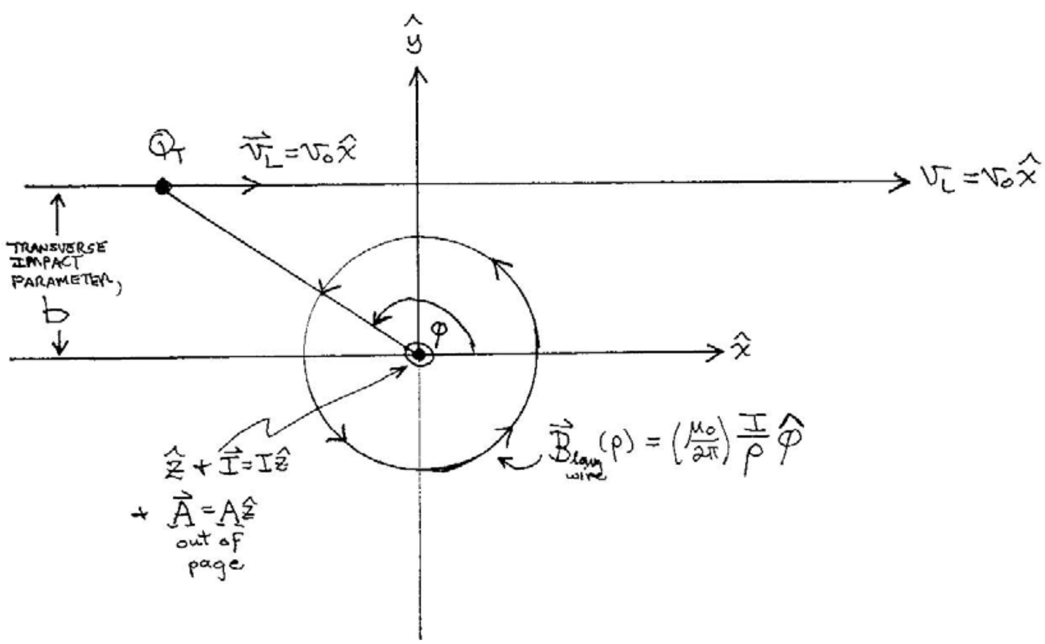
<u>But</u>: $\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}$ (in cylindrical coordinates)

$$\therefore \quad \vec{F}_L = Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\hat{x}\times[-\sin\varphi\hat{x} + \cos\varphi\hat{y}]\right) = Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\left(\sin\varphi\underbrace{\left(\hat{x}\times\hat{x}\right)}_{=0} + \cos\varphi\underbrace{\left(\hat{x}\times\hat{y}\right)}_{=\hat{z}}\right)$$

i.e. $$\boxed{\vec{F}_L = Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\cos\varphi\hat{z}}$$

$\therefore$ This result <u>*is*</u> <u>identical</u> to that obtained from $\boxed{\vec{F}_L = \vec{F}_L^{(4)} = -Q_T\left(\vec{v}_L\bullet\vec{\nabla}\right)\vec{A}_L}$!!!

Now let us look at an end view of this physics problem, in the lab reference frame:



The expression for the force on the test charge, Q_T (as seen from the lab reference frame) is:

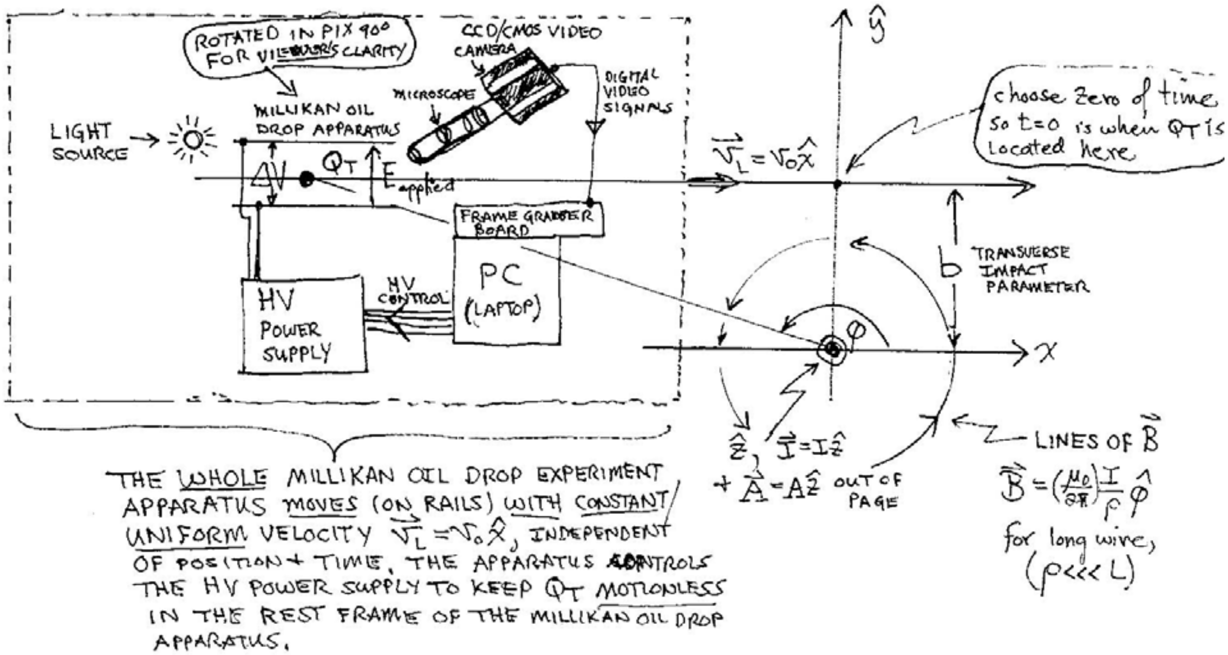
$$\vec{F}_L = Q_T\vec{v}_L \times \vec{B}_{\substack{long\\wire}} = -Q_T\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L = Q_T\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\cos\varphi\hat{z} \ \underline{\text{with}}\ \vec{v}_L = v_o\hat{x},\ \text{indep't of time},\ v_o \ll c\ .$$

The above force acting on Q_T wants to bend it in the $+\hat{z}$ direction (i.e. out of the page, in the above figure). As soon as this happens, the test charge is no longer maintaining its initial direction. Since we know that for $\vec{v}_L = v_o\hat{x}$, that $\vec{F} = F\hat{z}$, then we can imagine carrying out a quasi-realistic experiment using e.g. a Millikan oil drop apparatus consisting of a pair of parallel plates with an applied potential with associated electric field which exactly cancels out the Lorentz force acting on the test charge Q_T, i.e.

$$\boxed{\vec{F}_{ToT} = F_{Lorentz} + F_{applied} = Q_T\vec{v}_L \times \vec{B}_L - Q_T\vec{E}_{applied}}$$

$$\Rightarrow \boxed{\vec{E}_{applied} = -\vec{v}_L \times \vec{B}_{\substack{long\\wire}} = +\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_{\substack{long\\wire}} = -\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho}\cos\varphi\hat{z}}$$

We can accomplish this e.g. using a Millikan oil drop apparatus, that moves with the test charge. The test charge Q_T is e.g. attached to an oil-drop of independently known mass. A light source and a video camera are used to monitor the position of the test charge between the electrodes of the Millikan apparatus and a fast laptop pc is used to read in/frame grab the video camera data and analyze it to determine the oil drop's position (e.g. at $\simeq 1/30^{th}$ second intervals (NTSC standard is 30 video frames/sec), kinematic software is used to apply a correction to the potential difference ΔV across the electrodes of the Millikan oil drop apparatus in order to keep the test charge centered (i.e. motionless) in the gap of the Millikan oil drop apparatus. In this manner, we can maintain $\vec{v}_L = v_o\hat{x}$ =constant independent of position of Q_T in the lab frame <u>and</u> in time. $\Leftarrow$ n.b. This is nothing more than a software-controlled feedback loop.



As an added benefit of using the pc-controlled Millikan oil drop apparatus, we can also simultaneously use the pc to record the *x*-position of the Millikan apparatus as a function of time as it moves in the lab frame with constant velocity, $\vec{v}_L = v_o\hat{x}$. We choose the zero of time to occur when the test charge is at $x = 0$, i.e. when $Q_T\,(t = 0)$ is located at $(x, y, z)\big|_{t=0} = (0, b, 0)$.

Then this setup will record: $\vec{F}_{applied}(x_L, t)$, lab position $x_L(t)$ and time, *t*, where

$$\boxed{\vec{F}_{applied}(x_L,t) = Q_T\vec{E}_{applied}(x_L,t) = -Q_T\left[\Delta V_{applied}(x_L,t)/d\right]\hat{z}}$$ where d = plate separation distance.

If the pc-controlled feedback loop works perfectly, then the test charge Q_T remains motionless/centered/at rest in the gap of the Millikan oil drop apparatus and $\vec{F}_{applied}(x_L,t)$ <u>exactly</u> cancels $\vec{F}_{Lorentz} = Q_T\vec{v}_L \times \vec{B}_{\substack{long\\wire}} = -Q_T\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_{\substack{long\\wire}}$ with $\vec{v}_L = v_o\hat{x} =$ constant, independent of position and time!

Because this apparatus is such that the total force acting on the test charge, Q_T is continuously nulled out, i.e. $\vec{F}_{ToT}(x_L,t) = F_{Lorentz}(x_L,t) + \vec{F}_{applied}(x_L,t) = 0 \;\; \forall \; x$ and t then

$$\boxed{\vec{F}_{applied}(x_L,t) = -F_{Lorentz}(x_L,t)} \;\; \forall \; x \text{ and } t$$

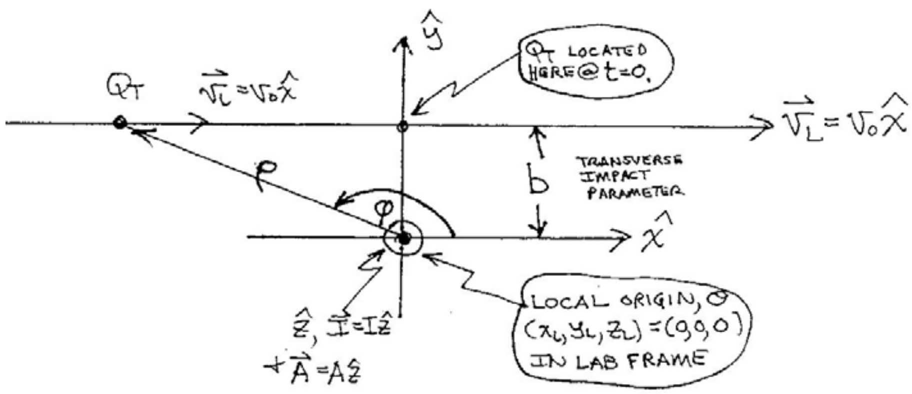
Also, because the pc-controlled Millikan oil-drop apparatus is in the rest frame of the test charge Q_T, the applied electric field is $\boxed{\vec{E}_{applied}\left(x_L,t\right)=+\frac{\partial\vec{A}_R\left(x_L,t\right)}{\partial t}}$.

Since: $\boxed{\vec{F}_{ToT}=\vec{F}_R+F_{applied}=0}$ in the rest frame of the test charge (and in the lab frame)

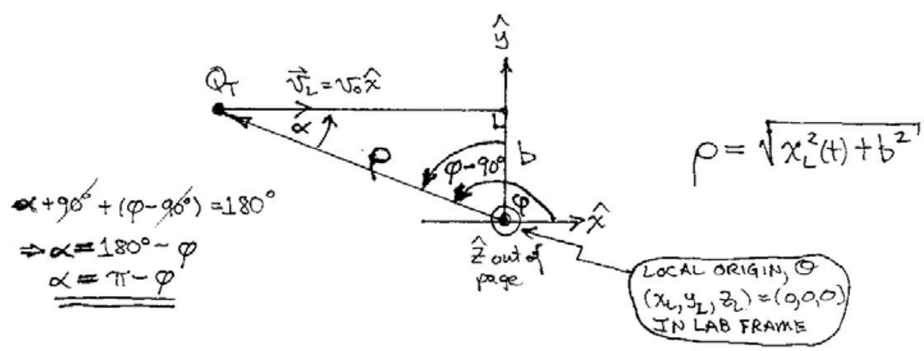
Then: $\boxed{\vec{F}_{applied}=-\vec{F}_R}$ or: $\boxed{Q_T\vec{E}_{applied}\left(x_L,t\right)=+Q_T\frac{\partial\vec{A}_R\left(x_L,t\right)}{\partial t}}$

But: $\boxed{\vec{E}_{applied}\left(x_L,t\right)=-\left[\frac{\Delta V_{applied}\left(x_L,t\right)}{d}\right]\hat{z}=+\frac{\partial\vec{A}_R\left(x_L,t\right)}{\partial t}}$

How does $\vec{A}_{\substack{long\\wire}}$ change with time, for $\left|\vec{v}_L\right|=\left|v_0\hat{x}\right|\ll c$ in this experimental setup??



n.b. drawn for $t<0,\ x_L=v_0t<0$:



The x_L - lab position of the test charge, Q_T as a function of time is:

$$\boxed{x_L\left(t\right)=0+v_ot=v_ot} \qquad \left(x_L\left(t=0\right)\equiv 0\right) \qquad \left(\text{for } v_L\ll c\right)$$

And: $\frac{-x_L\left(t\right)}{\rho}=\cos\left(\alpha\right)=\cos\left(\pi-\varphi\right)=\overbrace{\cos\pi}^{=-1}\cos\varphi+\overbrace{\sin\pi}^{=0}\sin\varphi=-\cos\varphi$

$$\boxed{\cos\varphi(x_L,t)=\frac{x_L(t)}{\rho(x_L,t)}=\frac{v_o t}{\sqrt{v_o^2t^2+b^2}}}$$ $\Leftarrow$ This expression works correctly for $\varphi(x_L,t)$ for $t<0$ and $t>0$.

with: $\rho(x_L,t)=\sqrt{x_L{}^2(t)+b^2}=\sqrt{v_o^2t^2+b^2}$

<u>Now if</u>: $\vec{A}_{\substack{long\\wire}}(\rho)\simeq\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{\rho}\right)\hat{z}$ for $\rho\ll L$

<u>And</u>: $\rho(x_L,t)=\sqrt{x_L^2(t)+b^2}=\sqrt{v_o^2t^2+b^2}$

<u>Then</u>: $\vec{A}_{\substack{long\\wire}}(x_L,t)\simeq\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{\rho(x_L,t)}\right)\hat{z}=\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{\sqrt{x_L^2(t)+b^2}}\right)\hat{z}$ for $\rho\ll L$

$$\vec{A}_{\substack{long\\wire}}(x_L,t)=\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{\sqrt{v_o^2t^2+b^2}}\right)\hat{z}\quad\text{for }\rho\ll L$$

<u>Then</u>: $\frac{\partial\vec{A}^{long}_{wire}(x_L,t)}{\partial t}=\left(\frac{\mu_o}{2\pi}\right)I\frac{\partial}{\partial t}\left\{\ln\left(\frac{2L}{\sqrt{v_o^2t^2+b^2}}\right)\right\}\hat{z}$

<u>Now</u>: $\frac{\partial}{\partial t}\left[\ln\left(\frac{2L}{\sqrt{v_o^2t^2+b^2}}\right)\right]=\frac{\sqrt{v_o^2t^2+b^2}}{2L}\frac{\partial}{\partial t}\left\{\frac{2L}{\sqrt{v_o^2t^2+b^2}}\right\}$

$$=\sqrt{v_o^2t^2+b^2}\,\frac{\partial}{\partial t}\left\{\frac{1}{\sqrt{v_o^2t^2+b^2}}\right\}=\sqrt{v_o^2t^2+b^2}\,\frac{\partial}{\partial t}\left\{\left[v_o^2t^2+b^2\right]^{-1/2}\right\}$$

$$=\frac{1}{2}*\left(2tv_o^2\right)\sqrt{v_o^2t^2+b^2}*\left[v_o^2t^2+b^2\right]^{-3/2}$$

$$=-v_o\left(v_ot\right)\left[v_o^2t^2+b^2\right]^{1/2}*\left[v_o^2t^2+b^2\right]^{-3/2}$$

$$=-v_o\left(v_ot\right)\left[v_o^2t^2+b^2\right]^{-1}=-v_o\frac{v_ot}{\left[v_o^2t^2+b^2\right]}$$

$$=-v_o\underbrace{\left\{\frac{v_ot}{\sqrt{v_o^2t^2+b^2}}\right\}}_{=+\cos\varphi(t)}\underbrace{\frac{1}{\sqrt{v_o^2t^2+b^2}}}_{=\rho(t)}=-\frac{v_o\cos\varphi(t)}{\rho(t)}\qquad\text{with: }\left(x_L=v_ot\right)$$

$$\therefore\ \boxed{\frac{\partial\vec{A}_{\substack{long\\wire}}(x_L,t)}{\partial t}=-\left(\frac{\mu_o}{2\pi}\right)Iv_o\frac{\cos\varphi(t)}{\rho(t)}\hat{z}}$$

$$\therefore\ \boxed{\vec{E}_R(t)=-\frac{\partial\vec{A}_{\substack{long\\wire}}(x_L,t)}{\partial t}=+\left(\frac{\mu_o}{2\pi}\right)Iv_o\frac{\cos\varphi(t)}{\rho(t)}\hat{z}}$$

With: $\cos\varphi(t) = \dfrac{x_L(t)}{\rho(t)} = \dfrac{v_o t}{\sqrt{v_o^2 t^2 + b^2}}$

$\rho(t) = \sqrt{x_L^2(t) + b^2} = \sqrt{v_o^2 t^2 + b^2}$

Then: $$\boxed{\vec{F}_R(t) = Q_L \vec{E}_R(t) = -Q_T \frac{\partial \vec{A}_{\substack{long\\wire}}}{\partial t} = +Q_T \left(\frac{\mu_o}{2\pi}\right) \frac{v_o I}{\rho(t)} \cos\varphi(t)\,\hat{z}}$$

Thus we see that: $\vec{F}_R(t) = \vec{F}_{Loretnz}(t)$!!! All three expressions agree with each other!!

i.e. $$\boxed{\vec{v}_L \times \vec{B}_L = -\left(\vec{v}_L \bullet \vec{\nabla}\right)\vec{A}_L = -\frac{\partial \vec{A}_R}{\partial t}} \text{ for } v_L \ll c$$

rest frame of Q_T

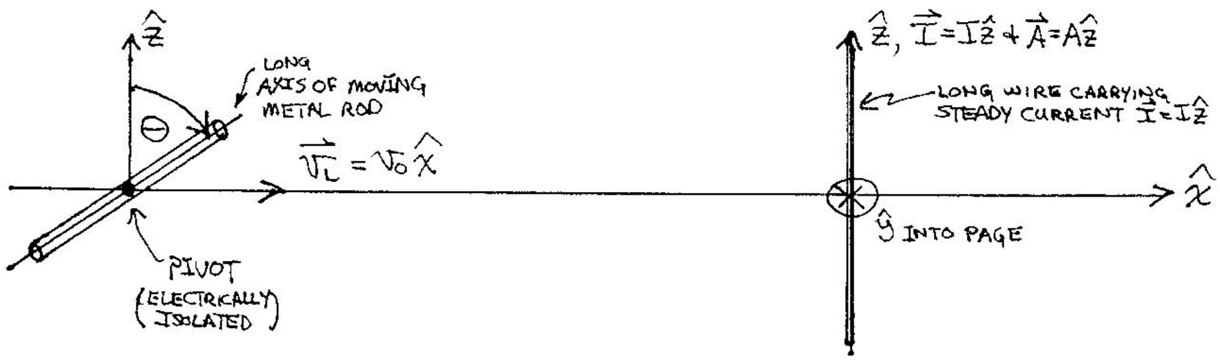
Experiment #2: Moving Metal Rod

Instead of using a single test charge Q_T moving with velocity $\vec{v}_L = v_o\hat{x}$ $(v_L \ll c)$ in the lab frame in proximity to a long wire carrying a steady current $\vec{I} = I\hat{z}$ (at rest in the lab frame), let us consider a conducting, non-magnetic (electrically neutral) uncharged metal rod (e.g. made of copper) of length ℓ and radius a (with volume $V = \pi a^2 \ell$ and mass $m = \rho_{cu} V = \rho_{cu} \pi a^2 \ell$) moving with constant / uniform velocity $\vec{v}_L = v_o\hat{x}$ $(v_o \ll c)$ in the lab frame, again in proximity to this same long wire carrying steady current $\vec{I} = I\hat{z}$.

Here (again), suppose e.g. that the center of mass of the conducting metal rod is mechanically constrained so that the center of mass of the metal rod maintains its initial velocity vector $\vec{v}_L = v_o\hat{x}$ at all time, independent of its position in the lab frame $\left(\vec{x}_L(t) = v_o t\hat{x}\right)$ with time.

However, the design of the mechanical fixture that imposes this constraint is such that it allows the conducting non-magnetic metal rod to rotate / orient itself freely (i.e. without any friction) at any arbitrary angle (θ, φ) about its center of mass. The metal fixture also electrically isolates (i.e. insulates) the rod from everything else.

When the metal rod is far away from the current carrying wire, the initial orientation of the metal rod is such that the long axis of the rod e.g. lies in the x-z plane, making an initial angle θ with respect to the $\hat{z}$ -axis:



Due to the conducting metal rod's motion relative to the long wire carrying steady current $\vec{I} = I\hat{z}$, an *EMF*, $\varepsilon(x_L, t)$ will be induced in the metal rod moving with $\vec{v}_L = v_o\hat{x}$ $(v_o \ll c)$:

$$EMF, \varepsilon(x_L, t) = \Delta V = V_B - V_A = \int_A^B \vec{f}_A \bullet d\vec{\ell}$$

Where: $$\vec{f}_A(x_L, t) \equiv \frac{\vec{F}(x_L, t)}{Q_T} = \vec{E}_R(x_L, t) = -\frac{\partial \vec{A}_{\substack{long\\wire}}(x_L, t)}{\partial t} = \left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho(x_L, t)}\cos\varphi(x_L, t)\,\hat{z}$$

And: $$\rho(x_L, t) = \sqrt{x_L^2(t) + b^2} = \sqrt{v_o^2 t^2 + b^2}$$

$$\cos\varphi(x_L, t) = \frac{x_L t}{\rho(x_L, t)} = \frac{v_o t}{\sqrt{v_o^2 t^2 + b^2}}$$

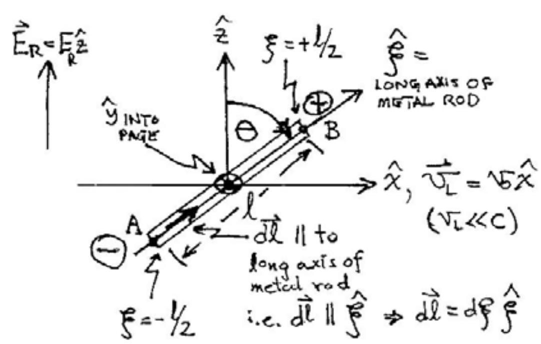
And where $d\vec{\ell}$ is || to the long axis of the metal rod.

$\hat{z}\bullet\hat{\xi} = \cos\theta$

n.b. $\vec{f}_A \bullet d\vec{\ell} = E_R \hat{z} \bullet d\xi\hat{\xi}$

$\qquad = E_R d\xi\left(\hat{z}\bullet\hat{\xi}\right) \neq 0$

Implies $\hat{\xi}$ *must* have some non-zero component || $\hat{z}$.



Now, for simplicity's sake, let us assume that the dimensions of the conducting metal rod are such that:

Rod radius $a \ll$ rod length $\ell \ll$ transverse impact parameter b (in $\hat{y}$ direction) $\leq \rho(x_L, t)$

$$= \frac{x_L(t)}{\sqrt{x_L^2(t) + b^2}}$$

$$= \frac{v_o t}{\sqrt{v_o^2 t^2 + b^2}}$$

Then $\vec{F}(x_L,t) = Q\vec{E}_R(x_L,t)$ and hence $\vec{E}_R(x_L,t)$ will be ≈ constant over the spatial extent / dimensions of the conducting rod, for a given location $x_L(t) = v_o t$ in the lab frame. (This simply makes the line integral $\int_A^B \vec{f}_A \bullet d\vec{\ell}$ easy to carry out):

$$EMF \quad \varepsilon(x_L,t) = \Delta V = V_B - V_A = -\int_A^B \vec{E}_R(x_L,t)\bullet d\vec{\ell} = -E_R(x_L,t)\cos\theta \int_{\xi=-\frac{\ell}{2}}^{\xi=+\frac{\ell}{2}} d\xi$$

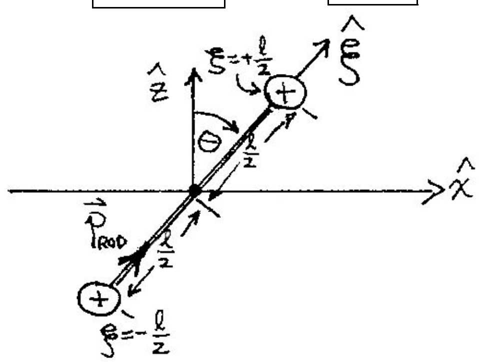
$$EMF \quad \varepsilon(x_L,t) = -\vec{E}_R(x_L,t)\,\ell\cos\theta(x_L,t) = \left(\frac{\mu_o}{2\pi}\right)\frac{v_o I\ell}{\rho(x_L,t)}\cos\theta(x_L,t)\cos\varphi(x_L,t)$$

$$V_A\left(\xi = -\frac{\ell}{2},(x_L,t)\right) = -\frac{1}{2}E_R(x_L,t)\,\ell\cos\theta(x_L,t) \Rightarrow \text{(−) charge builds up at point A } \left(\xi = -\ell/2\right)$$

$$V_B\left(\xi = +\frac{\ell}{2},(x_L,t)\right) = +\frac{1}{2}E_R(x_L,t)\,\ell\cos\theta(x_L,t) \Rightarrow \text{(+) charge builds up at point B } \left(\xi = +\ell/2\right)$$

Once an *EMF*, $\varepsilon(x_L,t)$ is induced in the conducting metal rod (−) charge at one end, (+) charge at the opposite end (see above figure), then the conducting metal rod then has an electric dipole moment $\vec{p}_{rod}(x_L,t)$ associated with it:

$$\vec{p}_{rod} = Q\vec{\ell} \text{ where } \vec{\ell} = \ell\hat{\xi} \Rightarrow p_{rod} = Q\ell\hat{\xi}$$



The magnitude of the charge Q induced at the ends of the conducting metal rod can be obtained from energy considerations:

$$U_{rod} = \frac{1}{2}QV = \frac{1}{2}Q\varepsilon = \int_{\substack{all\\space}} u\,d\tau' = \frac{1}{2}\int_{\substack{all\\space}} \vec{D}_{rod}\bullet\vec{E}_{rod}\,d\tau'$$

Where $\vec{E}_{rod} \approx \left(\frac{\varepsilon}{\ell}\right)\hat{\xi}$ is the *EMF*-induced electric field in the conducting metal rod

and $\vec{D}_{rod} = \varepsilon_{rod}\vec{E}_{rod}$ is the electric displacement associated with this $\vec{E}$-Field and

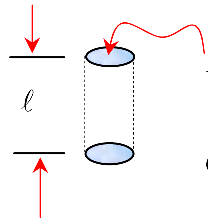
$\varepsilon_{rod} \simeq 3\varepsilon_o$ for e.g. copper metal, ∴ $U_{rod} \simeq \frac{3\varepsilon_o}{2}\int_{rod}\left(\frac{\varepsilon}{\ell}\right)^2 d\tau' \simeq \frac{3\varepsilon_o}{2}\left(\frac{\varepsilon}{\ell}\right)^2\left(\pi a^2\ell\right) = \frac{1}{2}Q\varepsilon$

$$\Rightarrow Q \simeq 3\varepsilon_o\frac{\pi a^2}{\ell}\varepsilon = \frac{\varepsilon_{rod}\pi a^2}{\ell}\varepsilon$$

But: $\boxed{Q = C_{rod}V = C_{rod}\varepsilon} \Rightarrow \boxed{C_{rod} \simeq \frac{\varepsilon_{rod}\pi a^2}{\ell}}$

(= "self-capacitance", analogous to self-inductance $\Phi_m = LI$!!!)

The long rod with induced charges Q @ ends behaves ≃ like parallel plate capacitor:



$A = \pi a^2$

$C \simeq \frac{\varepsilon A}{\ell} = \frac{\varepsilon\pi a^2}{\ell}$

Then: $\boxed{\vec{p}_{rod} = Q\vec{\ell} = Q\ell\hat{\xi} = \frac{\varepsilon_{rod}\pi a^2}{\ell}\ell\,\varepsilon mf\,\hat{\xi} = \varepsilon_{rod}\pi a^2\varepsilon mf\,\hat{\xi}}$

More explicitly: $\boxed{\vec{p}_{rod}(x_L,t) = \varepsilon_{rod}\pi a^2\varepsilon mf(x_L,t)\hat{\xi}(x_L,t)}$

The conducting metal rod thus has an induced electric dipole moment $\vec{p}_{rod}(x_L,t)$ associated with it. The external $\vec{E}$-field: $\boxed{\vec{E}_R(x_L,t) = -\frac{\partial\vec{A}_{\substack{long\\wire}}(x_L,t)}{\partial t} = \left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho(x_L,t)}\cos\varphi(x_L,t)\hat{z}}$

exerts a torque on the induced electric dipole moment of the conducting metal rod:

$$\boxed{\vec{\tau}(x_L,t) = \vec{p}_{rod}(x_L,t)\times\vec{E}_R(x_L,t)}$$

$$= \varepsilon_{rod}\pi a^2\,\underbrace{\varepsilon mf(x_L,t)}\,\hat{\xi}\times\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho(x_L,t)}\cos\varphi(x_L,t)\hat{z}$$

$$\varepsilon mf(x_L,t) = \left(\frac{\mu_o}{2\pi}\right)\frac{v_o I\ell}{\rho(x_L,t)}\cos\theta(x_L,t)\cos\varphi(x_L,t)$$

$$\boxed{\vec{\tau}(x_L,t) = \varepsilon_{rod}\pi a^2\ell\left[\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho(x_L,t)}\cos\varphi(x_L,t)\right]^2\cos\theta(x_L,t)\left(\hat{\xi}\times\hat{z}\right)}$$

Now: $\hat{\xi} = \cos\theta\hat{z} + \sin\theta\hat{x}$, $\quad \hat{\xi}\times\hat{z} = \cos\theta\,\overbrace{\hat{z}\times\hat{z}}^{=0} + \sin\theta\underbrace{\hat{x}\times\hat{z}}_{=-y} = -\sin\theta\hat{y}$

$\hat{x}\times\hat{y} = \hat{z} \qquad \hat{y}\times\hat{z} = \hat{x} \qquad \hat{z}\times\hat{x} = \hat{y}$ etc.

$$\therefore \quad \boxed{\vec{\tau}(x_L,t) = -\varepsilon_{rod}\pi a^2\ell\left[\left(\frac{\mu_o}{2\pi}\right)\frac{v_o I}{\rho(x_L,t)}\cos\varphi(x_L,t)\right]^2\cos\theta(x_L,t)\sin\theta(x_L,t)\hat{y}}$$

With: $\rho(x_L,t) = \sqrt{x_L(t)^2 + b^2} = \sqrt{v_o^2t^2 + b^2}$ $\qquad \vec{v}_L = v_o\hat{x}$ with $v_o \ll c$.

$\cos\varphi(x_L,t) = \frac{x_L(t)}{\rho(x_L,t)} = \frac{v_o t}{\sqrt{v_o^2t^2 + b^2}}$ $\qquad$ for $x_L(t) = v_o t$

This torque on the conducting rod is in the $-\hat{y}$ direction, which acts to try to align the induced electric dipole moment $\vec{p}_{rod}(x_L,t)$ with the electric field $\vec{E}_R(x_L,t) = -\dfrac{\partial \vec{A}(x_L,t)}{\partial t}$ (pointing in the $+\hat{z}$ direction). Thus, the conducting metal rod will rotate due to this torque. The rotational motion will be $\simeq$ fairly complex because:

Torque, $\boxed{\vec{\tau}(x_L,t) = \mathrm{I}_{rod}\vec{\alpha}(x_L,t)}$

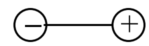
I_{rod} = moment of inertia of rod about pivot axis (center of mass) $\boxed{\mathrm{I}_{rod} = \frac{1}{12}m\left(\ell^2 + 3a^2\right)}$

$\boxed{\vec{\alpha}(x_L,t)}$ = angular acceleration, $\boxed{\vec{\omega}(x_L,t)}$ = angular velocity

$$\boxed{\theta(x_L,t) = \underbrace{\theta_o}_{@t=0} + \overset{=0}{\cancel{\omega_o}}\, t + \frac{1}{2}\alpha(x_L,t)t^2 = \theta_o + \frac{1}{2}\alpha(x_L,t)t^2} = \text{angular position}$$

$$\boxed{\theta(x_L,t) = \theta_o + \frac{1}{2}\frac{\left|\vec{\tau}(x_L,t)\right|}{\mathrm{I}_{rod}}t^2} \quad \text{use above expression for } \vec{\tau}(x_L,t)$$

Note that we could have instead accomplished the same results
e.g. using a real electric dipole (with electric dipole moment $\vec{p} = Q\vec{\ell}$)



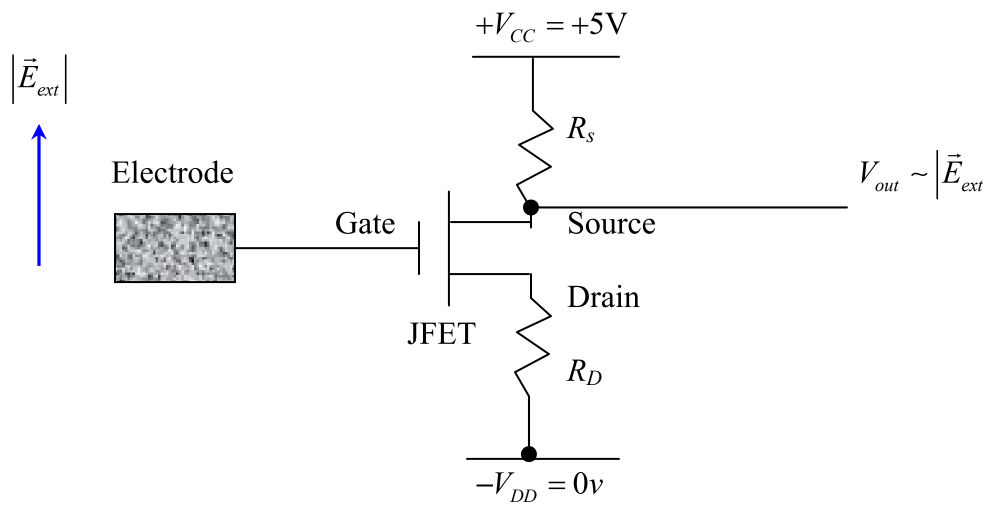
(e.g. a permanently polarized, thin electret-type dielectric rod, or a real electric dipole)

One could also use an initially unpolarized thin dielectric rod, which would then become polarized in the presence of the external electric field

$\vec{E}_R(x_L,t) = -\dfrac{\partial \vec{A}(x_L,t)}{\partial t} \underset{\substack{long\\wire}}{=} \left(\dfrac{\mu_o}{2\pi}\right)\dfrac{v_o I}{\rho(x_L,t)}\cos\varphi(x_L,t)\hat{z}$ with polarization $\vec{\mathrm{P}}(x_L,t)$ and total electric dipole moment $\boxed{\vec{p}_{rod}(x_L,t) = \mathrm{P}(x_L,t)\Delta V_{rod} = \mathrm{P}(x_L,t)\pi a^2\ell}$

(assuming uniform polarization i.e. uniform $\vec{E}_R(x_L,t)$)

One could instead simply directly measure the electric field $\vec{E}_R(x_L, t)$ – a variety of devices / techniques exist to do this / accomplish this. A very simple method is to use a JFET (junction FET) to measure the $\vec{E}$-field (A JFET has extremely high input impedance)



These "tools" could be used to have fun exploring situations associated with the other "force" terms are involved

$$\vec{E}_R = \underbrace{\vec{\nabla}\left(\vec{A}\cdot\vec{v}_L\right)}_{(1)} - \underbrace{\vec{A}\left(\vec{A}\times\vec{v}_L\right)}_{(2)} - \underbrace{\left(\vec{A}\cdot\vec{\nabla}\right)\vec{v}_L}_{(3)} - \underbrace{\left(\vec{v}_L\cdot\vec{\nabla}\right)\vec{A}}_{(4)}$$

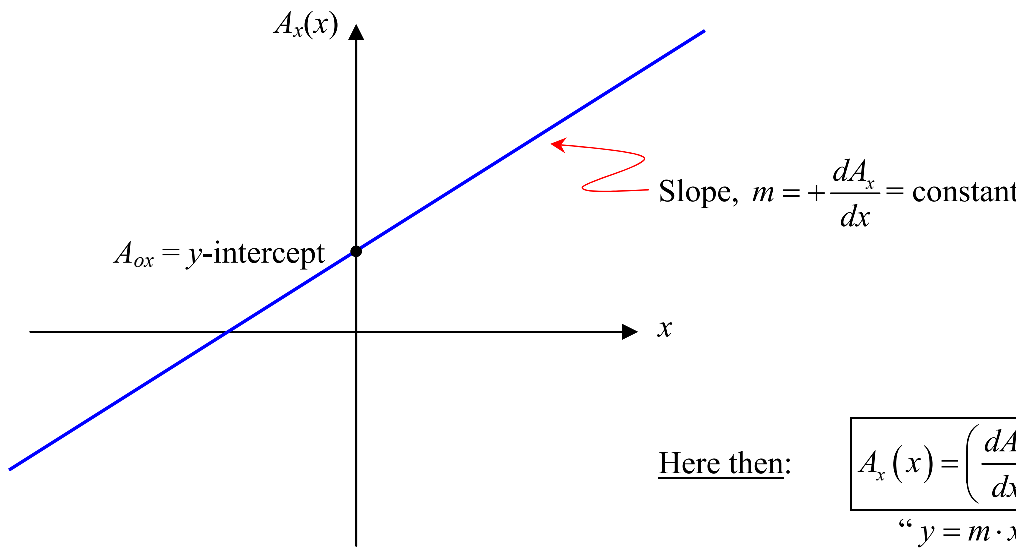
We have examined just this one term, here in these lecture notes…

Appendix A

Let us imagine a region of space where a static gradient of the magnetic vector potential $\vec{A}(\vec{r})$ exists, i.e.

In Cartesian Coordinates: $\vec{\nabla}\,\vec{A}(\vec{r}) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$

For example, consider a simple one-dimensional case where $A_x(x)$ increases linearly with x (only):



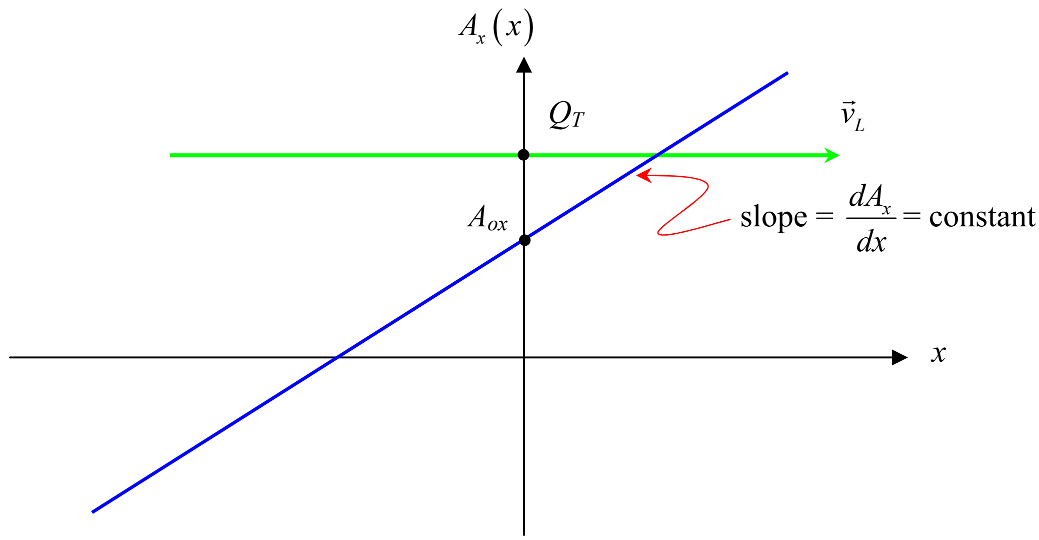
Here then: $$A_x(x) = \left(\frac{dA_x}{dx}\right)\cdot x + A_{ox}$$
"$y = m \cdot x + b$"

In general, A_y and A_z could also similarly depend on x (as well as y and z) and also, A_x could also similarly depend on y and z.

Now this magnetic vector potential $\vec{A}_L(\vec{r})$ is defined in e.g. the lab frame of reference (where $\vec{A}$ is at rest) i.e. it is a static vector field in this non-inertial reference frame, which varies spatially from one place to the next, as defined by the above formula.

Now imagine an electric test charge Q_T moving at constant relative velocity $\vec{v}_L$ in the lab frame $\left(|v_L| \ll c\right)$ moving through this spatially varying $\vec{A}$-field. In the rest frame of the electric charge, the $\vec{A}_Q$-field seen by the electric charge will appear to be varying in time!!!

In our simple one-dimensional example, with $\vec{v}_L = v_{ox}\hat{x}$ and $\vec{x}_L(t) = \vec{v}_L t = v_{ox}t\hat{x}$ or simply: $x_L(t) = v_{ox}t$



The charged particle sees (in its rest frame):

$$A_x(t)_Q = \left(\frac{dA_x}{dx}\right)_{lab} \cdot x_L(t) + A_{ox_{lab}} = \left(\frac{dA_x}{dx}\right)_{lab} \cdot v_{ox} t + A_{ox_{lab}}$$

Then here: $$\left.\frac{\partial A_x}{\partial t}\right|_Q = v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}$$ (for $v_{ox} \ll c$)

Generalizing these 2 equations:

$$\vec{A}_Q(t) = \left(\vec{v}_L t \cdot \vec{\nabla}\right)\vec{A}_{lab}(\vec{r}) + \vec{A}_{o_{lab}} \qquad \vec{r}_L = \vec{r}_o + \vec{v}_L t$$

$$\frac{\partial \vec{A}_Q(t)}{\partial t} = \left(\vec{v}_L \cdot \vec{\nabla}\right)\vec{A}_{lab}(\vec{r}) \qquad (\vec{r}_o = 0)$$

If the charged particle is also accelerating: $$\vec{r}_L(t) = v_o t\hat{x} + \frac{1}{2}\vec{a}_o t^2 \qquad (\vec{r}_o = 0)$$

$$\vec{A}_Q(t) = \left(\left(v_o t\hat{x} + \frac{1}{2}\vec{a}_o t^2\right)\cdot\vec{\nabla}\right)\vec{A}_{lab}(\vec{r}) + \vec{A}_{o_{lab}}$$

$$\frac{\partial \vec{A}_Q(t)}{\partial t} = \left(\left(v_o\hat{x} + \vec{a}_o t\right)\cdot\vec{\nabla}\right)\vec{A}_{lab}(\vec{r})$$

Then the electric field in the rest frame of the electric charge:

$$\vec{E}_Q = -\frac{\partial \vec{A}_Q(t)}{\partial t} \equiv \vec{f}_A = \frac{\vec{F}_A}{Q_T} = \frac{1}{Q_T}\frac{\partial \vec{p}_A}{\partial t}$$

The induced EMF, $$\varepsilon = \Delta V = V_B - V_A = \int_A^B \vec{f}_A \cdot d\vec{\ell} = \int_A^B \frac{\partial \vec{A}_Q(t)}{\partial t} \cdot d\vec{\ell}$$ (Volts)

Let us consider a conducting metal rod of length ℓ moving with constant velocity $\vec{v}_L = v_{ox}\hat{x}$ in a one-dimensional magnetic vector potential $\boxed{\vec{A}_L(\vec{r}) = A_{x_{lab}}(x)\hat{x} = \left(\frac{dA_x}{dx}\right)_{lab} x\hat{x} + A_{ox_{lab}}\hat{x}}$ in the lab frame. Note that (here), the long-axis of the conducting rod is || to $\vec{v}_L = v_{ox}\hat{x}$.

Then in the rest frame of the metal rod, the free charges in the metal rod see an $\vec{A}$-field:

$$\boxed{\vec{A}_Q(t) = \vec{v}_L t \bullet \left(\frac{dA_x}{dx}\right)_{lab} \hat{x}\hat{x} + A_{ox_{lab}}\hat{x} = v_{ox} t \left(\frac{dA_x}{dx}\right)_{lab} \hat{x} + A_{ox_{lab}}\hat{x}}$$

And:

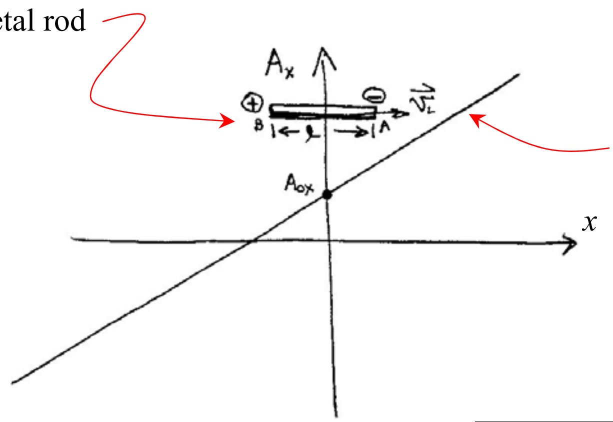
$$\boxed{\frac{\partial \vec{A}_Q(t)}{\partial t} = \frac{\partial A_x(t)}{\partial t}\hat{x} = v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\hat{x}}$$

Then:

$$\boxed{\vec{E}_R \equiv \vec{f}_A = \frac{\vec{F}_A}{Q} = \frac{d\vec{p}}{dt} = -\frac{\partial \vec{A}_Q}{\partial t} = -v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\hat{x}}$$

Moving metal rod

$A_x(x) = \left(\frac{dA_x}{dx}\right)_{lab} x + A_{ox}$ in lab frame



Slope $\left(\frac{dA_x}{dx}\right)_{lab}$ = constant

Here, the force acting on each free charge Q is: $\boxed{\vec{F} = Q\vec{E} = -Q\frac{\partial \vec{A}}{\partial t} = -Qv_{ox}\left(\frac{dA_x}{dx}\right)\hat{x}}$

If $\left(\frac{dA_x}{dx}\right)_{lab} > 0$, then $\vec{F} = (\ \ )(-\hat{x})$ direction, for charge $Q > 0$.

i.e. $+ve$ charges pushed to $-\hat{x}$ end of rod.
-ve charges pulled toward $+\hat{x}$ end of rod.

EMF $\boxed{\varepsilon = \Delta V = V_B - V_A = \int_A^B \vec{f}_A \bullet d\vec{\ell} = -v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\int_A^B d\vec{\ell}}$ where $d\vec{\ell} = -dx\hat{x}$ (here)

EMF $\boxed{\varepsilon = +v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\ell}$ Volts (here) $\quad = \int_{x=-\ell/2}^{x=+\ell/2}(-dx\hat{x}) = -\ell\hat{x}$ (here)

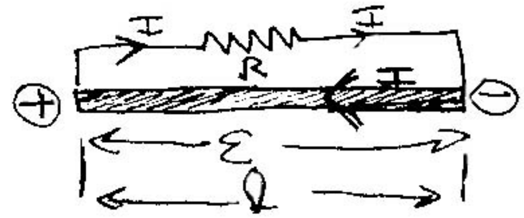
We see that the axis of moving metal rod must be <u>parallel</u> to the gradient of the magnetic vector potential in order to obtain a (maximally) non-zero EMF!!

∴ A moving metal rod can be used to detect <u>spatial gradients</u> in $\vec{A}$!!!

Again, if one connect the ends of the moving metal rod to an external electrical circuit (e.g. a resistor), an induced current will flow:

Current $\boxed{I = \frac{\varepsilon}{R} = \frac{\Delta V}{R} = v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\ell}$ Amps (here)

Power $\boxed{\mathrm{P} = \frac{\Delta V^2}{R} = \frac{\left(v_{ox}\left(\frac{dA_x}{dx}\right)_{lab}\ell\right)^2}{R}}$ Watts (here)



Note that the power P, induced current I, induced EMF ε all vanish when $v_{ox} \to 0$.