Supplemental Handout #3

Comments on the Auxilliary Field Relations

$\vec{D}(\vec{r}) = \varepsilon_o \vec{E}(\vec{r}) + \vec{\mathrm{P}}(\vec{r})$, $\vec{H}(\vec{r}) = \frac{1}{\mu_o}\vec{B}(\vec{r}) - \vec{\mathrm{M}}(\vec{r})$ and Deceptive Parallels

For electro<u>statics</u> / magneto<u>statics</u> problem, one must always keep in mind that in order <u>to fully specify any</u> vector field $\vec{F}(\vec{r})$, <u>both</u> the <u>divergence</u> and the <u>curl</u> of the vector field must be known (or specified):

i.e. $\vec{\nabla}\bullet\vec{F}(\vec{r}) =$ _______ and $\vec{\nabla}\times\vec{F}(\vec{r}) =$ _______

<u>For Electrostatics</u>:

$$\vec{D}(\vec{r}) = \varepsilon_o \vec{E}(\vec{r}) + \vec{\mathrm{P}}(\vec{r})$$

$$\underbrace{\vec{\nabla}\bullet\vec{D}(\vec{r})}_{=\rho_{free}(\vec{r})} = \underbrace{\varepsilon_o\vec{\nabla}\bullet\vec{E}(\vec{r})}_{=\rho_{TOT}(\vec{r})} + \underbrace{\vec{\nabla}\bullet\vec{\mathrm{P}}(\vec{r})}_{=\rho_{bound}(\vec{r})}$$

i.e. $\rho_{Tot}(\vec{r}) = \rho_{free}(\vec{r}) + \rho_{Bound}(\vec{r})$

$$\vec{\nabla}\bullet\vec{E}(\vec{r}) = \frac{1}{\varepsilon_o}\rho_{Tot}(\vec{r})$$

$$\vec{\nabla}\bullet\vec{\mathrm{P}}(\vec{r}) = -\rho_{bound}(\vec{r})$$

$$\vec{\nabla}\bullet\vec{D}(\vec{r}) = \rho_{free}(\vec{r})$$

<u>*But*</u>: $\vec{\nabla}\times\vec{D}(\vec{r}) = \varepsilon_o \underbrace{\vec{\nabla}\times\vec{E}(\vec{r})}_{=0} + \vec{\nabla}\times\vec{\mathrm{P}}(\vec{r})$

Always in electro<u>statics</u>

$\Rightarrow$ $\vec{\nabla}\times\vec{D}(\vec{r}) = \vec{\nabla}\times\vec{\mathrm{P}}(\vec{r})$

but $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$

(in electrostatics)

$\therefore$ $\vec{D}(\vec{r})$ *<u>cannot</u>* be "just like" $\vec{E}(\vec{r})$

<u>since</u>:

$\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ because

$\vec{E}(\vec{r})$ is a conservative field, i.e.

$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$; and $\vec{\nabla}\times\vec{\nabla}V(\vec{r}) = 0$ (always)

$\vec{\nabla}\times\vec{D}(\vec{r}) \neq 0$ necessarily (often <u>not</u>!) because $\vec{\nabla}\times\vec{\mathrm{P}}(\vec{r}) \neq 0$ (in general).

<u>For Magnetostatics</u>:

$$\vec{H}(\vec{r}) = \frac{1}{\mu_o}\vec{B}(\vec{r}) - \vec{\mathrm{M}}(\vec{r})$$

$$\underbrace{\vec{\nabla}\times\vec{H}(\vec{r})}_{=\vec{J}_{free}(\vec{r})} = \underbrace{\frac{1}{\mu_o}\vec{\nabla}\times\vec{B}(\vec{r})}_{=\vec{J}_{TOT}(\vec{r})} - \underbrace{\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r})}_{=\vec{J}_{bound}(\vec{r})}$$

i.e. $\vec{J}_{Tot}(\vec{r}) = \vec{J}_{free}(\vec{r}) + \vec{J}_{Bound}(\vec{r})$

$$\vec{\nabla}\bullet\vec{B}(\vec{r}) = \mu_o\vec{J}_{TOT}(\vec{r})$$

$$\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}) = \vec{J}_{bound}(\vec{r})$$

$$\vec{\nabla}\bullet\vec{H}(\vec{r}) = \vec{J}_{free}(\vec{r})$$

$\vec{\nabla}\bullet\vec{H}(\vec{r}) = \frac{1}{\mu_o}\underbrace{\vec{\nabla}\bullet\vec{B}(\vec{r})}_{=0} - \vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r})$

Always

$\Rightarrow$ $\vec{\nabla}\bullet\vec{H}(\vec{r}) = -\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r})$

but $\vec{\nabla}\bullet\vec{B}(\vec{r})$ (always)

$\therefore$ $\vec{H}(\vec{r})$ *<u>cannot</u>* be "just like" $\vec{B}(\vec{r})$

<u>since</u>:

$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_o\vec{J}_{TOT}(\vec{r})$

$\Rightarrow$ $\vec{B}(\vec{r})$ is <u>not</u> a conservative field

$\therefore$ $\vec{D}(\vec{r})$ is *<u>not</u>* (in general) a <u>conservative</u> field!!! $\Rightarrow \exists$ <u>no</u> Coulomb's Law for $\vec{D}(\vec{r})$!!!

The differential forms of these relations can be very misleading and confusing / deceptive!!!

In particular:
$\vec{\nabla}\bullet\vec{D}(\vec{r}) = \rho_{free}(\vec{r})$ does NOT mean $\vec{D}(\vec{r})$ depends ONLY on $\rho_{free}(\vec{r})$!!!
($\vec{D}(\vec{r})$ also depends on $\vec{P}(\vec{r})$ - see below *)

Likewise:
$\vec{\nabla}\times\vec{H}(\vec{r}) = \vec{J}_{free}(\vec{r})$ does NOT mean $\vec{H}(\vec{r})$ depends ONLY on $\vec{J}_{free}(\vec{r})$!!!
($\vec{H}(\vec{r})$ also depends on $\vec{M}(\vec{r})$ - see below *)

Using the { divergences / curls } of { $\vec{E}, \vec{P}, \vec{D}$ / $\vec{B}, \vec{M}, \vec{H}$ }, we implicitly assume the existence of space derivatives.

However, at the interface between { dielectric / magnetic } media, such space derivatives often do *not* exist!!!

Whenever we have to deal with the interface between two { dielectric / magnetic } media, we *must* use the integral forms of these relations, which are more general:

Electrostatics:

$$\varepsilon_o \int_{v'} \vec{\nabla}\bullet\vec{E}(\vec{r}')\,d\tau' = \varepsilon_o \oint_{S'} \vec{E}(\vec{r}')\bullet d\vec{a}' = Q_{TOT}^{encl}$$

Gives: $\left[E_2^{\perp} - E_1^{\perp}\right]\Big|_{\text{interface}} = \frac{1}{\varepsilon_o}\sigma_{TOT}\Big|_{\text{interface}}$

$$\int_{v'} \vec{\nabla}\bullet\vec{D}(\vec{r}')\,d\tau' = \oint_{S'} \vec{D}(\vec{r}')\bullet d\vec{a}' = Q_{free}^{encl}$$

Gives: $\left[D_2^{\perp} - D_1^{\perp}\right]\Big|_{\text{interface}} = \sigma_{free}\Big|_{\text{interface}}$

$$\int_{v'} \vec{\nabla}\bullet\vec{P}(\vec{r}')\,d\tau' = \oint_{S'} \vec{P}(\vec{r}')\bullet d\vec{a}' = Q_{bound}^{encl}$$

Gives: $\left[P_2^{\perp} - P_1^{\perp}\right]\Big|_{\text{interface}} = -\sigma_{bound}\Big|_{\text{interface}}$

$$\int_{S'} \vec{\nabla}\times E(\vec{r}')\bullet d\vec{a}' = \oint_{C'} E(\vec{r}')\bullet d\vec{\ell}' = 0$$

Gives: $E_2^{\parallel} = E_1^{\parallel}\Big|_{\text{interface}} \quad \left(\left[E_2^{\parallel} - E_1^{\parallel}\right]\Big|_{\text{interface}} = 0\right)$

$$\int_{S'} \vec{\nabla}\times\vec{D}(\vec{r}')\bullet d\vec{a}' = \oint_{C'} \vec{D}(\vec{r}')\bullet d\vec{\ell}'$$
$$= \int_{S'} \vec{\nabla}\times\vec{P}(\vec{r}')\bullet d\vec{a}' = \oint_{C'} \vec{P}(\vec{r}')\bullet d\vec{\ell}'$$

* Gives: $\left[D_2^{\parallel} - D_1^{\parallel}\right]\Big|_{\text{interface}} = \left[P_2^{\parallel} - P_1^{\parallel}\right]\Big|_{\text{interface}}$

Magnetostatics:

$$\frac{1}{\mu_o}\int_{S'} \vec{\nabla}\times\vec{B}(\vec{r}')\bullet d\vec{a}' = \frac{1}{\mu_o}\oint_{C'} \vec{B}(\vec{r}')\bullet d\vec{\ell}' = I_{TOT}^{encl}$$

Gives: $\frac{1}{\mu_o}\left[B_2^{\parallel} - B_1^{\parallel}\right]\Big|_{\text{interface}} = \vec{K}_{TOT}\times\hat{n}\Big|_{\text{interface}}$

$$\int_{S'} \vec{\nabla}\times\vec{H}(\vec{r}')\bullet d\vec{a}' = \oint_{C'} \vec{H}(\vec{r}')\bullet d\vec{\ell}' = I_{free}^{encl}$$

Gives: $\left[H_2^{\parallel} - H_1^{\parallel}\right]\Big|_{\text{interface}} = \vec{K}_{free}\times\hat{n}\Big|_{\text{interface}}$

$$\int_{S'} \vec{\nabla}\times\vec{M}(\vec{r}')\bullet d\vec{a}' = \oint_{C'} \vec{M}(\vec{r}')\bullet d\vec{\ell}' = I_{bound}^{encl}$$

Gives: $\left[M_2^{\parallel} - M_1^{\parallel}\right]\Big|_{\text{interface}} = \vec{K}_{bound}\times\hat{n}\Big|_{\text{interface}}$

$$\int_{v'} \vec{\nabla}\bullet\vec{B}(\vec{r}')\,d\tau' = \oint_{S'} \vec{B}(\vec{r}')\bullet d\vec{a}'$$

Gives: $B_2^{\perp} = B_1^{\perp}\Big|_{\text{interface}} \left(\left[B_2^{\perp} - B_1^{\perp}\right]\Big|_{\text{interface}} = 0\right)$

$$\int_{v'} \vec{\nabla}\bullet\vec{H}(\vec{r}')\,d\tau' = \oint_{S'} \vec{H}(\vec{r}')\bullet d\vec{a}'$$
$$= -\int_{v'} \vec{\nabla}\bullet\vec{M}(\vec{r}')\,d\tau' = -\oint_{S'} \vec{M}(\vec{r}')\bullet d\vec{a}'$$

* Gives: $\left[H_2^{\perp} - H_1^{\perp}\right]\Big|_{\text{interface}} = -\left[M_2^{\perp} - M_1^{\perp}\right]\Big|_{\text{interface}}$

We have derived / discussed these sets of boundary conditions on { $\vec{E}, \vec{P}, \vec{D}$ / $\vec{B}, \vec{M}, \vec{H}$ } before in previous lectures / lecture notes. We simply collected/summarized them here – all in one place.

For magnetic media, note that we <u>also</u> have the relations (previously derived in P435 Lecture Notes 20, page 8):

$$\boxed{\sigma_m^{bound} \equiv \vec{\mathrm{M}}\bullet\hat{n}\Big|_{\text{interface}} = \begin{Bmatrix}\text{Effective bound surface density of magnetic pole strength}\\ \text{Effective bound surface density of magnetic charge}\end{Bmatrix}}$$

and:

$$\boxed{\rho_m^{bound}(\vec{r}) \equiv -\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}) = \begin{Bmatrix}\text{Effective bound volume density of magnetic pole strength}\\ \text{Effective bound volume density of magnetic charge}\end{Bmatrix}}$$

The SI units of magnetic charge $\left(g_m = "qv"\right)$ = Ampere-meters (= Coulombs*meters/sec)

<u>But</u>: $\boxed{\int_{v'}\vec{\nabla}\bullet\vec{H}(\vec{r}')\,d\tau' = -\int_{v'}\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}')\,d\tau' = +\int_{v'}\rho_m^{bound}(\vec{r}')\,d\tau' = Q_m^{encl}}$ (S.I. Units Amps/m^2)

<u>So</u>: $\int_{v'}\vec{\nabla}\bullet\vec{H}(\vec{r}')\,d\tau' = Q_m^{encl}$?? Huh???

That's right! If $\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}) \neq 0$, $\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}) = -\rho_m(\vec{r})$ and $\int_{v'}\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r})\,d\tau' = -Q_m^{encl}$

$$\therefore \int_{v'}\vec{\nabla}\bullet\vec{H}(\vec{r}')\,d\tau' = -\int_{v'}\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}')\,d\tau' = +\int_{v'}\rho_m^{bound}(\vec{r}')\,d\tau' = Q_m^{encl}$$

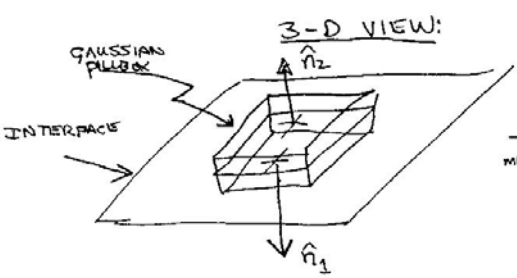
<u>Furthermore</u>:

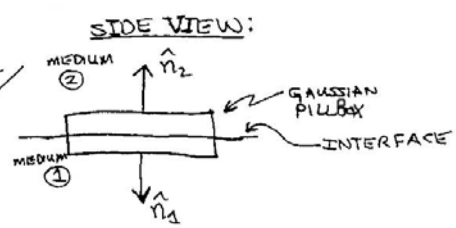
$$\int_{v'}\vec{\nabla}\bullet\vec{H}(\vec{r}')\,d\tau' = \oint_{S'}\vec{H}(\vec{r}')\bullet d\vec{a}' = \oint_{S'}\vec{H}(\vec{r}')\bullet\hat{n}\,da'$$

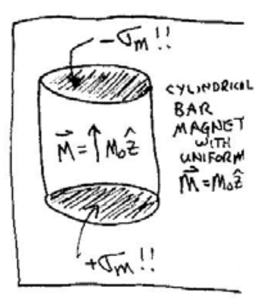
$$= \int_{v'}\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}')\,d\tau' = \oint_{S'}\vec{\mathrm{M}}(\vec{r}')\bullet d\vec{a}' = \oint_{S'}\vec{\mathrm{M}}(\vec{r}')\bullet\hat{n}\,da'$$

$$\therefore \quad \oint_{S'}\vec{H}(\vec{r}')\bullet\hat{n}\,da' = -\oint_{S'}\vec{\mathrm{M}}(\vec{r}')\bullet\hat{n}\,da' = -\oint_{S'}\sigma_m^{bound}\,da' = -Q_m^{encl}$$

This means that an interface between two magnetic media:







** Shrink height of Gaussian Pillbox to infinitesimally above and below interface.

Then, only the top and bottom surfaces of the Gaussian Pillbox contribute to the surface integral:

<u>Gives</u>: $\boxed{\left[\vec{H}_2\bullet\hat{n}_2 + \vec{H}_1\bullet\hat{n}_1\right]\Big|_{\text{interface}} = -\left[\vec{\mathrm{M}}_2\bullet\hat{n}_2 + \vec{\mathrm{M}}_1\bullet\hat{n}_2\right] = -\sigma_m\Big|_{\text{interface}}}$

<u>But</u>: $\hat{n}_1 = -\hat{n}_2$, $\perp$ to interface.

$$\therefore \quad \boxed{\left[H_2^{\perp} - H_1^{\perp}\right]\Big|_{\text{interface}} = -\left[\mathrm{M}_2^{\perp} - \mathrm{M}_1^{\perp}\right]\Big|_{\text{interface}} = -\sigma_m\Big|_{\text{interface}}} \;!!!$$

Equivalent surface magnetic charge density!