Supplemental Handout # 6

Symmetry Properties of Electromagnetism

The various field and source quantities, such as $\vec{E}, \vec{D}, \vec{\mathrm{P}}, \vec{H}, \vec{B}, \vec{\mathrm{M}}$ e.g. $\vec{J}_e, \vec{J}_m, \vec{v}, \varepsilon_o, \mu_o, \vec{L}, \vec{S}, \ldots$ etc. have various symmetry properties under symmetry operations such as:

$P \equiv$ Parity (Space-Inversion, $\vec{r} \to -\vec{r}$) → Reflection in a mirror
$T \equiv$ Time Reversal (e.g. particle motion, but run backwards in time)
$C \equiv$ Electric Charge Conjugation (Charge of Particle → Charge of Antiparticle, e.g. $e^- \to e^+$)
M = Magnetic Charge Conjugation (Magnetically Charged Particle → Magnetically charged antiparticle, e.g. $g_N \to g_S$)

The field and source quantities mentioned above fall into various generic mathematical classes of objects, or quantities:

1) Scalar quantities under a given symmetry transformation, designated ϕ.
2) Pseudoscalar quantities under a given symmetry transformation, designated *p*.
3) Polar Vector quantities under a given symmetry transformation, designated $\vec{V}$.
4) Axial, or Pseudo-Vector quantities under a given symmetry transformation, designated $\vec{A}$.
5) Nth rank covariant / contravariant tensors under a given symmetry transformation, $T_{\mu\nu}, T^{\mu\nu}, T_{\mu}^{V}, T_{\sigma\mu\nu}, T_{\alpha\beta\gamma}^{\gamma\mu\nu}, \ldots$

Note that, e.g.:
Electric charge q is ODD under electric charge conjugation: $Ce^- = e^+$
(i.e. q behaves as a pseudoscalar quantity p under C)

Magnetic charge g_m is ODD under magnetic charge conjugation: $Mg_m^- = g_m^+$
(i.e. g_m behaves as a pseudoscalar quantity p under M)

However note also that, e.g.:
Electric charge is EVEN under magnetic charge conjugation: $Me^- = e^-$
(i.e. q behaves as a scalar quantity ϕ under M.)

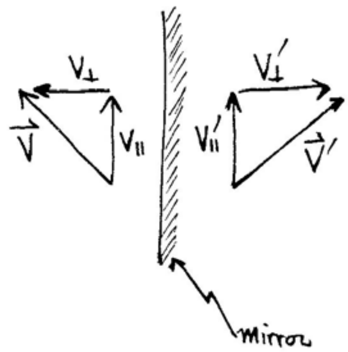
Magnetic charge is EVEN under electric charge conjugation: $Cg_m^- = g_m^-$
(i.e. g_m behaves as a scalar quantity ϕ under C.)

Note also that (if $\exists$ no magnetic charges) the combined operations $CPT = 1$ (in any order) (i.e. CPT = identity operator). If have magnetic charges, then $CPTM = 1$ (in any order).

In order to understand the distinction between Polar Vectors and Axial (or Pseudo)-Vectors under a specific symmetry transformation, consider parity P (i.e. mirror-reflection) operation:

Polar Vector $\vec{V}$

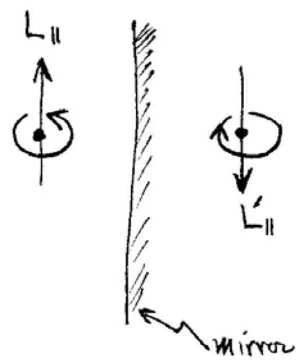
e.g. direction vector $\vec{r}$
P, Parity is space-inversion
i.e. $\vec{r} \rightarrow -\vec{r}$
$x \rightarrow -x$
$y \rightarrow -y$
$z \rightarrow -z$



$\parallel$-component of $\vec{V}$ unchanged under Parity
$\perp$-component of $\vec{V}$ changes sign under Parity

i.e. $PV_\parallel = V_\parallel$
$PV_\perp = V_\perp$

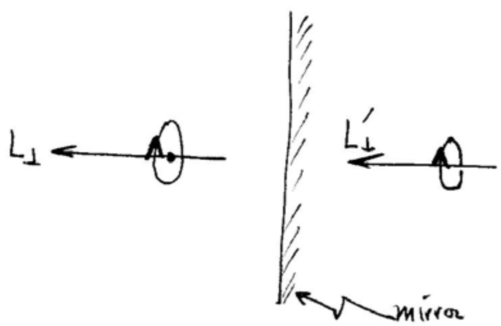
Axial (or Pseudo) Vector $\vec{A}$
e.g. spinning top –
angular momentum $\vec{L}$



$\parallel$-components of $\vec{A}$ reversed under Parity
$\perp$-components of $\vec{A}$ unchanged under Parity

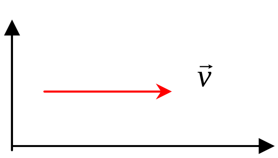
i.e. $PA_\parallel = -A_\parallel$

$PA_\perp = A_\perp$



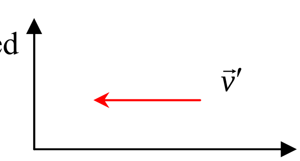
Time Reversal:

Particle Moving with Velocity $\vec{v}$



$\vec{v}$ time

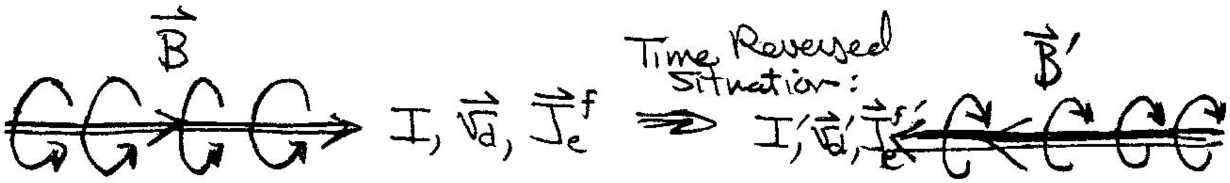
Time-reversed situation:



$\vec{v}'$ time

$\vec{v}' = T\vec{v} = -\vec{v}$ (velocity is odd under Time reversal)

Electric Current Flowing in a Long Wire:



$\therefore \quad T\,\vec{v}_d = -\vec{v}_d \qquad\qquad \vec{J}_e^{\,free} = n_e q \vec{v}_d \qquad\qquad Te^- = e^-$ (q is even under T)

$T\,\vec{J}_e^{\,free} = -\vec{J}_e^{\,free}$ — all — $I = \vec{J}_e^{\,free} \bullet \vec{A}rea$

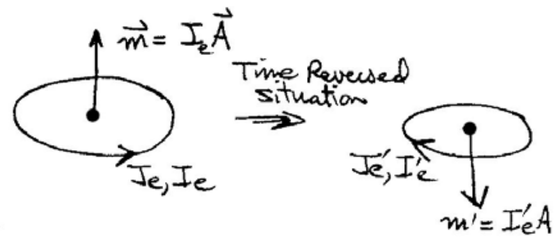
$TI = -I$ — odd — $\vec{B} = \dfrac{\mu_o}{4\pi}\displaystyle\int \dfrac{\vec{I} \times d\vec{\ell}}{r^2}$

$T\vec{B} = -\vec{B}$ — under — $\vec{B} = \mu\vec{H} \qquad\qquad T\mu = \mu$ (μ is even under T)

$T\vec{H} = -\vec{H}$ — T — $\vec{B} = \mu_o\vec{H} + \vec{\mathrm{M}}$

$T\vec{\mathrm{M}} = -\vec{\mathrm{M}}$

Current Flowing in a Loop:



$\therefore \quad T\vec{m} = -\vec{m}$ (mag. dipole moment)

And $\quad T\vec{\mathrm{M}} = -\vec{\mathrm{M}}$ (magnetization)

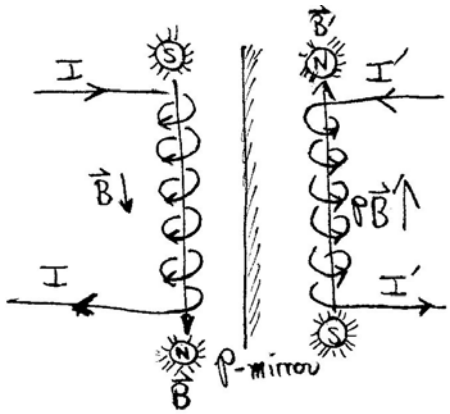
By considering a parallel-plate capacitor, it can be seen that $\vec{E}, \vec{D}$ and $\vec{\mathrm{P}}$ fields, $\vec{p}$ = electric dipole moment, ε = permittivity, etc. are all <u>even</u> under time reversal.

$T\vec{E} = \vec{E}$ (e.g. $\vec{E} = \dfrac{1}{4\pi\varepsilon_o}\dfrac{q}{r^2}\hat{r}$) $\qquad Tq = +q$ (even under T)

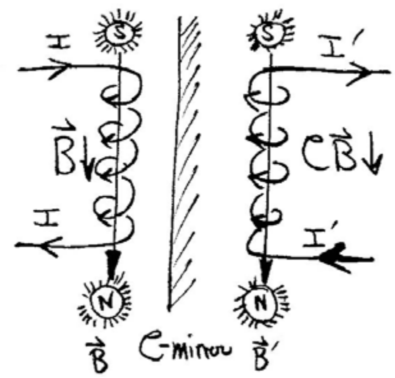
$T\vec{\mathrm{P}} = \vec{\mathrm{P}}$

$T\vec{D} = \vec{D} = \varepsilon\vec{E} \qquad\qquad T\varepsilon = \varepsilon$

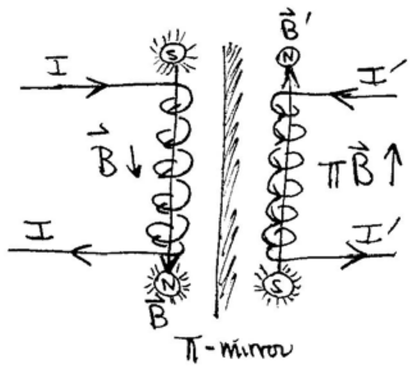
Parity and Magnetic Fields



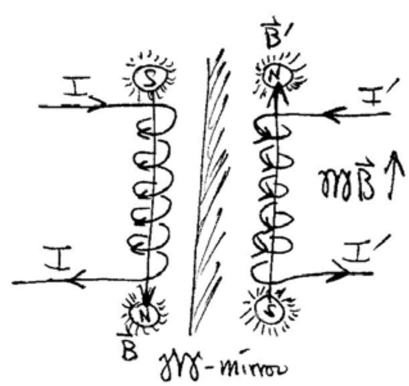
Electric Charge Conjugation & Magnetic Fields



Time Reversal & Magnetic Fields



Magnetic Charge Conjugation & Magnetic Fields



Summary of Symmetry Properties of Kinematic & Electromagnetic Quantities

ϕ ≡ Scalar Quantity p ≡ Pseudoscalar Quantity $\vec{V}$ = Polar Vector $\vec{A}$ = Axial Vector (Pseudo-Vector)

	KINEMATIC AND/OR ELECTROMAGNETIC QUANTITY	PARITY (SPACE INVERSION) $\mathcal{P}(\vec{r} \to -\vec{r})$		CHARGE (ELECTRIC) CONJUGATION $\mathcal{C}(q_e \to -q_e)$		TIME REVERSAL $\mathbb{T}(t \to -t)$		MAGNETIC CHARGE CONJUGATION $\mathfrak{M}(g_m \to -g_m)$	
KINEMATICAL	$\vec{r}$	−	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$\vec{v} = \frac{d\vec{r}}{dt}$; $\vec{p} = m\vec{v}$	−	$\vec{V}$	+	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$
	$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$; $\vec{F}_{mech} = m\vec{a}$	−	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$\vec{L} = \vec{r} \times \vec{p}$	+	$\vec{A}$	+	$\vec{V}$	−	$\vec{A}$	+	$\vec{V}$
	$\vec{T} = \vec{r} \times \vec{F}$	+	$\vec{A}$	+	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$KE = p^2/2m$, $PE = \frac{1}{2}kx^2$, $WORK = \vec{F}\cdot\vec{x}$	+	ϕ	+	ϕ	+	ϕ	+	ϕ
E+M CHARGES	$e, \lambda_e, \sigma_e, \rho_e$ (ALSO: $e g_m = nh$)	+	ϕ	−	p	+	ϕ	+	ϕ
	$g_m, \lambda_m, \sigma_m, \rho_m$ $g_m = $ "$q_e v$"	−	p	+	ϕ	−	p	−	p
VACUUM + DIELECTRIC/MAGNETIC MATERIALS	$\epsilon_o, \epsilon, \chi_e, K_e \equiv \frac{\epsilon}{\epsilon_o} = \chi_e - 1$	+	ϕ	+	ϕ	+	ϕ	+	ϕ
	$\mu_o, \mu, \chi_m, K_m \equiv \frac{\mu}{\mu_o} = \chi_m - 1$	+	ϕ	+	ϕ	+	ϕ	+	ϕ
E+M CURRENTS	$\vec{I}_e, \vec{K}_e, \vec{J}_e = n q_e \vec{v}$	−	$\vec{V}$	−	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$
	$\vec{I}_m, \vec{K}_m, \vec{J}_m = n_m g_m \vec{v}$	−	$\vec{A}$	+	$\vec{A}$	+	$\vec{A}$	−	$\vec{A}$
ELECTRIC FIELD QUANTITIES	V_E	−	ϕ	−	ϕ	+	ϕ	+	ϕ
	GRADIENT, CURL, DIVERGENCE $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$	+	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$\vec{E} = -\vec{\nabla}V_E = \frac{1}{4\pi\epsilon_o}\frac{q_e}{r^2}\hat{r}$	−	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$\vec{D} = \epsilon\vec{E}$	−	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
	$\vec{P} = \vec{D} - \epsilon_o\vec{E} = \epsilon_o\chi_e\vec{E}$; $\vec{p} = Q_e\vec{d}$	−	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$	+	$\vec{V}$
MAGNETIC FIELD QUANTITIES	$\vec{B} = \frac{\mu_o}{4\pi}\frac{g_m}{r^2}\hat{r}$, $\frac{\mu_o}{4\pi}I_e\oint_c \frac{d\vec{l}\times\hat{r}}{r^2}$, $\vec{\nabla}\times\vec{A}$	−	$\vec{A}$	− for I_e (+ for g_m)	$\vec{A}$	−	$\vec{A}$	−	$\vec{A}$
	$\vec{H} = \frac{1}{\mu}\vec{B} = -\vec{\nabla}V_m$	−	$\vec{A}$	− for I_e (+ for g_m)	$\vec{A}$	−	$\vec{A}$	−	$\vec{A}$
	$\vec{M} = \vec{H} - \frac{1}{\mu_o}\vec{B} = \chi_m\vec{H}$, $\vec{m} = I\vec{a}_\perp = g_m\vec{d}$	−	$\vec{A}$	−	$\vec{A}$	−	$\vec{A}$	−	$\vec{A}$
	$\vec{A} = \frac{\mu_o}{4\pi}I_e\oint_c\frac{d\vec{l}}{r}$ MAGNETIC VECTOR POTENTIAL	−	$\vec{A}$	− (+ for g_m)	$\vec{A}$	−	$\vec{A}$	−	$\vec{A}$
	V_m	−	p	− (+ for g_m)	p	−	p	−	p
MAXWELL'S EQUATIONS	$\vec{\nabla}\cdot\vec{D} = \rho_e^{free}$	+	ϕ	−	p	+	ϕ	+	ϕ
	$\vec{\nabla}\cdot\vec{B} = \rho_m^{free}$	−	p	+	ϕ	−	p	−	p
	$\vec{\nabla}\times\vec{E} = -\vec{J}_m^{free} - \frac{\partial\vec{B}}{\partial t}$	−	$\vec{A}$	−	$\vec{A}$	+	$\vec{A}$	−	$\vec{A}$
	$\vec{\nabla}\times\vec{H} = \vec{J}_e^{free} + \frac{\partial\vec{D}}{\partial t}$	−	$\vec{V}$	−	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$
EM ENERGY FLOW, etc.	$\vec{S} = \vec{E}\times\vec{H}$ POYNTING'S VECTOR	+	$\vec{V}$	+ (− for g_m)	$\vec{V}$	−	$\vec{V}$	+	$\vec{V}$
	$T_{\mu\nu}$ MAXWELL'S STRESS TENSOR (RANK)	+	$\overleftrightarrow{T}$	+	$\overleftrightarrow{T}$	−	$\overleftrightarrow{T}$	+	$\overleftrightarrow{T}$