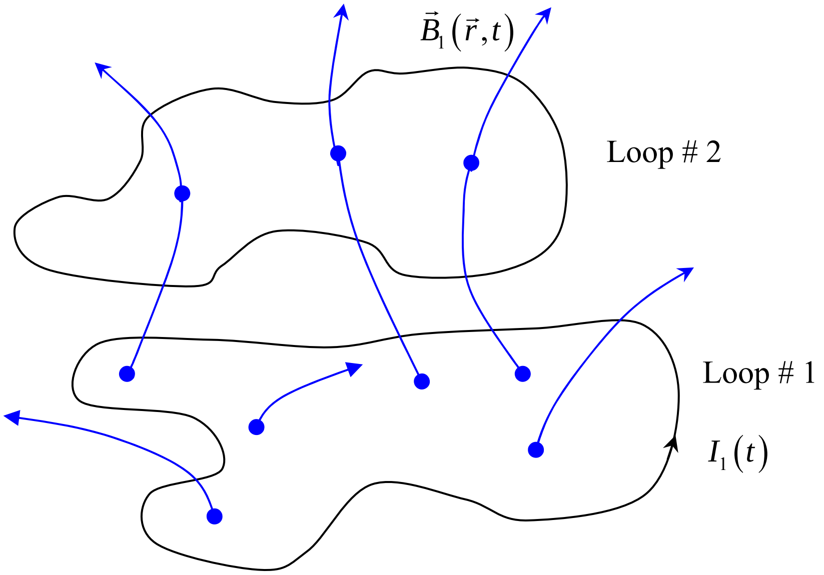
LECTURE NOTES 22

Inductance: Mutual Inductance and Self-Inductance

Inductance is the magnetic analog of capacitance in electric phenomena. Like capacitance, inductance has to do with the geometry of a magnetic device and the magnetic properties of the materials making up the magnetic device. The capacitance *C* of an electric device is associated with the ability to store energy in the electric field of that device. The inductance *L* of a magnetic device is associated with the ability to store energy in the magnetic field of that device.

Mutual Inductance:

Suppose we have two arbitrary shaped loops of wire (both at rest) in proximity to each other, as shown in the figure below. Suppose Loop # 1 carries a current $I_1(t)$.



The current $I_1(t)$ flowing in Loop # 1 produces a magnetic field $\vec{B}_1(\vec{r},t)$, and some of these magnetic field lines will pass through Loop # 2, linking it.

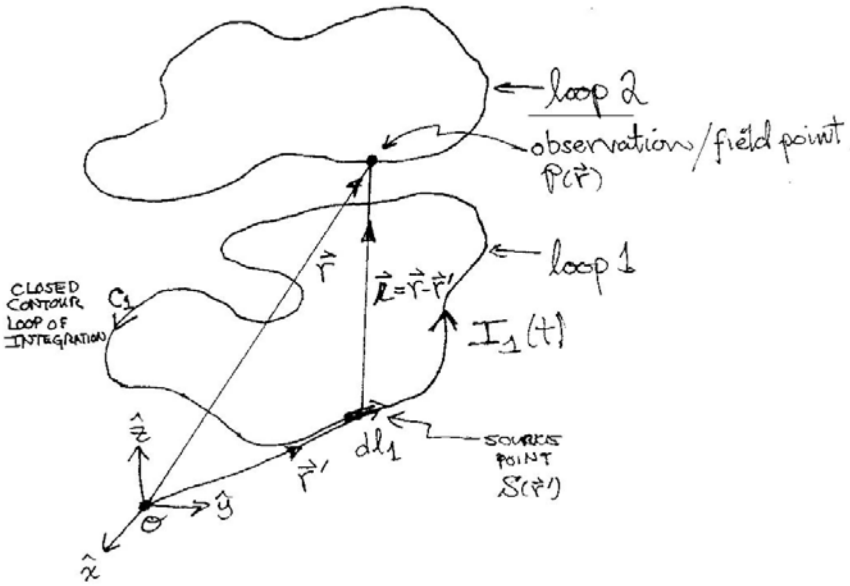
The magnetic flux from Loop # 1 linking Loop # 2 is:

$$\boxed{\Phi_m^{12}(t)=\int_{S_2}\vec{B}_1(\vec{r},t)\cdot d\vec{a}_{\perp_2}}$$

Where:

$$\boxed{\vec{B}_1(\vec{r},t)=\left(\frac{\mu_o}{4\pi}\right)I_1(t)\oint_{C_1}\frac{d\vec{\ell}_1'\times\hat{\mathcal{r}}}{\mathcal{r}^2}}$$

Note that $\vec{B}_1(\vec{r},t)$ is linearly proportional to $I_1(t)$.



<u>Then</u>:
$$\boxed{\Phi_m^{\overrightarrow{12}}(t)=\int_{S_2}\vec{B}_1(\vec{r},t)\bullet d\vec{a}_{\perp_2}=\left(\frac{\mu_o}{4\pi}\right)I_1(t)\int_{S_2}\left(\oint_{C_1}\frac{d\vec{\ell}_1'\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\bullet d\vec{a}_{\perp_2}}$$

Now irrespective of the details of doing the double integral in the above formula, we know that that $\vec{B}_1(\vec{r},t)$ is linearly proportional to $I_1(t)$ and thus to is $\Phi_m^{\overrightarrow{12}}(t)$, i.e.:

The magnetic flux from Loop 1 linking Loop 2 is: $\boxed{\Phi_m^{\overrightarrow{12}}(t)\equiv M_{\overrightarrow{12}}I_1(t)}$

Where the constant of proportionality:
$$\boxed{M_{\overrightarrow{12}}=\left(\frac{\mu_o}{4\pi}\right)\underbrace{\int_{S_2}\left(\oint_{C_1}\frac{d\vec{\ell}_1\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\bullet d\vec{a}_{\perp_2}}_{\text{what is this physically?}}}=\boxed{\text{\underline{Mutual} Inductance of Loop 1 and Loop 2}}$$

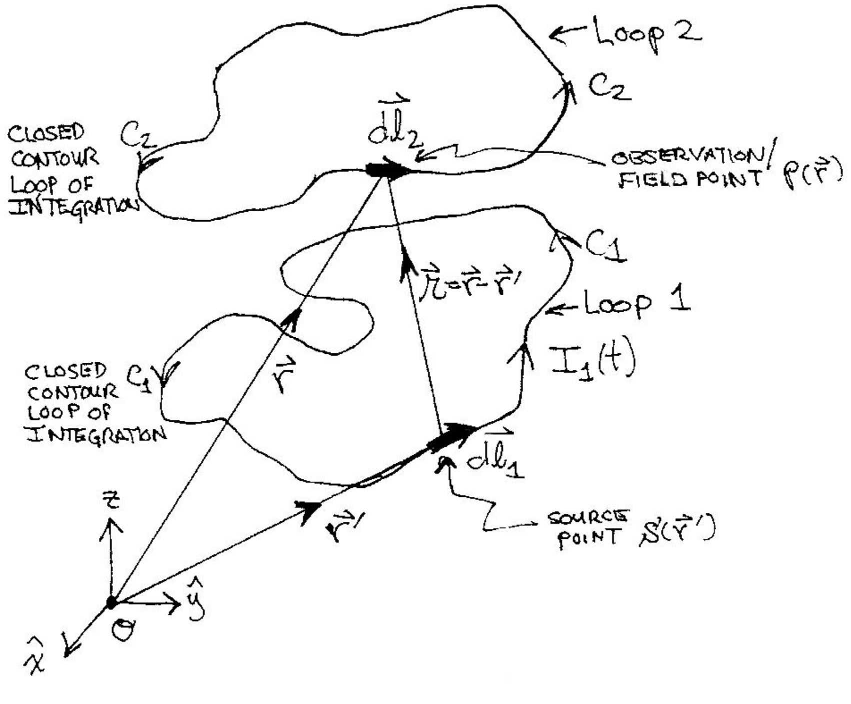
Equivalently, note that we can also obtain $M_{\overrightarrow{12}}$ using: $\boxed{\Phi_m^{\overrightarrow{12}}(t)=\oint_{C_2}\vec{A}_1(\vec{r},t)\bullet d\vec{\ell}_2}$

{obtained previously, using $\vec{B}=\vec{\nabla}\times\vec{A}$ }

Where: $\boxed{\vec{A}_1(\vec{r},t)=\left(\frac{\mu_o}{4\pi}\right)I_1(t)\oint_{C_1}\frac{d\vec{\ell}_1}{\mathscr{r}}}$ ← Note $\vec{A}_1(\vec{r},t)$ is also linearly proportional to $I_1(t)$

<u>Then</u>:
$$\boxed{\Phi_m^{\overrightarrow{12}}(t)=\oint_{C_2}\vec{A}_1(\vec{r},t)\bullet d\vec{\ell}_2=\left(\frac{\mu_o}{4\pi}\right)I_1(t)\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1\bullet d\vec{\ell}_2}{\mathscr{r}}}$$

<u>Then if</u>: $\boxed{\Phi_m^{\overrightarrow{12}}(t)=M_{\overrightarrow{12}}I_1(t)}$ <u>Then</u>: $\boxed{M_{\overrightarrow{12}}=\left(\frac{\mu_o}{4\pi}\right)\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1\bullet d\vec{\ell}_2}{\mathscr{r}}}$ ⇐ $\boxed{\text{Known as Neumann's Formula}}$



Thus we see that: $$\boxed{M_{\overrightarrow{12}} = \left(\frac{\mu_o}{4\pi}\right) \underbrace{\oint_{C_2}\oint_{C_1} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{\mathscr{r}}}_{\substack{\text{Depends only on geometry;}\\ \text{Has dimensions of length.}}} = \left(\frac{\mu_o}{4\pi}\right) \underbrace{\int_{S_2}\left(\oint_{C_1} \frac{d\vec{\ell}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right)\cdot d\vec{a}_{\perp_2}}_{\substack{\text{Depends only on geometry;}\\ \text{Has dimensions of length.}}}}$$

We also see that: $$\boxed{\oint_{C_2}\oint_{C_1} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{\mathscr{r}} = \int_{S_2}\left(\oint_{C_1} \frac{d\vec{\ell}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right)\cdot d\vec{a}_{\perp_2}}$$

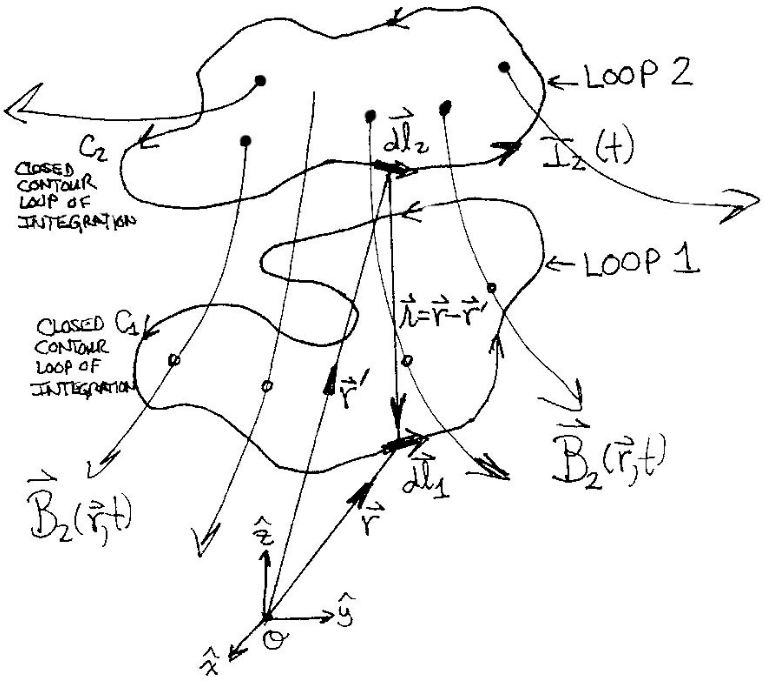
<u>The SI units of mutual inductance</u>: $\boxed{M_{\overrightarrow{12}} = \text{Henrys} = \left(\frac{Webers}{Ampere}\right)}$

$$\boxed{\Phi_m^{\overrightarrow{12}}(t) = M_{\overrightarrow{12}} I_1(t) \;\rightarrow\; \frac{\Phi_m^{\overrightarrow{12}}(t)}{I_1(t)} = M_{\overrightarrow{12}} = \frac{Flux}{Current} = \frac{Webers}{Ampere} = \frac{Tesla - m^2}{Ampere} = Henrys}$$

Note also that: $\boxed{\mu_o = 4\pi\times 10^{-7}\,\frac{N}{A^2} = 4\pi\times 10^{-7}\, Henrys\!\Big/\!meter \;\Rightarrow\; M_{\overrightarrow{12}} = Henrys}$

Why is $M_{\overrightarrow{12}}$ called the <u>mutual</u> inductance of a two-circuit system?

Let's <u>reverse</u> the roles of Loop 1 and Loop 2 – i.e. Loop 2 carries current I_2 (I_2 not necessarily = I_1 in original situation).



The magnetic flux from Loop 2 linking Loop 1: $\boxed{\Phi_m^{\overrightarrow{21}}(t) = \oint_{C_1} \vec{A}_2(\vec{r},t)\cdot d\vec{\ell}_1 = \int_{S_1} \vec{B}_2(\vec{r},t)\cdot d\vec{a}_{\perp_1}}$

With: $\boxed{\vec{B}_2(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right) I_2(t) \oint_{C_2} \frac{d\vec{\ell}_2 \times \hat{\mathscr{r}}}{\mathrm{r}^2}}$ and $\boxed{\vec{A}_2(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right) I_2(t) \oint_{C_2} \frac{d\vec{\ell}_2}{\mathscr{r}}}$

Then: $\boxed{\Phi_m^{\vec{21}}(t)=\left(\frac{\mu_o}{4\pi}\right)I_2(t)\left[\int_{S_1}\left(\oint_{C_2}\frac{d\vec{\ell}_2\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\bullet d\vec{a}_{\perp_1}\right]=\left(\frac{\mu_o}{4\pi}\right)I_2(t)\left[\oint_{C_1}\oint_{C_2}\frac{d\vec{\ell}_2\bullet d\vec{\ell}_1}{\mathscr{r}}\right]}$

Again, define $\boxed{\Phi_m^{\vec{21}}(t)\equiv M_{\vec{21}}I_2(t)}$ with constant of proportionality = mutual inductance $M_{\vec{21}}$

$\Phi_m^{\vec{21}}(t)$ Loop 2→ Loop 1: $\boxed{M_{\vec{21}}=\left(\frac{\mu_o}{4\pi}\right)\left[\oint_{C_1}\oint_{C_2}\frac{d\vec{\ell}_2\bullet d\vec{\ell}_1}{\mathscr{r}}\right]=\left(\frac{\mu_o}{4\pi}\right)\left[\int_{S_1}\left(\oint_{C_2}\frac{d\vec{\ell}_2\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\bullet d\vec{a}_{\perp_1}\right]}$

<u>But</u>:

$\Phi_m^{\vec{12}}(t)$ Loop 1→ Loop 2: $\boxed{M_{\vec{12}}=\left(\frac{\mu_o}{4\pi}\right)\left[\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1\bullet d\vec{\ell}_2}{\mathscr{r}}\right]=\left(\frac{\mu_o}{4\pi}\right)\left[\int_{S_2}\left(\oint_{C_1}\frac{d\vec{\ell}_1\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\bullet d\vec{a}_{\perp_2}\right]}$

Thus we see that $\boxed{M_{\vec{12}}\equiv M_{\vec{21}}}$!!! Hence why it is known as the <u>mutual</u> inductance of Loop # 1 with Loop # 2 (or vice versa)!! Thus, we don't need subscripts $\boxed{M_{\vec{12}}=M_{\vec{21}}=M}$

If one thinks about it, the fact that $M_{\vec{12}}\equiv M_{\vec{21}}$ is *<u>not</u>* a trivial consequence. If $I_1=I_2=I$, *<u>independent</u>* of the geometrical shapes / configurations of the two loop circuits, it is <u>not</u> immediately obvious that:

[magnetic flux $\Phi_m^{\vec{12}}(t)$ (due to current $I_1 = I(t)$ in Loop 1) linking Loop 2]

= [magnetic flux $\Phi_m^{\vec{21}}(t)$ (due to current $I_2 = I(t)$ in Loop 2) linking Loop 1].

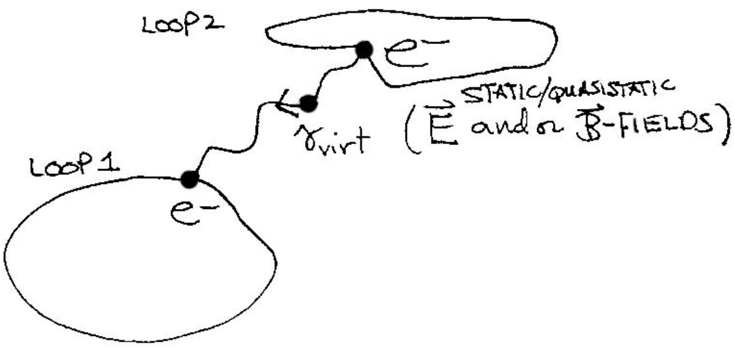
If $I_1=I_2=I$ then:

$$\boxed{\Phi_m^{\vec{12}}(t)=M_{\vec{12}}I_1(t)=M_{\vec{12}}I(t)}$$

$$\boxed{\Phi_m^{\vec{21}}(t)=M_{\vec{21}}I_2(t)=M_{\vec{21}}I(t)}$$

And since $\boxed{M_{\vec{12}}=M_{\vec{21}}=M}$ then $\boxed{\Phi_m^{\vec{12}}(t)\equiv\Phi_m^{\vec{21}}(t)}$.

This is a consequence of the <u>Reciprocity Theorem</u>. The underlying physics of the Reciprocity Theorem has to do with the intrinsic / fundamental properties of the electromagnetic interaction at the microscopic / particle physics level – i.e. exchange of virtual photons between electrically-charged particles, as well as the fundamental symmetry principles obeyed by the *EM*-interaction:



Charge Conjugation (*C*)
Parity (Space Inversion) (*P*)
Time Reversal (*T*)

The *EM* interaction is *<u>invariant</u>* under *C*, *P* and *T*.

Also, the intrinsic properties / nature of our 3-space dimensional and 1-time dimensional universe is also important e.g. 3-*D* space is isotropic – not anisotropic.

The Reciprocity Theorem has many consequences in all branches of physics / chemistry / science. It is also relevant e.g. in optics (i.e. visible light/real photons). Nearly everything in the "everyday world" deals with the *EM* interaction at the microscopic scale…

For two magnetically-coupled / flux linked circuits, the Reciprocity Theorem also predicts that:

$$\boxed{\varepsilon mf\ \ \varepsilon_2(t)=-\frac{\partial\Phi_m^{\overrightarrow{12}}(t)}{\partial t}=-M_{\overrightarrow{12}}\frac{\partial I_1(t)}{\partial t}}\equiv\boxed{\varepsilon mf\ \ \varepsilon_1(t)=-\frac{\partial\Phi_m^{\overrightarrow{21}}(t)}{\partial t}=-M_{\overrightarrow{21}}\frac{\partial I_2(t)}{\partial t}}$$

If $I_1(t)=I_2(t)=I(t)$ then: $\boxed{\frac{\partial I_1(t)}{\partial t}=\frac{\partial I_2(t)}{\partial t}=\frac{\partial I(t)}{\partial t}}$, then since: $\boxed{M_{\overrightarrow{12}}=M_{\overrightarrow{21}}=M}$ **by the Reciprocity Theorem**

Then:
$$\boxed{\varepsilon mf\ \ \varepsilon_2(t)=-M\frac{\partial I(t)}{\partial t}}\leftarrow\text{in Loop 1}$$
$$=\boxed{\varepsilon mf\ \ \varepsilon_1(t)=-M\frac{\partial I(t)}{\partial t}}\leftarrow\text{in Loop 2}$$

i.e. a $\frac{\partial I(t)}{\partial t}=$ 1 Amp/sec change in the current flowing in Loop 1 with $M=1$ Henry of mutual inductance between Loop 1 and Loop 2 will produce an *EMF* ε_2 = 1 Volt in Loop 2, which is = to a $\frac{\partial I(t)}{\partial t}=$ 1 Amp/sec change in the current flowing in Loop 2 with $M=1$ Henry of mutual inductance between Loop 1 and Loop 2 will produce an *EMF*, ε_1 = 1 Volt in Loop 1.

Thus, the *EMF* induced in a loop *b* due to a time-varying current flowing in loop *a* producing a time-varying magnetic field linking both loops via their mutual inductance, *M* is:

$$\boxed{\varepsilon mf\ \ \varepsilon_b(t)=-\frac{\partial\Phi_m(t)}{\partial t}=-M\frac{\partial I_a(t)}{\partial t}}$$

This phenomenon then provides a convenient way for us to measure / determine the mutual inductance, *M* of two circuits – i.e. we can compute the mutual inductance from:

$$\boxed{M=\frac{\varepsilon_b(t)\ \leftarrow\text{measure in loop } b}{\dfrac{\partial I_a(t)}{\partial t}\ \leftarrow\text{put in known (or measured) }\frac{\partial I_a(t)}{\partial t}\text{ into loop } a}}$$

As one might realize, e.g. from our previous experience(s) with dielectric and magnetic media, if loops / circuits 1 and 2 are both uniformly embedded inside a magnetically permeable material (with magnetic permeability $\mu = \mu_o(1+\chi_m) = K_m\mu_o$) then for linear magnetic materials with $\mu = K_m\mu_o$ we would expect:

$$\boxed{\Phi^b_{m_\mu}(t) = \left(\frac{\mu}{4\pi}\right) I_a(t)\left[\oint_{C_b}\oint_{C_a}\frac{d\vec{\ell}_a \bullet d\vec{\ell}_b}{r}\right] = M_\mu I_a(t)}$$

where: $$\boxed{M_\mu = \left(\frac{\mu}{4\pi}\right)\left[\oint_{C_b}\oint_{C_a}\frac{d\vec{\ell}_a \bullet d\vec{\ell}_b}{r}\right]}$$

vs. that for non-magnetic materials:

$$\boxed{\Phi^b_{m_{\mu_o}}(t) = \left(\frac{\mu_o}{4\pi}\right) I_a(t)\left[\oint_{C_b}\oint_{C_a}\frac{d\vec{\ell}_a \bullet d\vec{\ell}_b}{r}\right] = M_{\mu_o} I_a(t)}$$

where: $$\boxed{M_{\mu_o} = \left(\frac{\mu_o}{4\pi}\right)\left[\oint_{C_b}\oint_{C_a}\frac{d\vec{\ell}_a \bullet d\vec{\ell}_b}{r}\right]}$$

Thus, we see that: $\boxed{M_\mu = K_m M_{\mu_o}}$.

For example, for soft/annealed iron $K_m^{Fe} \equiv \mu^{Fe}/\mu_o \simeq 1000$. Then for two circuits embedded in soft/annealed iron, their mutual inductance is also correspondingly increased by this same factor:

$$\boxed{M_\mu = K_m M_{\mu_o} \simeq 1000 M_{\mu_o}}.$$

This also implies that magnetic fluxes are also correspondingly increased: $\boxed{\Phi^b_{m_\mu}(t) = K_m \Phi_{m_{\mu_o}}(t)}$ and the induced *EMF*'s in the loops are also increased by the same factor, since:

$$\boxed{\varepsilon^b_\mu(t) = -\frac{\partial\Phi^b_{m_\mu}(t)}{\partial t} = K_m\varepsilon^b_{\mu_o}(t) = -K_m\frac{\partial\Phi^b_{m_{\mu_o}}(t)}{\partial t}} \Rightarrow \boxed{\varepsilon^b_\mu(t) = K_m\varepsilon^b_{\mu_o}(t) = 1000\varepsilon^b_{\mu_o}(t)}$$

Of course, the above results implicitly assume that the flux-linking geometries of the two circuits are identical – with and without the presence of the magnetically permeable material.

Thus, we see that the use of highly magnetically permeable materials ($\mu \gg \mu_o$) can dramatically improve the magnetic coupling between two circuits, over and above the free-space μ_o value!!

For the long solenoid, this also means that a high magnetic permeability flux return (external to solenoid) from one end to other must also be provided to "complete" the magnetic circuit.

If only a magnetic core is used inside the solenoid, then: $\boxed{\left.\frac{M_\mu}{M_{\mu_o}}\right|_{\substack{core\\only}} \simeq 1.3\times\left(\frac{L}{D}\right)^{1.7}}$ where $D = 2R$

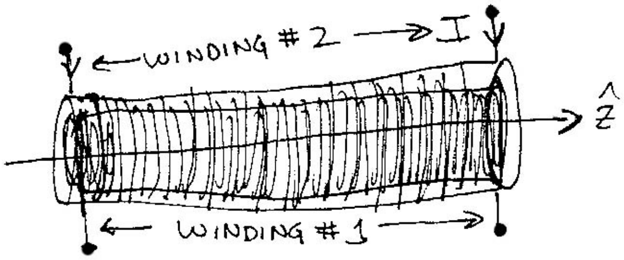
The Mutual Inductance Between Two Long *Coaxial* Solenoids

Take a long air-core solenoid of length L and radius R wound with $n_1 = N_{TOT_1}/L$ turns/meter.

Then wind a second winding over the first winding of the solenoid with $n_2 = N_{TOT_2}/L$ turns/meter.

If we put a steady current I through the 2nd (i.e. outer) winding of the long solenoid, the magnetic field inside the bore of the outer solenoid is:

$$\boxed{\vec{B}_2^{inside} = \mu_o n_2 I\hat{z}} \text{ (Using Ampere's Circuital Law } \oint \vec{B}\bullet d\vec{\ell} = \mu_o I_{Tot}^{enclosed}\text{)}$$



The magnetic flux through the bore of the outer solenoid is:

$$\boxed{\Phi_2 = \vec{B}_2\bullet\vec{A}_{2_\perp} = B_2\pi R^2 = \mu_o n_2 \pi R^2 I} \quad \text{where} \quad \boxed{\vec{A}_{2_\perp} = A_{2_\perp}\hat{z} = \pi R^2\hat{z}}$$

However, this same magnetic flux links each and every one of the N_{TOT_1} turns of the inner solenoid winding (if the two windings are close-packed / carefully wound).

Thus, the magnetic flux (arising from I flowing in the outer solenoid winding) that links one turn of the inner solenoid winding is:

$$\boxed{\Phi_1^{\text{one turn}} = \vec{B}_2\bullet\vec{A}_{\perp_1} = \mu_o n_2 \pi R^2 I = \Phi_2}$$

But the inner solenoid has N_{TOT_1} total number of turns, thus the total magnetic flux (arising from current I flowing in the outer solenoid winding) linking N_{TOT_1} total number of turns of the inner solenoid is:

$$\boxed{\Phi_1^{TOT} = N_{TOT_1}\Phi_1^{\text{one turn}} = \mu_o N_{TOT_1} n_2 \pi R^2 I} \quad \text{but} \quad \boxed{n_1 = \frac{N_{TOT_1}}{L}}$$

$$\therefore \boxed{\Phi_1^{TOT} = \mu_o\left[n_1 n_2 \pi R^2 L\right] I}$$

Total magnetic flux linking inner solenoid winding

Current I flowing in outer solenoid winding

But $\boxed{\Phi_1^{TOT} = MI}$ $\Rightarrow$ mutual inductance of two long coaxial solenoids: $\boxed{M = \mu_o\left[n_1 n_2 \pi R^2 L\right]}$

Notice that:

$\left[n_1 n_2 \pi R^2 L\right]$ has dimensions of length, and $= \frac{1}{4\pi}\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1\bullet d\vec{\ell}_2}{r} = \frac{1}{4\pi}\int_{S_2}\left(\oint_{C_1}\frac{d\vec{\ell}_1\times\hat{r}}{r^2}\right)\bullet d\vec{a}_{\perp_2}$!!!

$\mu_o = 4\pi\times 10^{-7}$ Henrys/meter → M in Henrys

In the real world, it is very difficult to build any two-coil device (e.g. a transformer!!) with perfect, 100.0000% magnetic flux linking / coupling between the two windings. It is always some fraction of 100%.

Can define an efficiency of magnetic coupling of the two windings to each other, $\in$: $\boxed{0 \leq \in \leq 1}$

Then: $\boxed{M_{actual} = \in M_{ideal} = \in \mu_o \left[n_1 n_2 \pi R^2 L \right]}$ and $\boxed{\Phi_{1_{actual}}^{TOT} = M_{actual} I = \in M_{ideal} I = \in \mu_o \left[n_1 n_2 \pi R^2 L \right] I}$

Then: $(1- \in)$ = inefficiency of magnetic coupling / flux linkage between the two windings

Define: $\Phi^{leakage} \equiv (1- \in) \mu_o \left[n_1 n_2 \pi R^2 L \right] I$ = Leakage flux not coupled from one winding to the other.

The value of $\in$ in an actual device depends on the details of the design (and who made it)... Generally speaking, everyone wants $\in \equiv 100.000\%$. Manufacturers try to achieve this, but the old adage: “Yous gits what yous payz for” is true . . .

Significant leakage flux in a transformer (a 2-winding magnetically-coupled circuit) will adversely affect the high-frequency response of the transformer output vs. input for a real transformer.

The mutual inductance for two coils tightly wound together on a long solenoid is:

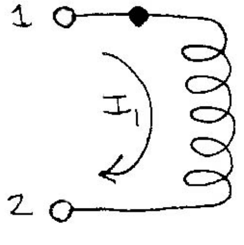
$$\boxed{M = \in \mu_o n_1 n_2 \pi R^2 L} \text{ (Henrys)}$$

Note that $M > 0$ i.e. the mutual inductance M is always a positive quantity.

Note that how the two separate coils are wired up into a larger circuit does matter – on the sign (i.e. polarity) of the voltage output from one coil relative to the voltage input of the other – i.e. the relative phase polarity of the two coils.

Schematically, this phase polarity is indicated by black dots on the circuit diagram for two magnetically coupled circuits (i.e. transformer) as follows:

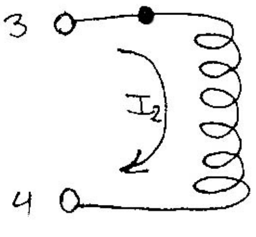
Coil winding # 1



The arrows indicate direction of positive current flow.

Current I_1 enters coil winding #1 through lead #1 (with black dot), leaves through lead #2

Coil Winding # 2



The direction of induced current flow in coil winding #2 is shown.

When the voltage at point 1 on coil # 1 goes positive, the voltage at point 3 on coil 2 also goes positive.

By the Reciprocity Theorem, either coil #1 or coil #2 could be viewed as the "primary" winding, the other coil would then be the "secondary" winding (or vice versa).

If two long solenoid coils are coaxially wound together on a magnetically permeable core $\left(\mu = \mu_o K_m = \mu_o \left(1 + \chi_m\right)\right)$, then:

$$\boxed{M_{\mu} = \in \mu n_1 n_2 \pi R^2 L = K_m M_{\mu_o} \gg M_{\mu_o}} \text{ if } \mu \gg \mu_o$$

n.b. adding μ also tends to improve $\in \simeq 100\%$!!

Self-Inductance – a.k.a. "Inductance":

As we have seen in two-coil magnetically coupled circuits, a time-varying current with $\partial I(t)/\partial t$ in one coil induces an *EMF* $\varepsilon = -M\left(\partial I_{free}(t)/\partial t\right)$ in the other coil due to their mutual inductance, *M*.

Similarly, a time-varying current with $\partial I_{free}(t)/\partial t$ in an isolated / single coil also induces a so-called *BACK EMF* (i.e. a reverse *EMF*) in itself, due to Lenz's Law (i.e. the coil tries to maintain constant magnetic flux – maintain the flux status quo).

The Self-Inductance of a Long Solenoid

For a long solenoid the magnetic flux in the bore (cross-sectional area) of the long solenoid (of length ℓ, radius *R* and $n = N_{TOT}/\ell$ turns/meter) is $\boxed{\Phi_m = \mu_o n\pi R^2 I_{free}}$ where $A_\perp^{solenoid} = \pi R^2$.

However, this magnetic flux links each and every turn of the solenoid (ideally):

Total magnetic flux linking all N_{TOT} turns

Magnetic flux linking one turn

Then: $\boxed{\Phi_m^{TOT} = N_{TOT}\Phi_m = \mu_o N_{TOT} n\pi R^2 I_{free}}$ with $\boxed{n = N_{TOT}/\ell}$

Thus: $\boxed{\Phi_m^{TOT} = \left[\mu_o n^2 \pi R^2 \ell\right] I_{free}}$

We see here again, for a single solenoid coil, that Φ_m^{TOT} is *linearly* proportional to I_{free}, i.e.:

$\boxed{\Phi_m^{TOT} \equiv L I_{free}}$ where the constant of proportionality (here) is: $\boxed{L = \mu_o n^2 \pi R^2 \ell}$ (Henrys)

The quantity *L* is known as the self-inductance of the long solenoid. SI units are (also) Henrys.

BACK EMF in a coil due to its self-inductance: $\boxed{\varepsilon(t) = -\frac{\partial \Phi_m^{TOT}(t)}{\partial t} = -L\frac{\partial I_{free}(t)}{\partial t}}$ Due to Lenz's Law

A two-lead device with many turns of wire – having much self-inductance is called an inductor. {Analogous to a two-lead device that can store much charge – called a capacitor.}

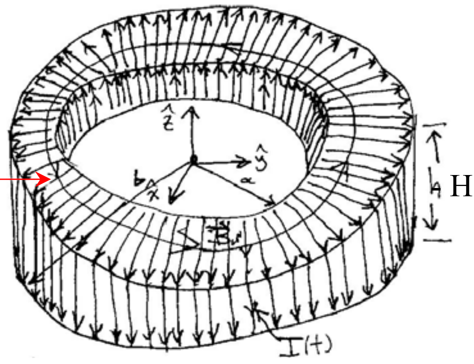
For the long solenoid, unless a good / high-permeability external magnetic flux return is provided, then using only a magnetic core:

$$\boxed{\left.\frac{L_\mu}{L_{\mu_o}}\right|_{\substack{core\\ only}} \simeq 1.3\left(\frac{\ell}{D}\right)^{1.7}}$$ where $D = 2R$ (Diameter)

The Self-Inductance of a Toroidal Coil

3-D View:

Direction of $\vec{B}_m$ and closed contour path of Integration, *C*

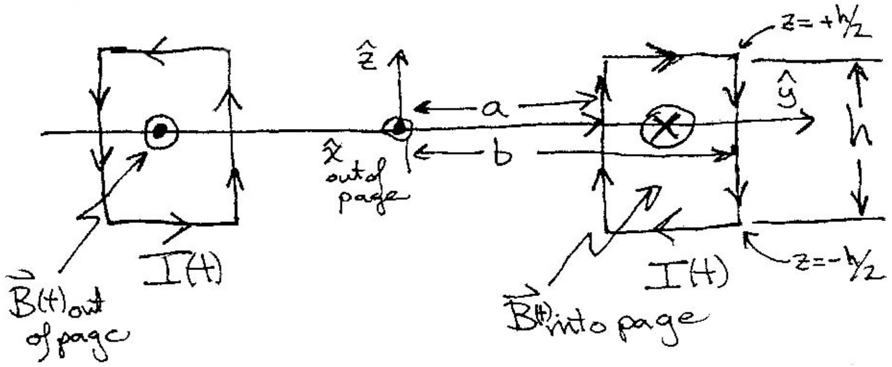


Dimensions of <u>Rectangular</u> Toroid:
- Inner radius *a*
- Outer radius *b*

Height *h*

- Toroid has N_{TOT} turns

Cross-Sectional View:



From Ampere's Circuital Law $\left(\oint_C \vec{B}(\vec{r},t)\bullet d\vec{\ell} = \mu_o I_{TOT}^{enclosed}(t)\right)$: $\boxed{I_{TOT}^{enclosed}(t) = N_{TOT} I_{free}(t)}$

The magnetic field inside the bore of the toroid: $\boxed{\vec{B}_{in}(\rho,t) = \left(\frac{\mu_o}{2\pi}\right)\frac{N_{TOT} I_{free}(t)}{\rho}\hat{\varphi}}$, $\rho = \sqrt{x^2+y^2}$

(in cylindrical coordinates)

The magnetic flux through the bore of the toroid is:

$$\begin{aligned}
\Phi_m(t) &= \int_{S_\perp} \vec{B}(\rho,t)\bullet d\vec{a}_\perp \text{ with } d\vec{a}_\perp = d\rho dz\hat{\varphi} \\
&= \left(\frac{\mu_o}{2\pi}\right) N_{TOT} I_{free}(t) \int_{z=-h/2}^{z=+h/2}\int_{\rho=a}^{\rho=b} \frac{1}{\rho}\, d\rho dz \\
&= \left(\frac{\mu_o}{2\pi}\right) N_{TOT} I_{free}(t) \int_{z=-h/2}^{z=+h/2} \underbrace{\left[\ln(b) - \ln(a)\right]}_{=\ln(b/a)} dz \\
&= \left(\frac{\mu_o}{2\pi}\right) N_{TOT} I_{free}(t) \ln(b/a)\left[h/2 + h/2\right]
\end{aligned}$$

<u>Thus</u>: $\boxed{\Phi_m(t) = \left(\frac{\mu_o}{2\pi}\right) N_{TOT} I_{free}(t)\, h \ln(b/a) = \Phi_m^1(t)}$ = magnetic flux linking *one* turn of the toroid.

But (here again) the magnetic flux inside the bore of the toroid links <u>each</u> and <u>every</u> turn of the toroid (ideally).

Thus: $\boxed{\Phi_m^{TOT}(t) = N_{TOT}\Phi_m^1(t) = \left(\frac{\mu_o}{2\pi}\right) N_{TOT}^2 I_{free}(t)\, h \ln\left(b/a\right)} = \boxed{\begin{array}{c}\underline{\text{total}}\text{ magnetic flux} \\ \text{linking all } N_{TOT} \text{ turns}\end{array}}$

But: $\boxed{\Phi_m^{TOT}(t) = L I_{free}(t)}$ $\Rightarrow$ Self-inductance of rectangular toroid: $\boxed{L = \left(\frac{\mu_o}{2\pi}\right) N_{TOT}^2 h \ln\left(b/a\right)}$

Back EMF in Toroid: $\boxed{emf\ \ \varepsilon(t) = -\frac{\partial \Phi_m^{TOT}(t)}{\partial t} = -L\frac{\partial I_{free}(t)}{\partial t} = -\left(\frac{\mu_o}{2\pi}\right) N_{TOT}^2 h \ln\left(b/a\right)\frac{\partial I_{free}(t)}{\partial t}}$

Due to the nature of the toroid's excellent magnetic <u>self-coupling</u> (toroid = solenoid bent back on itself), if the toroid coil is wound on magnetically permeable core (of magnetic permeability $\mu = K_m\mu_o = \mu_o(1+\chi_m)$), then:

Air core toroid

$$\boxed{\Phi_{m_\mu}^{TOT}(t) = \left(\frac{\mu}{2\pi}\right) N_{TOT}^2 h \ln\left(b/a\right) I_{free}(t) = K_m \Phi_{m_{\mu_o}}^{TOT}(t)}$$

Magnetic core toroid

$$\boxed{L_\mu = \left(\frac{\mu}{2\pi}\right) N_{TOT}^2 h \ln\left(b/a\right) = K_m L_{\mu_o}}$$

$$\boxed{emf\ \ \varepsilon_\mu(t) = -L_\mu \frac{\partial I_{free}(t)}{\partial t} = -K_m L_{\mu_o}\frac{\partial I_{free}(t)}{\partial t} = K_m \varepsilon_{\mu_o}(t)}$$

For inductors with magnetically permeable cores, the magnetization $\vec{M}$ and hence μ, χ_m are often (not <u>all</u>) <u>frequency dependent</u>!! Frequency dependence can be significant (100 % or more) (over wide frequency range) and depends on the microscopic (and (eddy currents) macroscopic) details of the magnetically permeable material(s) being used.

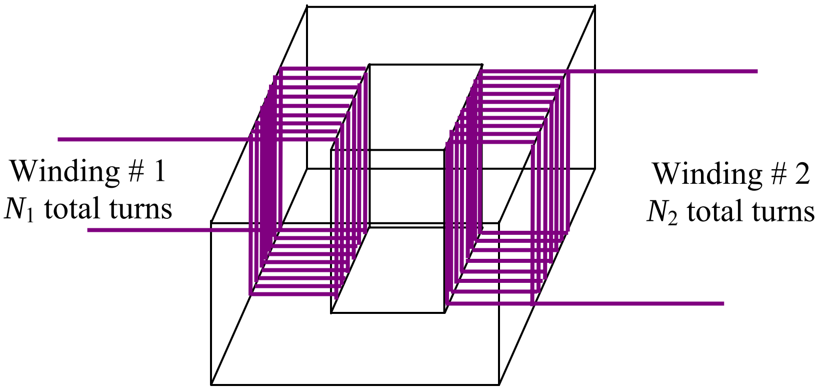
Magnetic materials have various magnetic dissipation mechanism(s) which can be/are frequency dependent…

<u>Furthermore</u>, magnetic dissipation mechanisms can be/are also dependent on the <u>strength</u> of $\vec{B}_{in}$

(Since $\vec{M} = \frac{\chi_m}{\mu}\vec{B}$ is a non-linear relationship, e.g. for soft iron) thus magnetic dissipation is frequently a non-linear function of the magnetization, $\vec{M}$.

The Ideal Transformer

An ideal transformer is a crude representation of a real transformer; nevertheless it is a very useful approximation to a real one. A simple (and very common version) of a real transformer is one which has two windings that are wound on a magnetically-permeable core (such as iron), such as is shown in the figure below:



For an idealized version of this transformer, we make the following simplifying assumptions:

(1) Each of the two coil windings have no resistance.

(2) There are no Eddy-current and/or hysteresis Joule-heating losses in the magnetic core of the transformer. The hysteresis loop for the core is then a straight line through the origin, then B is linearly proportional to H, which is in turn proportional to I, thus B is then proportional to I

(3) All of the magnetic flux is confined to the magnetic core – i.e. there is no leakage flux, and thus the magnetic flux through one winding is the same as that through the other winding.

Since the two windings are perfectly magnetically coupled to each other in the ideal transformer, then $\Phi^1_{m_1}(t) = \Phi^1_{m_2}(t) \equiv \Phi^1_m(t) =$ magnetic flux passing through one loop of winding # 1 (or # 2).

The mutual inductance of the two windings of the ideal transformer is M, and if e.g. a function generator is connected to winding # 1, then a potential difference (i.e. a voltage) $\Delta V_1(t)$ is present and thus a free current $I_1(t)$ will flow in winding # 1; thus a magnetic field $B_1(t)$ will be present in the magnetic core of the ideal transformer. If the magnetic core of the ideal transformer has a cross sectional area A_{core}, then a magnetic flux of $\Phi^1_{m_1}(t) = \left[B_1(t) A_{core}\right]$ is present in the core of the ideal transformer.

Then the *total* magnetic flux through winding # 1 is: $\boxed{\Phi^{Tot}_{m_1}(t) = N_1 \Phi^1_{m_1}(t) = N_1\left[B_1(t) A_{core}\right]}$

But: $\boxed{\Delta V_1(t) = -\frac{\partial \Phi^{Tot}_{m_1}(t)}{\partial t} = N_1 \frac{\partial \Phi^1_{m_1}(t)}{\partial t} = \varepsilon_1(t)}$

However since the two windings of the ideal transformer are perfectly magnetically coupled, i.e.:

$$\boxed{\Phi^1_{m_1}(t) = \Phi^1_{m_2}(t) \equiv \Phi^1_m(t)} \text{ then: } \boxed{\frac{\partial \Phi^1_{m_1}(t)}{\partial t} = \frac{\partial \Phi^1_{m_2}(t)}{\partial t} = \frac{\partial \Phi^1_m(t)}{\partial t}}.$$

Then the resulting *EMF* induced in winding # 2 is: $\boxed{\varepsilon_2(t) = -\frac{\partial \Phi_{m_2}^{Tot}(t)}{\partial t} = N_2 \frac{\partial \Phi_{m_2}^{1}(t)}{\partial t} = \Delta V_2(t)}$

Since: $\boxed{\frac{\partial \Phi_{m_1}^{1}(t)}{\partial t} = \frac{\partial \Phi_{m_2}^{1}(t)}{\partial t} = \frac{\partial \Phi_{m}^{1}(t)}{\partial t}}$ and $\boxed{\frac{\partial \Phi_{m_2}^{1}(t)}{\partial t} = \frac{\Delta V_2(t)}{N_2}}$ and $\boxed{\frac{\partial \Phi_{m_1}^{1}(t)}{\partial t} = \frac{\Delta V_1(t)}{N_1}}$

Thus: $\boxed{\frac{\partial \Phi_{m_1}^{1}(t)}{\partial t} = \frac{\partial \Phi_{m_2}^{1}(t)}{\partial t} = \frac{\partial \Phi_{m}^{1}(t)}{\partial t} = \frac{\Delta V_2(t)}{N_2} = \frac{\Delta V_1(t)}{N_1}}$ or: $\boxed{\frac{\Delta V_2(t)}{N_2} = \frac{\Delta V_1(t)}{N_1}}$

<u>Or</u>: $\boxed{\frac{\varepsilon_2(t)}{\varepsilon_1(t)} = \frac{\Delta V_2(t)}{\Delta V_1(t)} = \frac{N_2}{N_1}}$ = <u>Turns Ratio</u> of ideal transformer = <u>Constant</u>

Each winding of the ideal transformer has its own associated self-inductance, L and for each winding, *i* = 1,2: $\boxed{\Phi_{m_i}^{TOT}(t) = L_i I_i(t)}$ and $\boxed{\varepsilon mf \quad \varepsilon_i(t) = -\frac{\partial \Phi_{m_i}^{TOT}(t)}{\partial t} = -L_i \frac{\partial I_i(t)}{\partial t} = \Delta V_i(t)}$.

As we saw in the case of the rectangular toroid (with soft-iron core), the self-inductances associated with each of the two windings of the ideal transformer are proportional to the <u>square</u> of the number of turns of their windings, i.e. $L_1 \sim N_1^2$ and $L_2 \sim N_2^2$. Then we can also see that:

$$\boxed{\frac{\varepsilon_2(t)}{\varepsilon_1(t)} = \frac{\Delta V_2(t)}{\Delta V_1(t)} = \frac{N_2}{N_1} = \sqrt{\frac{L_2}{L_1}}}$$

If the ideal transformer is <u>lossless</u>, then electrical power in winding # 1 can be transferred with 100% efficiency to winding # 1 (and vice-versa) (n.b. microscopically, all energy/power is transferred from one winding to the other via virtual photons), and thus:

$\boxed{P_1(t) = \Delta V_1(t) I_1(t) = \Delta V_2(t) I_2(t) = P_2(t)}$ or: $\boxed{\frac{\Delta V_2(t)}{\Delta V_1(t)} = \frac{I_1(t)}{I_2(t)}}$

Then we see that: $\boxed{\frac{\varepsilon_2(t)}{\varepsilon_1(t)} = \frac{\Delta V_2(t)}{\Delta V_1(t)} = \frac{I_1(t)}{I_2(t)} = \frac{N_2}{N_1} = \sqrt{\frac{L_2}{L_1}}}$ and that $\boxed{N_1 I_1(t) = N_2 I_2(t)}$

Thus for an ideal transformer, e.g. if winding # 1 has <u>many</u> turns and winding # 2 has <u>few</u> turns, the above formulae tell us that for $N_1 \gg N_2$ (i.e. a so-called <u>“step-down” transformer</u>):

$\boxed{\Delta V_1(t) \gg \Delta V_2(t)}$ and $\boxed{I_1(t) \ll I_2(t)}$ with $\boxed{P_1(t) = \Delta V_1(t) I_1(t) = \Delta V_2(t) I_2(t) = P_2(t)}$;

whereas if winding # 1 has <u>few</u> turns and winding # 2 has <u>many</u> turns, then for $N_1 \ll N_2$ (i.e. a so-called <u>“step-up” transformer</u>):

$\boxed{\Delta V_1(t) \ll \Delta V_2(t)}$ and $\boxed{I_1(t) \gg I_2(t)}$ with $\boxed{P_1(t) = \Delta V_1(t) I_1(t) = \Delta V_2(t) I_2(t) = P_2(t)}$.

n.b. By the Reciprocity Theorem: A step-up transformer <u>run backwards</u> = a step down transformer!!!