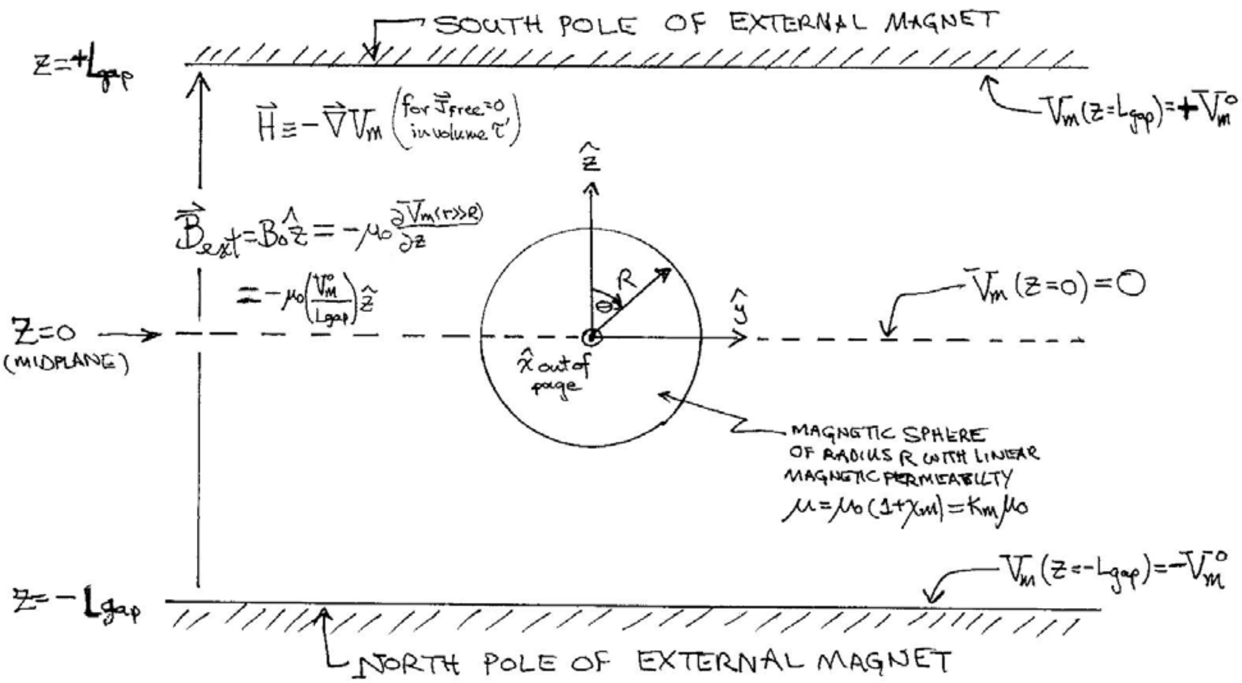
LECTURE NOTES 20.5

Magnetostatic Boundary Value Problems in Magnetic Media: Examples, Applications and Uses

Example # 1:
Use the magnetic scalar potential V_m for a magnetic sphere in a uniform external magnetic field.

Consider a sphere of radius R made of (arbitrary / unspecified) linear magnetic material of magnetic permeability $\mu = \mu_o\left(1+\chi_m\right)$ placed in the gap of a big electromagnet that produces a uniform external magnetic field $\vec{B}_{ext} = B_o\hat{z}$ as shown in the figure below:



Note that this magnetostatics problem of a linear magnetic sphere of radius R and magnetic permeability $\mu = \mu_o\left(1+\chi_m\right) = K_m\mu_o$ placed in an external uniform magnetic field $\vec{B}_{ext} = B_o\hat{z}$ is highly analogous to the electrostatics boundary value problem that we solved earlier (Griffiths Example 4.7, pp. 186-8) with a linear dielectric sphere of radius R and dielectric permittivity $\varepsilon = \varepsilon_o\left(1+\chi_e\right) = K_e\varepsilon_o$ placed in an external uniform electric field $\vec{E}_{ext} = E_o\hat{z}$. Here, we will use the magnetic scalar potential V_m to solve this magnetic boundary value problem. Note (aforehand) that this problem (like that of the dielectric sphere) is manifestly azimuthally symmetric – i.e. it has no explicit φ-dependence.

Since there are no free currents/free current densities anywhere in the volume v' of interest (i.e. $\vec{J}_{free}(\vec{r}) = 0$), then $\vec{\nabla} \times \vec{H}(\vec{r}) = 0$, and thus we can write $\boxed{\vec{H}(\vec{r}) \equiv -\vec{\nabla} V_m(\vec{r})}$ where $V_m(\vec{r})$ = magnetic scalar potential.

Again, note that since the SI units of $\vec{H}(\vec{r})$ are <u>Amperes/meter</u> → then the SI units of $V_m(\vec{r})$ are <u>Amperes</u>!!

n.b. This sort of makes "nice" sense, since for <u>electrostatics</u>, the SI units of $V_E(\vec{r})$ = <u>Volts</u>

Then since $\boxed{\vec{\nabla} \times \vec{H}(\vec{r}) = -\vec{\nabla} \times \vec{\nabla} V_m(\vec{r}) \equiv 0}$ here in this problem, and

$$\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = -\vec{\nabla} \cdot \vec{\nabla} V_m(\vec{r}) = -\nabla^2 V_m(\vec{r}) = -\vec{\nabla} \cdot \vec{M}(\vec{r}) = -\rho_m(\vec{r})}$$

Now in the volume v', we have uniform magnetization: $\boxed{\vec{M}(\vec{r}) = M_o \hat{z}}$ (here)

$\therefore$ $\boxed{\vec{\nabla} \cdot \vec{M}(\vec{r}) = \vec{\nabla} \cdot M_o \hat{z} = M_o \vec{\nabla} \cdot \hat{z} = 0}$, i.e. $\boxed{\rho_m(\vec{r}) = 0}$ (here)

Again, extreme caution must be used here, for we <u>know</u> that differential relations will <u>fail</u> on the boundaries / interfaces of dissimilar materials.

Nevertheless, <u>away</u> from these boundaries / interfaces:

$$\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = -\vec{\nabla} \cdot \vec{M}(\vec{r}) = -\nabla^2 V_m(\vec{r}) = 0}$$

$\boxed{\nabla^2 V_m(\vec{r}) = 0}$ is Laplace's Equation for the Magnetic Scalar Potential $V_m(\vec{r})$

We can/will use all the tools that we developed for solving <u>electrostatics boundary-value problems</u> here too, for solving <u>magnetostatics boundary-value problems</u>!!!

We will use the magnetostatic boundary conditions (derived/obtained from) the integral relations (given below) at the interface(s)/boundar(ies) of the magnetic material, in order to constrain the allowed form of the magnetic scalar potential $V_m(\vec{r})$ in various regions of v' as well as at boundaries / interfaces.

$$\boxed{\int_S \left(\vec{\nabla} \times \vec{H}(\vec{r})\right) \cdot da = \oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{free}^{enclosed}} \quad \text{and:} \quad \boxed{\int_v \left(\vec{\nabla} \cdot \vec{H}(\vec{r})\right) \cdot d\tau = \oint_S \vec{H}(\vec{r}) \cdot d\vec{a}}$$

$$\boxed{\int_S \left(\vec{\nabla} \times \vec{M}(\vec{r})\right) \cdot da = \oint_C \vec{M}(\vec{r}) \cdot d\vec{\ell} = I_{Bound}^{enclosed}} \quad \boxed{= \int_v \left(\vec{\nabla} \cdot \vec{M}(\vec{r})\right) \cdot d\tau = \oint_S \vec{M}(\vec{r}) \cdot d\vec{a} \neq 0} \text{ in general}$$

$$\boxed{\frac{1}{\mu_o} \int_S \left(\vec{\nabla} \times \vec{B}(\vec{r})\right) \cdot da = \frac{1}{\mu_o} \oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = I_{Tot}^{enclosed}} \quad \boxed{\int_v \left(\vec{\nabla} \cdot \vec{B}(\vec{r})\right) \cdot d\tau = \oint_S \vec{B}(\vec{r}) \cdot d\vec{a} = \Phi_m^{enclosed} = 0}$$

where $\boxed{I_{Tot}^{enclosed} = I_{free}^{enclosed} + I_{Bound}^{enclosed}}$

The most <u>general</u> solution for the magnetostatic version of Laplace's Equation ($\nabla^2 V_m(\vec{r}) = 0$) for the magnetic scalar potential, for problems with spherical symmetry and additionally ones that also have manifest / explicit <u>azimuthal symmetry</u> (i.e. no φ-dependence), with $\cos\Theta' \equiv \hat{r} \cdot \hat{r}'$, and choosing the origin (here) to be at the center of the magnetic sphere, then $\cos\Theta' \to \cos\theta$ (where θ = usual polar angle) and $r = |\vec{r} - \vec{r}'| \to r$, then $V_m(r,\theta)$ is given by:

$$\boxed{V_m(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

where $P_\ell(\cos\theta)$ is the "ordinary" Legendré Polynomial of order ℓ.

The <u>boundary conditions</u> for this magnetostatics problem parallel those (not identically though!!) for the analogous electrostatics problem – that of a linear dielectric sphere of radius R and linear electric permittivity $\varepsilon = \varepsilon_o(1+\chi_e) = K_e\varepsilon_o$ - in uniform electric field $\vec{E}_{ext} = E_o\hat{z}$ (see / refer to P435 Lecture Notes 11).

The <u>boundary conditions</u> that we have here for the magnetic sphere of radius R and linear magnetic permeability $\mu = \mu_o(1+\chi_m) = K_m\mu_o$ in a uniform magnetic field $\vec{B}_{ext} = B_o\hat{z}$ are:

0) $V_m(\vec{r})$ = finite $\forall\ \vec{r}$ in the volume v'

1) $V_m^{inside}(r=R) = V_m^{outside}(r=R) \Leftarrow V_m$ is continuous / single-valued at / across interface / boundary at $r = R$.

2) $V_m^{inside}(z=0) = V_m^{outside}(z=0) = 0$ (i.e. the x-y mid-plane = magnetic scalar equipotential due to the symmetry of problem (see picture on page 1))

Because $z = r\cos\theta$ this BC also says: $V_m^{inside}\left(r, \theta = \frac{\pi}{2}\right) = V_m^{outside}\left(r, \theta = \frac{\pi}{2}\right) = 0$

3) $V_m^{outside}(z = \pm L_{gap}) = V_m^{outside}(r\cos\theta = \pm L_{gap}) = \pm V_m^o$ $\left\{\begin{array}{l}\text{upper (south!)}\\ \text{lower (north!)}\end{array}\right\}$ magnetic poles of the <u>external</u> magnet

4) Far away from the magnetic sphere ($r >> R$) we demand:

$\vec{B}^{outside}(r \gg R) = \vec{B}_{ext} = B_o\hat{z}$ where $\hat{z} = \left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]$ in spherical polar coordinates.

In the region exterior to the magnetized sphere ($r > R$):

$$\frac{1}{\mu_o}\vec{B}^{outside}(r > R) = \vec{H}^{outside}(r > R) = -\vec{\nabla}V_m^{out}(r > R)$$

<u>Thus</u>:
$$\underbrace{\frac{1}{\mu_o}\vec{B}^{outside}}(r \gg R) = \underbrace{\frac{1}{\mu_o}\vec{B}_{ext}} = \underbrace{\frac{1}{\mu_o}B_o}\hat{z} = \underbrace{\frac{1}{\mu_o}B_o}\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]$$

$$= \vec{H}^{outside}(r \gg R) = \vec{H}_{ext} = H_o\hat{z} = H_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] = -\vec{\nabla}V_m^{out}(r \gg R)$$

<u>Then</u>: $\vec{H}^{outside}(r \gg R) = \frac{1}{\mu_o}\vec{B}^{outside}(r \gg R)$

$$= \vec{H}_{ext} \qquad = \frac{1}{\mu_o}\vec{B}_{ext}$$

$$= H_o\hat{z} \qquad = \frac{1}{\mu_o}B_o\hat{z} \qquad \leftarrow \qquad \left(H_o \equiv \frac{1}{\mu_o}B_o\right)$$

$$= H_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] = \frac{1}{\mu_o}B_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]$$

$$= -\vec{\nabla}V_m^{outside}(r \gg R)$$

$$\therefore \qquad -\vec{\nabla}V_m^{outside}(r \gg R) = H_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] = \frac{1}{\mu_o}B_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]$$

We see that: $\boxed{V_m^{outside}(r \gg R) = -H_o r\cos\theta}$ satisfies this requirement / condition.

<u>Explicit check</u>:

$$-\vec{\nabla}V_m^{outside}(r \gg R) = -\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right]V_m^{outside}(r \gg R)$$

$$= +\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right]H_o r\cos\theta$$

$$= H_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] = H_o\hat{z}$$

5) $B^{\perp}_{outside}(r=R) = B^{\perp}_{inside}(r=R) \Rightarrow B_r^{outside}(r=R) = B_r^{inside}(r=R)$

($\perp = \hat{r}$ direction at $r = R$ interface / boundary)

6) $H^{\parallel}_{outside}(r=R) - H^{\parallel}_{inside}(r=R) = \vec{K}_{free}\times\hat{n}\big|_{surface} = \vec{K}_{free}\times\hat{r}\big|_{surface} = 0$ ($\vec{K}_{free} = 0$ here)

7) $B^{\parallel}_{outside}(r=R) - B^{\parallel}_{inside}(r=R) = \mu_o\vec{B}_{Tot}\times\hat{n}\big|_{surface} = \mu_o\vec{K}_{TOT}\times\hat{r}\big|_{surface} = \mu_o\vec{K}_{bound}\times\hat{r}\big|_{surface}$

8) $\left.\begin{array}{l}\left(H^{\perp}_{outside}(r=R) - H^{\perp}_{inside}(r=R)\right) = -\left(\mathrm{M}^{\perp}_{outside}(r=R) - \mathrm{M}^{\perp}_{inside}(r=R)\right) \\ \left(H_r^{outside}(r=R) - H_r^{inside}(r=R)\right) = -\left(\mathrm{M}_r^{outside}(r=R) - \mathrm{M}_r^{inside}(r=R)\right)\end{array}\right\}$ ($\perp = \hat{r}$ direction at $r = R$ interface / boundary)

Note also that because of the manifest / intrinsic <u>*odd* reflection symmetry</u> associated with this problem (as we saw for the dielectric sphere in uniform external electric field problem) about the $z = 0$ midplane (i.e. $z \to -z$), namely that $V_m(-z) = -V_m(+z)$ {i.e. because of the corresponding $\theta \to -\theta$ reflection symmetry properties associated with the Legendré Polynomials themselves – $\mathrm{P}_\ell(-\theta) = (-1)^\ell\,\mathrm{P}_\ell(\theta)$} we anticipate / know in advance / expect that all even-ℓ $P_\ell(\cos\theta)$ terms <u>must</u> vanish – i.e. only odd-ℓ $P_\ell(\cos\theta)$ terms will be present in $V_m(r,\theta)$ due to the manifest / intrinsic odd reflection symmetry associated with this problem!

Again, the general solution for the magnetostatic version of Laplace's Equation (in spherical-polar coordinates) is:

$$\boxed{V_m(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

<u>Apply BC 0):</u> *<u>Inside</u>* the magnetic sphere ($r \le R$) we demand:

$V_m^{inside}(r \le R)$ must be finite $\forall \; r \le R$

$\rightarrow$ All $B_\ell = 0 \;\; \forall \;\; \ell$ in the region $r \le R$ (inside the magnetic sphere)

<u>Thus:</u> $$\boxed{V_m^{inside}(r \le R)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(cos\theta)}$$

<u>Apply BC 0):</u> *<u>Outside</u>* the magnetic sphere ($r \ge R$) we *<u>must</u>* allow both r^ℓ and $\dfrac{1}{r^{\ell+1}}$ terms, because the region $r = \infty$ is formally excluded in this problem (when $x, y \rightarrow \infty$ at the midplane region, simultaneously $\theta \rightarrow \dfrac{\pi}{2}$ and $V_m\left(\theta=\dfrac{\pi}{2}\right)=0$, automatically satisfied for all odd-ℓ $P_\ell(\cos\theta)$ terms)!!!

<u>Thus:</u> $$\boxed{V_m^{outside}(r \ge R)=\sum_{\ell=0}^{\infty}\left(A'_\ell+\frac{B'_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

Next, we apply BC 4), namely that for $r \gg R$, i.e. far from the magnetic sphere:

$$\vec{B}^{out}(r \gg R) \equiv B_{ext} = B_o\hat{z} \qquad \vec{H}^{outside}(r \gg R)=\vec{H}_{ext}=H_o\hat{z}=-\vec{\nabla}V_m^{outside}(r \gg R)$$

$$\begin{aligned}\vec{B}^{outside}(r \gg R) &= \mu_o\vec{H}^{outside}(r \gg R)\\ =\vec{B}_{ext}(r \gg R) &= \mu_o\vec{H}_{ext}(r \gg R)\\ =B_o\hat{z} &= \mu_o H_o\hat{z}\end{aligned}$$

We showed that $\boxed{V_m^{out}(r \gg R)=-\tfrac{1}{\mu_o}B_o r\cos\theta=-\tfrac{1}{\mu_o}B_o z=-H_o r\cos\theta=-H_o z}$ satisfies this boundary condition ($z = r\cos\theta$ in spherical coordinates).

<u>Apply BC 3):</u> We also want: $V_m^{out}(z=\pm L_{gap})=\pm V_m^o$ on $\left\{\begin{matrix}\text{upper (south!)}\\ \text{lower (north!)}\end{matrix}\right\}$ poles of external magnet

<u>Thus:</u> $$V_m(r \gg R)=\left(\frac{V_m^o}{L_{gap}}\right)z=\left(\frac{V_m^o}{L_{gap}}\right)r\cos\theta=\left(\frac{V_m^o}{L_{gap}}\right)rP_1(\cos\theta)$$

<u>Thus:</u> $$\boxed{H_o=-\left(\frac{V_m^o}{L_{gap}}\right)=\frac{1}{\mu_o}B_o} \qquad \underline{\text{or:}} \qquad \boxed{V_m^o=-H_o L_{gap}}$$

$$\boxed{B_o=-\mu_o\left(\frac{V_m^o}{L_{gap}}\right)} \qquad \underline{\text{or:}} \qquad \boxed{V_m^o=-\frac{1}{\mu_o}B_o L_{gap}}$$ (we'll need these later…)

So, if $V_m^{outside}(r \geq R) = \sum_{\ell=0}^{\infty}\left(A'_\ell r^\ell + \frac{B'_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

If for $r >> R$, $V_m^{outside}(r \gg R) = \left(\frac{V_m^o}{L_{gap}}\right) r P_1(\cos\theta)$

then we know that / demand that all A'_ℓ vanish (i.e. all $A'_\ell = 0$)

except the $\ell = 1$ term: $\boxed{A'_1 = \frac{V_m^o}{L_{gap}}}$

So now: $V_m^{outside}(r \geq R) = \left(\frac{V_m^o}{L_{gap}}\right) r\cos\theta + \sum_{\ell=0}^{\infty}\left(\frac{B'_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

<u>Now apply BC 1):</u> $V_m^{inside}(r = R) = V_m^{outside}(r = R) \Leftarrow V_m(r)$ is continuous across the boundary / interface

$$\therefore \text{ at } r = R: \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \left(\frac{V_m^o}{L_{gap}}\right) R \underbrace{\cos\theta}_{P_1(\cos\theta)} + \sum_{\ell=0}^{\infty}\frac{B'_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

Thus by the method of inspection, we see that, <u>because of the orthogonality properties of the</u> $P_\ell(\cos\theta)$, all A_ℓ and B'_ℓ coefficients must vanish <u>except</u> the $\ell = 1$ terms:

i.e. $\boxed{A_1 R\cos\theta = \left(\frac{V_m^o}{L_{gap}}\right) R\cos\theta + \frac{B'_1}{R^2}\cos\theta}$

or: $\boxed{A_1 = \left(\frac{V_m^o}{L_{gap}}\right) + \frac{B'_1}{R^3}}$ all other $A_\ell = B'_\ell = 0$

<u>Then</u>:
$$\boxed{\begin{aligned} &V_m^{inside}(r \leq R) = A_1 r\cos\theta \\ &V_m^{outside}(r \geq R) = \left(\frac{V_m^o}{L_{gap}}\right) r\cos\theta + \frac{B'_1}{r^2}\cos\theta \end{aligned}}$$

We still have one remaining unknown – e.g. B'_1. Thus, we need to apply one more boundary condition in order to obtain an independent relationship between A_1 and B'_1.

Let's choose BC 5): $B_r^{outside}(r = R) = B_r^{inside}(r = R)$

(i.e. here radial normal components of $\vec{B}$ are continuous across an interface)

<u>Now</u>: $\vec{B}^{outside}(r \geq R) = \mu_o \vec{H}^{outside}(r \geq R)$

$\vec{B}^{inside}(r \leq R) = \mu \vec{H}^{inside}(r \leq R)$ and $\mu = \mu_o(1 + \chi_m) = \mu_o K_m$

$\therefore$ BC 5) also says: $\mu_o H_r^{outside}(r=R) = \mu H_r^{outside}(r=R)$

But $\vec{H}(\vec{r}) = -\vec{\nabla} V_m \qquad H_r = -\nabla_r V_m = -\dfrac{\partial V_m}{\partial r}$ in spherical polar coordinates

$$\therefore \text{ BC 5) also says: } -\mu_o \left.\frac{\partial V_m^{outside}(r)}{\partial r}\right|_{r=R} = -\mu \left.\frac{\partial V_m^{inside}(r)}{\partial r}\right|_{r=R}$$

Now: $V_m^{inside}(r \le R) = A_1 r\cos\theta$ and $V_m^{outside}(r \ge R) = \left(\dfrac{V_m^o}{L_{gap}}\right) r\cos\theta + \dfrac{B_1'}{r^2}\cos\theta$

$$\therefore \quad -\mu_o\left[\left(\frac{V_m^o}{L_{gap}}\right)\cos\theta - \frac{2B_1'}{R^3}\cos\theta\right] = -\mu A_1\cos\theta \qquad \underline{\text{or}}: \qquad \mu_o\left(\frac{V_m^o}{L_{gap}}\right) - \mu_o\frac{2B_1'}{R^3} = \mu A_1$$

$$\therefore \quad A_1 = \frac{\mu_o}{\mu}\left[\left(\frac{V_m^o}{L_{gap}}\right) - \frac{2B_1'}{R^3}\right] \quad \underline{\text{but}}: \quad A_1 = \left(\frac{V_m^o}{L_{gap}}\right) + \frac{B_1'}{R^3} \qquad \text{(from BC 1))}$$

$$\therefore \left(\frac{V_m^o}{L_{gap}}\right) + \frac{B_1'}{R^3} = \left(\frac{\mu_o}{\mu}\right)\left(\frac{V_m^o}{L_{gap}}\right) - 2\left(\frac{\mu_o}{\mu}\right)\frac{B_1'}{R^3} \quad \text{Solve for } B_1'.$$

$$\frac{B_1'}{R^3} + 2\left(\frac{\mu_o}{\mu}\right)\frac{B_1'}{R^3} = \left[\left(\frac{\mu_o}{\mu}\right) - 1\right]\left(\frac{V_m^o}{L_{gap}}\right)$$

$$\left[1 + 2\left(\frac{\mu_o}{\mu}\right)\right]\frac{B_1'}{R^3} = \left[\left(\frac{\mu_o}{\mu}\right) - 1\right]\left(\frac{V_m^o}{L_{gap}}\right)$$

$$B_1' = \frac{\left(\dfrac{\mu_o}{\mu} - 1\right)}{\left(1 + 2\dfrac{\mu_o}{\mu}\right)}\left(\frac{V_m^o}{L_{gap}}\right)R^3 = \left(\frac{\mu_o - \mu}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right)R^3 = -\underbrace{\left(\frac{\mu - \mu_o}{\mu + 2\mu_o}\right)}\left(\frac{V_m^o}{L_{gap}}\right)R^3$$

We assume $\mu > \mu_o$ {doesn't really matter...}

$$\boxed{B_1' = -\left(\frac{\mu - \mu_o}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right)R^3}$$

$\underline{\text{Then}}$: $A_1 = \left(\dfrac{V_m^o}{L_{gap}}\right) + \dfrac{B_1'}{R^3} = \left(\dfrac{V_m^o}{L_{gap}}\right)\left[1 - \dfrac{(\mu - \mu_o)}{(\mu + 2\mu_o)}\right]$

$$A_1 = \left[\frac{\cancel{\mu} + 2\mu_o - \cancel{\mu} + \mu_o}{\mu + 2\mu_o}\right]\left(\frac{V_m^o}{L_{gap}}\right) = \left(\frac{3\mu_o}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right)$$

$$\boxed{A_1 = \left(\frac{3\mu_o}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right)}$$

Thus, finally we now have the fully-specified magnetic scalar potentials:

$$\boxed{V_m^{inside}(r \le R) = \left(\frac{3\mu_o}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right) r\cos\theta = \left(\frac{3}{K_m + 2}\right)\left(\frac{V_m^o}{L_{gap}}\right) z}$$ SI Units = Amperes for V_m? Yes!

since $z = r\cos\theta$ and $K_m \equiv \mu/\mu_o = (1+\chi_m)$

$$\boxed{\begin{aligned} V_m^{outside}(r \ge R) &= \left(\frac{V_m^o}{L_{gap}}\right) r\cos\theta - \left(\frac{\mu - \mu_o}{\mu + 2\mu_o}\right)\left(\frac{V_m^o}{L_{gap}}\right)\left(\frac{R^3}{r^2}\right)\cos\theta \\ &= \left(\frac{V_m^o}{L_{gap}}\right)\left[z - \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R^3}{r^2}\right)\cos\theta\right] \end{aligned}}$$ SI Units = Amperes for V_m? Yes!

Thus, we see that inside the magnetic sphere the magnetic scalar potential $V_m^{inside}(r \le R)$ increases linearly with *z,* whereas outside the magnetic sphere the magnetic scalar potential $V_m^{outside}(r \ge R)$ is the sum {i.e. linear superposition} of two terms, one which increases linearly with *z*, and another term which corresponds to the {magnetic scalar} potential associated with a {magnetic} dipole. The linear dependence of the magnetic scalar potential arises from the uniform external magnetic field $\vec{B}_{ext} = B_o\hat{z}$, and the dipole term in the external magnetic scalar potential arises simply from the magnetic dipole moment $\vec{m} = \frac{4}{3}\pi R^3\vec{\mathrm{M}}$ associated with the magnetized sphere! Note that for $z = r\cos(\theta = \pi/2) = 0$ that $V_m^{inside}(z=0) = V_m^{outside}(z=0) = 0$, i.e. the magnetic scalar potential $V_n(z=0)$ on the horizontal *x-y* plane in the middle of the gap of the electromagnet is an equi-"potential" of 0 Amperes. We also see that on the surface of the sphere, $V_m^{inside}(r=R) = V_m^{outside}(r=R) = \left(\frac{3}{K_m+2}\right)\left(\frac{V_m^o}{L_{gap}}\right) R\cos\theta$

and that $V_m^{outside}(z = \pm L_{gap}) = \pm V_m^o$ for $r \gg R$.

<u>Now recall that</u>: $\boxed{\left(\frac{V_m^o}{L_{gap}}\right) = -H_o = -\frac{1}{\mu_o} B_o}$ $\Leftarrow$ (for $\vec{B}_{ext} = B_o\hat{z}$) SI Units = Amps/meter for $\vec{H}$

<u>And since</u>: $\boxed{\vec{H}(\vec{r}) \equiv -\vec{\nabla} V_m(\vec{r}) = -\left\{\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right\} V_m(\vec{r})}$

<u>Then</u>:

$$\vec{H}^{inside}(r \le R) = -\vec{\nabla} V_m^{inside}(r \le R) = -\left(\frac{3}{K_m+2}\right)\left(\frac{V_m^o}{L_{gap}}\right)\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] = -\left(\frac{3}{K_m+2}\right)\left(\frac{V_m^o}{L_{gap}}\right)\hat{z}$$

$$\boxed{\vec{H}^{inside}(r \le R) = -\left(\frac{3}{K_m+2}\right)\left(\frac{V_m^o}{L_{gap}}\right)\hat{z} = +\left(\frac{3}{K_m+2}\right) H_o\hat{z} = \frac{1}{\mu_o}\left(\frac{3}{K_m+2}\right) B_o\hat{z}} \quad \boxed{K_m \equiv \mu/\mu_o = (1+\chi_m)}$$

And:

$$\vec{H}^{outside}(r \geq R) = -\vec{\nabla}V_m^{outside}(r \geq R)$$

$$= -\left(\frac{V_m^o}{L_{gap}}\right)\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{V_m^o}{L_{gap}}\right)\left[\frac{-2R^3}{r^3}\hat{r}\cos\theta - \frac{R^3}{r^3}\hat{\theta}\sin\theta\right]$$

$$\vec{H}^{outside}(r \geq R) = -\left(\frac{V_m^o}{L_{gap}}\right)\left\{\hat{z} + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{r}\right)^3\left[2\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]\right\}$$

But: $\boxed{\left(\frac{V_m^o}{L_{gap}}\right) = -H_o = -\frac{1}{\mu_o}B_o}$

$$\boxed{\vec{H}^{outside}(r \geq R) = +H_o\hat{z} + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{r}\right)^3 H_o\left[2\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]}$$ or:

$$\boxed{\vec{H}^{outside}(r \geq R) = \left(\frac{1}{\mu_o}\right)B_o\hat{z} + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{r}\right)^3\left(\frac{1}{\mu_o}\right)B_o\left[2\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]}$$ SI units: Amps/meter

Thus, we see that the *H*-field inside the magnetized sphere is constant/uniform, whereas the *H*-field outside the magnetized sphere is the linear superposition of the *H*-field associated with the constant/uniform externally-applied magnetic field $\vec{H}_{ext}(\vec{r}) = \frac{1}{\mu_o}\vec{B}_{ext}(\vec{r}) = \frac{1}{\mu_o}B_o\hat{z}$ and the *H*-field associated with the magnetic dipole moment $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$ of the magnetized sphere!

Then: $\boxed{\vec{B}^{inside}(r \leq R) = \mu\vec{H}^{inside}(r \leq R) = \left(\frac{3}{K_m + 2}\right)\left(\frac{\mu}{\mu_o}\right)B_o\hat{z} = \left(\frac{3K_m}{K_m + 2}\right)B_o\hat{z}}$ but note that:

$$\boxed{\vec{B}^{inside}(r \leq R) = \left(\frac{3K_m}{K_m + 2}\right)B_o\hat{z} = \frac{3(1+\chi_m)}{(3+\chi_m)}B_o\hat{z} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)B_o\hat{z}}$$ SI units = Teslas

n.b. This *is* the same answer as Griffiths Problem 6.18 – it better be the same!!!

And: $\vec{B}^{outside}(r \geq R) = \mu\vec{H}^{outside}(r \geq R) = B_o\hat{z} + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{r}\right)^3 B_o\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]$

$$\boxed{\vec{B}^{outside}(r \geq R) = B_o\hat{z} + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{r}\right)^3 B_o\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$ but: $K_m = \left(\frac{\mu}{\mu_o}\right) = (1+\chi_m)$

Or: $\boxed{\vec{B}^{outside}(r \geq R) = B_o\hat{z} + \left(\frac{1}{3}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)\left(\frac{R}{r}\right)^3 B_o\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$ SI units = Teslas

Thus, again we see that the *B*-field inside the magnetized sphere is constant/uniform, whereas the *B*-field outside the magnetized sphere is the linear superposition of the *B*-field associated with the constant/uniform externally-applied magnetic field $\vec{B}_{ext}(\vec{r}) = B_o\hat{z}$ and the *B*-field associated with the magnetic dipole moment $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$ of the magnetized sphere!

<u>Then since</u>:

$$\vec{H}^{inside}(\vec{r}) = \frac{1}{\mu_o}\vec{B}^{inside}(\vec{r}) - \vec{\mathrm{M}}(\vec{r}) \text{ or: } \vec{\mathrm{M}}(\vec{r}) = \frac{1}{\mu_o}\vec{B}^{inside}(\vec{r}) - \vec{H}^{inside}(\vec{r}) \text{ and: } \vec{B}^{inside}(\vec{r}) = \mu\vec{H}^{inside}(\vec{r})$$

$$\therefore \quad \boxed{\vec{\mathrm{M}}(\vec{r}) = \left(\frac{\mu}{\mu_o} - 1\right)\vec{H}^{inside}(\vec{r}) = (K_m - 1)\vec{H}^{inside}(\vec{r}) = \chi_m\vec{H}^{inside}(\vec{r})}$$

i.e. $\boxed{\vec{\mathrm{M}}(r \le R) = \chi_m\vec{H}^{inside}(r \le R)}$ {Note that $\vec{\mathrm{M}}(r > R) \equiv 0$ }

<u>But</u>: $\boxed{\vec{H}^{inside}(r \le R) = +\left(\frac{3}{K_m + 2}\right)H_o\hat{z} = \frac{1}{\mu_o}\left(\frac{3}{K_m + 2}\right)B_o\hat{z}}$ since $\boxed{H_o = \frac{1}{\mu_o}B_o = -\left(\frac{V_m^o}{L_{gap}}\right)}$

<u>Thus</u>: $\boxed{\vec{\mathrm{M}}(r \le R) = \left(\frac{1}{\mu_o}\right)\left(\frac{3\chi_m}{K_m + 2}\right)B_o\hat{z} = \left(\frac{1}{\mu_o}\right)\left(\frac{3\chi_m}{\chi_m + 3}\right)B_o\hat{z} = \mathrm{M}_o\hat{z}}$ SI units = Amps/meter

i.e. $\boxed{\mathrm{M}_o = \left(\frac{1}{\mu_o}\right)\left(\frac{3\chi_m}{\chi_m + 3}\right)B_o = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right)B_o}$

Thus, we see that the magnetization (magnetic dipole moment per unit volume) of the magnetized sphere is constant/uniform, and is aligned parallel (anti-parallel) with the applied external magnetic field for $\chi_m > 0$ ($\chi_m < 0$) respectively.

The magnetic dipole moment of the magnetized sphere is therefore:

$$\boxed{\vec{m} = \tfrac{4}{3}\pi R^3\vec{\mathrm{M}} = \tfrac{4}{3}\pi R^3\mathrm{M}_o\hat{z} = \tfrac{4}{3}\pi R^3\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o\hat{z} = 4\pi R^3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right)B_o\hat{z}}$$

Again, note that $\vec{m} = \tfrac{4}{3}\pi R^3\vec{\mathrm{M}}$ is parallel (anti-parallel) to the applied external magnetic field $\vec{B}_{ext}(\vec{r}) = B_o\hat{z}$ for $\chi_m > 0$ ($\chi_m < 0$) {i.e. $K_m = (1 + \chi_m) > 1$ ($K_m = (1 + \chi_m) < 1$) } respectively.

Let us now investigate / explicitly check out the boundary conditions that we <u>*didn't*</u> actually use:

BC 6): $\left(H_{outside}^{\parallel} - H_{inside}^{\parallel}\right)\Big|_{r=R} = \vec{K}_{free} \times \hat{n}\Big|_{surface}$

BC 7): $\left(B_{outside}^{\parallel} - B_{inside}^{\parallel}\right)\Big|_{r=R} = \mu_o\vec{K}_{Tot} \times \hat{n}\Big|_{surface}$

BC 8): $\left(H_{outside}^{\perp} - H_{inside}^{\perp}\right)\Big|_{r=R} = -\left(\mathrm{M}_{outside}^{\perp} - \mathrm{M}_{inside}^{\perp}\right)\Big|_{r=R}$

n.b. Because we had many more BC relations than # of unknown coefficients that needed to be determined in <u>this</u> problem, we see / realize that this problem is in fact <u>over-determined</u>!!!

BC 6): $\left(H^{\parallel}_{outside} - H^{\parallel}_{inside}\right)\big|_{r=R} = \underbrace{\vec{K}_{free}}_{=0} \times \underbrace{\hat{n}}_{=\hat{r}}$ (Tangential $\vec{H}$ @ $r = R$)

$\Rightarrow \left(H^{out}_{\theta} - H^{in}_{\theta}\right)\big|_{r=R} = 0\,??$ $\hat{z} = \underbrace{\hat{r}\cos\theta}_{} - \underbrace{\hat{\theta}\sin\theta}_{}$

Normal component @ $r = R$ tangential component @ $r = R$

$$= \left\{-\frac{1}{\mu_o}B_o\hat{\theta}\sin\theta + \left(\frac{K_m - 1}{K_m + 2}\right)\left(\frac{R}{R}\right)^{\cancel{3}}\frac{1}{\mu_o}B_o\hat{\theta}\sin\theta\right\} - \frac{3}{(K_m+2)}\left(\frac{1}{\mu_o}\right)B_o\hat{\theta}\sin\theta$$

$$= \left\{-1 + \left(\frac{K_m - 1}{K_m + 2}\right) - \frac{3}{(K_m+2)}\right\}\left(\frac{1}{\mu_o}\right)B_o\hat{\theta}\sin\theta$$

$$= \left\{-\frac{K_m - 2 + K_m - 1 - 3}{K_m + 2}\right\}\left(\frac{1}{\mu_o}\right)B_o\hat{\theta}\sin\theta = 0 \text{ !!! Yes!!!}$$

i.e. $\boxed{\begin{array}{ll} H^{\parallel}_{outside}(r=R) = H^{\parallel}_{inside}(r=R) & \text{Tangential-}H\text{ \underline{is} continuous across} \\ H^{outside}_{\theta}(r=R) = H^{inside}_{\theta}(r=R) & \text{\underline{this} interface / boundary at } r = R\text{!!} \end{array}}$

BC 7): $\left(B^{\parallel}_{outside} - B^{\parallel}_{inside}\right)\big|_{r=R} = \mu_o\vec{K}_{Tot} \times \overset{\hat{r}}{\hat{n}}\big|_{r=R} = \mu_o\overbrace{\cancel{\vec{K}_{free}}}^{=0} \times \hat{r}\big|_{r=R} + \mu_o\vec{K}_{Bound} \times \hat{r}\big|_{r=R}$

$$\Rightarrow \left(B^{outside}_{\theta} - B^{inside}_{\theta}\right)\big|_{r=R} = \mu_o\vec{K}_{Bound} \times \hat{r}\big|_{r=R}$$

$$= \left\{-B_o\hat{\theta}\sin\theta + \left(\frac{\chi_m}{\chi_m + 3}\right)\left(\frac{R}{R}\right)^{\cancel{3}} B_o\hat{\theta}\sin\theta\right\} + \left(\frac{3(1+\chi_m)}{\chi_m + 3}\right)B_o\hat{\theta}\sin\theta$$

$$= \left\{-1 + \left(\frac{\chi_m}{\chi_m + 3}\right) + \left(\frac{3 + 3\chi_m}{\chi_m + 3}\right)\right\}B_o\hat{\theta}\sin\theta$$

$$= \left\{\frac{-\cancel{\chi_m} - \cancel{3} + \cancel{\chi_m} + \cancel{3} + 3\chi_m}{\chi_m + 3}\right\}B_o\hat{\theta}\sin\theta = \left(\frac{3\chi_m}{\chi_m + 3}\right)B_o\hat{\theta}\sin\theta$$

$$= \left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o\hat{\theta}\sin\theta \neq 0$$

Now <u>what</u> is $\vec{K}_{Bound}$? $\boxed{\vec{K}_{Bound} \equiv \vec{\mathrm{M}} \times \hat{n}\big|_{surface}}$ $\hat{n}$ = <u>outward</u> normal from surface.

<u>Now</u>: $\vec{\mathrm{M}}(r \le R) = \mathrm{M}_o\hat{z} = \left[\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o\right]\hat{z} = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right)B_o\hat{z}$

<u>And</u>: $\hat{z} \times \hat{r} = \left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right] \times \hat{r} = -\sin\theta\left(\hat{\theta} \times \hat{r}\right) = \hat{\varphi}\sin\theta$ $\{\hat{r} \times \hat{\theta} = \hat{\varphi}$ and $\hat{\theta} \times \hat{r} = -\hat{\varphi}\}$

<u>Thus</u>: $\vec{K}_{Bound}(r = R) = \vec{\mathrm{M}}(r = R) \times \hat{r}\big|_{r=R} = \mathrm{M}_o\hat{z} \times \hat{r} = M_o\hat{\varphi}\sin\theta = \left[3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right)B_o\right]\sin\theta\hat{\varphi}$

Then: $\vec{K}_{Bound} \times \hat{r}\big|_{r=R} = \vec{\mathrm{M}}(r=R) \times \hat{r}\big|_{r=R} = \mathrm{M}_o \hat{z} \times \hat{r} = \mathrm{M}_o \hat{\varphi}\sin\theta$

$$\mathrm{M}_o = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)B_o = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o$$

Then: $\vec{K}_{Bound} \times \hat{r}\big|_{r=R} = \mathrm{M}_o \sin\theta \underbrace{\hat{\varphi}\times\hat{r}}_{=+\hat{\theta}}\big|_{r=R}$ $\{\hat{r}\times\hat{\theta} = \hat{\varphi}, \quad \hat{\theta}\times\hat{\varphi} = \hat{r}$ and $\hat{\varphi}\times\hat{r} = \hat{\theta}\}$

$$\vec{K}_{Bound} \times \hat{r}\big|_{r=R} = \mathrm{M}_o \hat{\theta}\sin\theta = \left[3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\right]\hat{\theta}\sin\theta$$

Then: $\left(B^{\|}_{outside} - B^{\|}_{inside}\right)\big|_{r=R} = \left(B_\theta^{outside} - B_\theta^{inside}\right)\big|_{r=R} = \mu_o \vec{K}_{Bound}\times\hat{r}\big|_{r=R}$

$$= 3\left(\frac{K_m-1}{K_m+2}\right)B_o\hat{\theta}\sin\theta = \cancel{\mu_o}\left[3\left(\frac{1}{\cancel{\mu_o}}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\right]\hat{\theta}\sin\theta = 3\left(\frac{K_m-1}{K_m+2}\right)B_o\hat{\theta}\sin\theta$$

Thus we see that BC 7) is indeed satisfied!

Finally, BC 8): $\left(H^{\perp}_{outside} - H^{\perp}_{inside}\right)\big|_{r=R} = -\left(\mathrm{M}^{\perp}_{outside} - \mathrm{M}^{\perp}_{inside}\right)\big|_{r=R}$ $\qquad \hat{z} = \underbrace{\hat{r}\cos\theta}_{\substack{\text{normal}\\ \text{component}\\ @r=R}} - \underbrace{\hat{\theta}\sin\theta}_{\substack{\text{tangential}\\ \text{component}\\ @r=R}}$

$$= \left[\left(\frac{1}{\mu_o}\right)B_o\hat{r}\cos\theta + \left(\frac{K_m-1}{K_m+2}\right)\left(\frac{\cancel{R}}{\cancel{R}}\right)^3\left(\frac{1}{\mu_o}\right)B_o\left(2\hat{r}\cos\theta\right) - \frac{3}{(K_m+2)}\left(\frac{1}{\mu_o}\right)B_o\hat{r}\cos\theta\right]$$

$$= -\left[0 - \mathrm{M}_o\cos\theta\right] = +\mathrm{M}_o\cos\theta = \left[3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\right]\cos\theta$$

$$= \left[1 + 2\left(\frac{K_m-1}{K_m+2}\right) - \frac{3}{(K_m+2)}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta = \left[3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\right]\cos\theta$$

$$= \left[\frac{K_m + \cancel{2} + 2K_m - \cancel{2} - 3}{K_m+2}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta = \left[\frac{3K_m-3}{K_m+2}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta$$

$$= 3\left[\frac{K_m-1}{K_m+2}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta \quad \text{with:} \quad K_m = \left(\frac{\mu}{\mu_o}\right) = 1+\chi_m \quad \text{or:} \quad K_m - 1 = \chi_m$$

$$= \left[\frac{3\chi_m}{\chi_m+3}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta = \left[\frac{\chi_m}{1+\chi_m/3}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta$$

Thus, we see that BC 8) is also indeed satisfied:

$$\boxed{\left(H^{\perp}_{outside} - H^{\perp}_{inside}\right)\big|_{r=R} = -\left(\mathrm{M}^{\perp}_{outside} - \mathrm{M}^{\perp}_{inside}\right)\big|_{r=R} = \left[\frac{3\chi_m}{\chi_m+3}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta = \left[\frac{\chi_m}{1+\chi_m/3}\right]\left(\frac{1}{\mu_o}\right)B_o\cos\theta}$$

Now let us examine and discuss these results in more detail:

- The magnetization (magnetic dipole moment per unit volume) $\vec{\mathrm{M}}$ inside the magnetic sphere is uniform / constant in the $\hat{z}$ -direction (n.b. same as the externally applied magnetic field $\vec{B}_{ext} = B_o\hat{z}$):

$$\boxed{\vec{\mathrm{M}}(r \le R) = \mathrm{M}_o\hat{z} = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)B_o\hat{z} = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\hat{z}}$$ SI Units = Amps/meter

- The corresponding magnetic dipole moment of the magnetized sphere (of radius R) is:

$$\boxed{\vec{m} = \vec{\mathrm{M}}\cdot V_{sphere} = \left(\tfrac{4}{3}\pi R^3\right)\vec{\mathrm{M}}}$$ SI Units = Ampere-meters^2 {recall "$\vec{m} = I\vec{a}$"}

$$\boxed{\vec{m} = \left(\tfrac{4}{3}\pi R^3\right)\vec{\mathrm{M}} = \tfrac{4}{3}\pi R^3\mathrm{M}_o\hat{z} = \tfrac{4}{3}\pi R^3\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)B_o\hat{z} = 4\pi R^3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m-1}{K_m+2}\right)B_o\hat{z}}$$

- The corresponding magnetic field inside the magnetized sphere (with $\vec{B}_{ext} = B_o\hat{z}$) is:

$$\boxed{\vec{B}^{inside}(r \le R) = \left(\frac{3K_m}{K_m+2}\right)B_o\hat{z} = \frac{3(1+\chi_m)}{(3+\chi_m)}B_o\hat{z} = \frac{(1+\chi_m)}{(1+\chi_m/3)}\vec{B}_{ext}}$$ SI units = Teslas

We can rearrange / manipulate this relation to further illuminate the physics of what is going on here, as follows:

$$\vec{B}^{inside}(r \le R) = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)B_o\hat{z} = \left(\frac{1+\frac{1}{3}\chi_m+\frac{2}{3}\chi_m}{1+\chi_m/3}\right)B_o\hat{z}$$

$$\vec{B}^{inside}(r \le R) = \underbrace{\left(\frac{1+\frac{1}{3}\chi_m}{1+\frac{1}{3}\chi_m}\right)}_{=1}B_o\hat{z} = \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o\hat{z}$$

$$\vec{B}^{inside}(r \le R) = B_o\hat{z} + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o\hat{z} = \vec{B}_{ext} + \left(\frac{2}{3}\right)\mu_o\overbrace{\left[\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o\hat{z}\right]}^{=\vec{\mathrm{M}}=\mathrm{M}_o\hat{z}}$$

This is identical with the result in Griffiths Example 6.1 pp. 266-67 (it better be!!!):

$$\boxed{\vec{B}^{inside}(r \le R) = \vec{B}_{ext} + \left(\frac{2}{3}\right)\mu_o\vec{\mathrm{M}}} \quad \text{with} \quad \boxed{\vec{\mathrm{M}} = \mathrm{M}_o\hat{z} = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\vec{B}_{ext}}$$

Thus we see that the magnetic field inside the sphere is the linear superposition of the externally applied magnetic field $\vec{B}_{ext} = B_o\hat{z}$ plus the internal $\vec{B}$ -field of the magnetized sphere (alone):

$$\boxed{\vec{B}^{inside}_{sphere}(r \le R) = \tfrac{2}{3}\mu_o\vec{\mathrm{M}}}$$!!!

Outside the magnetized sphere, the magnetic field is:

$$\boxed{\vec{B}^{outside}(r \ge R) = B_o\hat{z} + \left(\frac{1}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\left(\frac{R}{r}\right)^3 B_o\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$ SI Units = Teslas

Again, we can rearrange / manipulate this relation further to elucidate the underlying physics:

$$\boxed{\vec{B}^{outside}(r \geq R) = \vec{B}_{ext} + \left(\frac{\mu_o}{4\pi}\right)\left[\left(\frac{4\pi}{3}R^3\right)\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o\right]\left(\frac{1}{r^3}\right)\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

<u>But</u>: $$\boxed{\vec{m} = \left(\tfrac{4}{3}\pi R^3\right)\vec{\mathrm{M}} = \tfrac{4}{3}\pi R^3 \mathrm{M}_o \hat{z} = \frac{4\pi}{3}R^3\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o\hat{z} = \left(\frac{4\pi}{3}R^3\right)\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\vec{B}_{ext}}$$

<u>Thus</u>: $$\boxed{m = |\vec{m}| = \tfrac{4}{3}\pi R^3 \mathrm{M}_o = \left(\frac{4\pi}{3}R^3\right)\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o}$$

$\therefore$ $$\boxed{\vec{B}^{outside}(r \geq R) = \vec{B}_{ext} + \left(\frac{\mu_o}{4\pi}\right)\left(\frac{m}{r^3}\right)\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

Which again can be seen as the linear superposition of the external magnetic field and the (external) magnetic field of a physical magnetic dipole, with magnetic dipole moment $\vec{m}$:

$$\boxed{\vec{B}^{dipole}(r \geq R) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

See / compare to Griffiths 5.86 p. 246
& also P435 Lecture Notes 16, p. 14

$\therefore$ $$\boxed{\vec{B}^{outside}(r \geq R) = \vec{B}_{ext} + \vec{B}^{dipole}(r \geq R)}$$

We have also shown that $\vec{B}^{dipole}(r \geq R)$ can be written in <u>coordinate-free form</u> as:

$$\boxed{\vec{B}^{dipole}(r \geq R) = \left(\frac{\mu_o}{4\pi}\right)\frac{1}{r^3}\left[3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m}\right]}$$ SI Units = Teslas

With: $$\boxed{\vec{m} = \left(\tfrac{4}{3}\pi R^3\right)\vec{\mathrm{M}} = \left(\frac{4\pi}{3}R^3\right)\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\vec{B}_{ext}}$$ SI Units = Ampere-meters^2 ("$\vec{m} = I\vec{a}$")

- Comments on the relative strengths of internal & external magnetic fields vs. the applied external field - we can gain some additional physics insight on the nature of this problem by taking <u>ratios</u> of $\left|\vec{B}^{inside}(r \leq R)\right|$ and $\left|\vec{B}^{outside}(r \geq R)\right|$ to $\left|\vec{B}_{ext}\right| = B_o$:

$$\boxed{\frac{\left|\vec{B}^{inside}(r \leq R)\right|}{\left|\vec{B}_{ext}\right|} = \frac{B_o + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)B_o}{B_o} = 1 + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)}$$ n.b. = constant

For the outside ratio, since this is polar angle dependent, let's simply do it for $\theta = 0$:

$$\boxed{\frac{\left|\vec{B}^{outside}(r \geq R, \theta = 0)\right|}{\left|\vec{B}_{ext}\right|} = \frac{B_o + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\left(\frac{R}{r}\right)^3 B_o}{B_o} = 1 + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1+\frac{1}{3}\chi_m}\right)\left(\frac{R}{r}\right)^3}$$

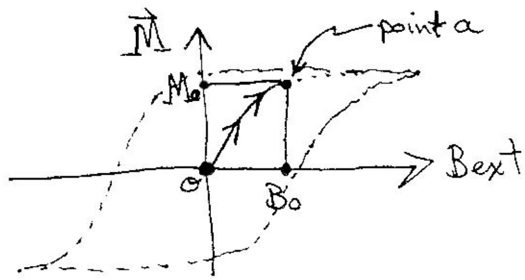
Note that at $r = R$, the inside ratio = outside ratio (i.e. normal component of B is continuous across an interface/boundary).

For either linear diamagnetic $(\chi_m < 0)$ or linear paramagnetic $(\chi_m > 0)$ materials, the typical values of magnetic susceptibilities associated with these materials are $|\chi_m| \sim 10^{-3} - 10^{-6}$. Thus for diamagnetic & paramagnetic materials with $|\chi_m| \ll 1$ we see that:

$$\boxed{\frac{\left|\vec{B}^{inside}(r \le R)\right|}{\left|\vec{B}_{ext}\right|} \sim 1} \quad \text{and} \quad \boxed{\frac{\left|\vec{B}^{outside}(r = R, \theta = 0)\right|}{\left|\vec{B}_{ext}\right|} \sim 1}$$

i.e. $\boxed{\left|\vec{B}^{inside}(r \le R)\right| \sim \left|\vec{B}_{ext}\right|}$ and $\boxed{\left|\vec{B}^{outside}(r = R, \theta = 0)\right| \sim \left|\vec{B}_{ext}\right|}$ simply because: $|\chi_m| \ll 1$

For ferromagnetic materials, where formally / technically speaking, the magnetization $\vec{M}$ is history-dependent, if we imagine that we have an initially unmagnetized sphere of ferromagnetic material and place it in our experimental apparatus and then slowly turn on the external magnetic field, from $\vec{B}_{ext} = 0$ (initially) to $\vec{B}_{ext} = B_o \hat{z}$ (finally) then we trace out a curve along the $\vec{M}$ vs. $\vec{B}_{ext}$ relation as shown below:



Let's suppose that at point *a* on this curve, the magnetization, $\vec{M}$ corresponds to a magnetic susceptibility $\chi_m = 1000$ (i.e. $\chi_m \gg 1$). Then for this ferromagnetic material we see that for $\chi_m \gg 1$:

$$\boxed{\frac{\left|\vec{B}^{inside}(r \le R)\right|}{\left|\vec{B}_{ext}\right|} = 1 + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1 + \frac{1}{3}\chi_m}\right) \approx 1 + 2 = 3} \quad \text{for } \chi_m \gg 1$$

Likewise: $$\boxed{\frac{\left|\vec{B}^{outside}(r = R, \theta = 0)\right|}{\left|\vec{B}_{ext}\right|} = 1 + \left(\frac{2}{3}\right)\left(\frac{\chi_m}{1 + \frac{1}{3}\chi_m}\right) \approx 1 + 2 = 3} \quad \text{for } \chi_m \gg 1$$

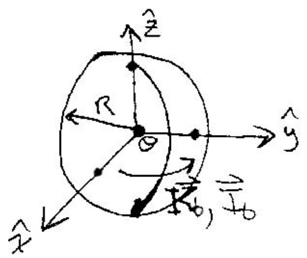
i.e. $\boxed{\left|\vec{B}^{inside}(r \le R)\right| \simeq 3\left|\vec{B}_{ext}\right|}$ for $\chi_m \gg 1$

$\boxed{\left|\vec{B}^{outside}(r = R, \theta = 0)\right| \simeq 3\left|\vec{B}_{ext}\right|}$ (at surface) for $\chi_m \gg 1$

For the uniformly-magnetized sphere of radius R, we have also seen that the magnetization $\vec{\mathrm{M}}(r \le R) = \mathrm{M}_o \hat{z} = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right) B_o \hat{z} = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right) B_o \hat{z}$ can be <u>replicated</u> by an <u>equivalent</u>, <u>bound</u> surface current density, $\vec{K}_{Bound}(r = R) = \vec{\mathrm{M}} \times \hat{n}\big|_{r=R} = \vec{\mathrm{M}} \times \hat{r}\big|_{r=R} = \mathrm{M}_o \sin\theta \hat{\varphi}$ circulating in the $+\hat{\varphi}$ direction on the surface of the sphere with magnitude:

$$\boxed{K_{Bound}(r = R, \theta) = \mathrm{M}_o \sin\theta = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right) B_o \sin\theta} \qquad \text{SI Units = Amperes/meter}$$

This corresponds to an <u>equivalent</u> bound current of: $\vec{I}_{bound}(r = R, \theta) = \int_{C_\perp} \vec{K}_{Bound}(r = R, \theta)\, d\ell_\perp$



$$\vec{I}_{bound}(r = R, \theta) = \int_{\substack{south\\pole}}^{\substack{north\\pole}} \vec{K}_{Bound}(r = R, \theta)\, d\ell_\perp$$

$$\vec{I}_{bound}(r = R, \theta) = \pi R \vec{K}_{Bound}(r = R, \theta)$$

$$\vec{I}_{bound}(r = R, \theta) = \pi R \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right) B_o \sin\theta \hat{\varphi}$$

SI Units = Amperes

Now, recall for the charged, spinning hollow conducting sphere of radius R (n.b. $\vec{B}_{ext} = 0$ there) with surface electric charge density σ (Coulombs per meter^2) and angular velocity of rotation ω (radians/sec)

The corresponding free surface current density $\vec{K}_{free} = K^o_{free}\hat{\varphi} = [\sigma\omega R]\sin\theta\hat{\varphi}$ (See Griffiths Example 5.11 pp. 236-37; P435 Lecture Notes 19 pp. 12-13; See also Griffiths Example 6.1 p. 264). This spinning free surface current density produced internal $(r \le R)$ and external $(r \ge R)$ magnetic fields: $\boxed{\vec{B}_{inside}(r \le R) = \frac{2}{3}\mu_o[\sigma\omega R]\hat{z}}$ and $\boxed{\vec{B}_{outside}(r \ge R) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$ with: $\boxed{m = \tfrac{4}{3}\pi R^3[\sigma\omega R]}$ which are identical to those of a permanently magnetized sphere, of uniform magnetization $\vec{\mathrm{M}} = M_o\hat{z}$ (i.e. $\vec{B}_{ext} = 0$ here) provided $\mathrm{M}_o \equiv [\sigma\omega R]$:

$$\boxed{\vec{B}^{inside}_{\substack{spinning\\sphere}}(r \le R) = \frac{2}{3}\mu_o\vec{\mathrm{M}} = \frac{2}{3}\mu_o\mathrm{M}_o\hat{z} = \frac{2}{3}\mu_o[\sigma\omega R]\hat{z}} \qquad \text{n.b. } \vec{B}_{ext} = 0 \text{ here!!}$$

$$\boxed{\vec{B}^{outside}_{\substack{spinning\\sphere}}(r \ge R) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]} \qquad \text{with } \boxed{m = \frac{4}{3}\pi R^3 \mathrm{M}_o = \frac{4}{3}\pi R^3[\sigma\omega R]}$$

Using the principle of linear superposition the magnetic field associated with a charged, spinning hollow conducting sphere of radius *R*, surface electric charge density σ and angular velocity of rotation ω that is additionally immersed in an external magnetic field $\vec{B}_{ext} = B_o\hat{z}$ is:

$$\boxed{\vec{B}^{inside}_{\substack{spinning\\sphere}}(r \le R) = \vec{B}_{ext} + \frac{2}{3}\mu_o\vec{\mathrm{M}}} \quad \text{where} \quad \boxed{\vec{\mathrm{M}} = \mathrm{M}_o\hat{z} = [\sigma\omega R]\hat{z}}$$

$$\boxed{\vec{B}^{outside}_{\substack{spinning\\sphere}}(r \ge R) = \vec{B}_{ext} + \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]} \quad \text{and} \quad \boxed{m = \frac{4}{3}\pi R^3 M_o = \frac{4}{3}\pi R^3[\sigma\omega R]}$$

We can investigate one more aspect of the uniformly-magnetized sphere in a uniform external magnetic field $\hat{B}_{ext} = B_o\hat{z}$. In P435 Lecture Notes 20 p. 8, we introduced the concepts(s) of effective bound surface and volume densities of magnetic pole strength (i.e. magnetic charge):

$\sigma_m^{Bound}(r = R) \equiv \vec{\mathrm{M}}\bullet\hat{n}\big\vert_{surface}$	SI Units = Amperes/meter
$\rho_m^{Bound}(\vec{r}) \equiv -\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r})$	SI Units = Amperes/meter^2

Recall here that the SI Units of magnetic charge g_m are Ampere-meters
($g_m = "qv"$ = Coulombs * meters/sec = Ampere-meters)

These relations for σ_m^{Bound} and ρ_m^{Bound} were / are defined in complete analogy to the bound surface and volume electric charge densities for electrostatic dielectric materials:

$\sigma_e^{Bound}(r = R) \equiv \vec{\mathrm{P}}\bullet\hat{n}\big\vert_{surface}$	SI Units = Coulombs/meter^2
$\rho_e^{Bound}(\vec{r}) \equiv -\vec{\nabla}\bullet\vec{\mathrm{P}}(\vec{r})$	SI Units = Coulombs/meter^3

Since the magnetization of the sphere is uniform/constant:

$$\boxed{\vec{\mathrm{M}}(r \le R) = \mathrm{M}_o\hat{z} = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o\hat{z} = 3\left(\frac{1}{\mu_o}\right)\left(\frac{K_m - 1}{K_m + 2}\right)B_o\hat{z}}$$

we see that the effective volume density of magnetic pole strength $\boxed{\rho_m^{Bound}(\vec{r}) \equiv -\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r}) = 0.}$

On the other hand, the effective surface density of magnetic pole strength is non-zero:

$$\boxed{\sigma_m^{Bound}(r = R) \equiv \vec{\mathrm{M}}\bullet\hat{n}\big\vert_{surface} = \mathrm{M}_o\hat{z}\bullet\hat{r} = \mathrm{M}_o\cos\theta = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o\cos\theta}$$

Thus we see that at the north pole on the surface of the magnetized sphere $(r = R, \theta = 0)$:

$$\boxed{\sigma_m^{Bound}(r = R, \theta = 0) = +\mathrm{M}_o = +\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o}$$

And at the south pole on the surface of the magnetized sphere $(r = R, \theta = \pi)$:

$$\boxed{\sigma_m^{Bound}(r = R, \theta = \pi) = -\mathrm{M}_o = -\left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1 + \chi_m/3}\right)B_o}$$

We can now also understand BC 8 in a new light. We can rewrite it, noting that $\hat{n} = \hat{r}$ here, as:

$$\boxed{\left(H_{outside}^{\perp} - H_{inside}^{\perp}\right)\Big|_{r=R} = -\left(M_{outside}^{\perp} - M_{inside}^{\perp}\right)\Big|_{r=R} = \left(\frac{1}{\mu_o}\right)\left[\frac{3\chi_m}{\chi_m+3}\right]B_o\cos\theta = \left(\frac{1}{\mu_o}\right)\left[\frac{\chi_m}{1+\chi_m/3}\right]B_o\cos\theta}$$

as:

$$\boxed{\begin{aligned}&\left(\vec{H}_{outside}(\vec{r}) - \vec{H}_{inside}(\vec{r})\right)\cdot\hat{n}\Big|_{surface} = -\left(\underbrace{\cancel{\vec{\mathrm{M}}_{outside}(\vec{r})}}_{\equiv 0\ here} - \vec{\mathrm{M}}_{inside}(\vec{r})\right)\cdot\hat{n}\Bigg|_{surface} = \vec{\mathrm{M}}_{inside}(\vec{r})\cdot\hat{n}\Big|_{surface} = \sigma_m^{Bound}\\ &= \sigma_m^{Bound} = \left(\frac{1}{\mu_o}\right)\left[\frac{\chi_m}{1+\chi_m/3}\right]B_o\cos\theta\end{aligned}}$$

i.e. the point here is that this boundary condition is actually:

$$\boxed{\left(H_{outside}^{\perp} - H_{inside}^{\perp}\right)\Big|_{r=R} = -\left(M_{outside}^{\perp} - M_{inside}^{\perp}\right)\Big|_{r=R} = \sigma_m^{Bound}} \quad !!!$$

More explicitly, this boundary condition actually is:

$$\boxed{\left(\vec{H}_{outside}(\vec{r}) - \vec{H}_{inside}(\vec{r})\right)\cdot\hat{n}\Big|_{surface} = -\left(\vec{\mathrm{M}}_{outside}(\vec{r}) - \vec{\mathrm{M}}_{inside}(\vec{r})\right)\cdot\hat{n}\Big|_{surface} = \sigma_m^{Bound}}$$

We also know that the *net* effective bound magnetic charge on surface of the magnetized sphere must be = 0, i.e.

$$\boxed{Q_{m}^{\overset{Bound}{NET}} = \oint_S \sigma_m^{bound}\,da = \oint_S \left(\vec{\mathrm{M}}\cdot\hat{n}\right)da = \oint_S \vec{\mathrm{M}}\cdot d\vec{a} = 0}$$

Explicit check:

$$\begin{aligned}Q_{m}^{\overset{Bound}{NET}} &= \oint_S \vec{\mathrm{M}}\cdot d\vec{a} = \oint_S \mathrm{M}_o\hat{z}\cdot\hat{n}da = \oint_S \mathrm{M}_o\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]\cdot\hat{r}\ da = \mathrm{M}_o\oint_S \cos\theta da\\ &= M_oR^2\int_{\varphi=0}^{\varphi=2\pi}d\varphi\int_{\theta=0}^{\theta=\pi}\cos\theta\sin\theta d\theta = 2\pi R^2M_o\int_{\theta=0}^{\theta=\pi}\underbrace{\cos\theta}_{=u}\underbrace{d\cos\theta}_{=du}\\ &= 2\pi R^2M_o\int_{-1}^{+1}u\ du = \pi R^2M_ou^2\Big|_{-1}^{+1} = 0\end{aligned}$$

$$\boxed{\begin{aligned}\theta &= \pi: \ u = \cos\pi = -1\\ \theta &= 0: \ \ u = \cos 0 = +1\end{aligned}}$$

$$\therefore \boxed{Q_{m}^{\overset{Bound}{NET}} = 0}$$

We can also compare the equivalent bound surface magnetic charge vs. the bound surface electric current $\sigma_m^{bound}(r=R,\theta)$ vs. $\vec{K}_{Bound}(r=R,\theta)$

$$\boxed{\sigma_m^{Bound}(r=R) \equiv \vec{\mathrm{M}}\cdot\hat{n}\Big|_{surface} = \mathrm{M}_o\hat{z}\cdot\hat{r} = \mathrm{M}_o\cos\theta = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)B_o\cos\theta}$$

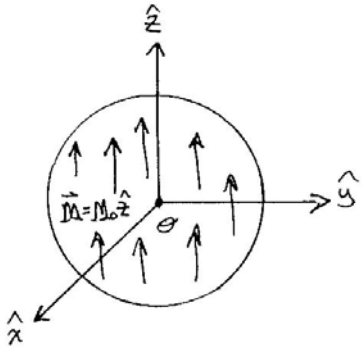
$$\boxed{\vec{K}_{Bound}(r=R) = \vec{\mathrm{M}}\times\hat{n}\Big|_{surface} = \vec{\mathrm{M}}\times\hat{r}\Big|_{r=R} = \mathrm{M}_o\sin\theta\hat{\varphi} = \left(\frac{1}{\mu_o}\right)\left(\frac{\chi_m}{1+\chi_m/3}\right)B_o\sin\theta\hat{\varphi}}$$

Both of these produce the exact same magnetization $\vec{\mathrm{M}}$ and associated / corresponding magnetic fields (internal and external)! They are simply two equivalent, but different ways / methods of viewing the same physics problem.

Example #2: Magnetic Fields Associated with a Uniformly Magnetized Sphere

Consider a permanently magnetized sphere of radius R that has uniform magnetization:

$\boxed{\vec{\mathrm{M}}(\vec{r}) = \mathrm{M}_o \hat{z} \quad (r \le R)}$



Since the magnetization $\boxed{\vec{\mathrm{M}}(\vec{r}) = \mathrm{M}_o \hat{z}}$ is constant, then $\boxed{\vec{\nabla} \cdot \vec{\mathrm{M}}(\vec{r}) = 0}$

But $\boxed{\vec{B}(\vec{r}) = \mu_o \left(\vec{H}(\vec{r}) + \vec{\mathrm{M}}(\vec{r}) \right)}$ and $\boxed{\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0}$ $\Rightarrow$ $\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = -\vec{\nabla} \cdot \vec{\mathrm{M}}(\vec{r})}$, $\therefore$ $\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = 0}$

And since $\boxed{\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}_{free}(\vec{r}) = 0}$ (here), since $\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = 0}$ and $\boxed{\vec{\nabla} \times \vec{H}(\vec{r}) = 0}$

Then we may write $\boxed{\vec{H}(\vec{r}) = -\vec{\nabla} V_m(\vec{r})}$

And thus $\boxed{\vec{\nabla} \cdot \vec{H}(\vec{r}) = \nabla^2 V_m = 0}$ (i.e. Laplace's Equation – Magnetic Scalar Potential $V_m(\vec{r})$)

Note that this problem has <u>azimuthal</u> symmetry (i.e. no φ-dependence):

Thus $\boxed{\nabla^2 V_m(r,\theta) = 0}$ has a general solution of the form: $\boxed{V_m(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)}$

Where $P_\ell(\cos\theta)$ = the ordinary Legendré Polynomial of order ℓ.

<u>Boundary Conditions:</u>

Sufficient relations to determine all coefficients A_ℓ and B_ℓ inside and outside the sphere

0) $V_m(r)$ = finite everywhere

1) $V_m^{OUT}(r=R) = V_m^{IN}(r=R)$

2) $B_{out}^{\perp}(r=R) = B_{in}^{\perp}(r=R) \Rightarrow B_r^{out}(r=R) = B_r^{in}(r=R)$

Normal component of $\vec{B}$ is continuous at the surface of sphere

($\perp = \hat{r}$ direction at $r = R$ interface / boundary)

$\hat{\theta}$-direction

Problem is actually over-determined / over-constrained

3) $H_{out}^{\parallel}(r=R) = H_{in}^{\parallel}(r=R) = \vec{K}_{free} \times \hat{n}\big|_{surface} = \vec{K}_{free} \times \hat{r}\big|_{surface} = 0$

($\vec{K}_{free} = 0$ here $\Rightarrow$ $\vec{K}_{TOT} = \cancel{\vec{K}_{free}} + \vec{K}_{bound} = \vec{K}_{bound}$)

4) $B_{out}^{\parallel}(r=R) - B_{in}^{\parallel} = \mu_o \vec{K}_{TOT} \times \hat{n}\big|_{surface} = \mu_o \vec{K}_{TOT} \times \hat{r}\big|_{surface} = \mu_o \vec{K}_{bound} \times \hat{r}\big|_{surface}$

5) $\left(H_{out}^{\perp}(r=R) - H_{in}^{\perp}(r=R) \right) = -\left(M_{out}^{\perp}(r=R) - M_{in}^{\perp}(r=R) \right)$

$\left(H_r^{out}(r=R) - H_r^{in}(r=R) \right) = -\left(M_r^{out}(r=R) - H_r^{in}(r=R) \right)$

($\perp = \hat{r}$ direction at $r = R$ interface / boundary)

General solutions for the magnetic scalar potential inside/outside of the sphere are of the form:

$$\boxed{V_m^{in}(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\quad (r\le R)}$$

$$\boxed{V_m^{out}(r,\theta)=\sum_{\ell=0}^{\infty}\left(A'_\ell r^l+\frac{B'_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\quad (r\ge R)}$$

Impose BC 0): $V_m(\vec{r})$ must be finite everywhere:

$\rightarrow$ for $V_m^{in}(r,\theta)$: $\boxed{B_\ell=0\quad \forall\ \ell\quad (r\le R)}$ $\Rightarrow$ $\boxed{V_m^{in}(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)\quad (r\le R)}$

$\rightarrow$ for $V_m^{out}(r,\theta)$: $\boxed{A'_\ell=0\quad \forall\ \ell\quad (r\ge R)}$ $\Rightarrow$ $\boxed{V_m^{out}(r,\theta)=\sum_{\ell=0}^{\infty}\frac{B'_\ell}{r^{\ell+1}}P_\ell(\cos\theta)\quad (r\ge R)}$

Impose BC 1): $V_m^{out}(r=R)=V_m^{in}(r=R)$

(i.e. $V_m(r=R)$ is continuous across the interface / boundary of sphere at $r=R$)

$\Rightarrow$ At $r=R$ we must have for each ℓ: $\boxed{A_\ell R^\ell=\frac{B'_\ell}{R^{\ell+1}}}$ or: $\boxed{B'_\ell=A_\ell R^{2\ell+1}}$

Impose BC 2): $B_r^{out}(r=R)=B_r^{in}(r=R)$ (Normal component of $\vec{B}$ is continuous across interface / boundary of sphere at $r=R$)

Now: $\vec{H}(\vec{r})\equiv-\vec{\nabla}V_m(\vec{r})$

$\Rightarrow$ radial component of H: $H_r(\vec{r})=-\partial V_m(\vec{r})/\partial r$ (in spherical polar coordinates)

But: $\vec{H}^{in}(r\le R)=\frac{1}{\mu_o}\vec{B}^{in}(r\le R)-\vec{\mathrm{M}}(r\le R)$

$\Rightarrow$ $\vec{H}^{in}(r\le R)=\frac{1}{\mu_o}\vec{B}^{in}(r\le R)-\vec{\mathrm{M}}(r\le R)$

where $\vec{\mathrm{M}}(\vec{r})=\mathrm{M}_o\hat{z}=\mathrm{M}_o\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]$ $(r\le R)$ since $\hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta$

$\therefore$ radial component of $\vec{H}^{in}(r\le R)$:

$$H_r^{in}(r\le R)=\frac{1}{\mu_o}B_r^{in}(r\le R)-\mathrm{M}_r(r\le R)$$

$$=\frac{1}{\mu_o}B_r^{in}(r\le R)-\mathrm{M}_o\cos\theta$$

$\therefore$ radial component of $\vec{B}^{in}(r\le R)$:

$B_r^{in}(r\le R)=\mu_o H_r^{in}(r\le R)+\mu_o\mathrm{M}_o\cos\theta$, but $H_r(\vec{r})=-\partial V_m(\vec{r})/\partial r$

$$\therefore\boxed{B_r^{in}(r\le R)=-\mu_o\frac{\partial V_m^{in}(r\le R)}{\partial r}+\mu_o\mathrm{M}_o\cos\theta}$$

Outside the sphere $(r \geq R)$: $\vec{\mathrm{M}}(r \geq R) = 0$ for $r > R$

$\therefore \quad \vec{H}^{out}(r \geq R) = \dfrac{1}{\mu_o}\vec{B}^{out}(r \geq R)$

$\therefore$ radial component of $\vec{H}^{out}(r \geq R)$: $H_r^{out}(r \geq R) = \dfrac{1}{\mu_o}B_r^{out}(r \geq R)$

$\Rightarrow \quad -\dfrac{\partial V_m^{out}(r \geq R)}{\partial r} = \dfrac{1}{\mu_o}B_r^{out}(r \geq R)$ or: $\boxed{B_r^{out}(r \geq R) = -\mu_o \dfrac{\partial V_m^{out}(r \geq R)}{\partial r}}$

Thus BC2 is: $\boxed{B_r^{out}(r = R) = B_r^{in}(r = R)}$

with: $\boxed{B_r^{in}(r,\theta) = -\mu_o \dfrac{\partial V_m^{in}(r,\theta)}{\partial r} + \mu_o \mathrm{M}_o \cos\theta}$ and $\boxed{B_r^{out}(r,\theta) = -\mu_o \dfrac{\partial V_m^{out}(r,\theta)}{\partial r}}$

gives: $-\mu_o \dfrac{\partial V_m^{out}(r,\theta)}{\partial r}\Big|_{r=R} = -\mu_o \dfrac{\partial V_m^{in}(r,\theta)}{\partial r}\Big|_{r=R} + \mu_o \mathrm{M}_o \cos\theta$

or: $\boxed{-\dfrac{\partial V_m^{out}(r,\theta)}{\partial r}\Bigg|_{r=R} = -\dfrac{\partial V_m^{in}(r,\theta)}{\partial r}\Bigg|_{r=R} + \mathrm{M}_o \cos\theta}$

(n.b. $\mathrm{M}_o \cos\theta$ only contains the $P_\ell(\cos\theta) = P_1(\cos\theta) = \cos\theta$ term)

Now: $\boxed{V_m^{out}(r,\theta) = \sum_{\ell=0}^{\infty} \dfrac{B'_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad (r \geq R)}$ and $\boxed{V_m^{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad (r \leq R)}$

Carrying out the radial differentiation on both sides of BC 2 relation, we obtain:

$$\boxed{+\sum_{\ell=0}^{\infty} \frac{(\ell+1)B'_\ell}{R^{\ell+2}} P_\ell(\cos\theta) = -\sum_{\ell=1}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) + \mathrm{M}_o \cos\theta}$$

This relation can *only* be satisfied term-by-term, i.e. for each ℓ-value in the infinite series, thus:

For $\ell = 0$: $\boxed{B'_0 = 0}$ (and therefore, from BC 1: $\boxed{B'_0 = A_0 R^{2\ell+1}}$ $\Rightarrow$ $\boxed{A_0 = 0}$)

For $\ell = 1$: $\boxed{+\dfrac{2B'_1}{R^3} = -A_1 + \mathrm{M}_o}$

For $\ell \geq 2$: $\dfrac{(\ell+1)B'_l}{R^{\ell+2}} = -\ell A_\ell R^{\ell-1}$ or: $\boxed{B'_\ell = -\left(\dfrac{\ell}{\ell+1}\right) A_\ell R^{2\ell+1}}$

But from BC 1: $\boxed{B'_\ell = A_\ell R^{2\ell+1}}$ $\therefore$ $A_\ell R^{2\ell+1} = -\left(\dfrac{\ell}{\ell+1}\right) A_\ell R^{2\ell+1} \Rightarrow \boxed{A_\ell = -\left(\dfrac{\ell}{\ell+1}\right) A_\ell}$

The relation $\boxed{A_\ell = -\left(\dfrac{\ell}{\ell+1}\right) A_\ell}$ can *only* be satisfied for each ℓ $(\ell \geq 2)$ if $\boxed{A_\ell = 0}$ $\therefore \boxed{B'_\ell = 0}$

$\therefore$ The only surviving term in the series expansion(s) for $V_m(r,\theta)$ is the $\ell = 1$ term!

Thus, the solutions for the magnetic scalar potential inside/outside the sphere, for *this* particular physics problem are:

$$\boxed{V_m^{in}(r,\theta) = A_1 r\cos\theta} \quad \text{and} \quad \boxed{V_m^{out}(r,\theta) = \frac{B_1'}{r^2}\cos\theta}$$

With: 1) $\boxed{B_1' = A_1 R^3}$ (from BC 1)

And: 2) $\boxed{\dfrac{2B_1'}{R^3} = -A_1 + \mathrm{M}_o}$ (from BC 2) $\Rightarrow$ 2') $\boxed{2B_1' = -A_1 R^3 + \mathrm{M}_o R^3}$

Simultaneously solve 1) and 2) above for A_1 and B_1':

Add 1) and 2'): (i.e. eliminate A_1):

$$B_1' + 2B_1' = A_1 R^3 - A_1 R^3 + \mathrm{M}_o R^3$$

$$\Rightarrow \boxed{3B_1' = M_o R^3} \quad \text{or:} \quad \boxed{B_1' = \frac{1}{3}\mathrm{M}_o R^3}$$

Plug this back into eq. 1) above: $\boxed{B_1' = \frac{1}{3}\mathrm{M}_o R^3 = A_1 R^3} \Rightarrow \boxed{A_1 = \frac{1}{3}\mathrm{M}_o}$

Thus, the <u>specific</u> solutions for the magnetic scalar potential inside/outside the sphere, unique for *this* particular physics problem are:

$$\boxed{V_m^{in}(r,\theta) = \frac{1}{3}\mathrm{M}_o r\cos\theta} \qquad (r \le R)$$

$$\boxed{V_m^{out}(r,\theta) = \frac{1}{3}\mathrm{M}_o R\left(\frac{R}{r}\right)^2 \cos\theta} \qquad (r \ge R)$$

<u>Now</u>: $\vec{H}(\vec{r}) = -\vec{\nabla}V_m(\vec{r})$ where $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$ (in spherical-polar coordinates)

<u>Thus</u>: $\vec{H}^{in}(r,\theta) = -\vec{\nabla}V_m^{in}(r,\theta) = -\frac{1}{3}\mathrm{M}_o\cos\theta\hat{r} + \frac{1}{3}\mathrm{M}_o\sin\theta\hat{\theta} = -\frac{1}{3}\mathrm{M}_o\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$

i.e. $$\boxed{\vec{H}^{in}(r,\theta) = -\frac{1}{3}\mathrm{M}_o\underbrace{\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]}_{\equiv\hat{z}} = -\frac{1}{3}\mathrm{M}_o\hat{z} = -\frac{1}{3}\vec{\mathrm{M}}}$$

Notice that here, in this problem, that $\vec{H}^{in}(r,\theta)$ points in the $-\hat{z}$ direction, <u>opposite</u> to the direction of the magnetization $\boxed{\vec{\mathrm{M}}(\vec{r}) = \mathrm{M}_o\hat{z} \quad (r \le R)}$!!!

Outside the magnetic sphere, $\boxed{\vec{H}^{out}(r,\theta) = -\vec{\nabla}V_m^{out}(r,\theta) = +\frac{2}{3}\mathrm{M}_o\left(\frac{R}{r}\right)^3\cos\theta\hat{r} + \frac{1}{3}\mathrm{M}_o\left(\frac{R}{r}\right)^3\sin\theta\hat{\theta}}$

i.e. $\boxed{\vec{H}^{out}(r,\theta) = +\frac{1}{3}\mathrm{M}_o\left(\frac{R}{r}\right)^3\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]} \Leftarrow \vec{H}$-field associated with a magnetic dipole!

<u>Thus</u>:

$$\boxed{\vec{H}^{in}(r,\theta) = -\frac{1}{3}\mathrm{M}_o\underbrace{\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]}_{\equiv\hat{z}} = -\frac{1}{3}\mathrm{M}_o\hat{z} = -\frac{1}{3}\vec{\mathrm{M}}} \text{ where } \boxed{\vec{\mathrm{M}}(\vec{r}) = \mathrm{M}_o\hat{z} \quad (r \le R)}$$

$$\boxed{\vec{H}^{out}(r,\theta) = +\frac{1}{3}\mathrm{M}_o\left(\frac{R}{r}\right)^3\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

<u>Now</u>: $\boxed{\vec{B}^{in}(r,\theta) = \mu_o\left(\vec{H}^{in}(r,\theta) + \vec{\mathrm{M}}\right)}$ and $\boxed{\vec{B}^{out}(r,\theta) = \mu_o\vec{H}^{out}(r,\theta)}$ (outside ($r > R$) M = 0)

$$\therefore\quad \frac{1}{\mu_o}\vec{B}^{in}(r,\theta) = -\frac{1}{3}\mathrm{M}_o\hat{z} + \mathrm{M}_o\hat{z} = +\frac{2}{3}\mathrm{M}_o\hat{z} = +\frac{2}{3}\mathrm{M}_o\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$$

$$\frac{1}{\mu_o}\vec{B}^{out}(r,\theta) = \frac{1}{3}\mathrm{M}_o\left(\frac{R}{r}\right)^3\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]$$

<u>Or</u>:

$$\boxed{\vec{B}^{in}(r,\theta) = +\frac{2}{3}\mu_o\mathrm{M}_o\hat{z} = \frac{2}{3}\mu_o\mathrm{M}_o\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) = \frac{2}{3}\mu_o\vec{\mathrm{M}}} \text{ where } \boxed{\vec{\mathrm{M}}(\vec{r}) = \mathrm{M}_o\hat{z} \quad (r \le R)}$$

$$\boxed{\vec{B}^{out}(r,\theta) = \frac{1}{3}\mu_o\mathrm{M}_o\left(\frac{R}{r}\right)^3\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

Déjà vu! We have seen before that the magnetic field associated with a magnetic dipole moment $\vec{m}$ (see Griffiths Equation 5.86 and/or P435 Lecture Notes 16, page14) is:

$$\boxed{\vec{B}^{dipole}(r \ge R) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)}$$

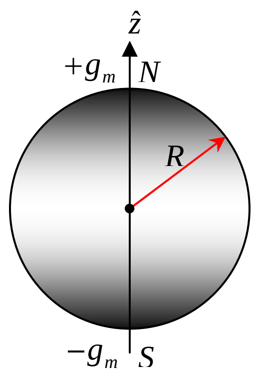
$$\boxed{\vec{H}^{dipole}(r \ge R) = \frac{1}{\mu_o}\vec{B}^{dipole}(r \ge R) = \left(\frac{1}{4\pi}\right)\frac{m}{r^3}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)}$$

Thus, we see here that for a uniformly, permanently magnetized sphere of radius R that:

$$\boxed{m = |\vec{m}| = \frac{|\vec{\mathrm{M}}|}{volume} = \frac{\mathrm{M}_o}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{\mathrm{M}_o}{R^3}} \quad \text{or:} \quad \boxed{\mathrm{M}_o = \frac{4\pi R^3}{3}m}$$

→ Compare these results with those of the previous magnetostatic boundary value problem example above, pages 9-18.

Note that the lines of $\vec{B}$ are continuous across the boundary / interface at $r = R$ (i.e. closed) whereas the lines of $\vec{H}$ are discontinuous across the boundary / interface at $r = R$. This is because the lines of $\vec{H}$ originate / terminate from / on the effective bound magnetic charges g_m on the surface of the magnetized sphere:



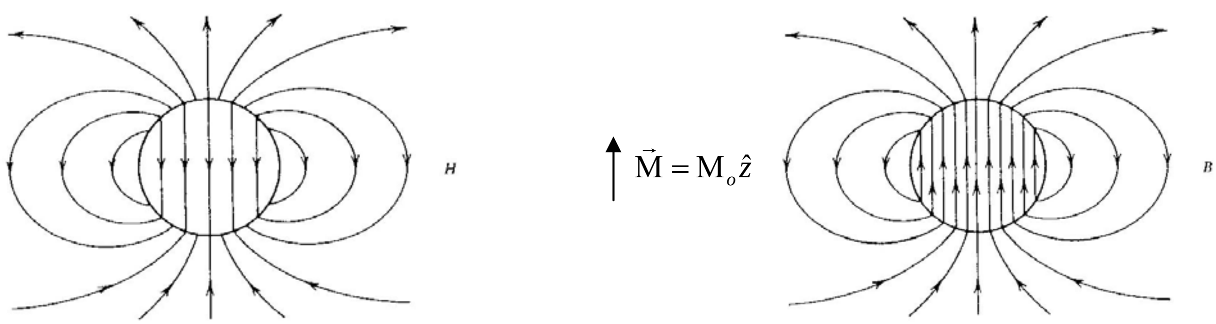
$$\sigma_m^{bound}(R,\theta) = \vec{\mathrm{M}} \cdot \hat{r}\big|_{r=R} = \mathrm{M}_o \hat{z} \cdot \hat{r} = \mathrm{M}_o \cos\theta$$

(See the previous BVP example, pages 17-18 above)

North Magnetic Poles (magnetic charges) have $+g_m$.

South Magnetic Poles (magnetic charges) have $-g_m$.

Lines of $\vec{H}$ and $\vec{B}$ Produced by Uniformly Magnetized Sphere ($\vec{\mathrm{M}} = \mathrm{M}_o \hat{z}$):



The lines of $\vec{H}$ and $\vec{B}$ produced by a uniformly magnetized sphere.

Again, we wish to emphasize the fact that $\vec{H}^{in}(\vec{r}) = -\frac{1}{3}\mathrm{M}_o \hat{z}$ points in the direction <u>opposite</u> to the magnetization $\vec{\mathrm{M}} = \mathrm{M}_o \hat{z}$ and also $\vec{B}^{in}(\vec{r}) = +\frac{2}{3}\mathrm{M}_o \hat{z}$. The lines of $\vec{H}$ emanate/terminate from the effective bound magnetic charge on the surface of the magnetized sphere. Note that the lines of $\vec{B}$ close on themselves – they do not terminate/emanate from the effective bound magnetic charges on the surface of the magnetized sphere.

Since the $\vec{H}^{in}(\vec{r})$ "bucks" the magnetization $\vec{\mathrm{M}}$, it results in a <u>demagnetizing</u> effect, which occurs over over a long period of time – e.g. $\simeq$ centuries, for *AlNiCo* materials at room temperature, $T \simeq$ 300 K. How fast depends on the nature of the magnetic material, and on the <u>geometry</u> of the magnetic material!

The demagnetization effect of having $\vec{H}$ antiparallel to $\vec{M}$ can be quantified / characterized by defining a quantity known as the <u>Demagnetization Factor γ</u>, defined as follows:

$$\boxed{V_m^{in}(r,\theta) \equiv \gamma \frac{M_o}{4\pi} r\cos\theta}$$

For the uniformly magnetized sphere, we found:

$$\boxed{V_m^{in}(r,\theta) = \frac{1}{3} M_o r\cos\theta} \text{ hence we see that } \boxed{\gamma_{sphere}\frac{M_o}{4\pi} = \frac{1}{3}M_o}$$

$\therefore$ The demagnetization factor for a uniformly magnetized sphere is $\boxed{\gamma_{sphere} = \frac{4\pi}{3} \approx 4.19 \sim 4.2}$

Different geometries of uniformly magnetized objects will have different values of $\vec{H}^{in}(r,\theta) = -[\_\_\_\_\_]M_o\hat{z}$ and hence different values of demagnetization factors γ.

e.g. For a large flat thin sheet lying in the *x-y* plane with uniform magnetization $\vec{M} = M_o\hat{z}$, $\gamma_{sheet} \simeq 4\pi \simeq 12.57$ (very unstable magnetization has $\gamma \to \infty$!!!)

For a very long, thin rod of radius $R <<$ length L with uniform magnetization $\vec{M} = M_o\hat{z} \parallel$ to the long axis of rod, $\gamma_{rod} \simeq 0$!! (i.e. very stable magnetization has $\gamma \to 0$)

Let us now also explicitly verify / show that the remaining boundary conditions (i.e. the ones we didn't use for determining the A_ℓ and B'_ℓ coefficients are indeed satisfied, i.e. that this particular physics problem <u>is</u> actually <u>over</u>-determined:

BC 4): $$B_{out}^{\parallel}(r=R) - B_{in}^{\parallel}(r=R) = \mu_o\vec{K}_{bound}\times\hat{r}\big|_{r=R}$$

i.e. $$= B_\theta^{out}(r=R) - B_\theta^{in}(r=R) = \mu_o\vec{K}_{bound}\times\hat{r}\big|_{r=R}$$

<u>Then</u>: $$\frac{1}{3}\mu_o M_o\sin\theta\hat{\theta} + \frac{2}{3}\mu_o M_o\sin\theta\hat{\theta} = \mu_o\vec{K}_{bound}\times\hat{r}\big|_{r=R}$$

<u>or</u>: $$\boxed{M_o\sin\theta\hat{\theta} = \vec{K}_{bound}\times\hat{r}\big|_{r=R}}$$

But we know that: $$\boxed{\vec{K}_{bound}(r=R,\theta) \equiv \vec{M}\times\hat{r}\big|_{r=R} = M_o\hat{z}\times\hat{r}\big|_{r=R} = -M_o\sin\theta\left(\hat{\theta}\times\hat{r}\right)}$$

<u>Since</u>: $\boxed{\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta}$ and $\boxed{\hat{r}\times\hat{\theta} = \hat{\varphi}}$ $\Rightarrow$ $\boxed{\hat{\theta}\times\hat{r} = -\hat{\varphi}}$

<u>Thus</u>: $\boxed{\vec{K}_{bound}(r=R,\theta) = +M_o\sin\theta\hat{\varphi}}$

$$\therefore \boxed{\vec{K}_{bound} \times \hat{r} = \mathrm{M}_o \sin\theta \underbrace{\left(\hat{\varphi} \times \hat{r}\right)}_{=\hat{\theta}} = \mathrm{M}_o \sin\theta\hat{\theta}} \quad \text{with } r \times \theta = \varphi \quad \text{and} \quad \theta \times \varphi = r \quad \text{and} \quad \varphi \times r = \theta$$

$$\therefore \boxed{B_\theta^{out}\left(r = R\right) - B_\theta^{in}\left(r = R\right) = \mu_o \vec{K}_{bound} \times \hat{r}\Big|_{r=R}} \quad \text{Yes!!!}$$

BC 5): $\left(H_r^{out}\left(r = R\right) - H_r^{in}\left(r = R\right)\right) = -\left(\cancel{\mathrm{M}_r^{out}\left(r = R\right)} - \mathrm{M}_r^{in}\left(r = R\right)\right)$

{n.b. originally derived from: $\frac{1}{\mu_o}\int_S \vec{B} \bullet d\vec{a} = 0 = \left(\int_S \vec{H} \bullet d\vec{a} + \int_S \vec{\mathrm{M}} \bullet d\vec{a}\right)$}

<u>Then</u>: $\frac{2}{3}\mathrm{M}_o \cos\theta + \frac{1}{3}\mathrm{M}_o \cos\theta = -\left(0 - \mathrm{M}_o \cos\theta\right)$

<u>But</u>: $\vec{\mathrm{M}} = \mathrm{M}_o \hat{z} = \mathrm{M}_o\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$ for $r < R$

$\therefore$ $\boxed{\mathrm{M}_o \cos\theta = +\mathrm{M}_o \cos\theta}$ Yes!!!

This boundary condition can be rewritten (with $\hat{n}$ = <u>outward</u> unit normal <u>here</u>!) as:

$$\boxed{\left(\vec{H}^{out} \bullet \hat{n} - \vec{H}^{in} \bullet \hat{n}\right)\Big|_{r=R} = -\left(\vec{\mathrm{M}}^{out} \bullet \hat{n} - \vec{\mathrm{M}}^{in} \bullet \hat{n}\right)\Big|_{r=R} = -\sigma_m^{bound}\left(r = R\right)}$$

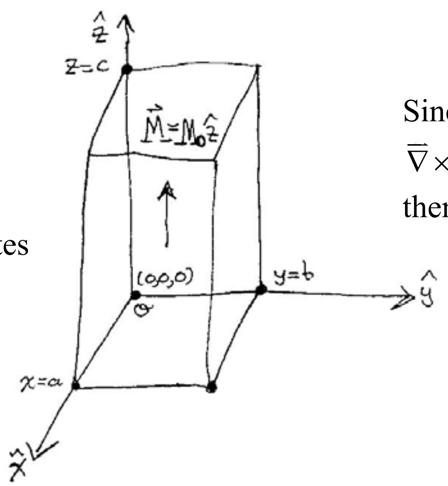
Bound effective surface magnetic charge / magnetic pole density:

$\boxed{\sigma_m^{bound}\left(r = R, \theta\right) = +\mathrm{M}_o \cos\theta \equiv \vec{\mathrm{M}} \bullet \hat{n}}$ (for $\hat{n}$ =radial <u>outward</u> normal unit vector here)

Example 3 – Magnetic Field of Uniformly Magnetized Bar Magnet

Consider a rectangular bar magnet of dimensions $x, y, z = a, b, c$ with uniform magnetization $\vec{M} = M_o \hat{z}$:

Since problem has manifest rectangular symmetry → use rectangular coordinates



Since $\vec{J}_{free} = 0$ everywhere in space, $\vec{\nabla} \times \vec{H} = 0$; and since $\vec{\nabla} \times \vec{M} = 0$ here, then $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{M} = 0$

$\therefore$ We may write $\vec{\nabla} \bullet \vec{H} = -\nabla^2 V_m = 0$

i.e. $\vec{H} = -\vec{\nabla} V_m$

So: $\boxed{\nabla^2 V_m(x,y,z) = 0 \text{ everywhere}}$

Then we need to solve $\nabla^2 V_m = 0$ (Laplace's Equation) for the magnetic scalar potential $V_m(\vec{r}) = V_m(x,y,z)$. In rectangular coordinates as in electrostatics case, try product solution of the form: $V_m(\vec{r}) = V_m(x,y,z) = X(x)Y(y)Z(z)$ i.e. use separation of variables technique)

$$\nabla^2 V_m(x,y,z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V_m(x,y,z) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) X(x)Y(y)Z(z) = 0$$

Give three separated equations:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\alpha^2 \rightarrow \text{general solution} \quad X(x) \sim \cos\alpha x + \sin\alpha x$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta^2 \rightarrow \text{general solution} \quad Y(y) \sim \cos\beta x + \sin\beta x$$

$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = \gamma^2 = \alpha^2 + \beta^2 \rightarrow \text{general solution} \quad Z(z) \sim e^{\gamma z} + e^{-\gamma z} \quad \text{or:} \quad \sim \cos\gamma x + \sin\gamma x$$

Or:

$$\left\{\begin{array}{c} \cos u = \frac{1}{2}\left(e^{iu} + e^{-iu}\right) \\ \sin u = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right) \\ i \equiv \sqrt{-1} \end{array}\right\} \qquad \left\{\begin{array}{c} \cosh u \equiv \frac{1}{2}\left(e^{u} + e^{-u}\right) \\ \sinh u \equiv \frac{1}{2}\left(e^{u} - e^{-u}\right) \end{array}\right\}$$

<u>What are the boundary conditions for this problem?</u>

Because $\vec{M} = M_o\hat{z}$ and from intrinsic geometrical symmetries associated with this problem, <u>and</u> from the fact that we know that we can <u>replace</u> the magnetization $\vec{M} = M_o\hat{z}$ with <u>effective</u> bound magnetic pole strength (magnetic charge) surface charge densities $\sigma_m = -\vec{M}\bullet\hat{n}$ ($\hat{n}$= outward unit normal here) on the top and bottom surfaces. Thus, this problem has many similarities to the electrostatics problem of a six-sided hollow, rectangular <u>conducting</u> box with 5 of its 6 sides at ground, and the top surface at potential $V_m(x,y,z=c) = +V_o$. Recall that only potential <u>differences</u> have physical significance, thus $\Delta V_m = V_m(x,y,z=c)+V_m(x,y,z=0)$ will ultimately need to be tied in with the magnetization $\vec{M} = M_o\hat{z}$.

Thus, on each of the six sides of the rectangular bar magnet, each side (i.e. face) is a <u>magnetic</u> equipotential, and from symmetry of this problem:

$$\text{Dirichlet Boundary Conditions}\quad\left\{\begin{array}{ll} 1)\; V_m^{LHS}(x,0,z) = V_m^{RHS}(x,b,z) = 0 & \\ 2)\; V_m^{back}(0,y,z) = V_m^{front}(a,y,z) = 0 & \\ 3)\; V_m^{bottom}(x,y,0) = 0 & 3')\; V_m^{top}(x,y,c) = +V_o \end{array}\right.$$

BC 0) Of course, $V_m(x,y,z)$ must be finite everywhere.

BC 4) $B_{out}^{\perp} = B_{in}^{\perp}$ at each surface

BC 5) $H_{out}^{\parallel} - H_{in}^{\parallel} = \vec{K}_{free}\times\hat{n}\Big|_{\substack{@each\\surface}} = 0$ (because $\vec{K}_{free} = 0$ here)

i.e. 5) $H_{out}^{\parallel} - H_{in}^{\parallel}$ at each surface

BC 6) $\left[H_{out}^{\perp} - H_{in}^{\perp}\right] = -\left[M_{out}^{\perp} - M_{in}^{\perp}\right] = {}^{0 \text{ on four sides}}_{\pm\sigma_m \text{on top(-) and bottom(+), respectively}}$

BC 7) $B_{out}^{\parallel} - B_{in}^{\parallel} = \mu_o\vec{K}_{TOT}\times\hat{n}\Big|_{\substack{@each\\surface}} = \mu_o\vec{K}_{bound}\times\hat{n}\Big|_{\substack{@each\\surface}}$ (because $\vec{K}_{free} = 0$ here)

<u>Inside the rectangular bar magnet:</u>

- The Dirichlet Boundary Conditions 1): $V_m^{LHS}(x,0,z) = V_m^{RHS}(x,b,z) = 0$ on y require $\sin\beta y$ solutions, with: $\sin\beta b = 0$ or: $\beta b = n\pi$, n = 1, 2, 3, . . . (i.e. $\beta n = \dfrac{n\pi}{b}$, n = 1, 2, 3, . .)

n.b. n = 0 and m = 0 solutions not allowed because then $V_m(x,y,z) = 0$ everywhere.

- The Dirichlet Boundary conditions 2): $V_m^{back}(0,y,z) = V_m^{front}(a,y,z) = 0$ on x require $\sin\alpha x$ solutions, with: $\sin\alpha a = 0$ or: $\alpha a = m\pi$, m = 1, 2, 3, . . . (i.e. $\alpha m = \dfrac{m\pi}{a}$, m = 1, 2, 3, . . .)

- The Dirichlet Boundary conditions 3): $V_m^{bottom}(x,y,0) = 0$ and 4): $V_m^{top}(x,y,c) = +V_o$ require $\sinh\gamma z$ solutions, with:

$$\gamma_{m,n} \equiv \sqrt{\alpha_m^2 + \beta_n^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

m = 1, 2, 3, . . . and n = 1, 2, 3, . . .

∴ Inside the rectangular bar magnet, the general solution for the magnetic scalar potential is of the form:

$$V_m^{in}(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}^{in} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\,z\right)$$

At $z = c$ we must have (BC 4)):

$$V_m^{in}(x,y,c) = +V_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}^{in} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\,c\right)$$

- Now, take inner products (i.e. use orthogonality properties of $\sin\alpha px$ and $\sin\beta qy$) to "project out" the p,q^{th} term (i.e. coefficient A_{pq} where p,q = 1, 2, 3, . . .)

→ Multiply both sides of above expression (BC 4) and $z = c$) by $\sin\left(\dfrac{p\pi x}{a}\right)\sin\left(\dfrac{q\pi y}{b}\right)$ and then integrate over $\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} dy$:

$$\int_{x=0}^{x=a}\int_{y=0}^{y=b} V_m^{in}(x,y,c)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy = +V_o\int_{x=0}^{x=a}\int_{y=0}^{y=b}\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}^{in}\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\,c\right)\int_{x=0}^{x=a}\int_{y=0}^{y=b}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi x}{a}\right)dxdy$$

Now define:

$$u \equiv \left(\frac{\pi}{a}\right)x \qquad v \equiv \left(\frac{\pi}{b}\right)y \qquad \rightarrow \qquad x = \left(\frac{a}{\pi}\right)u \qquad y = \left(\frac{b}{\pi}\right)v$$

$$du = \left(\frac{\pi}{a}\right)dx \qquad dv = \left(\frac{\pi}{b}\right)dy \qquad \rightarrow \qquad dx = \left(\frac{a}{\pi}\right)du \qquad dy = \left(\frac{b}{\pi}\right)dv$$

<u>When</u>: $\begin{pmatrix} x = 0 \rightarrow u = 0 \\ x = a \rightarrow u = \pi \end{pmatrix}$ <u>when</u>: $\begin{pmatrix} y = 0 \rightarrow v = 0 \\ y = b \rightarrow v = \pi \end{pmatrix}$

Then: $\int_{x=0}^{x=a} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx \Rightarrow \frac{a}{\pi}\int_{u=0}^{u=\pi} \sin(mu)\sin(pu)\,du = \frac{a}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2}\right)\delta_{m,p} = \frac{a}{2}\delta_{m,p}$

$\int_{y=0}^{y=b} \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi y}{b}\right)dy \Rightarrow \frac{b}{\pi}\int_{v=0}^{v=\pi} \sin(nv)\sin(qv)\,dv = \frac{b}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2}\right)\delta_{n,q} = \frac{b}{2}\delta_{n,q}$

Where: $\delta_{i,j}$ = <u>Kroenecker δ-function</u> = $\begin{Bmatrix} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{Bmatrix}$ $i, j = 1, 2, 3, \ldots$

Then: $+V_o\int_{x=0}^{x=a}\int_{y=0}^{y=b} \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy = A_{p,q}^{in}\sinh\left(\sqrt{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2}\,c\right)\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)$

Again, let: $u \equiv \left(\frac{p\pi}{a}\right)x \qquad v \equiv \left(\frac{q\pi}{b}\right)y \quad \rightarrow \quad x = \left(\frac{a}{p\pi}\right)u \qquad y = \left(\frac{b}{q\pi}\right)v$

$du = \left(\frac{p\pi}{a}\right)dx \qquad dv = \left(\frac{q\pi}{b}\right)dy \quad \rightarrow \quad dx = \left(\frac{a}{p\pi}\right)du \qquad dy = \left(\frac{b}{q\pi}\right)dv$

When: $\begin{pmatrix} x = 0 \rightarrow u = 0 \\ x = a \rightarrow u = p\pi \end{pmatrix}$ <u>when</u>: $\begin{pmatrix} y = 0 \rightarrow v = 0 \\ y = b \rightarrow v = q\pi \end{pmatrix}$

Then: $\left(\frac{b}{q\pi}\right)\left(\frac{a}{p\pi}\right)V_o\int_{u=0}^{u=p\pi}\int_{v=0}^{v=q\pi}\sin(u)\sin(v)\,dudv$

$= +V_o\left(\frac{a}{p\pi}\right)\left(\frac{b}{q\pi}\right)\left[\cos u\right]\Big|_0^{p\pi}\left[\cos v\right]\Big|_0^{q\pi}$

$= +V_o\left(\frac{a}{p\pi}\right)\left(\frac{b}{q\pi}\right)\left[\cos(p\pi)-1\right]\left[\cos(q\pi)-1\right]$ with: $p = 1, 2, 3, 4, \ldots$

$q = 1, 2, 3, 4, \ldots$

when p or q = <u>odd</u> integer (1, 3, 5, . . .):

$\cos p_{odd}\pi = \cos q_{odd}\pi = -1 \Rightarrow$ above expression $= +V_o\left(\frac{2a}{p_{odd}\pi}\right)\left(\frac{2b}{q_{odd}\pi}\right)$

But when p or q = <u>even</u> integer (2, 4, 6, . . .):

$\cos p_{even}\pi = \cos q_{even}\pi = +1 \Rightarrow$ above expression <u>vanishes</u> for either p = even integer <u>or</u> q = even integer!!

$\therefore$ Only odd integer values of p and q give non-zero values for above expression (due to manifest symmetry of problem in x and y directions!!)

$$\therefore +V_o\left(\frac{2a}{p_{odd}\pi}\right)\left(\frac{2b}{q_{odd}\pi}\right)=A^{in}_{p_{odd},q_{odd}}\sinh\left(\sqrt{\left(\frac{p_{odd}\pi}{a}\right)^2+\left(\frac{q_{odd}\pi}{b}\right)^2}\,c\right)\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)$$

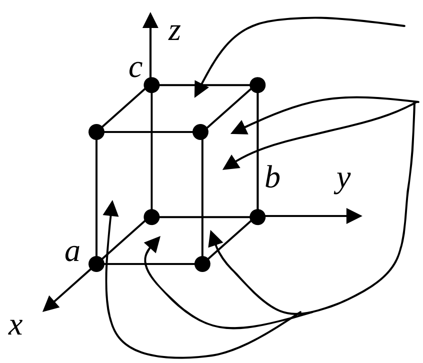
Or: $$A^{in}_{p_{odd},q_{odd}}=+V_o\left(\frac{4}{p_{odd}\pi}\right)\left(\frac{4}{q_{odd}\pi}\right)\frac{1}{\sinh\left(\sqrt{\left(\frac{p_{odd}\pi}{a}\right)^2+\left(\frac{q_{odd}\pi}{b}\right)^2}\,c\right)}\qquad \begin{matrix}p_{odd}=1,3,5,...\\ q_{odd}=1,3,5,...\end{matrix}$$

Therefore, inside the rectangular bar magnet, the specific solution for the magnetic scalar potential is of the form:

$$V^{in}_m(x,y,z)=\sum_{\substack{m=\text{odd}\\ \text{integers}}}^{\infty}\ \sum_{\substack{n=\text{odd}\\ \text{integers}}}^{\infty}\mathrm{A}^{in}_{m,n}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\ z\right)$$

With: $$A^{in}_{m,n}\equiv+V_o\left(\frac{4}{m\pi}\right)\left(\frac{4}{n\pi}\right)\frac{1}{\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\ \mathrm{c}\right)}\qquad \begin{pmatrix}m=1,3,5,7,...\\ n=1,3,5,7,...\end{pmatrix}$$

Physically, these terms represent the 3-D spatial Fourier Harmonic Amplitudes associated with a 3-D rectangular "wave" – i.e. a 3-D rectangular box potential (here) an infinite series of such terms is required in order to properly mathematically define the abrupt / sharp edges of this object (in 3-D):



$+V_o$ on top

$V = 0$ everywhere else on remaining 5 sides (LHS, RHS, front, back and bottom)

Outside the rectangular bar magnet, we require solutions which either vanish or constant value (at least) when $x \to \pm\infty$, $y \to \pm\infty$ and / or when $z \to \pm\infty$, i.e. when an observer is infinitely far away from the bar magnet, because for either $V^{out}_m(\vec{r})$ = constant or ? when $\vec{r}\to\infty$, since $\vec{H}^{out}(\vec{r})\equiv-\vec{\nabla}V^{out}_m(\vec{r})$, then $\vec{H}^{out}(\vec{r})\to 0$ when $\vec{r}\to\infty$ (hence $\vec{B}^{out}(\vec{r})=\mu_o\vec{H}^{out}(\vec{r})\to 0$ when $\vec{r}\to\infty$.

However, we <u>also</u> require <u>continuity</u> of the magnetic scalar potential at / on <u>each</u> of the six sides of the rectangular bar magnet, i.e.:

Back and Front surfaces:	$V_m^{out}(0,y,z)=V_m^{in}(0,y,z)=0$	$V_m^{out}(a,y,z)=V_m^{in}(a,y,z)=0$
LHS and RHS surfaces:	$V_m^{out}(x,0,z)=V_m^{in}(x,0,z)=0$	$V_m^{out}(x,b,z)=V_m^{in}(x,b,z)=0$
Bottom and Top surfaces:	$V_m^{out}(x,y,0)=V_m^{in}(x,y,0)=0$	$V_m^{out}(x,y,c)=V_m^{in}(x,y,c)=?$

The "natural" choice for general form solutions to $V_m^{out}(\vec{r})$ would be e.g. $e^{\pm kx}$ however we <u>cannot</u> choose such exponential type solutions for <u>all of</u> *x* and *y* and *z* because of the constraint $\gamma^2=\alpha^2+\beta^2$ - i.e. at least one solution in <u>*x*</u> or <u>*y*</u> or <u>*z*</u> <u>must</u> be oscillatory (i.e. sine or cosine), because of this constraint.

Let us re-examine $\nabla^2 V_m^{out}(x,y,z)=0$ again. We still want product-type solutions of the form $V_m^{out}(x,y,z)=X^{out}(x)Y^{out}(y)Z^{out}(z)$ with:

$$\left.\begin{aligned}\frac{d^2X^{out}(x)}{dx^2}&=-A^2X^{out}(x)\\ \frac{d^2Y^{out}(y)}{dy^2}&=-B^2Y^{out}(y)\\ \frac{d^2Z^{out}(z)}{dz^2}&=+C^2Z^{out}(z)\end{aligned}\right\}$$

$$\left(\underbrace{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}\right)V_m^{out}(x,y,z)=0$$

and with: $C^2=A^2+B^2$ $\qquad \left(\overbrace{-A^2-B^2+C^2=0}\right)$

However, <u>here</u> we will define:

$$A_1^2\equiv(\alpha+i\beta)^2=\alpha^2+2i\alpha\beta-\beta^2\Rightarrow -A^2=\left[i(\alpha+i\beta)\right]^2=-(\alpha+i\beta)^2$$
$$B_1^2\equiv(\gamma+i\delta)^2=\gamma^2 2i\gamma\delta-\delta^2\Rightarrow -B^2=\left[i(\gamma+i\delta)\right]^2=-(\gamma+i\delta)^2$$
$$C_1^2\equiv(\mu+i\nu)^2=\mu^2+2i\mu\nu-\nu^2$$

With: $\quad C_1^2=A_1^2+B_1^2\Rightarrow(\alpha^2-\beta^2)+(\gamma^2-\delta^2)=(\mu^2-\nu^2)$

And: $\quad \alpha\beta+\gamma\delta=\mu\nu$

Solutions are then of the form:

$$X(x)\sim e^{i(\alpha+i\beta)x}=e^{(i\alpha-\beta)x}$$
$$Y(y)\sim e^{i(\gamma+i\delta)y}=e^{(i\gamma-\delta)y}$$
$$Z(z)\sim e^{(\mu+i\nu)z}=e^{(\mu+i\nu)z}$$

$$\frac{dX(x)}{dx} = i(\alpha + i\beta)e^{(i\alpha-\beta)x} \qquad \frac{dY(y)}{dy} = i(\gamma + i\delta)e^{i(\gamma+i\delta)y} \qquad \frac{dZ(z)}{dz} = (\mu + iv)e^{(\mu+iv)}$$

$$\frac{d^2X(x)}{dx^2} = -(\alpha+\beta)^2 e^{i(\alpha-i\beta)x} \qquad \frac{d^2Y(y)}{dy^2} = -(\gamma+i\delta)^2 e^{i(\gamma+i\delta)y} \qquad \frac{d^2Z(z)}{dz^2} = (\mu+iv)^2 e^{(\mu+iv)}$$

Or: $$\frac{d^2X(x)}{dx^2} = -A_1^2 X(x) \qquad \frac{d^2Y(y)}{dy^2} = -B_1^2 Y(y) \qquad \frac{d^2Z(z)}{dz^2} = +C_1^2 Z(z)$$

However, most / more generally there are actually four possible acceptable relations for each of A, B and C (simply changing ± signs):

$A_1^2 \equiv (\alpha + i\beta)^2$	with	$B_1^2 \equiv (\gamma + i\delta)^2$	and with	$C_1 \equiv (\mu + iv)^2$
$A_2 \equiv (-\alpha + i\beta)^2$	with	$B_2^1 \equiv (-\gamma + i\delta)^2$	and with	$C_2 \equiv (-\mu + iv)^2$
$A_3 \equiv (\alpha - i\beta)^2$	with	$B_3 \equiv (\gamma - i\delta)^2$	and with	$C_3 \equiv (\mu - iv)^2$
$A_4 \equiv (-\alpha - i\beta)^2$	with	$B_4 \equiv (-\gamma - i\delta)^2$	and with	$C_4 \equiv (-\mu - iv)^2$

With: $C_1^2 = A_1^2 + B_1^2 \qquad C_2^2 = A_2^2 + B_2^2 \qquad C_3^2 = A_3^2 + B_3^2 \qquad C_4^2 = A_4^2 + B_4^2$

$$(\alpha^2 - \beta^2) + (\gamma^2 - \delta^2) = (\mu^2 - v^2) \qquad (\alpha^2 - \beta^2) + (\gamma^2 - \delta^2) = (\mu^2 - v^2)$$

$$\alpha\beta + \gamma\delta = \mu v \qquad \alpha\beta + \gamma\delta = \mu v$$

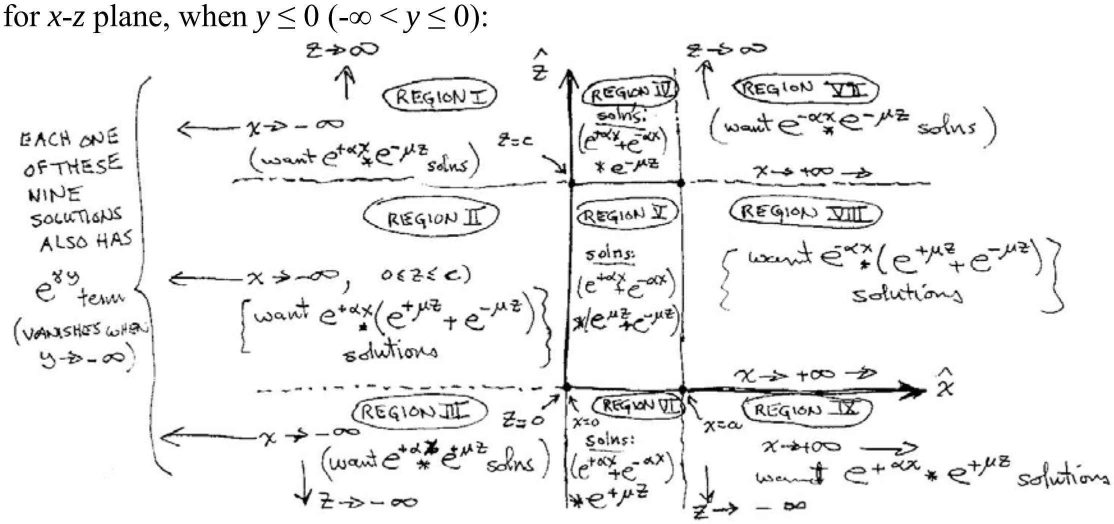
n.b all relations the same for i = 1, 2, 3, 4

Thus, the most general solution for $V_m^{out}(x,y,z)$ will be of the form:

$$V_m^{out}(x,y,z) = Ke^{\pm i(\alpha \pm i\beta)x} e^{\pm i(\gamma \pm i\delta)y} e^{\pm(u \pm iv)z}$$

$$= Ke^{\pm(i\alpha \pm \beta)x} e^{\pm(i\gamma \pm \delta)y} e^{\pm(u \pm iv)z} \qquad K = \text{constant}$$

For each variable / in each direction x, y and z, we will have to match nine separate solutions, e.g. for x-z plane, when $y \le 0$ ($-\infty < y \le 0$):



Then this also must be completed / repeated for $a \le y \le b$ and completely repeated again for $y \ge b$ region. This gives a total of 27 – 1 separate solutions for $V_m^{out}(x,y,z)$, one for each of 9 x 3 = 27 – 1 = 26 regions each with unknown coefficients, and in general, we will again require infinite odd-integer series solutions, once we start matching $V_m^{out}(x,y,z) = V_m^{m}(x,y,z)$ at surfaces / boundaries of rectangular bar magnet. Lots of equations / constraints to simultaneously solve!! Doable, but with much, much work!!

Assuming we succeeded in uniquely and correctly determining the solution(s) $V_m^{out}(x,y,z)$ in all 26 regions exterior to the bar magnet, we would then e.g. apply BC 4) $B_{out}^{\perp} = B_{in}^{\perp}$ at each surface and / or BC 5) $H_{out}^{\parallel} = H_{in}^{\parallel}$ at each surface to then formally connect $V_m^{out}(x,y,z)$ solution(s) to $V_m^{in}(x,y,z)$.

Even though we do not explicitly have solution(s) for $V_m^{out}(x,y,z)$, we can still easily determine the fields inside the rectangular bar magnet, because $\vec{H}^{in}(x,y,z) \equiv -\vec{\nabla} V_m^{in}(x,y,z)$ and $V_m^{in}(x,y,z)$ is explicitly known.

$$\vec{H}^{in}(x,y,z) \equiv -\vec{\nabla} V_m^{in}(x,y,z) = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) V_m^{in}(x,y,z)$$

With: $$V_m^{in}(x,y,z) = \sum_{\text{odd } m}^{\infty} \sum_{\text{odd } n}^{\infty} A_{m,n}^{in} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

And with: $$A_{m,n}^{in} = +V_o \left(\frac{4}{m\pi}\right)\left(\frac{4}{n\pi}\right) \frac{1}{\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; c\right)} \qquad \vec{H} \equiv H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$$

$$\therefore H_x^{in}(x,y,z) = -\sum_{\text{odd } m}^{\infty} \sum_{\text{odd } n}^{\infty} \left(\frac{m\pi}{a}\right) A_{m,n}^{in} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

$$H_y^{in}(x,y,z) = -\sum_{\text{odd } m}^{\infty} \sum_{\text{odd } n}^{\infty} \left(\frac{n\pi}{b}\right) A_{m,n}^{in} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

$$H_z^{in}(x,y,z) = -\sum_{\text{odd } m}^{\infty} \sum_{\text{odd } n}^{\infty} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, A_{m,n}^{in} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cosh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

** Note that H_z^{in} is anti-parallel to magnetization $\vec{M} = M_o \hat{z}$

Then since: $\vec{H}^{in} = \frac{1}{\mu_o}\vec{B}^{in} - \vec{M} \Rightarrow \vec{B}^{in} = \mu_o\left(\vec{H}^{in} + \vec{M}\right)$ $\quad\quad \vec{M} = M_o\hat{z} = M_z\hat{z}$

Then: $\vec{B}^{in} = B_x^{in}\hat{x} + B_y^{in}\hat{y} + B_z^{in}\hat{z} = \mu_o\left(H_x^{in} + \overset{=0}{\cancel{M_x}}\right)\hat{x} + \mu_o\left(H_y^{in} + \overset{=0}{\cancel{M_y}}\right)\hat{y} + \mu_o\left(H_z^{in} + M_z\right)\hat{z}$

$$= \mu_o H_x^{in}\hat{x} + \mu_o H_y^{in}\hat{y} + \mu_o\left(H_z^{in} + M_o\right)\hat{z}$$

$$\therefore B_x^{in}(x,y,z) = \mu_o H_x^{in}(x,y,z) = -\mu_o \sum_{\text{odd } m}^{\infty}\sum_{\text{odd } n}^{\infty}\left(\frac{m\pi}{a}\right)A_{m,n}^{in}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

$$B_y^{in}(x,y,z) = \mu_o H_y^{in}(x,y,z) = -\mu_o \sum_{\text{odd } m}^{\infty}\sum_{\text{odd } n}^{\infty}\left(\frac{n\pi}{b}\right)A_{m,n}^{in}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right)$$

$$B_z^{in}(x,y,z) = \mu_o\left(H_z^{in}(x,y,z) + M_o\right)$$

$$= -\mu_o \sum_{\text{odd } m}^{\infty}\sum_{\text{odd } n}^{\infty}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\,A_{m,n}^{in}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cosh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right) + \mu_o M_o$$

With: $$A_{m,n}^{in} = +V_o\left(\frac{4}{m\pi}\right)\left(\frac{4}{n\pi}\right)\frac{1}{\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; c\right)}$$

$$\left.\begin{array}{l}\text{Now we can use}\\ \text{BC 6) to connect}\\ \text{to connect } V_o \text{ to } M_o\end{array}\right\} \left(H_{out}^{\perp} - H_{in}^{\perp}\right) = -\left(\overset{=0}{\cancel{M_{out}^{\perp}}} - M_{in}^{\perp}\right) = \left\{\begin{array}{l}0 \text{ on 4 sides}\\ +\sigma_m \text{ on bottom surface } (@\, z=0)\\ -\sigma_m \text{ on top surface } (@\, z=0)\end{array}\right.$$

i.e.

$$\left.\begin{array}{ll}@\, z=0: & \left(H_z^{out} - H_z^{in}\right)\big|_{z=0} = M_o = +\sigma_m\\ @\, z=c: & \left(H_z^{out} - H_z^{in}\right)\big|_{z=c} = -M_o = -\sigma_m\end{array}\right\}$$

$\pm\sigma_m$ = bound magnetic surface charge densities ("pole strength" surface charge densities)

Obviously, we need to explicitly solve $H_z^{out}(x,y,z)$ <u>first</u> in order to carry this out . . .
However, we can also turn this around, so that:

$$H_z^{out}\big|_{z=0} = H_z^{in}\big|_{z=0} + M_o \qquad\qquad H_z^{out}\big|_{z=c} = H_z^{in}\big|_{z=c} - M_o$$

From symmetry arguments, we also know that: $H_z^{out}\big|_{z=0} = -H_z^{out}\big|_{z=c}$

More generally: $\vec{H}^{out}(x,y,z \le 0) = -\vec{H}^{out}(x,y,z \ge c)$

Δz by same amounts

If we go back to BC 3'):

$$V_m^{in}\left(x,y,z=c\right)=+V_o=\sum_{m=odd}^{\infty}\sum_{n=odd}^{\infty}A_{m,n}^{in}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\;c\right)$$

With: $$A_{m,n}^{in}=+V_o\left(\frac{4}{m\pi}\right)\left(\frac{4}{n\pi}\right)\frac{1}{\sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\;c\right)}$$

Due to orthonormality properties of sine functions → we realize that:

$$1=\sum_{m=odd}^{\infty}\sum_{n=odd}^{\infty}\left(\frac{4}{m\pi}\right)\left(\frac{4}{n\pi}\right)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)=1*1$$

Because: $$1=\sum_{m=odd}^{\infty}\left(\frac{4}{m\pi}\right)\sin\left(\frac{m\pi}{a}x\right)\quad \text{and} \quad 1=\sum_{n=odd}^{\infty}\left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{b}y\right)$$