LECTURE NOTES 24

MAXWELL'S EQUATIONS

Thus far, we have the following four Maxwell equations (in differential form):

Divergence and curl of both $\vec{E}$ and $\vec{B}$ specified $\Rightarrow$ nature of $\vec{E}$ and $\vec{B}$ is fully defined

$$\vec{\nabla}\bullet\vec{E}(\vec{r}) = \frac{1}{\varepsilon_0}\rho_{ToT}(\vec{r}) \quad (\text{Gauss' Law})$$

$$\vec{\nabla}\bullet\vec{B}(\vec{r}) = 0 \quad \left\{\text{no magnetic monopoles} \big/ \text{no magnetic charges}\right\}$$

$$\vec{\nabla}\times\vec{E}(\vec{r}) = -\frac{\partial\vec{B}(\vec{r})}{\partial t} \quad (\text{Faraday's Law})$$

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}_{ToT}(\vec{r}) \quad (\text{Ampere's Law})$$

However, there is a problem with this set of equations…
Recall that $\vec{\nabla}\bullet\left(\vec{\nabla}(\vec{r})\times\vec{F}(\vec{r})\right) = 0$ always for an arbitrary vector field, $\vec{F}(\vec{r})$.

Apply this to Faraday's Law:

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\bullet\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{B}\right) = 0 \qquad \underline{\underline{\text{OK}}}$$

Apply this to Ampere's Law:

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\bullet\left(\mu_0\vec{J}_{ToT}\right) = \mu_0\left(\vec{\nabla}\bullet\vec{J}_{ToT}\right)$$

For steady total currents: $\left(\vec{\nabla}\bullet\vec{J}_{ToT}\right) = 0$ because $\frac{\partial\rho_{ToT}}{\partial t} = 0$

However, for *time-varying* situations the continuity equation (total charge conservation)

$$\vec{\nabla}\bullet\vec{J}_{ToT} = -\frac{\partial\rho_{ToT}}{\partial t} \neq 0 \quad \text{BIG PROBLEM!!!}$$

Let us investigate Ampere's Law (in integral form) for the case of a parallel-plate capacitor:

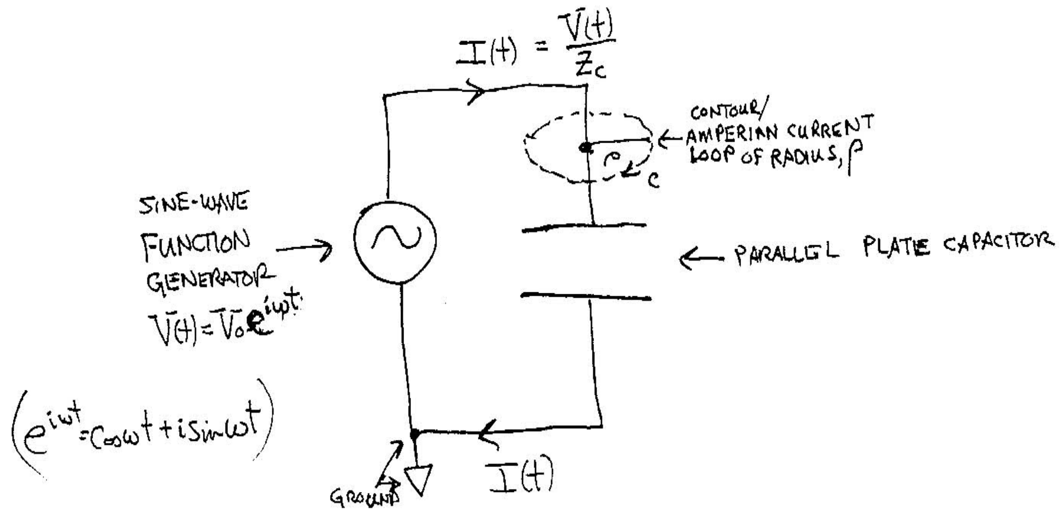
$$\int_S\left(\vec{\nabla}\times\vec{B}\right)\bullet d\vec{a} = \oint_C\vec{B}\bullet d\vec{\ell} = \mu_0\int_S\vec{J}\bullet d\vec{a} = \mu_0 I_{\text{enclosed}}$$

$$\oint_C\vec{B}\bullet d\vec{\ell} = \mu_0 I_{\text{enclosed}}$$

Suppose we have an electric circuit consisting of sine wave function generator that supplies/generates a time dependent voltage $V(t) = V_0 e^{i\omega t}$ and a parallel-plate capacitor:

Complex impedance of a capacitor: $\boxed{Z_C = \frac{1}{i\omega c}}$ $\quad \omega = 2\pi f \quad i \equiv \sqrt{-1} \quad \begin{pmatrix} i^* - i = +1 \\ i^* i = -1 \end{pmatrix}$

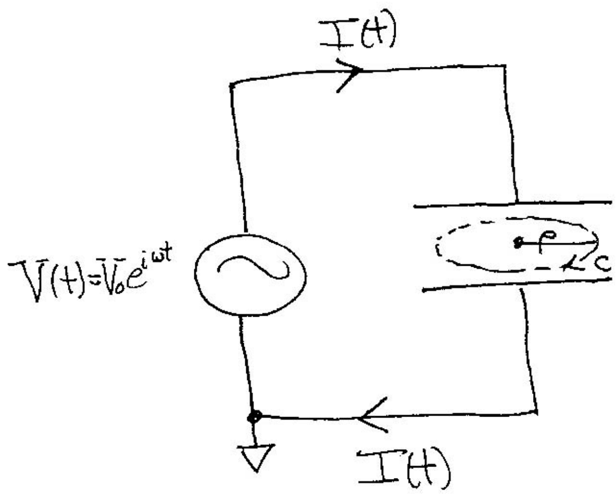
Complex form of Ohm's law: $\boxed{V(t) = I(t) Z_C}$ n.b. $V(t)$, $I(t)$ and Z_C are complex quantities.



<u>For contour loop shown in above figure</u>: $\oint_c \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = \mu_0 I$

Get: $B(\rho) = \frac{\mu_0 I}{2\pi\rho}$ as usual – so no problem with this…

<u>What about the following contour loop</u>:



$\oint_c \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl} = 0$!!!

<u>Cain't be true!!!</u>

We're missing something! <u>*Energy*</u> flows across the gap between plates of parallel-plate capacitor… - virtual photons associated with electric field of ||-plate capacitor!

Let's look again at the continuity equation:

$$\vec{\nabla}\bullet\vec{J}_{ToT}(\vec{r},t) = -\frac{\partial\rho_{ToT}(\vec{r},t)}{\partial t}$$

But: $\rho_{ToT}(\vec{r},t) = \varepsilon_0\vec{\nabla}\bullet\vec{E}(\vec{r},t)$ (Gauss' Law)

$$\therefore \quad -\frac{\partial\rho_{ToT}(\vec{r},t)}{\partial t} = -\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{E}(\vec{r},t)\right) = -\varepsilon_0\vec{\nabla}\bullet\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$$

or: $$\vec{\nabla}\bullet\vec{J}_{ToT}(\vec{r},t) = -\varepsilon_0\vec{\nabla}\bullet\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$$

$\Rightarrow$ If we combine $\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$ with $\vec{J}_{ToT}(\vec{r},t)$ this "cures" the problem with Ampere's Law:

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_0\left(\vec{J}_{ToT}(\vec{r},t) + \varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right) = \mu_0\vec{J}_{ToT}(\vec{r},t) + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}(\vec{r},t)$$

Then: $$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{B}(\vec{r},t)\right) = \mu_0\left(\vec{\nabla}\bullet\vec{J}_{ToT}(\vec{r},t)\right) + \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{E}(\vec{r},t)\right)$$

$$= \mu_0\left(-\frac{\partial\rho_{ToT}(\vec{r},t)}{\partial t}\right) + \cancel{\varepsilon_0}\mu_0\frac{\partial}{\partial t}\left[\left(\frac{1}{\cancel{\varepsilon_0}}\right)\rho_{ToT}(\vec{r},t)\right]$$

$$= -\mu_0\frac{\partial\rho_{ToT}(\vec{r},t)}{\partial t} + \mu_0\frac{\partial\rho_{ToT}(\vec{r},t)}{\partial t} = \underline{0!!!} \quad \text{YES INDEED!!}$$

Note the aesthetic/pleasing symmetry:

If $\vec{J}_{ToT}(\vec{r},t) = 0$, then $\vec{\nabla}\times\vec{B}(\vec{r},t) = \varepsilon_0\mu_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t} = \frac{1}{c^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$ "new" Ampere's Law

$\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$ (Faraday's Law)

$\Rightarrow$ A changing electric field produces a magnetic field!

$\Rightarrow$ A changing magnetic field produces a electric field!

New term is known as Maxwell's displacement current: $\vec{J}_D(\vec{r},t) \equiv \varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$

New Ampere's Law:

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_0\vec{J}_{ToT}(\vec{r},t) + \mu_0\vec{J}_D(\vec{r},t)$$
$$= \mu_0\vec{J}_{ToT}(\vec{r},t) + \mu_0\left(\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)$$

For the ||-plate capacitor, the electric field $\boxed{E(t) = \dfrac{\sigma(t)}{\varepsilon_0}}$

Where: $\sigma(t) = Q(t)/A$ where A = area of one plate

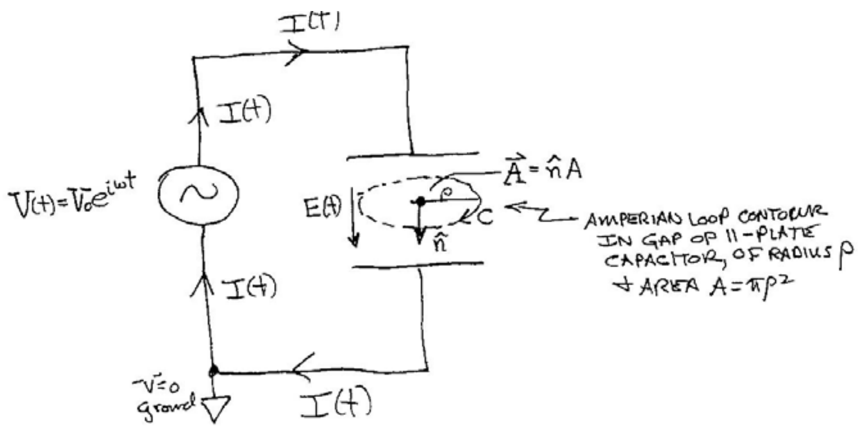
Thus: $E(t) = \dfrac{\sigma(t)}{\varepsilon_0} = \dfrac{Q(t)}{\varepsilon_0 A}$

Thus: $\dfrac{\partial E(t)}{\partial t} = \dfrac{1}{\varepsilon_0 A}\dfrac{\partial Q(t)}{\partial t} = \dfrac{1}{\varepsilon_0 A} I(t)$ where $I(t) \equiv \dfrac{\partial Q(t)}{\partial t}$

Then: $\int_S \left(\vec{\nabla}\times\vec{B}\right)\bullet d\vec{a} = \oint_C \vec{B}\bullet d\vec{\ell} = \mu_0 \int_S \vec{J}_{ToT}\bullet d\vec{a} + \mu_0 \int_S \vec{J}_D \bullet d\vec{a}$

Thus: $\boxed{\oint_c \vec{B}\bullet d\vec{\ell} = \mu_0 I_{ToT}^{encl} + \mu_0\varepsilon_0 \int_s \dfrac{\partial \vec{E}}{\partial t}\bullet d\vec{a}}$

For ||-plate capacitor circuit with contour C taken inside the gap of the ||-plate capacitor:



$$\oint_c \vec{B}\bullet d\vec{\ell} = \mu_0 \overset{=0}{\cancel{I_{ToT}^{enclosed}}} + \mu_0\varepsilon_0 \int_s \frac{\partial \vec{E}}{\partial t}\bullet d\vec{a}$$

$$\vec{B}_{\substack{in\\gap}}(\rho) = \frac{\mu_0}{2\mu}\cancel{\varepsilon_0}\left(\frac{1}{\cancel{\varepsilon_0}\cancel{A}} I(t) * \cancel{A}\right) = \frac{\mu_0}{2\pi\rho} I(t) \Leftarrow \text{Same answer!!!}$$

Thus the four "new" Maxwell equations are:

$$\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=\frac{1}{\varepsilon_0}\rho_{ToT}(\vec{r},t)}$$ (Gauss' Law)

$$\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0}$$ (no magnetic monopoles/magnetic charges)

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$$ (Faraday's Law)

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{B}(\vec{r},t)&=\mu_0\vec{J}_{ToT}(\vec{r},t)+\mu_0\vec{J}_D(\vec{r},t)\\&=\mu_0\vec{J}_{ToT}(\vec{r},t)+\mu_0\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\end{aligned}}$$ (Ampere's new Law)

Where: $\boxed{\vec{J}_D(\vec{r},t)=\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$ (Maxwell's displacement current)

Force Law: $\boxed{\vec{F}_{ToT}(\vec{r},t)=\vec{F}_E(\vec{r},t)+\vec{F}_m(\vec{r},t)=q\vec{E}(\vec{r},t)+q\vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)}$

Continuity equation: $\boxed{\vec{\nabla}\bullet\vec{J}_{ToT}(\vec{r},t)=-\frac{\partial\rho_{toT}(\vec{r},t)}{\partial t}}$ ⇐ can now be derived from Maxwell's eqns!!

(charge conservation)

Note that *if* magnetic charges g_m existed, then Maxwell's equations would become more "symmetrical":

$$\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=\frac{1}{\varepsilon_0}\rho^E_{ToT}(\vec{r},t)}$$ (Gauss' Law for electric charges)

$$\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=\mu_0\rho^m_{ToT}(\vec{r},t)}$$ (Gauss' Law for magnetic charges)

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\mu_0 J^m_{ToT}(\vec{r},t)-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$$ (Faraday's Law)

$$\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_0\vec{J}^E_{ToT}(\vec{r},t)+\underbrace{\mu_0\varepsilon_0}_{\equiv\frac{1}{c^2}}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$$ (Ampere's Law)

$$\boxed{\vec{F}^E_{ToT}(\vec{r},t)=q\left(\vec{E}(\vec{r},t)+\vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}$$

$$\boxed{\vec{F}^m_{ToT}(\vec{r},t)=g_m\left(\vec{B}(\vec{r},t)-\frac{1}{c^2}\vec{v}(\vec{r},t)\times\vec{E}(\vec{r},t)\right)}$$

Please see/read Phy435 lecture notes #18 for more details about magnetic monopoles….

Maxwell's Equations in Matter

In dielectric and magnetic media we have:

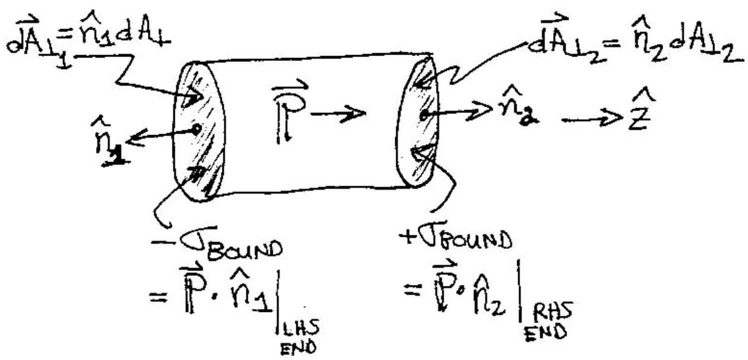
Electric polarization $\vec{P}$ = electric dipole moment per unit volume.
Magnetization $\vec{M}$ = magnetic dipole moment per unit volume.

For static fields: $\boxed{\rho_{bound}(\vec{r}) = -\vec{\nabla}\cdot\vec{P}(\vec{r})}$

$\boxed{\vec{J}_{bound}(\vec{r}) = \vec{\nabla}\times\vec{M}(\vec{r})}$

For non-static fields, a change in electric polarization $\Delta\vec{P}$ has associated with it a flow (i.e. a current) of bound charge – a bound polarization current density $\vec{J}_P(\vec{r})$ which *must* be included in the total current, I_{ToT}.

Consider a tiny chunk/block of polarized dielectric material



The electric polarization $\vec{P}$ induces bound surface charge densities

$$-\sigma_{bound} = \vec{P}\cdot\hat{n}_1\Big|_{\substack{LHS\\END}} \quad \text{and} \quad +\sigma_{bound} = \vec{P}\cdot\hat{n}_2\Big|_{\substack{RHS\\END}}$$

If the electric polarization $\vec{P}$ increases an infinitesimal amount $\Delta\vec{P}$ in time Δt, then the bound surface charges on each end also increase accordingly, giving a net Polarization bound current:

$$\Delta I_{P_{bound}} = \frac{\Delta\sigma_{bound}}{\Delta t}dA_\perp = \frac{\Delta\vec{P}}{\Delta t}\cdot d\vec{A}_\perp$$

OR: $$dI_{P_{bound}} = \frac{\partial\sigma_{bound}}{\partial t}dA_\perp = \frac{\partial\vec{P}}{\partial t}\cdot d\vec{A}_\perp = \vec{J}_{P_{bound}}\cdot dA_\perp$$

$$\boxed{\vec{J}_{P_{bound}}(\vec{r},t) = \vec{J}_P(\vec{r},t) \equiv \frac{\partial\vec{P}(\vec{r},t)}{\partial t}}$$ = electric polarization current density (SI units: Amps/m^2)

Must *not* be confused with: $\vec{J}_{m_{bound}}(\vec{r},t) \equiv \vec{\nabla}\times\vec{M}(\vec{r},t)$ = bound magnetic current density!!!

Because charge is charge is charge, there (of course) exists a continuity equation (expressing electric charge conservation) for bound electric polarization current density $\vec{J}_{P_{bound}}(\vec{r},t)$:

$$\boxed{\vec{\nabla}\cdot\vec{J}_P = \vec{\nabla}\cdot\frac{\partial\vec{P}}{\partial t} = \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{P}\right) = -\frac{\partial\rho^E_{bound}}{\partial t}}$$ $\vec{J}_P$ is essential to conservation of overall electric charge…

Total electric charge density: $\rho_{ToT}^{E}(\vec{r},t) = \rho_{free}^{E}(\vec{r},t) + \rho_{bound}^{E}(\vec{r},t) = \rho_{free}^{E}(\vec{r},t) - \vec{\nabla}\bullet\vec{\mathrm{P}}(\vec{r},t)$

Total electric current density:

$$\vec{J}_{ToT}^{E}(\vec{r},t) = \vec{J}_{free}^{E}(\vec{r},t) + \vec{J}_{bound}^{m}(\vec{r},t) + \vec{J}_{P_{bound}}^{E}(\vec{r},t)$$
$$= \vec{J}_{free}^{E}(\vec{r},t) + \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t) + \frac{\partial\vec{\mathrm{P}}}{\partial t}(\vec{r},t)$$

Then Gauss' Law becomes:

$$\vec{\nabla}\bullet\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho_{Tot}(\vec{r},t) = \frac{1}{\varepsilon_0}\left(\rho_{free}(\vec{r},t) + \rho_{bound}(\vec{r},t)\right)$$
$$= \frac{1}{\varepsilon_0}\rho_{free}^{E}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{\nabla}\bullet\vec{\mathrm{P}}(r,t)$$

$$\vec{\nabla}\bullet\vec{D}(\vec{r},t) \equiv \varepsilon_0\vec{\nabla}\bullet\vec{E}(\vec{r},t) + \vec{\nabla}\bullet\vec{\mathrm{P}}(\vec{r},t) = \vec{\nabla}\left(\varepsilon_0\vec{E}(\vec{r},t) + \vec{\mathrm{P}}(\vec{r},t)\right)$$

$$\vec{\nabla}\bullet\vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) \quad \text{and} \quad D(\vec{r},t) \equiv \varepsilon_0\vec{E}(\vec{r},t) + \vec{\mathrm{P}}(\vec{r},t)$$

(New) Ampere's Law (with Maxwell's displacement current term)

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_0\vec{J}_{ToT}(\vec{r},t) + \mu_0\vec{J}_D(\vec{r},t)$$
$$= \mu_0\left(\vec{J}_{free}^{E}(\vec{r},t) + \vec{J}_{bound}^{m}(\vec{r},t) + \vec{J}_{bound}^{E}(\vec{r},t)\right) + \mu_0\left(\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)$$
$$= \mu_0\left(\vec{J}_{free}^{E}(\vec{r},t) + \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t) + \frac{\partial\vec{\mathrm{P}}(\vec{r},t)}{\partial t}\right) + \mu_0\left(\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)$$
$$= \mu_0\left(\vec{J}_{free}^{E}(\vec{r},t) + \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)\right) + \mu_0\left(\varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t} + \frac{\partial\vec{\mathrm{P}}(\vec{r},t)}{\partial t}\right)$$
$$= \mu_0\left(\vec{J}_{free}^{E}(\vec{r},t) + \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)\right) + \mu_0\left(\frac{\partial}{\partial t}\underbrace{\left[\varepsilon_0\vec{E}(\vec{r},t) + \vec{\mathrm{P}}(\vec{r},t)\right]}_{\equiv\vec{D}(\vec{r},t)}\right)$$

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_0\left(\vec{J}_{free}^{E}(\vec{r},t) - \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)\right) + \mu_0\frac{\partial\vec{D}(\vec{r},t)}{\partial t}$$

<u>Then</u>: $\vec{\nabla}\times\vec{H}(\vec{r},t) \equiv \left[\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}(\vec{r},t) - \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)\right] = \mu_0\vec{J}_{free}^{E}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t}$

$$\vec{\nabla}\times\vec{H}(\vec{r},t) = \mu_0\vec{J}_{free}^{E}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t} \quad \text{and} \quad \vec{H}(\vec{r},t) \equiv \frac{1}{\mu_0}\vec{B}(\vec{r},t) - \vec{\mathrm{M}}(\vec{r},t)$$

Faraday's Law: $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$ and $\vec{\nabla}\bullet\vec{B}(\vec{r},t) = 0$ (no magnetic monopoles)

are unaffected by separation of electric charge and electric current into free and bound parts.

The four Maxwell's Equations <u>for free charges and free currents</u> <u>only</u>:

free charges & currents only

$$\begin{cases} \vec{\nabla}\bullet\vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) & \text{(Gauss' Law)} \\ \vec{\nabla}\bullet\vec{B}(\vec{r},t) = 0 & \text{(no magnetic charges)} \\ \vec{\nabla}\times\vec{E}(\vec{r},t) = -\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t} & \text{(Faraday's Law)} \\ \vec{\nabla}\times\vec{H}(\vec{r},t) = \vec{J}^{E}_{free}(\vec{r},t) + \dfrac{\partial\vec{D}(\vec{r},t)}{\partial t} & \text{(Ampere's Law)} \end{cases}$$

The four Maxwell equations for matter (i.e. dielectric *and* magnetic materials) are:

<u>Gauss' Law</u>: $$\vec{\nabla}\bullet\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho^{E}_{ToT}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho^{E}_{free}(\vec{r},t) + \rho^{E}_{bound}(\vec{r},t)$$ $$\rho^{E}_{bound}(\vec{r},t) \equiv -\vec{\nabla}\bullet\vec{P}(\vec{r},t)$$

<u>Auxilliary Relation</u>: $$\vec{D}(\vec{r},t) \equiv \varepsilon_0\vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$$

$$\vec{\nabla}\bullet\vec{D}(\vec{r},t) = \varepsilon_0\vec{\nabla}\bullet\vec{E}(\vec{r},t) + \vec{\nabla}\bullet\vec{P}(\vec{r},t) = \rho^{E}_{free}(\vec{r},t)$$

No magnetic monopoles: $$\vec{\nabla}\bullet\vec{B}(\vec{r},t) = 0$$

Faraday's Law: $$\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$$

Ampere's Law: $$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_0\left(\vec{J}_{ToT}(\vec{r},t) + \vec{J}_D(\vec{r},t)\right)$$

$$= \mu_0\left(\vec{J}^{E}_{free}(\vec{r},t) + \vec{J}^{m}_{bound}(\vec{r},t) + \vec{J}_{P_{bound}}(\vec{r},t) + \vec{J}_D(\vec{r},t)\right)$$

$$= \mu_0\left(\vec{J}^{E}_{free}(\vec{r},t) + \vec{\nabla}\times\vec{M}(\vec{r},t) + \frac{\partial\vec{P}(\vec{r},t)}{\partial t} + \varepsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)$$

With $\vec{J}_D(\vec{r},t) \equiv \varepsilon_0\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$ and $\vec{J}^{m}_{bound}(\vec{r},t) \equiv \vec{\nabla}\times\vec{M}(\vec{r},t)$ and $\vec{J}_{P_{bound}}(\vec{r},t) \equiv \dfrac{\partial\vec{P}(\vec{r},t)}{\partial t}$

<u>Auxilliary Relation</u>: $$\vec{H}(\vec{r},t) \equiv \frac{1}{\mu_o}\vec{B}(\vec{r},t) - \vec{M}(\vec{r},t)$$

$$\vec{\nabla}\times\vec{H}(\vec{r},t) = \mu_0\left(\vec{J}_{free}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\right)$$

For *linear* dielectric and/or magnetic media:

$$\vec{P}(\vec{r},t) = \varepsilon_o\chi_e\vec{E}(\vec{r},t) \qquad \vec{M}(\vec{r},t) = \chi_m\vec{H}(\vec{r},t)$$

$$\varepsilon = \varepsilon_o(1+\chi_e) \qquad K_e \equiv \varepsilon/\varepsilon_o = (1+\chi_e) \qquad \mu = \mu_o(1+\chi_m) \qquad K_m \equiv \mu/\mu_o = (1+\chi_m)$$

$$\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t) \qquad \vec{H}(\vec{r},t) = \vec{B}(\vec{r},t)/\mu$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

As we have seen previously, in order to obtain relations between normal (i.e. perpendicular) and tangential (i.e. parallel) components of $\{\vec{E},\vec{D},\vec{\mathrm{P}}\}$ and/or $\{\vec{B},\vec{H},\vec{\mathrm{M}}\}$ at an interface/boundary between dielectric and/or magnetic media, we must use the integral form(s) of Maxwell's equations, because spatial derivatives of $\{\vec{E},\vec{D},\vec{\mathrm{P}}\}$ and/or $\{\vec{B},\vec{H},\vec{\mathrm{M}}\}$ are not defined at an interface/boundary, for Maxwell's equations in differential form:

Gauss' Law:

$$\int_v \vec{\nabla}\cdot\vec{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \rho_{Tot}^{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v\left(\rho_{free}^{E}(\vec{r},t)+\rho_{bound}^{E}(\vec{r},t)\right)d\tau'$$

$$= \oint_S \vec{E}(\vec{r},t)\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{ToT}^{enclosed}(t) = \frac{1}{\varepsilon_0}\left(Q_{free}^{enclosed}(t)+Q_{bound}^{enclosed}(t)\right)$$

$$\oint_S \vec{D}(\vec{r},t)\,d\vec{a} = Q_{free}^{enclosed}(t) \qquad \oint_S \vec{\mathrm{P}}(\vec{r},t)\cdot d\vec{a} \equiv -Q_{bound}^{enclosed}(t)$$

Auxiliary Relation:

$$\vec{D}(\vec{r},t) = \varepsilon_o\vec{E}(\vec{r},t)+\vec{\mathrm{P}}(\vec{r},t) \qquad \rho_{Bound}(\vec{r},t) \equiv -\vec{\nabla}\cdot\vec{\mathrm{P}}(\vec{r},t) \qquad \sigma_{Bound}(\vec{r},t) \equiv \vec{\mathrm{P}}(\vec{r},t)\cdot\hat{n}\big|_{intf}$$

No Magnetic Monopoles:

$$\int_v \vec{\nabla}\cdot\vec{B}(\vec{r},t)\,d\tau' = \oint_S \vec{B}(\vec{r},t)\cdot d\vec{a} = 0$$

Faraday's Law:

$$\int_S \vec{\nabla}\times\vec{E}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{E}(\vec{r},t)\cdot d\vec{\ell} = -\int_S \frac{\partial\vec{B}(\vec{r},t)}{\partial t}\cdot d\vec{a} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t)\cdot d\vec{a}\right]$$

$$\varepsilon mf\;\;\varepsilon(t) \equiv \oint_C \vec{E}(\vec{r},t)\cdot d\vec{\ell} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t)\cdot d\vec{a}\right] = -\frac{d\Phi_M^{enclosed}(t)}{dt}$$

Ampere's Law:

$$\int_S \vec{\nabla}\times\vec{B}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0\int_S\left(\vec{J}_{ToT}(\vec{r},t)+\vec{J}_D(\vec{r},t)\right)\cdot d\vec{a}$$

$$= \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0\left(I_{ToT}^{encl}(t)+I_D^{encl}(t)\right) = \mu_0\left(I_{free}^{encl}(t)+I_{bound}^{m\;encl}(t)+I_{P_{bound}}^{encl}(t)+I_D^{encl}(t)\right)$$

Auxiliary Relation:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{B}(\vec{r},t)-\vec{\mathrm{M}}(\vec{r},t)$$

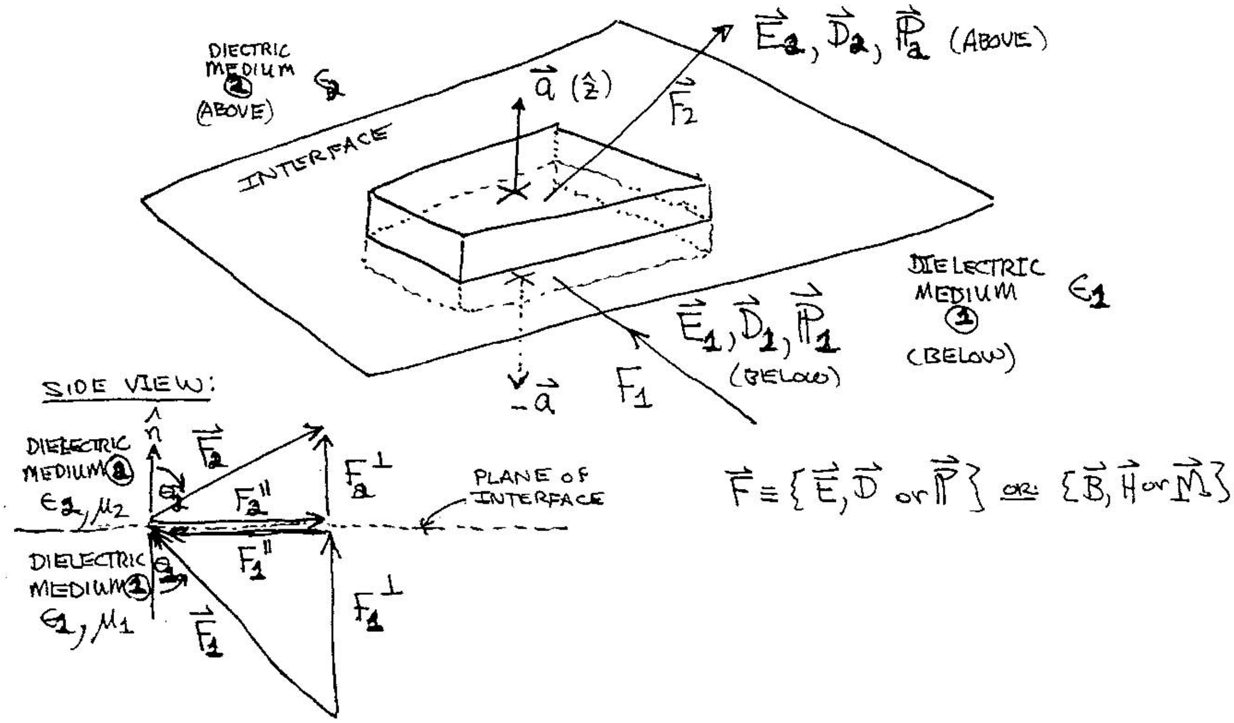
$$\vec{J}_{bound}^{m}(\vec{r},t) \equiv \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t) \qquad \vec{K}_{bound}^{m}(\vec{r},t) \equiv \vec{\mathrm{M}}(\vec{r},t)\times\hat{n}\big|_{intf}$$

$$\vec{J}_{P_{bound}}(\vec{r},t) \equiv \frac{\partial\vec{P}(\vec{r},t)}{\partial t}$$

$$\rho_m^{Bound}(\vec{r},t) \equiv -\vec{\nabla}\cdot\vec{\mathrm{M}}(\vec{r},t) \qquad \sigma_m^{Bound}(\vec{r},t) \equiv \vec{\mathrm{M}}(\vec{r},t)\cdot\hat{n}\big|_{intf}$$

$$\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{H}(\vec{r},t)\cdot d\vec{\ell} = I_{free}^{enclosed}(t)+\int_S \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\cdot d\vec{a} = I_{free}^{enclosed}(t)+\frac{d}{dt}\left[\int_S \vec{D}(\vec{r},t)\cdot d\vec{a}\right]$$

1) Apply the integral form of Gauss' Law at a dielectric interface/boundary using infinitesimally thin Gaussian pillbox extending slightly into dielectric material on either side of interface:



$$\oint_S \vec{E}\bullet d\vec{a} = \frac{1}{\varepsilon_0} Q_{ToT}^{enclosed} = \frac{1}{\varepsilon_0} Q_{free}^{enclosed} + \frac{1}{\varepsilon_0} Q_{bound}^{enclosed} = \frac{1}{\varepsilon_0}\oint_S \sigma_{free} da + \frac{1}{\varepsilon_0}\oint_S \sigma_{bound} da$$

<u>Gives</u>: $\underset{above}{\vec{E}_2}\bullet\vec{a} - \underset{below}{\vec{E}_1}\bullet\vec{a} = \frac{1}{\varepsilon_0}\sigma_{free} a + \frac{1}{\varepsilon_0}\sigma_{bound} a = \frac{1}{\varepsilon_0}\sigma_{ToT} a$ (<u>at interface</u>)

<u>or</u>: $\boxed{\underset{above}{E_2^{\perp}} - \underset{below}{E_1^{\perp}} = \frac{1}{\varepsilon_0}\sigma_{ToT} = \frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$ (<u>at interface</u>)

<u>Here</u>, the positive direction is <u>from</u> medium 2 (below) <u>to</u> medium 1 (above)

$$\oint_S \vec{D}\bullet d\vec{a} = Q_{free}^{enclosed} = \oint_S \sigma_{free} da \;\Rightarrow\; \boxed{\underset{above}{D_2^{\perp}} - \underset{below}{D_1^{\perp}} = \sigma_{free}}$$ (<u>at interface</u>)

<u>Likewise</u>:

$$\oint_S \vec{P}\bullet d\vec{a} = Q_{bound}^{enclosed} = -\oint_S \sigma_{bound} da \Rightarrow \boxed{\underset{above}{P_2^{\perp}} - \underset{below}{P_1^{\perp}} = \sigma_{bound}}$$ (<u>at interface</u>)

<u>Since</u>: $\vec{E} \equiv -\vec{\nabla} V$

$$\boxed{\left.\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\right|_{\text{interface}} = -\frac{1}{\varepsilon_0}\sigma_{ToT} = -\frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$$ (<u>at interface</u>)

Since: $\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}V$

$$\boxed{\left.\left(\varepsilon_2\frac{\partial V_2^{above}}{\partial n} - \varepsilon_1\frac{\partial V_1^{below}}{\partial n}\right)\right|_{\text{interface}} = -\sigma_{free}}$$ (at interface)

2) Similarly, for $\int_v \vec{\nabla}\cdot\vec{B}d\tau' = \oint_S \vec{B}\cdot d\vec{a} = 0$ (no magnetic monopoles), then at an interface:

$\vec{B}_2^{above}\cdot\vec{a} - \vec{B}_1^{above}\cdot\vec{a} = 0 \Rightarrow \boxed{\underset{above}{B_2^\perp} - \underset{below}{B_1^\perp} = 0}$ or: $\boxed{\underset{above}{B_2^\perp} = \underset{below}{B_1^\perp}}$ (at interface)

Since: $\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{M}$ Then: $\vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)$

$\oint_S \vec{B}\cdot d\vec{a} = \mu_o\oint_S\left(\vec{H} + \vec{M}\right)\cdot d\vec{a} = 0$ or: $\oint_S \vec{H}\cdot d\vec{a} = -\oint_S \vec{M}\cdot d\vec{a}$

Then: $\vec{H}_2^{above}\cdot\vec{a} - \vec{H}_1^{below}\cdot\vec{a} = -\left(\vec{M}_2^{above}\cdot\vec{a} - \vec{M}_1^{below}\cdot\vec{a}\right)$ (at interface)

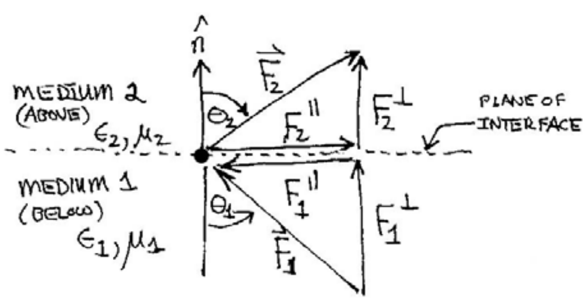
Or: $\boxed{\left(\underset{above}{H_2^\perp} - \underset{below}{H_1^\perp}\right) = -\left(\underset{above}{M_2^\perp} - \underset{below}{M_1^\perp}\right) = -\sigma_{magnetic}^{bound}}$ (at interface)

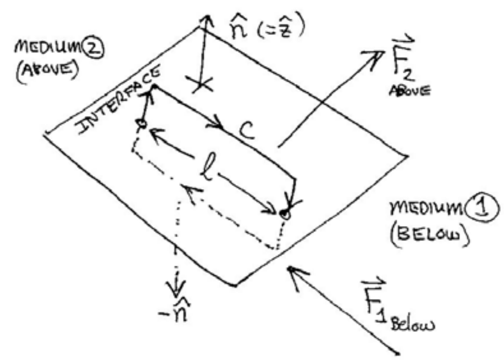
Effective bound magnetic charge at interface

3) For Faraday's Law: EMF, $\varepsilon = \oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\left(\oint_S \vec{B}\cdot d\vec{a}\right) = -\frac{d\Phi_m}{dt}$ at an interface / boundary between two different media, taking a closed contour C of width ℓ extending slightly (i.e. infinitesimally) into the material on either side of interface, as shown below:

$\vec{F} \equiv \left\{\vec{E}, \vec{D} \text{ or } \vec{P}\right\}$ or: $\left\{\vec{B}, \vec{H} \text{ or } \vec{M}\right\}$

Side View:





$\vec{E}_2^{above}\cdot\vec{\ell} - \vec{E}_1^{below}\cdot\vec{\ell} = -\frac{d}{dt}\oint_S \vec{B}\cdot d\vec{a} = 0$ (in limit area of contour loop → 0, magnetic flux enclosed → 0)

Thus: $\boxed{\underset{above}{E_2^{\|}} - \underset{below}{E_1^{\|}} = 0}$ (at interface) or: $\boxed{\underset{above}{E_2^{\|}} = \underset{below}{E_1^{\|}}}$ (at interface)

Since: $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ And: $\varepsilon_o \vec{E} = \vec{D} - \vec{P}$

Thus: $$\left(\vec{E}_2^{above} \cdot \vec{\ell} - \vec{E}_1^{below} \cdot \vec{\ell}\right) = \left(\vec{D}_2^{above} \cdot \vec{\ell} - \vec{D}_1^{below} \cdot \vec{\ell}\right) - \left(\vec{P}_2^{above} \cdot \vec{\ell} - \vec{P}_1^{below} \cdot \vec{\ell}\right) = 0$$

In limit area of contour loop → 0 magnetic flux enclosed → 0

$$\Rightarrow \boxed{\left(\underset{above}{\vec{D}_2^{\parallel}} - \underset{below}{\vec{D}_1^{\parallel}}\right) = \left(\underset{above}{\vec{P}_2^{\parallel}} - \underset{below}{\vec{P}_1^{\parallel}}\right)}$$ (at interface)

4) Finally, for Ampere's Law: $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o \left(I_{TOT}^{encl} + I_D^{encl}\right)$

$$\vec{B}_2^{above} \cdot \vec{\ell} - \vec{B}_1^{below} \cdot \vec{\ell} = \mu_o I_{TOT}^{encl} + \mu_o I_D^{encl}$$

$$I_D^{encl} = \int_S \vec{J}_D \cdot d\vec{a} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

$$I_{TOT}^{encl} = I_{free}^{encl} + I_{bound}^{encl} + I_{P_{bound}}^{encl}$$

$$I_{P_{bound}}^{encl} = \int_S \vec{J}_{P_{bound}} \cdot d\vec{a} = \int_S \frac{\partial \vec{P}}{\partial t} \cdot d\vec{a}$$

$$I_{bound}^{encl} = \int_S \vec{J}_m^{bound} \cdot d\vec{a} = \int_S \vec{\nabla} \times \vec{M} \cdot d\vec{a}$$

- Where I_{TOT}^{encl} = *TOTAL* current (free + bound + polarization) passing through enclosing Amperian loop contour *C*

- No volume current density $\vec{J}_{TOT}, \vec{J}_{free}, \vec{J}_{bound}^m$ or $\vec{J}_P$ contributes to I_{TOT}^{encl} in the limit area of contour loop → 0, however a surface current $\vec{K}_{TOT}, \vec{K}_{free}, \vec{K}_{bound}^m = \vec{M} \times \hat{n}$ can contribute!

- In the limit that the enclosing Amperian loop contour C shrinks to zero height above/below interface, the enclosed area of loop contour → 0, and

Then: $$I_D^{encl} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \varepsilon_o \frac{d}{dt}\left[\int_S \vec{E} \cdot d\vec{a}\right] = \varepsilon \frac{d\Phi_E}{dt} \to 0$$

($\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$ = enclosed flux of electric field lines)

Similarly: $$I_{P_{bound}}^{encl} = \int_S \frac{\partial \vec{P}}{\partial t} \cdot d\vec{a} = \frac{d}{dt}\left[\int_S \vec{P} \cdot d\vec{a}\right] = \frac{d\Phi_P}{dt} \to 0$$

($\Phi_P \equiv \int_S \vec{P} \cdot d\vec{a}$ = enclosed flux of electric polarization field lines)

- If $\hat{n}$ is unit normal/perpendicular to interface (pointing from medium 1 (below) into medium 2 (above), note that $\left(\hat{n} \times \vec{\ell}\right)$ is normal/perpendicular to plane of the Amperian loop contour.

Thus:
$$I_{TOT}^{encl} = \vec{K}_{TOT} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{TOT} \times \hat{n}\right) \cdot \vec{\ell}$$
$$I_{free}^{encl} = \vec{K}_{free} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{free} \times \hat{n}\right) \cdot \vec{\ell}$$
$$I_{bound}^{encl} = \vec{K}_{bound} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{bound}^m \times \hat{n}\right) \cdot \vec{\ell}$$

Using: $$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) \left\{= \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}\right\}$$

$$I_{TOT} = I_{free} + I_{bound} \qquad \vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{bound}$$

In the limit that the enclosing Amperian loop contour C (of width ℓ) shrinks to zero height above/below interface, causing area of enclosed loop contour → 0, then:

$$\vec{B}_2^{above}\cdot\vec{\ell} - \vec{B}_1^{below}\cdot\vec{\ell} = \mu_o I_{TOT}^{encl} + \overbrace{\cancel{\mu_o I_D^{encl}}}^{=0} = \mu_o I_{TOT}^{encl} = \left(\vec{K}_{TOT}\times\hat{n}\right)\cdot\vec{\ell}$$

$$\Rightarrow \boxed{\underset{above}{B_2^{\|}} - \underset{below}{B_1^{\|}} = \mu_o\vec{K}_{TOT}\times\hat{n} = \mu_o\left(\vec{K}_{free} + \vec{K}_{bound}^m\right)\times\hat{n}} \quad \text{(\underline{at interface})}$$

Since: $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$ and: $\frac{1}{\mu_o}\vec{B} = \vec{H} + \vec{M}$ then:

$$\boxed{\frac{1}{\mu_o}\left(\vec{B}_2^{above}\cdot\vec{\ell} - \vec{B}_1^{below}\cdot\vec{\ell}\right) = \left(\vec{H}_2^{above}\cdot\vec{\ell} - \vec{H}_1^{below}\cdot\vec{\ell}\right) + \left(\vec{M}_2^{above}\cdot\vec{\ell} - \vec{M}_1^{below}\cdot\vec{\ell}\right) = \left[\left(\vec{K}_{free}\times\hat{n}\right) + \left(\vec{K}_{bound}\times\hat{n}\right)\right]}$$

(<u>at interface</u>)

We also see that: $\boxed{\underset{above}{H_2^{\|}} - \underset{below}{H_1^{\|}} = \vec{K}_{free}\times\hat{n}}$ (<u>at interface</u>)

and: $\boxed{\underset{above}{M_2^{\|}} - \underset{below}{M_1^{\|}} = \vec{K}_{bound}^m\times\hat{n}}$ (<u>at interface</u>)

|| - components of $\vec{B}$ are discontinuous at interface by $\mu_o\vec{K}_{TOT}\times\hat{n}$
|| - components of $\vec{H}$ are discontinuous at interface by $\vec{K}_{free}\times\hat{n}$
|| - components of $\vec{M}$ are discontinuous at interface by $\vec{K}_{bound}^m\times\hat{n}$

If $\vec{B} = \vec{\nabla}\times\vec{A}$ where $\vec{A}$ is the magnetic vector potential, then:

$$\boxed{\left(\frac{1}{\mu_0}\right)\left[\underset{above}{B_2^{\|}} - \underset{below}{B_1^{\|}}\right] = \vec{K}_{TOT}\times\hat{n}}$$ (<u>at interface</u>) is equivalent to:

$$\boxed{\left(\frac{1}{\mu_o}\right)\left(\frac{\partial\vec{A}_2^{above}}{\partial n} - \frac{\partial\vec{A}_1^{below}}{\partial n}\right)\Bigg|_{\text{interface}} = -\vec{K}_{TOT}}$$ (<u>at interface</u>)

For <u>linear</u> magnetic media: $\vec{B} = \mu\vec{H}$ or: $\vec{H} = \frac{1}{\mu}\vec{B}$

<u>Then</u>: $\boxed{\left[\underset{above}{H_2^{\|}} - \underset{below}{H_1^{\|}}\right] = \vec{K}_{free}\times\hat{n}}$ (<u>at interface</u>) is <u>equivalent</u> to:

$$\boxed{\left(\frac{1}{\underset{above}{\mu_2}}\right)\frac{\partial\vec{A}_2^{above}}{\partial n}\Bigg|_{\text{interface}} - \left(\frac{1}{\underset{below}{\mu_1}}\right)\frac{\partial\vec{A}_1^{below}}{\partial n}\Bigg|_{\text{interface}} = -\vec{K}_{free}}$$ (<u>at interface</u>)

Summary of Maxwell's Equations In Differential and Integral Forms
(Suppress Explicit $(\vec{r},t)$ Dependence)

Gauss' Law: $$\vec{\nabla}\bullet\vec{E} = \frac{1}{\varepsilon_o}\rho_{TOT} = \frac{1}{\varepsilon_o}\left(\rho_{free} + \rho_{bound}\right)$$ $$\rho_{bound} \equiv -\vec{\nabla}\bullet\vec{\mathrm{P}}$$

Use of the auxiliary relation: $\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$

Yields: $$\vec{\nabla}\bullet\vec{D} = \varepsilon_o\vec{\nabla}\bullet\vec{E} + \vec{\nabla}\bullet\vec{\mathrm{P}} = \rho_{free}$$

No magnetic monopoles: $$\vec{\nabla}\bullet\vec{B} = 0$$

Faraday's Law: $$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's Law: $$\vec{\nabla}\times\vec{B} = \mu_o\left(\vec{J}_{TOT} + \vec{J}_D\right)$$ $$\vec{J}_{TOT} = \vec{J}_{free} + \vec{J}^{m}_{bound} + \vec{J}_{\mathrm{P}_{bound}}$$

Use of auxiliary relation: $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}$ with $\vec{J}_D = \varepsilon_o\frac{\partial\vec{E}}{\partial t}$

Yields: $$\vec{\nabla}\times\vec{H} = \mu_o\vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}$$

Gauss' Law: $$\int_v \vec{\nabla}\bullet\vec{E}\, d\tau' = \oint_S \vec{E}\bullet d\vec{a} = \frac{1}{\varepsilon_o}Q^{encl}_{tot} = \frac{1}{\varepsilon_o}\left[Q^{encl}_{free} + Q^{encl}_{bound}\right] \equiv \Phi_{\mathrm{E}}$$ (Electric Flux)

Use of the auxiliary relation: $\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$

Yields: $$\int_v \vec{\nabla}\bullet\vec{D}\, d\tau' = \oint_S \vec{D}\bullet d\vec{a} = Q^{encl}_{free} \equiv \Phi_D$$ (Electric Displacement Flux)

$$\int_v \vec{\nabla}\bullet\vec{\mathrm{P}}\, d\tau' = \oint_S \vec{\mathrm{P}}\bullet d\vec{a} = -Q^{encl}_{bound} \equiv \Phi_{\mathrm{P}}$$ (Electric Polarization Flux)

No magnetic monopoles: $$\int_v \vec{\nabla}\bullet\vec{B}\, d\tau' = \oint_S \vec{B}\bullet d\vec{a} = 0$$

Faraday's Law: EMF $$\varepsilon = \int_S \vec{\nabla}\times\vec{E}\bullet d\vec{a} = \oint_C \vec{E}\bullet d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\bullet d\vec{a} = -\frac{d\Phi^{encl}_m}{dt}$$

Ampere's Law: $$\int_S\left(\vec{\nabla}\bullet\vec{B}\right)\bullet d\vec{a} = \oint_C \vec{B}\bullet d\vec{\ell} = \mu_o\int_S\left(\vec{J}_{TOT} + \vec{J}_D\right)\bullet d\vec{a}$$ $$= \mu_o\left(I^{encl}_{TOT} + I^{encl}_D\right) = \mu_o\left(I^{encl}_{free} + I^{encl}_{bound} + I^{encl}_{\mathrm{P}_{bound}} + I^{encl}_D\right)$$

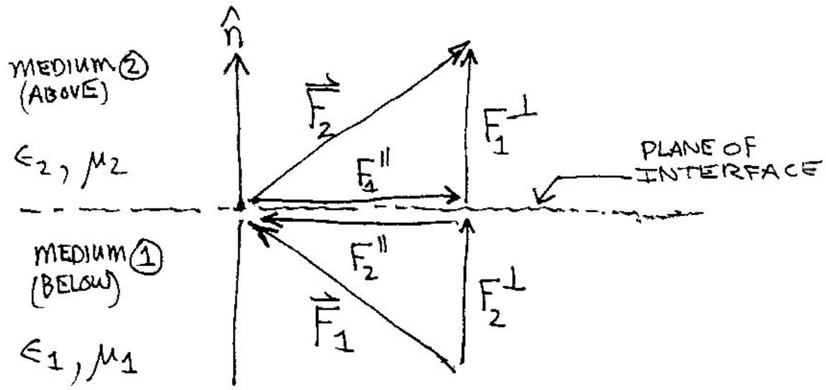
Use of auxiliary relation(s): $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}$ and $\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$

Yields: $$\int_S\left(\vec{\nabla}\times\vec{H}\right)\bullet d\vec{a} = \oint_C \vec{H}\bullet d\vec{\ell} = I^{encl}_{free} + \frac{d}{dt}\left[\int_S \vec{D}\bullet d\vec{a}\right]$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

Suppressing explicit $(\vec{r},t)$ dependence and defining $\hat{n}$ (unit normal) pointing from medium 1 (below) into medium 2 (above) as shown in the figure below:

Side View:



$\vec{F} \equiv \{\vec{E}, \vec{D} \text{ or } \vec{\mathrm{P}}\}$

or:

$\vec{F} \equiv \{\vec{B}, \vec{H}, \text{ or } \vec{\mathrm{M}}\}$

Then at the interface:

Gauss' Law:

$$\underset{above}{E_2^{\perp}} - \underset{below}{E_1^{\perp}} = \frac{1}{\varepsilon_o}\sigma_{TOT} = \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right) \qquad \vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$$

$$\underset{above}{D_2^{\perp}} - \underset{below}{D_1^{\perp}} = \sigma_{free} \qquad \underset{above}{\mathrm{P}_2^{\perp}} - \underset{below}{\mathrm{P}_1^{\perp}} = -\sigma_{bound}$$

$$\left.\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\right|_{\text{interface}} = -\frac{1}{\varepsilon_o}\sigma_{TOT} = -\frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\left.\left(\varepsilon_2\frac{\partial V_2^{above}}{\partial n} - \varepsilon_1\frac{\partial V_1^{below}}{\partial n}\right)\right|_{\text{interface}} = -\sigma_{free}$$

No Magnetic Monopoles:

$$\underset{above}{B_2^{\perp}} - \underset{below}{B_1^{\perp}} = 0 \Rightarrow \left(\underset{above}{H_2^{\perp}} - \underset{below}{H_1^{\perp}}\right) = -\left(\underset{above}{\mathrm{M}_2^{\perp}} - \underset{below}{\mathrm{M}_1^{\perp}}\right) = -\sigma_{magnetic}^{bound}$$

$$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{\mathrm{M}}$$

Faraday's Law:

$$\underset{above}{E_2^{\parallel}} - \underset{below}{E_1^{\parallel}} = 0 \Rightarrow \left(\underset{above}{D_2^{\parallel}} - \underset{below}{D_1^{\parallel}}\right) = \left(\underset{above}{\mathrm{P}_2^{\parallel}} - \underset{below}{\mathrm{P}_1^{\parallel}}\right)$$

Ampere's Law:

$$\underset{above}{B_2^{\parallel}} - \underset{below}{B_1^{\parallel}} = \mu_o\vec{K}_{TOT}\times\hat{n} = \mu_o\left(\vec{K}_{free} + \vec{K}_{bound}^{m}\right)\times\hat{n} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\Rightarrow \left.\left(\frac{1}{\mu_o}\right)\left(\frac{\partial\vec{A}_2^{above}}{\partial n} - \frac{\partial\vec{A}_1^{below}}{\partial n}\right)\right|_{\text{interface}} = -\vec{K}_{TOT}$$

$$\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} = \vec{K}_{free}\times\hat{n} \qquad \underset{above}{\mathrm{M}_2^{\parallel}} - \underset{below}{\mathrm{M}_1^{\parallel}} = \vec{K}_{bound}^{m}\times\hat{n}$$

$$\Rightarrow \left(\frac{1}{\mu_2}\right)\left.\frac{\partial\vec{A}_2}{\partial n}\right|_{\substack{\text{above}\\ \text{interface}}} - \left(\frac{1}{\mu_1}\right)\left.\frac{\partial\vec{A}_1}{\partial n}\right|_{\substack{\text{below}\\ \text{interface}}} = -\vec{K}_{free}$$

BC's Specific to *Linear* Homogeneous Isotropic Dielectric and/or Magnetic Media:

$$\vec{P} = \varepsilon_o \chi_e \vec{E} \qquad \vec{M} = \chi_m \vec{H} \qquad \vec{D} = \varepsilon \vec{E} = \varepsilon_o \vec{E} + \vec{P}$$

$$\varepsilon = \varepsilon_o (1+\chi_e) \qquad \mu = \mu_o (1+\chi_m) \qquad \vec{B} = \mu \vec{H} \text{ or } \vec{H} = \frac{1}{\mu}\vec{B}$$

$$K_e = \frac{\varepsilon}{\varepsilon_o} = 1+\chi_e \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) \qquad \vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$$

The boundary conditions at the interface between linear dielectric or magnetic media become:

Gauss' Law:

$$\left(\underset{above}{E_2^\perp} - \underset{below}{E_1^\perp}\right) = \frac{1}{\varepsilon_o}\sigma_{TOT} = \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\Rightarrow \left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{\text{interface}} = -\frac{1}{\varepsilon_o}\sigma_{TOT} = -\frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\left(\underset{above}{D_2^\perp} - \underset{below}{D_1^\perp}\right) = \sigma_{free} \Rightarrow \left(\varepsilon_2 \underset{above}{E_2^\perp} - \varepsilon_1 \underset{below}{E_1^\perp}\right) = \sigma_{free}$$

$$\left(\underset{above}{P_2^\perp} - \underset{below}{P_1^\perp}\right) = -\sigma_{bound} \Rightarrow \varepsilon_o\left(\chi_{e_2} \underset{above}{E_2^\perp} - \chi_{e_1} \underset{below}{E_1^\perp}\right) = -\sigma_{bound}$$

$$\varepsilon_2 \frac{\partial V_2}{\partial n}\Bigg|_{\substack{\text{above}\\ \text{interface}}} - \varepsilon_1 \frac{\partial V_1}{\partial n}\Bigg|_{\substack{\text{below}\\ \text{interface}}} = -\sigma_{free}$$

No magnetic monopoles:

$$\left(\underset{above}{B_2^\perp} - \underset{below}{B_1^\perp}\right) = 0 \Rightarrow \left(\underset{above}{H_2^\perp} - \underset{below}{H_1^\perp}\right) = -\left(\underset{above}{M_2^\perp} - \underset{below}{M_1^\perp}\right) = -\sigma_{magnetic}^{bound}$$

$$= \left(\underset{above}{\mu_2} H_2^\perp - \underset{below}{\mu_1} H_1^\perp\right) = 0 \qquad = \left(\frac{1}{\underset{above}{\mu_2}} B_2^\perp - \frac{1}{\underset{below}{\mu_1}} B_1^\perp\right)$$

Faraday's Law:

$$\left(\underset{above}{E_2^{||}} - \underset{below}{E_1^{||}}\right) = 0 \qquad \Rightarrow \qquad \left(\underset{above}{D_2^{||}} - \underset{below}{D_1^{||}}\right) = \left(\underset{above}{P_2^{||}} - \underset{below}{P_1^{||}}\right)$$

$$= \left(\frac{1}{\underset{above}{\varepsilon_2}} D_2^{||} - \frac{1}{\underset{below}{\varepsilon_1}} D_1^{||}\right) = 0 \Rightarrow \left(\underset{above}{\varepsilon_2} E_2^{||} - \underset{below}{\varepsilon_1} E_1^{||}\right) = -\varepsilon_o\left(\underset{above}{\chi_{e_2}} E_2^{||} - \underset{below}{\chi_{e_1}} E_1^{||}\right)$$

Ampere's Law:

$$\underset{above}{B_2^{||}} - \underset{below}{B_1^{||}} = \mu_o \vec{K}_{TOT} \times \hat{n} = \mu_o\left(\vec{K}_{free} + \vec{K}_{bound}^m\right) \times \hat{n} \Rightarrow \left(\frac{1}{\mu_o}\right)\left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n}\right)\Bigg|_{\text{interface}} = -\vec{K}_{TOT}$$

$$= \left(\underset{above}{\mu_2} H_2^{||} - \underset{below}{\mu_1} H_1^{||}\right)$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\boxed{\begin{aligned} &\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} = \vec{K}_{free} \times \hat{n} \\ &= \frac{1}{\underset{above}{\mu_2}} \underset{above}{B_2^{\parallel}} - \frac{1}{\underset{below}{\mu_1}} \underset{below}{B_1^{\parallel}} \end{aligned}}$$

and

$$\boxed{\begin{aligned} &\left(\underset{above}{M_2^{\parallel}} - \underset{below}{M_1^{\parallel}} \right) = \vec{K}_{bound}^{m} \times \hat{n} \\ &= \left(\chi_{\underset{above}{m_2}} \underset{above}{H_2^{\parallel}} - \chi_{\underset{below}{m_1}} \underset{below}{H_1^{\parallel}} \right) \\ &= \left(\frac{\chi_{\underset{above}{m_2}}}{\underset{above}{\mu_2}} \underset{above}{B_2^{\parallel}} - \frac{\chi_{\underset{below}{m_1}}}{\underset{below}{\mu_1}} \underset{below}{B_1^{\parallel}} \right) \end{aligned}}$$

$$\boxed{\left(\frac{1}{\mu_2} \right) \frac{\partial \vec{A}_2}{\partial n} \bigg|_{\substack{\text{above} \\ \text{interface}}} - \left(\frac{1}{\mu_1} \right) \frac{\partial \vec{A}_1}{\partial n} \bigg|_{\substack{\text{below} \\ \text{interface}}} = -\vec{K}_{free}} \quad \text{using} \quad \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$