Supplemental Handout #2

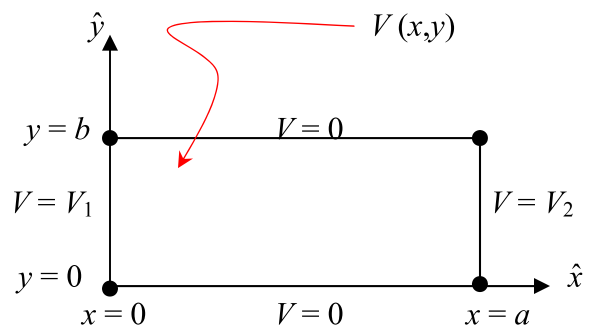
EIGHT MORE EXAMPLES OF SERIES SOLUTIONS TO LAPLACE'S EQUATION

$$\boxed{\nabla^2 V = 0}$$

1st Example:

Solve Laplace's equation for infinitely long rectangular box: 2 sides at ground, 2 sides at V_1, V_2.

NOTE: Problem has no z-dependence. $\Rightarrow$ This is a 2-D problem in rectangular coordinates.



$$\nabla^2 V = 0 \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$V(x,y) = X(x)\,Y(y)$
use separation of variables – try product sol'n

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{\text{LHS depends only on } x} = -\underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{\text{RHS depends only on } y} = \text{constant} = C$$

$$\frac{d^2 X(x)}{dx^2} - CX(x) = 0 \quad \text{and} \quad \frac{d^2 Y(y)}{dy^2} + CY(y) = 0$$

(Dirichlet) boundary conditions:

1) @ $y = 0$ $\quad V(x,0) = 0 \quad 0 \le x \le a$
2) @ $y = b$ $\quad V(x,b) = 0 \quad 0 \le x \le a$

$\}$ $V(x,0) = V(x,b) = 0$ implies that we need $Y(y)$-solutions of the form: $Y(y) \sim \sin \alpha y$

3) @ $x = 0$ $\quad V(0,y) = V_1 \quad 0 \le y \le b$
4) @ $x = a$ $\quad V(a,y) = V_2 \quad 0 \le y \le b$

If $Y(y) \sim \sin \alpha y$ then B.C. 1) $V(x,0) = 0$ is automatically satisfied.

Then: $\displaystyle \frac{d^2 Y(y)}{dy^2} + CY(y) = 0 \Rightarrow -\alpha^2 \sin \alpha y + C \sin \alpha y = 0 \qquad \therefore\; C = \alpha^2$

So: $\displaystyle \frac{d^2 X(x)}{dx^2} - \alpha^2 X(x) = 0 \quad \text{and} \quad \frac{d^2 Y(y)}{dy^2} + \alpha^2 Y(y) = 0$

Now impose B.C. 2): $V(x, y = b) = 0$ i.e. $\sin \alpha b = 0$ $\Rightarrow \alpha b = n\pi, \quad n = 1, 2, 3, \ldots$

$\therefore\ \alpha = \frac{n\pi}{b},\quad n = 1,2,3....$ ($n = 0$ is <u>trivial solution</u> - i.e. $V(x,y) \equiv 0$ <u>everywhere</u> – useless!)

So *<u>both</u>* boundary conditions #'s 1) and 2) are satisfied by $\sin\left(\frac{n\pi y}{b}\right)$ functions for y.

<u>Then</u>: $\frac{d^2X(x)}{dx^2} - \alpha^2 X(x) = 0 \qquad \alpha = \frac{n\pi}{b}$

Solutions for the $X(x)$ differential equation are $e^{\pm\alpha x}$ type functions: $X(x) = Ae^{-\alpha x} + Be^{+\alpha x}$

Plug this back into $X(x)$ differential equation and explicitly check if satisfied:

$$+\alpha^2 Ae^{-\alpha x} + \alpha^2 Be^{+\alpha x} - \left[\alpha^2 Ae^{-\alpha x} + \alpha^2 Be^{+\alpha x}\right] = 0 \checkmark \text{ (yup!)}$$

Then the most general solution (which also satisfies these 2 boundary conditions) is of the form:

$$V(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{-\frac{n\pi x}{b}} + B_n e^{+\frac{n\pi x}{b}}\right)\sin\frac{n\pi y}{b},\quad n = 1,2,3,....$$

Now impose B.C. 3): @ $x = 0 \qquad V(0,y) = V_1 = \sum_{n=1}^{\infty}(A_n + B_n)\sin\frac{n\pi y}{b}$

Determine the coefficients A_n and B_n by multiplying above relation by $\sin\frac{p\pi y}{b}$ and integrating over $0 \le y \le b$. Because of <u>orthogonality</u> properties of $\sin\frac{p\pi y}{b}$, only the A_p and B_p terms will survive!

$$\int_{y=0}^{y=b} V(0,y)\sin\frac{p\pi y}{b}dy = \int_{y=0}^{y=b} \underbrace{V_1}_{\text{constant}} \sin\frac{p\pi y}{b}dy = \int_{y=0}^{y=b}\sum_{n=1}^{\infty}(A_n + B_n)\sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}$$

$$= V_1\underbrace{\int_{y=0}^{y=b}\sin\frac{p\pi y}{b}dy}_{\substack{=\frac{2b}{p\pi} \text{ if } p = \text{odd integer} \\ = 0 \text{ if } p = \text{even integer}}} = \sum_{n=1}^{\infty}\left\{(A_n + B_n)\underbrace{\int_{y=0}^{y=b}\sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}dy}_{\substack{=\frac{b}{2} \text{ if } p = n \\ = 0 \text{ if } p \ne n}}\right\}$$

$\therefore\ \frac{2bV_1}{p\pi} = \frac{b}{2}(A_p + B_p)$ for p = odd integer only

or: $(A_p + B_p) = \frac{4V_1}{p\pi}$ for p = odd integer

$= 0$ for p = even integer

Now impose B.C. 4): @ $x = a$ $\quad V(a,y) = V_2 = \sum_{n=1}^{\infty}\left(A_n e^{-\frac{n\pi a}{b}} + B_n e^{+\frac{n\pi a}{b}}\right)\sin\frac{n\pi y}{b}$

Multiply by $\sin\frac{p\pi y}{b}$, integrate over $0 \le y \le b$ to project out p^{th} components:

$$\int_{y=0}^{y=b} V(a,y)\sin\frac{p\pi y}{b}dy = \int_{y=0}^{y=b} V_2 \sin\frac{p\pi y}{b}dy = \int_{y=0}^{y=b}\sum_{n=1}^{\infty}\left(A_n e^{-\frac{n\pi a}{b}} + B_n e^{+\frac{n\pi a}{b}}\right)\sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}$$

$$= V_2 \underbrace{\int_{y=0}^{y=b} \sin\frac{p\pi y}{b}dy}_{\substack{=\frac{2b}{p\pi}\text{ for } p \text{ odd} \\ =0 \text{ for } p \text{ even}}} = \sum_{n=1}^{\infty}\underbrace{\left(A_n e^{-\frac{n\pi a}{b}} + B_n e^{+\frac{n\pi a}{b}}\right)}_{\text{all constants!}}\underbrace{\int_{y=0}^{y=b} \sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}dy}_{=\frac{b}{2}\delta_{pn}}$$

$\delta_{pn} = 1$ if $p = n$
$\quad\;\; = 0$ if $p \ne n$

$$\therefore \quad A_p e^{-\frac{p\pi a}{b}} + B_p e^{+\frac{p\pi a}{b}} = \frac{4V_2}{p\pi} \quad \text{for } p = \text{ odd integer}$$
$$= 0 \quad \text{for } p = \text{even integer}$$

From B.C. 3) we found: $\quad A_p + B_p = \frac{4V_1}{p\pi}$ for p odd

$\quad = 0$ for p even

We have 2 equations and 2 unknowns ($A_p + B_p$). Solve simultaneously to get:

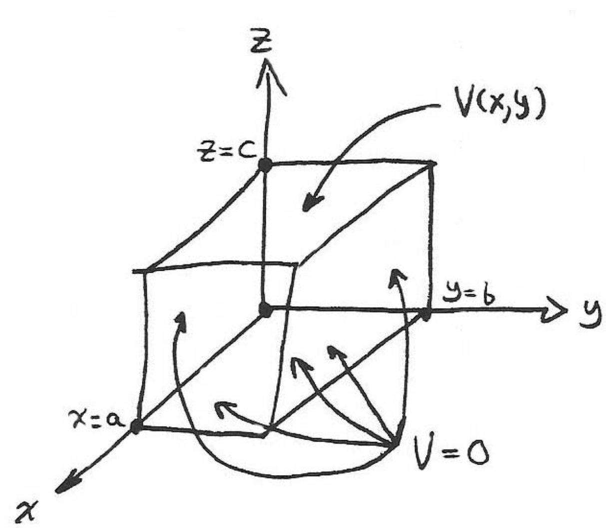
$$\left.\begin{aligned} A_p &= \frac{4}{p\pi}\left(\frac{V_1 - V_2 e^{-\frac{p\pi a}{b}}}{1 - e^{-\frac{2p\pi a}{b}}}\right) \\ B_p &= \frac{4e^{-\frac{p\pi a}{b}}}{p\pi}\left(\frac{V_2 - V_1 e^{-\frac{p\pi a}{b}}}{1 - e^{-\frac{2p\pi a}{b}}}\right)\end{aligned}\right\} \quad \begin{aligned}&p = \text{odd integer} \\ &(A_p = B_p = 0 \text{ for } p \text{ even})\end{aligned}$$

$$\boxed{\therefore \quad V(x,y) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\left\{\frac{V_1 - V_2 e^{-\frac{n\pi a}{b}}}{1 - e^{-\frac{2n\pi a}{b}}}\right\}\left\{e^{-\frac{n\pi x}{b}} + e^{-\frac{n\pi(a-x)}{b}}\right\}\sin\frac{n\pi y}{b}}$$ Valid solution for *inside* of box.

2nd Example: Laplace's Equation in Rectangular Coordinates

Hollow, Rectangular Box, Five Sides @ Ground, Top @ $V(x,y)$

This is a 3-D problem: $\nabla^2 V(x,y,z) = 0$ inside/outside box



(Dirichlet) Boundary Conditions:

1) @ $x = 0$	$V(0,y,z) = 0$	4) @ $x = a$	$V(a,y,z) = 0$
2) @ $y = 0$	$V(x,0,z) = 0$	5) @ $y = b$	$V(x,b,z) = 0$
3) @ $z = 0$	$V(x,y,0) = 0$	6) @ $z = c$	$V(x,y,c) = V(x,y)$

Use separation of variables technique - try product solution of the form: $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

Then: $\nabla^2 V = 0 \rightarrow \dfrac{1}{X(x)}\dfrac{d^2X(x)}{dx^2} = -\alpha^2$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta^2$$

$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = \gamma^2 = \alpha^2 + \beta^2$$

B.C. 1) $\rightarrow$ $X(0) = 0$ $\rightarrow$ $X(x) \sim \sin \alpha x$

B.C. 2) $\rightarrow$ $Y(0) = 0$ $\rightarrow$ $Y(y) \sim \sin \beta y$

B.C. 3) $\rightarrow$ $Z(0) = 0$ $\rightarrow$ $Z(z) \sim \sinh \gamma z = \sinh\left(\sqrt{\alpha^2+\beta^2}\,z\right)$

B.C. 4) $\rightarrow$ $X(a) = 0$ $\rightarrow$ $\alpha = \dfrac{n\pi}{a}$, $n = 1, 2, 3, \ldots$

B.C. 5) $\rightarrow$ $Y(b) = 0$ $\rightarrow$ $\beta = \dfrac{m\pi}{b}$, $m = 1, 2, 3, \ldots$

$$\therefore \quad \gamma = \sqrt{\alpha^2 + \beta^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$\therefore$ General solution for $V(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, z$

Last boundary condition 6): $V(x,y,c) = V(x,y)$ $\leftarrow$ some arbitrary potential (unspecified) on top surface of box.

$$\therefore \quad V(x,y,c) = V(x,y) = \underbrace{\sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c}_{\text{Double Fourier Series}}$$

Multiply above relation by $\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)$ and integrate over $\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} dy$ to project out the p, q^{th} term (i.e. use orthogonality properties of $\sin\left(\frac{n\pi}{a}\right)$ etc. to obtain A_{nm} coefficients):

$$\int_0^a \int_0^b V(x,y) \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right) dxdy$$

$$= \sum_{n,m=1}^{\infty} A_{nm} \sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c \int_0^a \int_0^b \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) * \underbrace{\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)}\, dxdy$$

For p <u>or</u> (and) q even integers: $= 0$

For p or q <u>both</u> odd integers: $= \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\delta_{pn}\delta_{qm}$

<u>Kroenecker delta-functions</u>: $\delta_{pn} = \begin{Bmatrix} 1 \text{ for } p = n \\ 0 \text{ for } p \neq n \end{Bmatrix}, \quad \delta_{qm} = \begin{Bmatrix} 1 \text{ for } q = m \\ 0 \text{ for } q \neq m \end{Bmatrix}$

<u>Thus</u>:

$\int_0^a \int_0^b V(x,y) \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right) dxdy = A_{pq}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\sinh\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2}\, c$ (for p, q <u>both</u> odd ints)

$= 0$ for p <u>or</u> (and) q even integers.

$$\therefore \quad A_{pq} = \left(\frac{4}{ab}\right) * \left[\frac{\int_0^a \int_0^b V(x,y) \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right) dxdy}{\sinh\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2}\, c}\right]$$ for p <u>and</u> q <u>both</u> odd integers.

$A_{pq} = 0$ for p <u>or</u> (and) q even integers.

Suppose $V(x,y) = V_o$ on top surface of box.

Then $A_{pq} = \left(\frac{4}{ab}\right) * \left[\frac{\int_0^a \int_0^b V_o \sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)dxdy}{\sinh\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2}\,c}\right]$ for p <u>and</u> q <u>both</u> odd integers.

$$= \left(\frac{4}{ab}\right)\left(\frac{2a}{p\pi}\right)\left(\frac{2b}{q\pi}\right)\frac{V_o}{\sinh\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2}\,c}$$

$\therefore \quad A_{pq} = \frac{16V_o}{pq\pi^2}\frac{1}{\sinh\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2}\,c}$ for p <u>and</u> q <u>both</u> odd integers,

$A_{pq} = 0$ for p <u>or</u> (and) q even integers.

Then $V(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{p\pi}{b}y\right)\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\,z$

$$V(x,y,z) = \sum_{n,m=1}^{\infty} \frac{16V_o}{nm\pi^2\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\,z$$

Summation is over $\left.\begin{matrix} n = \text{odd integers} \\ m = \text{odd integers}\end{matrix}\right\}$ only!!

Suppose all SIX sides of box have potentials $\neq 0$!!

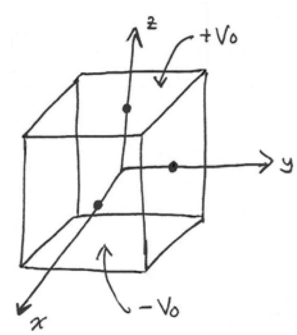
Then $V(x,y,z) = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$ where V_i ($i = 1, 2, \ldots 6$) represents solution for $V(x,y,z)$ for that surface - i.e. make *<u>linear superposition</u>* of six particular solutions (one for each surface) to generate solution for *<u>this</u>* 3-*D* problem.

3rd Example: Laplace's Equation in Rectangular Coordinates
Cubical Box, Sides at ground, Top at $+V_o$, Bottom at $-V_o$, Origin at Center of Box.

B.C. 1) $x = \pm \frac{L}{2}$: $V\left(\pm L/2, y, z\right) = 0$

2) $y = \pm \frac{L}{2}$: $V\left(x, \pm L/2, z\right) = 0$

3) $z = \pm \frac{L}{2}$: $V\left(x, y, \pm L/2\right) = \pm V_o$



Note spatial reflection symmetry of problem is
Even under: $x \to -x$ and $y \to -y$
Odd under: $z \to -z$

$\therefore \Rightarrow$ *even* functions for x, y (i.e. cosines) but need *odd* function for z: (i.e. sines)

General solution of $\nabla^2 V = 0$ is of the form:
(Again, use separation of variables technique: $V(x,y,z) = X(x)Y(y)Z(z)$)

$$V(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \cos(\alpha_n x)\cos(\beta_m y)\sinh(\gamma_{nm} z) \quad \text{where} \quad \alpha_n^2 + \beta_m^2 = \gamma_{nm}^2$$

B.C. 1):
$$V\left(\pm L/2, y, z\right) = 0 = \sum_{n,m=1}^{\infty} A_{nm} \cos\left(\pm\alpha_n L/2\right)\cos(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Rightarrow \pm\alpha_n L/2 = \frac{n\pi}{2} \Rightarrow \alpha_n = \frac{n\pi}{L} \quad \text{for } n = \text{odd integers}$$

Similarly:

B.C. 2):
$$V\left(x, \pm L/2, z\right) = 0 = \sum_{n,m=1}^{\infty} A_{nm} \cos(\alpha_n x)\cos\left(\beta_m\left(\pm L/2\right)\right)\sinh(\gamma_{nm} z)$$

$$\Rightarrow \pm\beta_m L/2 = \frac{m\pi}{2} \Rightarrow \beta_m = \frac{m\pi}{L} \quad \text{for } m = \text{odd integers}$$

All $A_{nm} \equiv 0$ with n *or* (and) m = even integers

$$\therefore \quad \alpha_n^2 + \beta_m^2 = \gamma_{nm}^2 \Rightarrow \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2} = \gamma_{nm}$$

B.C. 3): {top, bottom}

$$V\left(x, y, \pm L/2\right) = \pm V_o = \underbrace{\sum_{n,m=1}^{\infty} A_{nm} \cos(\alpha_n x)\cos(\beta_m y)\sinh\left(\gamma_{nm}\left(\pm L/2\right)\right)}_{\text{Double Fourier Series in } x \text{ and } y} \quad \text{for } n, m \text{ both odd integers}$$

Multiply both sides of above relation by $\cos\alpha_p x \cos\beta_q y$ and integrate from $\int_{-L/2}^{+L/2}\int_{-L/2}^{+L/2} dxdy$:

$$\pm V_o \int_{-L/2}^{+L/2}\int_{-L/2}^{+L/2} \cos\left(\frac{p\pi}{L}x\right)\cos\left(\frac{q\pi}{L}y\right)dxdy$$

$$= \sum_{n,m=1}^{\infty} A_{nm}\sinh\left(\gamma_{nm}\left(\pm L/2\right)\right)\int_{-L/2}^{+L/2}\int_{-L/2}^{+L/2}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}y\right)\cos\left(\frac{p\pi}{L}x\right)\cos\left(\frac{q\pi}{L}y\right) * dxdy$$

$$= \pm V_0\left(\frac{2L}{\rho\pi}\right)\left(\frac{2L}{q\pi}\right) = A_{pq}\sinh\left(\gamma_{pq}\left(+\frac{L}{2}\right)\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\delta_{n,p}\ \delta_{m,q}$$ for p and q both odd integers

$$\therefore\quad A_{pq} = \frac{\pm 16V_0}{pq\pi\sinh\left(\gamma_{pq}\left(\pm L/2\right)\right)} \qquad\qquad \gamma_{nm} = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}$$

n.b. $\pm$ cancel each other – sinh(x) (like sin(x)) is an odd fcn(x)!!!

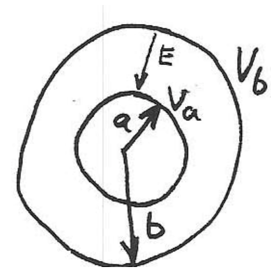
So that: $A_{pq} = \dfrac{16V_0}{pq\pi^2\sinh\left(\gamma_{pq} L/2\right)}$ for p and q both odd integers

$$\therefore\quad V(x,y,z) = \sum_{\substack{n,m=1\\ n,m\ odd\\ integers\\ only}}^{\infty} A_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}y\right)\sinh\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,z$$

where: $A_{nm} = \dfrac{16V_0}{nm\pi^2\sinh\left(\gamma_{nm} L/2\right)}$ for n and m both odd integers

4th Example: Laplace's Equation in Cylindrical Coordinates
Two long coaxial conductors of radius a & b ($b > a$), charged to potentials V_a, V_b respectively. Find potential V and electric field intensity $\vec{E}$ in between cylinders

Note: This is a *two* dimensional problem (has no z-dependence).



B.C.'s 1) $V(r=a)=V_a$

2) $V(r=b)=V_b$

Note that V also has no φ-dependence!

Try production solution of the form: $V(r,\varphi)=R(r)Q(\varphi)$

Solutions to Laplace's Equation $\nabla^2 V = 0$ for 2-D circular-type problem are *Zonal Harmonics*.

General solution is of the form:

$$V(r,\varphi)=V_0+V_1\ln r+\sum_{n=1}^{\infty}\left[A_n r^n\cos(n\varphi)+B_n r^{-n}\cos(n\varphi)+C_n r^n\sin(n\varphi)+D_n r^{-n}\sin(n\varphi)\right]$$

Since this problem has *no* φ-dependence, $\Rightarrow$ series in sines and cosines have all coefficients = 0!!

Then solution for *this* problem is of the form $V(r,\varphi)=V_0+V_1\ln r$

Impose boundary conditions to solve for V_0 & V_1:

1) @ $r=a$: $V(a,\varphi)=V_0+V_1\ln a=V_a$

2) @ $r=b$: $V(b,\varphi)=V_0+V_1\ln b=V_b$

Thus: $V_1\ln a-V_1\ln b=V_a-V_b$ or: $V_1\left(\ln\left(a/b\right)\right)=V_a-V_b \Rightarrow V_1=\left[V_a-V_b\right]/\ln\left(a/b\right)$

$\therefore\ V_0=V_a-(V_a-V_b)\ln a$ and thus $V(r,\varphi)=V_0+V_1\ln r$

gives: $\boxed{V(r,\varphi)=V_a-\dfrac{(V_a-V_b)}{\ln\left(a/b\right)}\ln a+\dfrac{(V_a-V_b)}{\ln\left(a/b\right)}\ln r}$ $\Leftarrow$ valid for $a\le r\le b$

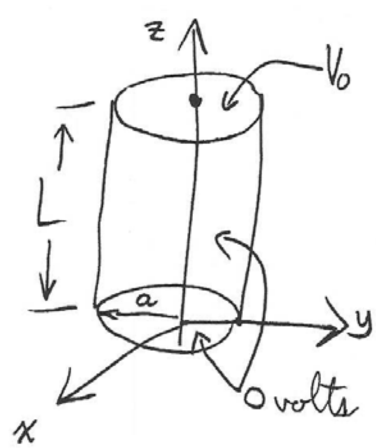
Then: $\boxed{\vec{E}(\vec{r})=-\vec{\nabla}V(\vec{r})=-\vec{\nabla}V(r,\varphi)=-\dfrac{\partial}{\partial r}V(r,\varphi)\hat{r}=-\dfrac{\left[V_a-V_b\right]}{\ln\left(a/b\right)}\dfrac{1}{r}\hat{r}}$

And: $\sigma_{free}=\varepsilon_o\vec{E}\cdot\hat{n}\Big|_{surface}$ Noting that $V_b > V_a$ and $b > a$

On inner surface $(\hat{n}=+\hat{r})$: $\boxed{\sigma_{free}^{inner}=-\dfrac{\varepsilon_o(V_b-V_a)}{a\ln\left(b/a\right)}}$

On outer surface $(\hat{n}=-\hat{r})$: $\boxed{\sigma_{free}^{outer}=+\dfrac{\varepsilon_o(V_b-V_a)}{b\ln\left(b/a\right)}}$

5th Example: Laplace's Equation in Cylindrical Coordinates
Cylinder of Radius a, Length L, Top Surface at V_0, Bottom & Sides at Ground.



$\nabla^2 V(\rho,\varphi,z)=0$ in cylindrical coordinates.

Try product solution of form: $V(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z)$

Solve for $V(\rho,\varphi,z)$ inside "can".
n.b. Origin *is* included, and $\rho=\infty$ is *excluded*.
Since there are no charge(s) at $\rho=0 \Rightarrow$ require potential $V(\rho,\varphi,z)$ to be finite at $\rho=0$.

$\therefore$ General Solution (for *this* problem) is of the form:

$$V(\rho,\varphi,z)=\sum_{\substack{m=0\\n=1}}^{\infty} J_m(k_{mn}\rho)\sinh(k_m z)\left[A_{mn}\sin m\varphi+B_{mn}\cos m\varphi\right]$$

where the $J_m(x_{mn})$ are Bessel functions of the 1st kind, of order m,
and $x_{mn}=k_{mn}\rho,\ \ n=1,2,3,....$ The x_{mn} are the roots (i.e. zeroes) of $J_m(x_{mn})=0$.

(Note that the Bessel functions of the 2nd kind $N_m(x_{mn})$ (= Neumann functions) are *not* allowed because these functions are *singular* (i.e. infinite) at $\rho=0$.)

B.C. 1): Potential vanishes at $\rho=a$: $V(\rho=a,\varphi,z)=0$
B.C. 2): Potential = V_0 at $z=L$: $V(\rho,\varphi,z=L)=V_0$

$$\boxed{V(\rho,\varphi,z=L)=\sum_{\substack{m=0\\n=1}}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin m\varphi+B_{mn}\cos m\varphi\right]}$$

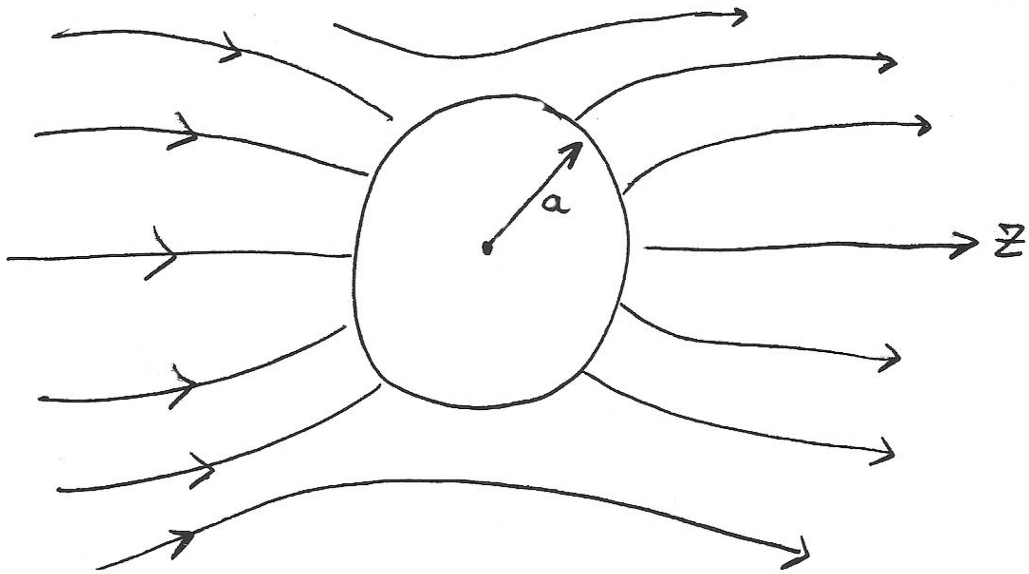
$\Rightarrow$ Fourier Series in φ and Fourier-Bessel Series in ρ.
Multiply above equation by $\sin p\varphi$ and $J_p(k_{\rho n}\rho)$, and integrate over $\int_0^{2\pi}d\varphi\int_0^a \rho d\rho$

Obtain: $$\boxed{A_{mn}=\frac{2\operatorname{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\varphi\int_0^a \rho d\rho V(\rho,\varphi) J_m(k_{mn}\rho)\sin(m\varphi)}$$ (Note: $A_{0n}=0$ for $m=0$.)

And:
$$\boxed{B_{mn}=\frac{2\operatorname{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\varphi\int_0^a \rho d\rho V(\rho,\varphi) J_m(k_{mn}\rho)\cos(m\varphi)}$$ (Note: for $m=0$ use ½ B_{0n} as given here.)

6th Example: Laplace's Equation in Spherical Coordinates

Initially uncharged conducting sphere (radius, a) placed in uniform electric field $\vec{E} = E_0\hat{z}$:



Note that this problem has azimuthal symmetry (i.e. no φ-dependence).

$\therefore$ Generalized Legendré Equation → Ordinary Legendré Equation

$P_l^m(cos\theta)$ with $m = 0$ → $P_l(\cos\theta)$

$V(r,\theta,\varphi) \longrightarrow V(r,\theta)$ Zonal Harmonics

External Field $E = E_0\hat{z}$ (Initially Uniform)

B.C. 1): At $r \to \infty$: $\vec{E}(r,\theta,\varphi) = E_0\hat{z}$ but: $\vec{E} = -\vec{\nabla}V$

Then: $V(r \to \infty,\theta,\varphi) = -E_0 z + \text{constant}$ where: $z = r\cos\theta$

$$= -\mathrm{E}_o r\cos\theta + \text{constant}$$

$$= -E_0 r P_1(\cos\theta) + \text{constant}$$

If the conducting sphere is initially uncharged, then after being placed in external $\vec{E}$-field, it will (still) have *net* charge = 0.

$\therefore$ B.C. 2): $V(r = a,\theta,\varphi) = 0$ surface of conducting sphere is at 0 volts (an equipotential).

General Solution for $V(r,\theta)$ in spherical polar coordinates, for no φ-dependence is of the form:

$$V(r,\theta) = \sum_{n=0}^{\infty}\left\{A_n r^n + B_n r^{-(n+1)}\right\}P_n(\cos\theta)$$

At $r = a$, B.C. 2) on surface of sphere is such that:

$$V(r,\theta) = \sum_{n=0}^{\infty}\left\{A_n a^n + B_n a^{-(n+1)}\right\} P_n(\cos\theta) = 0$$

Multiply above expression on both sides by $P_m(\cos\theta)$ and integrate over $-1 \le \cos\theta \le +1$:

$$\int_{-1}^{+1}\left\{A_n a^n + B_n a^{-(n+1)}\right\} P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta) = 0$$

Now: $\int_{-1}^{1} P_n(\cos\theta) P_m(\cos\theta)\, d(\cos\theta) = \frac{2}{2m+1}\delta_{nm} \Leftarrow$ orthogonality condition $P_l(\cos\theta)$

$$= \left\{A_m a^m + B_m a^{-(m+1)}\right\} \frac{2}{2m+1}$$

$$\therefore B_m = -A_m a^{2m+1}$$

Now for $r \to \infty$, must have uniform E-field and potential $V(r,\theta)$

$$V(r \to \infty, \theta) = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

but: $V(r \to \infty, \theta) = \sum_{n=0}^{\infty}\left\{A_n r^n + B_n r^{-(n+1)}\right\} P_n(\cos\theta)$

$\therefore$ All terms in this series other than $n = 1$ *must* VANISH!!!

Then: $V(r \to \infty, \vartheta) = \left(A_1 r + \underbrace{B_1 r^{-2}}_{\text{vanishes as } r \to \infty} \right) P_1(\cos\theta)$

$\therefore$ $A_1 = -E_0$. All $A_n = 0$ for $n \neq 1$.

Then if $A_1 = -E_0$, since $B_m = -A_m a^{2m+1}$ then for $m = 1$, $B_1 = -A_1 a^3 = +E_0 a^3$

$$\therefore \boxed{V(r,\theta) = -E_0 r\cos\theta + E_0 \frac{a^3\cos\theta}{r^2} = -E_0\left(1 - \frac{a^3}{r^3}\right) r\cos\theta = -E_0\left(1 - \left[\frac{a}{r}\right]^3\right) r\cos\theta}$$

Now $\vec{E} = -\vec{\nabla} V$

$$\therefore \boxed{E_r = -\frac{\partial V}{\partial r} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta}$$

$$\boxed{E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta}$$

The surface charge density $\sigma(r = a, \theta, \varphi)$ on the surface of sphere can be obtained via the relation:

$\sigma = \varepsilon_0 E_\perp\big|_{surface} = \varepsilon_0 E_r\big|_{r=a}$ (by Gauss' Law for $\vec{E}$)

$\therefore$ $\sigma(r = a, \theta, \varphi) = 3\varepsilon_0 E_0 \cos\theta$ (Note: σ has no explicit φ-dependence due to azimuthal symmetry)

Note also that this problem is also equivalent to a *point* electric dipole of strength $p = 4\pi\varepsilon_0 E_0 a^3$ located at the origin (i.e. center of the sphere) in an externally-applied electric field $\vec{E} = E_0\hat{z}$.

7th Example: Laplace's Equation in Spherical Coordinates
Two Conducting Hemispheres at $\pm V_0$

Problem has no φ-dependence (azimuthally symmetric)

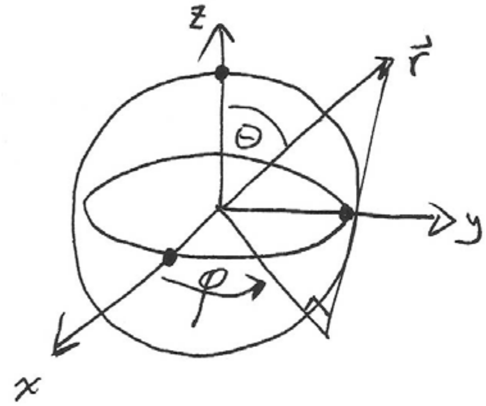
Boundary Conditions:

$$V(r=a,\vartheta)=\begin{cases}+V_0 & \text{for } 0\le\theta\le\frac{\pi}{2}\\ -V_0 & \text{for } \frac{\pi}{2}\le\theta\le\pi\end{cases}$$

Note problem is odd when change $z\to -z$:
$\therefore$ We will find that only odd $P_n(\cos\theta)$ will work

General solution for $\nabla^2 V=0$ in spherical coordinates (for problems with no explicit φ-dependence):

$$V(r,\theta)=\sum_{n=0}^{\infty}\left[A_n r^n+B_n r^{-(n+1)}\right]P_n(\cos\theta)$$



Inside spherical shell: $(r\le a)$

$V(r,\theta)$ must be finite! $\rightarrow$ All $B_n\equiv 0$ inside $(r\le a)$

$$\therefore\ V_{in}(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

B.C. at $r=a$: $\quad V_{in}(a,\theta)=\sum_{n=0}^{\infty}A_n a^n P_n(\cos\theta)=\left\{\begin{array}{ll}+V_o & 0\le\theta\le\pi/2\\ & \text{and}\\ -V_o & \pi/2\le\theta\le\pi\end{array}\right\}$

Multiply above expression on both sides by $P_m(\cos\theta)$, integrate over $\int_{-1}^{1}d(\cos\theta)$:

$$\sum_{n=0}^{\infty}A_n a^n\int_{-1}^{1}P_n(\cos\theta)P_m(\cos\theta)\,d(\cos\theta)=-V_o\int_{-1}^{0}P_m(\cos\theta)d(\cos\theta)+V_o\int_{0}^{1}P_m(\cos\theta)d(\cos\theta)$$

$$\sum_{n=0}^{\infty}A_n a^n\underbrace{\int_{-1}^{1}P_n(\cos\theta)P_m(\cos\theta)\,d(\cos\theta)}_{=\frac{2}{2m+1}\delta_{nm}}=V_o\underbrace{\int_{0}^{1}P_m(\cos\theta)d(\cos\theta)-\int_{-1}^{0}P_m(\cos\theta)d(\cos\theta)}$$

= 0 (for m = even integers)

$$=\left(-\frac{1}{2}\right)^{\frac{(m-1)}{2}}\frac{(m-2)!!}{\left(\frac{m+1}{2}\right)!}\quad\text{(for } m \text{ = odd int)}$$

{Obtained using Rodriguez' Formula for the $P_l(\cos\theta)$ }

$$\therefore \quad A_m a^m \left(\frac{2}{2m+1}\right) = V_o \left(-\frac{1}{2}\right)^{\frac{(m-1)}{2}} \frac{(m-2)!!}{\left(\frac{m+1}{2}\right)!} \quad \text{for } m = \underline{\text{odd}} \text{ integers}$$

$$\therefore \quad A_m = V_o \left(-\frac{1}{2}\right)^{\frac{(m-1)}{2}} \frac{(2m+1)(m-2)!!}{2a^m \left(\frac{m+1}{2}\right)!} \quad \text{for } m = \underline{\text{odd}} \text{ integers}$$

$A_m = 0$ for m = even integers.

$$\therefore \quad \boxed{V_m(r,\theta) = \sum_{\substack{n=\text{odd}\\ \text{integer}}}^{\infty} V_o \left(-\frac{1}{2}\right)^{\left(\frac{n-1}{2}\right)} \frac{(2n+1)(n-2)!!}{2\left(\frac{n+1}{2}\right)!} \left(\frac{r}{a}\right)^n P_n(\cos\theta)}$$

i.e. $$\boxed{V_{in}(r,\theta) = V_o \left(\frac{3}{2}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) + \ldots\right)}$$

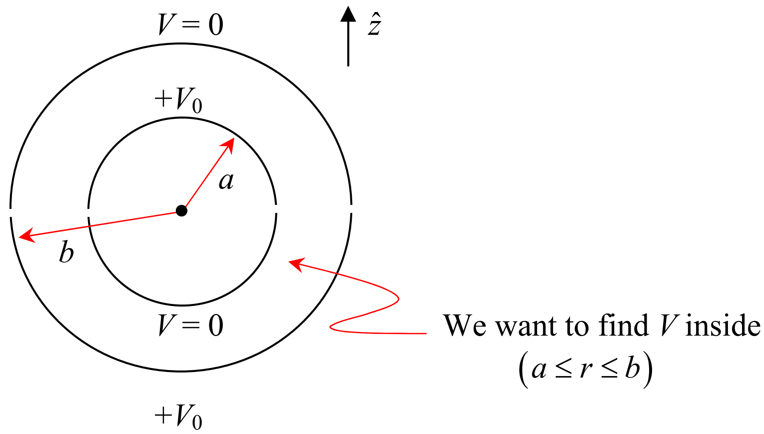
For $r > a$, simply replace $\left(\frac{r}{a}\right)^n$ in above expression by $\left(\frac{a}{r}\right)^{n+1}$ (!!!)

<u>Then</u>: $$\boxed{V_{out}(r,\theta) = \sum_{\substack{n=\text{odd}\\ \text{integer}}}^{\infty} V_o \left(-\frac{1}{2}\right)^{\left(\frac{n-1}{2}\right)} \frac{(2n+1)(n-2)!!}{2\left(\frac{n+1}{2}\right)!} \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta)}$$

Can also easily see that $V_{in}(r = a,\theta) = V_{out}(r = a,\theta)$ - i.e. the potential is continuous across the radial boundary of sphere (i.e. on a radial trajectory)!

8th Example: Laplace's Equation in Spherical Coordinates
The Double Spherical Capacitor.

This problem has azimuthal symmetry (i.e. no φ-dependence)



B.C.'s:

@ $r=a$: $\quad V=+V_o \quad 0 \le \theta \le \dfrac{\pi}{2}$

$\quad V=0 \quad \dfrac{\pi}{2} \le \theta \le \pi$

@ $r=b$: $\quad V=0 \quad 0 \le \theta \le \dfrac{\pi}{2}$

$\quad V=+V_o \quad \dfrac{\pi}{2} \le \theta \le \pi$

General solution is of the form: $V(r,\vartheta)=\sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta)$

In the region $a \le r \le b$ we have (note: the origin and $r=\infty$ are both excluded in this problem):

@ $r=a$: $\quad V(a,\theta)=\sum_{n=0}^{\infty}\left[A_n a^n + B_n a^{-(n+1)}\right]P_n(\cos\theta)=\begin{cases} +V_o & 0\le\theta\le\frac{\pi}{2} \\ 0 & \frac{\pi}{2}\le\theta\le\pi \end{cases}$

@ $r=b$: $\quad V(b,\theta)=\sum_{n=0}^{\infty}\left[A_n b^n + B_n b^{-(n+1)}\right]P_n(\cos\theta)=\begin{cases} 0 & 0\le\theta\le\frac{\pi}{2} \\ +V_o & \frac{\pi}{2}\le\theta\le\pi \end{cases}$

To determine the coefficients A_n and B_n, multiply above expressions on both sides by $P_m(\cos\theta)$, integrate over $\int_{-1}^{1} d(\cos\theta)$ to project out m^{th} term:

$$r=a:\quad \sum_{n=0}^{\infty}\left[A_n a^n + B_n a^{-(n+1)}\right]\int_{-1}^{1} P_n(\cos\theta)P_m(\cos\theta)\,d(\cos\theta)=\int_0^1 V_o P_m(\cos\theta)\,d(\cos\theta)$$

$$r=b:\quad \sum_{n=0}^{\infty}\left[A_n b^n + B_n b^{-(n+1)}\right]\underbrace{\int_{-1}^{1} P_n(\cos\theta)P_m(\cos\theta)\,d(\cos\theta)}_{}=\int_{-1}^{0} V_o P_m(\cos\theta)\,d(\cos\theta)$$

$$=\frac{2}{2m+1}\delta_{nm}\quad (\delta_{nm}=1 \text{ for } n=m,\ =0 \text{ for } n\ne m)$$

$$\int_0^1 P_m(\cos\theta)d(\cos\theta) = 1 \quad \text{for } n = 0$$

$$= \left(-\frac{1}{2}\right)^{\frac{(n-1)}{2}} \frac{(n-2)!!}{\left(\frac{n+1}{2}\right)!} \quad \text{for } n = 1,3,5,7,\ \ldots \text{ odd integers}$$

$$= 0 \text{ for } n = 2,4,6,8,\ \ldots \text{ even integers}$$

$$r = a\text{:} \quad \left[A_m a^m + B_m a^{-(m+1)}\right]\left(\frac{2}{2m+1}\right) = \left(-\frac{1}{2}\right)^{\left(\frac{m-1}{2}\right)} \frac{(m-2)!!}{\left(\frac{m+1}{2}\right)!} V_o \leftarrow \text{for } m > 0,\ m \text{ \underline{odd} integers only}$$

$$r = b\text{:} \quad \left[A_m b^m + B_m b^{-(m+1)}\right]\left(\frac{2}{2m+1}\right) = \left(-\frac{1}{2}\right)^{\left(\frac{m-1}{2}\right)} \frac{(m-2)!!}{\left(\frac{m+1}{2}\right)!} V_o \leftarrow \text{for } m > 0,\ m \text{ \underline{odd} integers only}$$

<u>for $m = 0$</u>: A_m, $B_m = 0$ for $m = 2,4,6,8,\ \ldots$ <u>even</u> integers

$$r = a\text{:} \quad A_0 + \frac{1}{a}B_0 = \frac{1}{2}V_0$$

$$r = b\text{:} \quad A_0 + \frac{1}{b}B_0 = \frac{1}{2}V_0 \quad \longrightarrow \quad B_0 = 0 \rightarrow A_0 = \frac{1}{2}V_0$$

<u>for $m = 1$</u>:

$$r = a\text{:} \quad A_1 a^3 + B_1 = \frac{3}{4}a^3 V_0$$

$$r = b\text{:} \quad A_1 b^3 + B_1 = -\frac{3}{4}b^3 V_0 \quad \longrightarrow \quad A_1 = \frac{3}{4}\left(\frac{a^2+b^2}{a^3-b^3}\right)V_0,\quad B_1 = -\frac{3}{4}a^2b^2V_0\left(\frac{a+b}{a^3-b^3}\right)$$

<u>for m = 3</u>:

$$r = a\text{:} \quad A_3 a^7 + B_3 = -\frac{7}{16}V_0 a^4$$

$$r = b\text{:} \quad A_3 b^7 + B_3 = \frac{7}{16}V_0 b^4 \quad \longrightarrow \quad A_3 = -\frac{7}{16}\left(\frac{a^4+b^4}{a^7-b^7}\right)V_0,\quad B_3 = \frac{7}{16}a^4b^4V_0\left(\frac{a^3+b^3}{a^7-b^7}\right)$$

<u>In general</u>:

$$A_n = \frac{(-1)^{(n+1)!!}(2n+1)}{2^{n+1}}V_0\left(\frac{a^{n+1}+b^{n+1}}{a^{2n+1}-b^{2n+1}}\right) \qquad \text{for } n = 1,3,5,7,\ \ldots. \text{ odd integers}$$

$$B_n = \frac{(-1)^{n!!}(2n+1)}{2^{n+1}}V_0\left(\frac{a^n+b^n}{a^{2n+1}-b^{2n+1}}\right)a^{n+1}b^{n+1} \qquad A_0 = \frac{1}{2}V_0,\quad B_0 = 0$$

$$A_n = B_n = 0 \quad \text{for } n = 2,4,6,8,\ \ldots. \text{ even integers}$$

and where $(-1)^{(n+1)!!} = (-1)^{(n+1)(n-1)(n-3)\ldots 1} = \pm 1$, i.e. $+1$ for $n = 1$, -1 for $n = 3$, $+1$ for $n = 5$, etc.

$\therefore$ Solution for $V(r,\theta)$ in region $a \le r \le b$: $\quad V(r,\theta)=\sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta)$

$$\boxed{\begin{aligned} V(r,\theta)=\frac{1}{2}V_0+\sum_{n=odd}^{\infty}\Bigg\{&\frac{(-1)^{(n+1)!!}(2n+1)}{2^{n+1}}V_0\left(\frac{a^{n+1}+b^{n+1}}{a^{2n+1}-b^{2n+1}}\right)r^n \\ &+\frac{(-1)^{n!!}(2n+1)a^{n+1}b^{n+1}}{2^{n+1}}V_o\left(\frac{a^n+b^n}{a^{2n+1}-b^{2n+1}}\right)r^{-(n+1)}\Bigg\}P_n(\cos\theta)\end{aligned}}$$

$$\boxed{\begin{aligned} V(r,\theta)=\frac{1}{2}V_0+\frac{3}{4}V_0&\left[\frac{(a^2+b^2)}{(a^3-b^3)}r-a^2b^2\left(\frac{a+b}{a^3-b^3}\right)\frac{1}{r^2}\right]P_1(\cos\theta) \\ -\frac{7}{16}V_o&\left[\frac{(a^4+b^4)}{a^7-b^7}r^3-a^4b^4\left(\frac{a^3+b^3}{a^7-b^7}\right)\frac{1}{r^4}\right]P_3(\cos\theta)+\ldots.\end{aligned}}$$

Note that if $b\to\infty$, then:

$$V(r,\theta)=\frac{1}{2}V_0+\frac{3}{4}V_0\left(\frac{a}{r}\right)^2 P_1(\cos\theta)-\frac{7}{16}V_0\left(\frac{a}{r}\right)^4 P_3(\cos\theta)+\ldots.$$

Potential outside two hemispheres held at potentials V_0, 0.

If $a\to 0$, then:

$$V(r,\theta)=\frac{1}{2}V_0+\frac{3}{4}V_0\left(\frac{r}{b}\right) P_1(\cos\theta)-\frac{7}{16}V_0\left(\frac{r}{b}\right)^3 P_3(\cos\theta)+\ldots.$$

Potential inside two hemispheres held at potentials V_0, 0.

Average value of potential seen at $r = 0$ (or $r = \infty$ also)

