# LECTURE NOTES 20

## Static Magnetic Fields in Matter II

We summarize here the relations that we have obtained thus far for the macroscopic magnetic vector potential $\vec{A}(\vec{r})$ and the magnetic field $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ associated with magnetic materials in terms of steady filamentary/line, surface and volume free and bound current densities, and with the two Maxwell's equations:

(1) No magnetic charges: $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$

(2) Ampere's Law: $\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_o \vec{J}_{TOT}(\vec{r}) = \mu_o \vec{J}_{free}(\vec{r}) + \mu_o \vec{J}_{Bound}(\vec{r})$.

Using the Principle of Linear Superposition:

| | | |
|---|---|---|
| Filamenary/Line Currents: | $I_{TOT} = I_{free} + I_{bound}$ | (Amps) |
| Surface Current Densities: | $\vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{Bound}$ | (Amps/m) $I = \int_{C_\perp} \vec{K} \cdot \hat{n}_\perp d\ell_\perp$ |
| Volume Current Densities: | $\vec{J}_{TOT} = \vec{J}_{free} + \vec{J}_{Bound}$ | (Amps/m$^2$) $I = \int_{S_\perp} \vec{J} \cdot d\vec{a}$ $d\vec{a} = \hat{n} da$ |

| | | |
|---|---|---|
| Filamentary/Line Currents: | $\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{C'} \frac{\vec{I}_{TOT}(\vec{r}')d\ell'}{\mathscr{r}}$ $= \left(\frac{\mu_o}{4\pi}\right) I_{TOT} \int_{C'} \frac{d\vec{\ell}'(\vec{r}')}{\mathscr{r}}$ (If $I_{TOT} \neq \text{fcn}(\vec{r}')$) | $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{C'} \frac{\vec{I}_{TOT}(\vec{r}')d\ell' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ $= \left(\frac{\mu_o}{4\pi}\right) I_{TOT} \int_{C'} \frac{d\vec{\ell}'(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ (If $I_{TOT} \neq \text{fcn}(\vec{r}')$) |
| Surface Current Densities: | $\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{S'} \frac{\vec{K}_{TOT}(\vec{r}')}{\mathscr{r}} da'$ | $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{S'} \frac{\vec{K}_{TOT}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da'$ |
| Volume Current Densities: | $\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{v'} \frac{\vec{J}_{TOT}(\vec{r}')}{\mathscr{r}} d\tau'$ | $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{v'} \frac{\vec{J}_{TOT}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$ |

SI Units of $\vec{B} =$ Tesla = Newtons/(Ampere-meter) and since $\vec{B} = \vec{\nabla} \times \vec{A}$, then:
SI Units of $\vec{A} =$ Tesla-meters = Newtons/Ampere (= force/unit current).

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**The Magnetic Field $\vec{B}(\vec{r})$ Inside a Magnetized Material with Macroscopic Magnetization (Magnetic Dipole Moment Per Unit Volume) $\vec{M}(\vec{r})$**

In the previous P435 Lecture Notes 19 (page 8), we derived the magnetic vector potential $\vec{A}(\vec{r})$ associated with a magnetized material with macroscopic magnetization (*a.k.a.* magnetic dipole moment per unit volume) $\vec{M}(\vec{r})$:

$$\boxed{\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \frac{\vec{M}(\vec{r}')\times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\, d\tau'}$$

The corresponding magnetic field $\vec{B}(\vec{r})$ associated with this magnetic vector potential $\vec{A}(\vec{r})$ is:

$$\boxed{\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{\nabla}\times\left(\frac{\vec{M}(\vec{r}')\times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right) d\tau' = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{\nabla}\times\left(\vec{M}(\vec{r}')\times\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right) d\tau'}$$

Using Griffiths Product Rule # 8: $\boxed{\vec{\nabla}\times\vec{A}\times\vec{B} = \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{B}\left(\vec{\nabla}\cdot\vec{A}\right) + \vec{B}\cdot\left(\vec{\nabla}\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\vec{B}\right)}$

<u>where</u>: $\vec{A} \equiv \vec{M}(\vec{r}')$   (<u>Note</u>: $\vec{M}(\vec{r}') \neq \text{fcn}(\vec{r})$, it is = fcr$(\vec{r}')$ only !!!)

<u>and</u>: $\vec{B} \equiv \frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} = \frac{\vec{\boldsymbol{\imath}}}{\boldsymbol{\imath}^3}\ \left(= \frac{\boldsymbol{\imath}\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^3}\right)$ <u>with</u>: $\vec{\boldsymbol{\imath}} = \vec{r} - \vec{r}'$ <u>and</u>: $\boldsymbol{\imath} = |\vec{r} - \vec{r}'|$

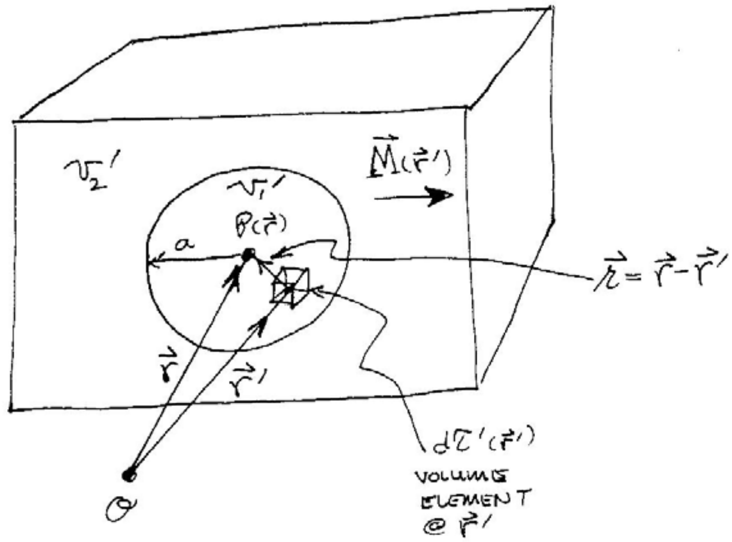
<u>then both</u>: $\left\{\begin{array}{c}\vec{\nabla}\cdot\vec{A} = 0\\ \vec{\nabla}\vec{A} = 0\end{array}\right\}$ <u>because</u>: $\left\{\begin{array}{c}\vec{\nabla}\cdot\vec{M}(\vec{r}') = 0\\ \vec{\nabla}\vec{M}(\vec{r}') = 0\end{array}\right\}$ <u>since</u>: $\left\{\begin{array}{l}\vec{\nabla} = \text{fcn }(\vec{r})\text{ only, and}\\ \vec{M}(\vec{r}') = \text{fcn }(\vec{r}')\text{ only}\end{array}\right\}$

<u>So therefore *here*</u>: $\boxed{\vec{\nabla}\times\vec{A}\times\vec{B} = \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{B}\left(\cancel{\vec{\nabla}\cdot\vec{A}}\right) + \vec{B}\cdot\left(\cancel{\vec{\nabla}\vec{A}}\right) - \vec{A}\cdot\left(\vec{\nabla}\vec{B}\right) = \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{A}\cdot\left(\vec{\nabla}\vec{B}\right)}$

<u>Then</u>: $$\boxed{\vec{\nabla}\times\left(\vec{M}(\vec{r}')\times\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right) = \vec{M}(\vec{r}')\left[\vec{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right] - \left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right)}$$

<u>Thus</u>: $$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{M}(\vec{r}')\left[\vec{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right] d\tau' - \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\right)\right) d\tau'}$$

Let us now divide the volume $v'$ of the magnetized magnetic material into two regions
A very small spherical region $v_1'$ of radius $a$, and a surrounding region of volume $v_2'$.
Then $v' = v_1' + v_2'$. We locate the observation / field point $P(\vec{r})$ at the center of spherical volume $v_1'$ as shown in figure below:




Then:

$$\begin{aligned}
\vec{B}^{inside}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' - \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)d\tau' \\
&\overset{(1)}{=} \left(\frac{\mu_o}{4\pi}\right)\int_{v_1'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' \overset{(2)}{-} \left(\frac{\mu_o}{4\pi}\right)\int_{v_1'}\left(\vec{M}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)d\tau' \\
&\overset{(3)}{+} \left(\frac{\mu_o}{4\pi}\right)\int_{v_2'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' \overset{(4)}{-} \left(\frac{\mu_o}{4\pi}\right)\int_{v_2'}\left(\vec{M}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)d\tau'
\end{aligned}$$

Recall that: $$\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{\nabla}\bullet\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = \vec{\nabla}\bullet\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}'\bullet\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}'\bullet\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = -\vec{\nabla}'\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

Or: $$\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -\vec{\nabla}'\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

Then terms (1) and (3) become:

$$\left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = \overset{(1)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_1'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' + \overset{(3)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_2'} \vec{M}(\vec{r}')\left[\vec{\nabla}\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'$$

$$= -\left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{M}(\vec{r}')\left[\vec{\nabla}'\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = -\overset{(1)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_1'} \vec{M}(\vec{r}')\left[\vec{\nabla}'\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' - \overset{(3)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_2'} \vec{M}(\vec{r}')\left[\vec{\nabla}'\bullet\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'$$

Now note that:

$$\int_{v'}\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = \int_{v'}\left[\vec{\nabla}'\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]d\tau' = -\int_{S'}\left[\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\cdot\hat{n}\right]da' = -\int_{S'}\left[\frac{(\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}\,'|^2}\cdot\hat{n}\right]da' = -\int_{S'}\left[\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\cdot\hat{n}\right]da'$$

$$= -\int_{S'}\left(\frac{\hat{\mathscr{r}}\cdot\hat{n}}{\mathscr{r}^2}\right)da' = -\int_{\Omega'} d\Omega' \Leftarrow \text{ solid } \measuredangle \text{ looking inside!!!}$$

Recall also that: $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$, i.e. $\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -4\pi\delta^3(\vec{\mathscr{r}})$ thus:

$$\int_{v'}\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = \int_{v'}\left[\vec{\nabla}'\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]d\tau' = -4\pi\int_{v'}\delta^3(\vec{\mathscr{r}})\,d\tau' = -\int_{\Omega'} d\Omega' = \begin{cases} -4\pi & \text{if } \vec{r} \text{ in volume } v' \\ 0 & \text{if } \vec{r} \textit{ not} \text{ in volume } v' \end{cases}$$

Similarly, we also see that for an arbitrary <u>scalar</u> point function $f(\vec{r}\,')$ that:

$$\int_{v'} f(\vec{r}\,')\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = -4\pi\int_{v'} f(\vec{r}\,')\delta^3(\vec{\mathscr{r}})\,d\tau' = \begin{cases} -4\pi f(\vec{r}) & \text{if } \vec{r} \textit{ is} \text{ in volume } v' \\ 0 & \text{if } \vec{r} \textit{ not} \text{ in volume } v' \end{cases}$$

$$= -\int_{S'} f(\vec{r}\,')\left(\frac{\hat{\mathscr{r}}\cdot\hat{n}}{\mathscr{r}^2}\right)da' = -\int_{\Omega'} f(\vec{r}\,')\,d\Omega'$$

Likewise, we also see that for an arbitrary <u>vector</u> point function $\vec{A}(\vec{r}\,')$ that:

$$\int_{v'} \vec{A}(\vec{r}\,')\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = -4\pi\int_{v'} \vec{A}(\vec{r}\,')\delta^3(\vec{\mathscr{r}})\,d\tau' = \begin{cases} -4\pi \vec{A}(\vec{r}) & \text{if } \vec{r} \textit{ is} \text{ in volume } v' \\ 0 & \text{if } \vec{r} \textit{ not} \text{ in volume } v' \end{cases}$$

$$= -\int_{S'} \vec{A}(\vec{r}\,')\left(\frac{\hat{\mathscr{r}}\cdot\hat{n}}{\mathscr{r}^2}\right)da' = -\int_{\Omega'} \vec{A}(\vec{r}\,')\,d\Omega'$$

However, note that the observation/field point $P(\vec{r})$, located at $\vec{r}$ is <u>*not*</u> contained within the volume $v_2'$ - it is located at the <u>center</u> of spherical volume $v_1'$ (see/refer to picture on page 3)!!

Thus, we see that term (3) vanishes, i.e. $\overset{(3)}{\left(\frac{\mu_o}{4\pi}\right)\int_{v_2'} \vec{\mathrm{M}}(\vec{r}\,')\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'} = 0$ and that term (1)

$$\left(\frac{\mu_o}{4\pi}\right)\int_{v'} \vec{\mathrm{M}}(\vec{r}\,')\left[\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau' = -\overset{(1)}{\left(\frac{\mu_o}{4\pi}\right)\int_{v_1'} \vec{\mathrm{M}}(\vec{r}\,')\left[\vec{\nabla}'\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'} = \left(\frac{\mu_o}{\cancel{4\pi}}\right)\cancel{4\pi}\,\vec{\mathrm{M}}(\vec{r}) = \mu_o\vec{\mathrm{M}}(\vec{r})$$

What about terms (2) and (4) ??

$$-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau' = -\overset{(2)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_1'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau' - \overset{(4)}{\left(\frac{\mu_o}{4\pi}\right)}\int_{v_2'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'$$

Note that: $$\vec{\nabla}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right) = \vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right) + \vec{M}(\vec{r}')\times\left(\overset{=0}{\cancel{\vec{\nabla}\times\left(\frac{\hat{r}}{r^2}\right)}}\right) = \vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)$$

The 2nd term vanishes because: $\vec{\nabla}\times\left(\frac{\hat{r}}{r^2}\right) = -\vec{\nabla}\times\vec{\nabla}\left(\frac{1}{r}\right) = 0$ {since: $\vec{\nabla}\times\vec{\nabla}f = \vec{\nabla}\times\vec{\nabla}\left(\frac{1}{r-r'}\right) \equiv 0$ }

Thus:

$$-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau' = -\left(\frac{\mu_o}{4\pi}\right)\vec{\nabla}\left[\int_{v'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right]$$
$$= -\overset{(2)}{\left(\frac{\mu_o}{4\pi}\right)}\vec{\nabla}\left[\int_{v_1'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right] - \overset{(4)}{\left(\frac{\mu_o}{4\pi}\right)}\vec{\nabla}\left[\int_{v_2'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right]$$

We now define a magnetic *scalar* potential $V_m(\vec{r})$ for use with magnetic materials:

$$V_m(\vec{r}) \equiv \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'$$

Then we see that:

$$-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau' = -\left(\frac{\mu_o}{4\pi}\right)\vec{\nabla}\left[\int_{v'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right] = -\vec{\nabla}V_m(\vec{r})$$
$$= -\overset{(2)}{\left(\frac{\mu_o}{4\pi}\right)}\vec{\nabla}\left[\int_{v_1'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right] - \overset{(4)}{\left(\frac{\mu_o}{4\pi}\right)}\vec{\nabla}\left[\int_{v_2'}\left(\vec{M}(\vec{r}')\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'\right] = -\vec{\nabla}V_m^{(2)[v_1']}(\vec{r}) - \vec{\nabla}V_m^{(4)[v_2']}(\vec{r})$$

Putting this all together now, finally we obtain:

$$\vec{B}^{inside}(\vec{r}) = \underbrace{\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\vec{M}(\vec{r}')\left[\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right]d\tau'}_{=\mu_o\vec{M}(\vec{r})} - \underbrace{\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)\right)d\tau'}_{=\vec{\nabla}V_m(\vec{r})}$$

i.e. $\vec{B}^{inside}(\vec{r}) = \mu_o\vec{M}(\vec{r}) - \vec{\nabla}V_m(\vec{r})$ where $V_m(\vec{r}) \equiv \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}')\cdot\frac{\hat{r}}{r^2}\right)d\tau'$

Note that there are in fact *two* contributions to the magnetic scalar potential – one contribution arises from the integral (2) over the small, spherical region $v_1'$ (containing the "near" magnetic dipoles) , the other contribution arises from the integral (4) over region $v_2'$ (containing the "far" magnetic dipoles):

$$V_m(\vec{r}) = V_m^{(2)[v_1']}(\vec{r}) + V_m^{(4)[v_2']}(\vec{r})$$
$$V_m(\vec{r}) \equiv \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\frac{\hat{\imath}}{\imath^2}\right)d\tau' = \overset{(2)}{}\left(\frac{\mu_o}{4\pi}\right)\int_{v_1'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\left(\frac{\hat{\imath}}{\imath^2}\right)\right)d\tau' + \overset{(4)}{}\left(\frac{\mu_o}{4\pi}\right)\int_{v_2'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\left(\frac{\hat{\imath}}{\imath^2}\right)\right)d\tau'$$

and:

$$-\nabla V_m(\vec{r}) = -\nabla\left[V_m^{(2)[v_1']}(\vec{r}) + V_m^{(4)[v_2']}(\vec{r})\right] = -\nabla V_m^{(2)[v_1']}(\vec{r}) - \nabla V_m^{(4)[v_2']}(\vec{r})$$
$$\equiv -\left(\frac{\mu_o}{4\pi}\right)\nabla\left[\int_{v'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\frac{\hat{\imath}}{\imath^2}\right)\right]d\tau' = -\overset{(2)}{}\left(\frac{\mu_o}{4\pi}\right)\nabla\left[\int_{v_1'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\left(\frac{\hat{\imath}}{\imath^2}\right)\right)d\tau'\right] - \overset{(4)}{}\left(\frac{\mu_o}{4\pi}\right)\nabla\left[\int_{v_2'}\left(\vec{\mathrm{M}}(\vec{r}')\cdot\left(\frac{\hat{\imath}}{\imath^2}\right)\right)d\tau'\right]$$

Note that the SI units for the above-defined magnetic scalar potential $V_m(\vec{r})$ are Tesla-meters (= Newtons/Ampere) which are the same units as that for the magnetic vector potential $\vec{A}(\vec{r})$:

$$\underset{=\left(\frac{N}{A-m}\right)}{\vec{B}(Teslas)} \sim \underset{=\left(\frac{N}{A-m}\right)}{-\vec{\nabla}V_m(Teslas)} \quad\Rightarrow\quad \underset{=\frac{N}{A-\not{m}}\not{m}=\frac{N}{A}}{V_m(Tesla-meters)}$$

Note that *this* magnetic scalar potential, $V_m(\vec{r})$ (SI units = Tesla-meters = Newtons/Ampere) as defined *here* for use with magnetic materials is *different* than the "*other*" magnetic scalar potential $\Phi_m(\vec{r})$ (SI units = Tesla-meters$^2$) that we had previously introduced (and defined) in P435 Lecture Notes 16 (page 2) in the context of gauge transformations and the arbitrary nature of the magnetic vector potential: $\vec{A}'(\vec{r}) \equiv \vec{A}(\vec{r}) + \vec{\mathbb{A}}(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})$ i.e. $\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\Phi_m(\vec{r})$.

## **Use(s) of the Magnetic Scalar Potential** $V_m(\vec{r})$

Surface & Volume Densities of Magnetic Pole Strength (i.e. Bound Magnetic Charge Densities) $\sigma_m(\vec{r})$ and $\rho_m(\vec{r})$

Inside a magnetic material with macroscopic magnetization (magnetic dipole moment per unit volume) $\vec{\mathrm{M}}(\vec{r})$ we have the constitutive relation: $\vec{H}^{inside}(\vec{r}) = \frac{1}{\mu_o}\vec{B}^{inside}(\vec{r}) - \vec{\mathrm{M}}(\vec{r})$

i.e. $\vec{B}^{inside}(\vec{r}) = \mu_o\vec{H}^{inside}(\vec{r}) + \mu_o\vec{\mathrm{M}}(\vec{r})$ or: $\vec{B}^{inside}(\vec{r}) = \mu_o\vec{\mathrm{M}}(\vec{r}) + \mu_o\vec{H}^{inside}(\vec{r})$.

However, we also have just derived: $\vec{B}^{inside}(\vec{r}) = \mu_o\vec{\mathrm{M}}(\vec{r}) \;\; -\vec{\nabla}V_m(\vec{r})$.

Thus we see that $\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}V_m(\vec{r})/\mu_o$.

However, we also need to consider here the previously-derived constraints: $\vec{\nabla}\cdot\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}\cdot\vec{\mathrm{M}}(\vec{r})$, obtained from $\vec{\nabla}\cdot\vec{B}^{inside}(\vec{r}) = 0$, and also that associated with Ampere's law for $\vec{H}^{inside}(\vec{r})$, namely: $\vec{\nabla}\times\vec{H}^{inside}(\vec{r}) = \vec{J}_{free}^{inside}(\vec{r})$.

But if $\boxed{\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}V_m(\vec{r})/\mu_o}$, then: $\boxed{\vec{\nabla}\times\vec{H}^{inside}(\vec{r}) = -\underbrace{\vec{\nabla}\times\vec{\nabla}V_m(\vec{r})}_{\equiv 0\ always!!!}/\mu_o = 0}$

Thus, we see that the (proper) use of the magnetic scalar potential $V_m(\vec{r})$ restricts us to use it only in cases where there are <u>*no*</u> free currents contained <u>anywhere</u> within the source volume $v'$, i.e. we <u>*must*</u> have (i.e. we demand that) $\vec{J}^{inside}_{free}(\vec{r}) = 0$ everywhere within the source volume $v'$, in order to be able to (properly/correctly) use the magnetic scalar potential $V_m(\vec{r})$.

If the condition $\vec{J}^{inside}_{free}(\vec{r}) = 0$ <u>*is*</u> satisfied everywhere within the source volume $v'$, then we also see from $\boxed{\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}V_m(\vec{r})/\mu_o}$, along with the condition $\boxed{\vec{\nabla}\cdot\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}\cdot\vec{M}(\vec{r})}$ that

$$\boxed{\vec{\nabla}\cdot\vec{H}^{inside}(\vec{r}) = -\vec{\nabla}\cdot\vec{\nabla}V_m(\vec{r})/\mu_o = -\nabla^2 V_m(\vec{r})/\mu_o = -\vec{\nabla}\cdot\vec{M}(\vec{r}) = \rho_m(\vec{r})}$$

i.e. we obtain (another kind of) Poisson's equation for <u>*this*</u> magnetic scalar potential $V_m(\vec{r})$ of the form: $\boxed{\nabla^2 V_m(\vec{r}) = \mu_o\vec{\nabla}\cdot\vec{M}(\vec{r}) = -\mu_o\rho_m(\vec{r})}$. Note that this also says that: $\boxed{\rho_m(\vec{r}) = -\vec{\nabla}\cdot\vec{M}(\vec{r})}$.

Again, we (already) know the form of the solutions to this magnetic Poisson's equation:

$$\boxed{V_m(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\rho_m^{Bound}(\vec{r}\,')}{r}d\tau' + \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\sigma_m^{Bound}(\vec{r}\,')}{r}da'}$$

where $\rho_m^{Bound}(\vec{r}\,')$ and $\sigma_m^{Bound}(\vec{r}\,')$ are <u>effective</u> bound volume and surface <u>*magnetic*</u> charge densities contained within the source volume $v'$ and/or on the surface $S'$ that encloses the volume $v'$.

What are the <u>effective</u> bound volume and surface <u>*magnetic*</u> charge densities $\rho_m^{Bound}(\vec{r}\,')$ and $\sigma_m^{Bound}(\vec{r}\,')$???

We have derived (i.e. defined) the magnetic scalar potential $V_m(\vec{r})$ as:

$$\boxed{V_m(\vec{r}) \equiv \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}\,')\cdot\frac{\hat{r}}{r^2}\right)d\tau' = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}\,')\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right)d\tau'} \quad \text{since} \quad \boxed{\frac{\hat{r}}{r^2} = \vec{\nabla}\left(\frac{1}{r}\right)}$$

<u>But</u>: $$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\vec{\nabla}'\left(\frac{1}{r}\right)}$$

<u>Thus</u>:

$$\boxed{V_m(\vec{r}) \equiv \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}\,')\cdot\frac{\hat{r}}{r^2}\right)d\tau' = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}\,')\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right)d\tau' = -\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{r}\right)\right)d\tau'}$$

However, from Griffiths Product Rule # 5: $\vec{\nabla}\cdot\left(f\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right)$

then: $\vec{A}\cdot\left(\vec{\nabla}f\right)=\vec{\nabla}\cdot\left(f\vec{A}\right)-f\left(\vec{\nabla}\cdot\vec{A}\right)$ thus if: $f=\left(\frac{1}{r}\right)$ and $\vec{A}=\vec{M}$ then we see that:

$$V_m\left(\vec{r}\right)=-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\vec{M}\left(\vec{r}'\right)\cdot\vec{\nabla}'\left(\frac{1}{r}\right)\right)d\tau'=\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\vec{\nabla}'\cdot\left(\frac{\vec{M}\left(\vec{r}'\right)}{r}\right)d\tau'-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{1}{r}\left(\vec{\nabla}'\cdot\vec{M}\left(\vec{r}'\right)\right)d\tau'$$

$$=-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\frac{\vec{\nabla}'\cdot\vec{M}\left(\vec{r}'\right)}{r}\right)d\tau'+\left(\frac{\mu_o}{4\pi}\right)\int_{S'}\left(\frac{\vec{M}\left(\vec{r}'\right)\cdot\hat{n}}{r}\right)da'$$

<u>We can therefore define:</u>

1) An <u>*effective*</u> bound surface density of magnetic pole strength (= <u>*effective*</u> bound surface magnetic charge density):

$\sigma_m^{Bound}\left(\vec{r}'\right)\equiv\vec{M}\left(\vec{r}'\right)\cdot\hat{n}\big|_{surface}$ SI Units of $\sigma_m^{Bound}$ = Magnetization $\vec{M}$ = Amperes/meter

Recall that the SI Units of magnetic charge $g_m=qv$ = Ampere-Meters

$$\sigma_m^{Bound}=g_m/Area,A=\frac{Ampere-meters}{(meters)^2}=Amperes/meter$$

2) The <u>*effective*</u> bound volume density of magnetic pole strength (= <u>*effective*</u> bound volume magnetic charge density):

$\rho_m^{Bound}\left(\vec{r}'\right)\equiv-\vec{\nabla}\cdot\vec{M}\left(\vec{r}'\right)$ SI Units of $\rho_m^{Bound}$ = Amperes / meter$^2$

$$\rho_m^{Bound}=g_m/Volume,V=\frac{Ampere-meters}{meters^3}=Amperes/meter^2$$

Thus, we see that the magnetic scalar potential $V_m\left(\vec{r}\right)$ can be written as:

$$V_m\left(\vec{r}\right)=-\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\left(\frac{\vec{\nabla}'\cdot\vec{M}\left(\vec{r}'\right)}{r}\right)d\tau'+\left(\frac{\mu_o}{4\pi}\right)\int_{S'}\left(\frac{\vec{M}\left(\vec{r}'\right)\cdot\hat{n}}{r}\right)da'$$

$$V_m\left(\vec{r}\right)=\left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\rho_m^{Bound}\left(\vec{r}'\right)}{r}d\tau'+\left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\sigma_m^{Bound}\left(\vec{r}'\right)}{r}da'$$

<u>with</u>: $\rho_m^{Bound}\left(\vec{r}'\right)\equiv-\vec{\nabla}'\cdot\vec{M}\left(\vec{r}'\right)$ and $\sigma_m^{Bound}\left(\vec{r}'\right)\equiv\vec{M}\left(\vec{r}'\right)\cdot\hat{n}\big|_{surface}$

<u>Important Note</u>:

$\sigma_m^{Bound}\left(\vec{r}'\right)$ and $\rho_m^{Bound}\left(\vec{r}'\right)$ are fixed/immovable <u>bound</u> <u>*magnetic*</u> charge densities!!!

The above is analogous to the electrostatic scalar potential $V_e(\vec{r})$ for dielectric materials:

Electrostatic Scalar Potential: $$V_e(\vec{r}) = -\frac{1}{4\pi\varepsilon_o}\int_{v'}\left(\frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{r}\right)d\tau' + \frac{1}{4\pi\varepsilon_o}\int_{S'}\left(\frac{\vec{P}(\vec{r}')\cdot\hat{n}}{r}\right)da'$$

$$V_e(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{Bound}(\vec{r}')}{r}d\tau' + \frac{1}{4\pi\varepsilon_o}\int_{S'}\frac{\sigma_{Bound}(\vec{r}')}{r}da'$$

with: $\rho_{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{P}(\vec{r}')$ and $\sigma_{Bound}(\vec{r}') \equiv \vec{P}(\vec{r}')\cdot\hat{n}\big|_{surface}$

Thus for a magnetic material with macroscopic magnetization $\vec{M}$, the magnetic field inside such a material is given by:

$\vec{B}^{inside}(\vec{r}) = \mu_o\vec{M}(\vec{r}) - \vec{\nabla}V_m(\vec{r})$ with: $V_m(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\rho_m^{Bound}(\vec{r}')}{r}d\tau' + \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\sigma_m^{Bound}(\vec{r}')}{r}da'$

and with: $\rho_m^{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{M}(\vec{r}')$ and $\sigma_m^{Bound}(\vec{r}') \equiv \vec{M}(\vec{r}')\cdot\hat{n}\big|_{surface}$

Therefore:

$$\vec{B}^{inside}(\vec{r}) = \mu_o\vec{M}(\vec{r}) - \vec{\nabla}V_m(\vec{r}) = \mu_o\vec{M}(\vec{r}) - \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\vec{\nabla}\left(\frac{\rho_m^{Bound}(\vec{r}')}{r}\right)d\tau' - \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\vec{\nabla}\left(\frac{\sigma_m^{Bound}(\vec{r}')}{r}\right)da'$$

Now: $\vec{\nabla} = \text{fcn}(\vec{r})$ only, $\rho_m = \text{fcn}(\vec{r}')$ only, $\sigma_m = \text{fcn}(\vec{r}')$ only.

$$\therefore \quad \vec{B}^{inside}(\vec{r}) = \mu_o\vec{M}(\vec{r}) - \left(\frac{\mu_o}{4\pi}\right)\int_{V'}\rho_m^{Bound}(\vec{r}')\vec{\nabla}\left(\frac{1}{r}\right)d\tau' - \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\sigma_m^{Bound}(\vec{r}')\vec{\nabla}\left(\frac{1}{r}\right)da'$$

But again: $\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}'\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

$$\therefore \quad \vec{B}^{inside}(\vec{r}) = \mu_o\vec{M}(\vec{r}) + \frac{\mu_o}{4\pi}\int_{v'}\rho_m^{Bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right)d\tau' + \frac{\mu_o}{4\pi}\int_{S'}\sigma_m^{Bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right)da'$$

with $\rho_m^{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{M}(\vec{r}')$ and $\sigma_m^{Bound}(\vec{r}') \equiv \vec{M}(\vec{r}')\cdot\hat{n}\big|_{surface}$

Similarly for a dielectric material, using $\vec{E}^{inside}(\vec{r})$ can be analogously shown to be:

$\vec{E}^{inside}(\vec{r}) = -\frac{1}{\varepsilon_o}\vec{P}(\vec{r}) - \vec{\nabla}V_e(\vec{r})$ (Note that $\vec{E}^{inside}$ and $\vec{P}$ point in opposite directions)

$$\vec{E}^{inside}(\vec{r}) = -\frac{1}{\varepsilon_o}\vec{P}(\vec{r}) - \frac{1}{4\pi\varepsilon_o}\int_{v'}\vec{\nabla}\left(\frac{\rho_{Bound}(\vec{r}')}{r}\right)d\tau' - \frac{1}{4\pi\varepsilon_o}\int_{S'}\vec{\nabla}\left(\frac{\sigma_{bound}(\vec{r}')}{r}\right)da'$$

Again: $\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}'\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ and $\vec{\nabla} = \text{fcn}(\vec{r})$ only, $\rho_{Bound} = \text{fcn}(\vec{r}')$ only, $\sigma_{Bound} = \text{fcn}(\vec{r}')$ only.

$$\vec{E}^{inside}(\vec{r}) = -\frac{1}{\varepsilon_o}\vec{P}(\vec{r}) + \frac{1}{4\pi\varepsilon_o}\int_{v'}\rho_{bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right)d\tau' + \frac{1}{4\pi\varepsilon_o}\int_{S'}\sigma_{bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right)da'$$

with $\rho_{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{P}(\vec{r}')$ and $\sigma_{Bound}(\vec{r}') \equiv \vec{P}(\vec{r}')\cdot\hat{n}\big|_{surface}$

If additionally, there exist free volume and/or free surface charge densities $\rho_{free}(\vec{r}'),\sigma_{free}(\vec{r}')$ and free volume and/or surface current densities $\vec{J}_{free}(\vec{r}')$, $\vec{K}_{free}(\vec{r}')$ within the volume $v'$ or residing on the enclosing surface $S'$, then the most general form(s) of $\vec{B}^{inside}(\vec{r})$ and $\vec{E}^{inside}(\vec{r})$ are given by:

$$\boxed{\vec{B}^{inside}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{free}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' + \mu_o\vec{M}(\vec{r}) - \vec{\nabla}V_m(\vec{r})}$$

with: $$\boxed{-\vec{\nabla}V_m(\vec{r}) = \frac{\mu_o}{4\pi}\int_{v'}\rho_m^{Bound}(\vec{r}')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' + \frac{\mu_o}{4\pi}\int_{S'}\sigma_m^{Bound}(\vec{r}')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)da'}$$

and with: $\boxed{\rho_m^{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{M}(\vec{r}')}$ and $\boxed{\sigma_m^{Bound}(\vec{r}') \equiv \vec{M}(\vec{r}')\cdot\hat{n}\big|_{surface}}$

$$\boxed{\vec{E}^{inside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\rho_{free}(\vec{r}')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' - \frac{1}{\varepsilon_o}\vec{P}(\vec{r}) - \vec{\nabla}V_e(\vec{r})}$$

with: $$\boxed{-\vec{\nabla}V_e(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\rho_{bound}(\vec{r}')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' + \frac{1}{4\pi\varepsilon_o}\int_{S'}\sigma_{bound}(\vec{r}')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)da'}$$

and with: $\boxed{\rho_{Bound}(\vec{r}') \equiv -\vec{\nabla}'\cdot\vec{P}(\vec{r}')}$ and $\boxed{\sigma_{Bound}(\vec{r}') \equiv \vec{P}(\vec{r}')\cdot\hat{n}\big|_{surface}}$

- Thus, $\vec{B}^{inside}(\vec{r})$ can be determined if $\vec{M}(\vec{r})$ and $\vec{J}_{free}(\vec{r}')$ are specified at all points $\vec{r}'$ in the source volume $v'$ enclosed by the bounding surface $S'$.
- Similarly, $\vec{E}^{inside}(\vec{r})$ can be determined if $\vec{P}(\vec{r})$ and $\rho_{free}(\vec{r}')$ are specified at all points $\vec{r}'$ in the source volume $v'$ enclosed by the bounding surface $S'$.

However, in most problems, $\vec{J}_{free}(\vec{r}')$ (and $\rho_{free}(\vec{r}')$) are specified (or can be specified / are analytically known), but $\vec{M}(\vec{r})$ (and $\vec{P}(\vec{r})$) themselves depend on $\vec{B}^{inside}(\vec{r})$ (and $\vec{E}^{inside}(\vec{r})$) respectively.

→ Even if the functional form(s) of $\vec{M}(\vec{r})$ and $\vec{P}(\vec{r})$ are known, i.e. $\vec{M}(\vec{r}') = \text{fcn}\left(\vec{B}^{inside}(\vec{r}')\right)$ and $\vec{P}(\vec{r}') = \text{fcn}\left(\vec{E}^{inside}(\vec{r}')\right)$, we still have (at best) an integral equation to solve!!!

This is the motivation for introducing the auxiliary fields $\vec{H}$ and $\vec{D}$ thereby obtaining the constitutive relations:

| | | |
|---|---|---|
| $\vec{H}$ = magnetic "displacement" | $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}}$ | $\Rightarrow \boxed{\vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)}$ |
| $\vec{D}$ = electric "displacement" | $\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}}$ | $\Rightarrow \boxed{\vec{E} = \frac{1}{\varepsilon_o}\left(\vec{D} - \vec{P}\right)}$ |

From the above constitutive relations, we see that:

$$\vec{H}^{inside}(\vec{r}) = \frac{1}{4\pi}\int_{v'} \frac{\vec{J}_{free}(\vec{r}') \times \hat{r}}{r^2} d\tau' - \frac{1}{\mu_o}\vec{\nabla} V_m(\vec{r})$$

with: $$-\vec{\nabla} V_m(\vec{r}) = \frac{\mu_o}{4\pi}\int_{v'} \rho_m^{Bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) d\tau' + \frac{\mu_o}{4\pi}\int_{S'} \sigma_m^{Bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) da'$$

and with: $\rho_m^{Bound}(\vec{r}') \equiv -\vec{\nabla}' \bullet \vec{M}(\vec{r}')$ and $\sigma_m^{Bound}(\vec{r}') \equiv \mathrm{M}(\vec{r}') \bullet \hat{n}\big|_{surface}$

$$\vec{D}^{inside}(\vec{r}) = \frac{1}{4\pi}\int_{v'} \rho_{free}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) d\tau' - \varepsilon_o \nabla V_e(\vec{r})$$

with: $$-\varepsilon_o \nabla V_e(\vec{r}) = \frac{1}{4\pi}\int_{v'} \rho_{bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) d\tau' + \frac{1}{4\pi}\int_{S'} \sigma_{bound}(\vec{r}')\left(\frac{\hat{r}}{r^2}\right) da'$$

and with: $\rho_{Bound}(\vec{r}') \equiv -\vec{\nabla}' \bullet \vec{P}(\vec{r}')$ and $\sigma_{Bound}(\vec{r}') \equiv \vec{P}(\vec{r}') \bullet \hat{n}\big|_{surface}$

The boundary conditions on *B, H, E, D,* M and P at interfaces/boundaries will constrain / determine $\vec{M}(\vec{r}')$ and $\vec{P}(\vec{r}')$ inside these materials, as well as the behavior of the fields inside/outside these materials.

<u>Electrostatic Boundary Conditions</u>

$$E_2^{\parallel} = E_1^{\parallel}$$

$$D_2^{\perp} - D_1^{\perp} = \sigma_{free}\big|_{surface}$$

$$E_2^{\perp} - E_1^{\perp} = \frac{1}{\varepsilon_o}\sigma_{TOT} = \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)\big|_{surface}$$

$$D_2^{\parallel} - D_1^{\parallel} = P_2^{\parallel} - P_1^{\parallel}$$

$$\varepsilon_2 \frac{\partial V_2}{\partial n}\bigg|_{\text{interface}} - \varepsilon_1 \frac{\partial V_1}{\partial n}\bigg|_{\text{interface}} = -\sigma_{free}$$

$$\varepsilon_0\left(\frac{\partial V_2}{\partial n}\bigg|_{\text{interface}} - \frac{\partial V_1}{\partial n}\bigg|_{\text{interface}}\right) = -\sigma_{TOT}$$

<u>Magnetostatic Boundary Conditions</u>

$$B_2^{\perp} = B_1^{\perp}$$

$$\vec{H}_2 - \vec{H}_1 = \vec{K}_{free} \times \hat{n}\big|_{surface}$$

$$\vec{B}_2 - \vec{B}_1 = \mu_o \vec{K}_{TOT} \times \hat{n}\big|_{surface} = \mu_o\left(\vec{K}_{free} + \vec{K}_{Bound}\right) \times \hat{n}\big|_{surface}$$

$$H_2^{\perp} - H_1^{\perp} = -\left(M_2^{\perp} - M_1^{\perp}\right)$$

$$\frac{1}{\mu_2}\frac{\partial \vec{A}_2}{\partial n}\bigg|_{\text{interface}} - \frac{1}{\mu_1}\frac{\partial \vec{A}_1}{\partial n}\bigg|_{\text{interface}} = -\vec{K}_{free}$$

$$\frac{1}{\mu_o}\left(\frac{\partial \vec{A}_2}{\partial n}\bigg|_{\text{interface}} - \frac{\partial \vec{A}_1}{\partial n}\bigg|_{\text{interface}}\right) = -\vec{K}_{TOT}$$

In P435 Lecture Notes 20.5, we will explicitly show/work through a few examples of this.

## The Langevín Equation for Paramagnetic Materials

** n.b. See / refer to P435 Lecture Notes #12 for a more complete derivation of the Langevín equation, for the case of polar dielectric materials.

A paramagnetic material is analogous to a polar dielectric in that it has permanent atomic / molecular magnetic dipole moments $\vec{m}_{mol}$ uniformly / randomly oriented (due to thermal internal energy / energy fluctuations ~ $k_B T$) in the absence of an externally applied magnetic field $\vec{B}_{ext}$. Thermal energy in the magnetic material (and fluctuations thereof) is responsible for macroscopically depolarizing the atomic / molecular magnetic dipole moments.

When a paramagnetic material is placed in an external $\vec{B}$-field, a partial macroscopic alignment of the atomic / molecular magnetic dipoles results, which is temperature dependent.

The (macroscopic) magnetic polarization – the magnetization (= magnetic dipole moment per unit volume) $\vec{M}$ of a paramagnetic material is well-described by the Langevín equation (as we found for the case of polar dielectrics, i.e. dielectric materials with permanent atomic / molecular electric dipole moments $\vec{p}_{mol}$) The same analogous physics applies - for the electrostatics case, the energy of an electric dipole in the macroscopic, space-and-time-averaged local electric field is $W_E = P.E._E = -\vec{p}_{mol} \cdot \langle \vec{E}_{loc} \rangle$ (see P435 Lecture Notes 12 for details on $\langle \vec{E}_{loc} \rangle$), thus the energy of an atomic/molecular magnetic dipole with magnetic dipole moment $\vec{m}_{mol}$ in the macroscopic, space-and-time-averaged local magnetic field $\langle \vec{B}_{loc} \rangle$ associated with a paramagnetic material is $W_M = P.E._M = -\vec{m}_{mol} \cdot \langle \vec{B}_{loc} \rangle$. The derivation of the Langevín equation for paramagnetic materials exactly parallels that for polar dielectrics, so we will not repeat this here – we'll just give the results:

$$\boxed{\mathrm{M} = n_{mol} m_{mol} \left( \coth u - \frac{1}{u} \right) = n_{mol} m_{mol} \left[ \coth\left( \frac{m_{mol} \langle B_{loc} \rangle}{k_B T} \right) - \left( \frac{k_B T}{m_{mol} \langle B_{loc} \rangle} \right) \right]}$$

where:

$n_{mol}$ = # density of magnetic dipoles = # of magnetic dipole moments per unit volume

$u \equiv \dfrac{m_{mol} \langle B_{loc} \rangle}{k_B T}$ $\qquad k_B$ = Boltzmann's constant = 1.381 x $10^{-23}$ Joules / Kelvin

$T$ = absolute temperature (Kelvin degrees)

At room temperature, $T \sim 300\ K$:

$u \equiv \dfrac{m_{mol} \langle B_{loc} \rangle}{k_B T} \simeq 2.5 \times 10^{-3} \ll 1$ for typical paramagnetic materials with

e.g. $\left| \langle \vec{B}_{loc} \rangle \right| \sim 0.1$ Tesla (= 1 K Gauss) ($10^4$ Gauss = 1 Tesla)

i.e. $k_B T \left( \approx \tfrac{1}{40} eV\ @\ T = 300 K \right) \gg m_{mol} \langle B_{loc} \rangle$ unless e.g. $B_{ext}$ is very large.

For $u \ll 1$ we have $\coth(u) \simeq \frac{1}{u} + \frac{u}{3} - \frac{u^3}{45} + \frac{2u^5}{945} + \ldots.$ and thus: $\boxed{\mathrm{M}(\vec{r}) \simeq \frac{n_{mol} m_{mol}^2}{3k_B T} \langle B_{loc}(\vec{r}) \rangle}$

Curie's Law for paramagnetic materials (valid only when $u \ll 1$) is: $\boxed{\mathrm{M}(\vec{r}) = \frac{C}{T} \langle B_{loc}(\vec{r}) \rangle}$

where the Curie constant $C$ of the paramagnetic material is: $\boxed{C \simeq \frac{n_{mol} m_{mol}^2}{3k_B}}$

For $u \equiv \frac{m_{mol} \langle B_{loc} \rangle}{k_B T} \ll 1$ then if $\mathrm{M}(\vec{r}) = \chi_m \vec{H}(\vec{r})$ or, since $\chi_m > 0$ for a paramagnetic material:

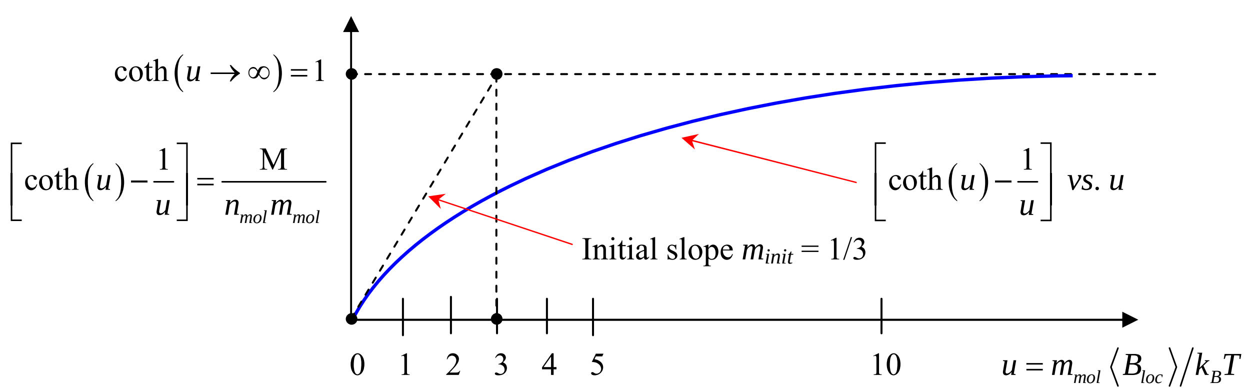
$$\chi_m = \frac{|\vec{\mathrm{M}}(\vec{r})|}{|\vec{H}(\vec{r})|} = \mu \frac{|\vec{\mathrm{M}}(\vec{r})|}{|\vec{B}(\vec{r})|} \simeq \mu_o \frac{|\vec{\mathrm{M}}(\vec{r})|}{|\langle \vec{B}_{loc}(\vec{r}) \rangle|} \simeq \mu_o \frac{n_{mol} m_{mol}^2}{3k_B T}$$

<u>Thus</u>: $\boxed{\chi_m \simeq \mu_o \left( \frac{n_{mol} m_{mol}^2}{3k_B T} \right) > 0}$ for $m_{mol} \langle B_{loc} \rangle \ll k_B T$

$\rightarrow \chi_m \sim 1/T$ for $m_{mol} \langle B_{loc} \rangle \ll k_B T$ (as we found for $\chi_e \sim 1/T$ for $p_{mol} \langle E_{loc} \rangle \ll k_B T$ )

In paramagnetic materials note that the magnetization $\vec{\mathrm{M}}_{para}$ is parallel to the externally-applied magnetic field $\vec{B}_{ext}$ and $\vec{H}$ -field, since $\vec{\mathrm{M}} = \chi_m \vec{H} \Rightarrow \chi_m^{para} > 0$.

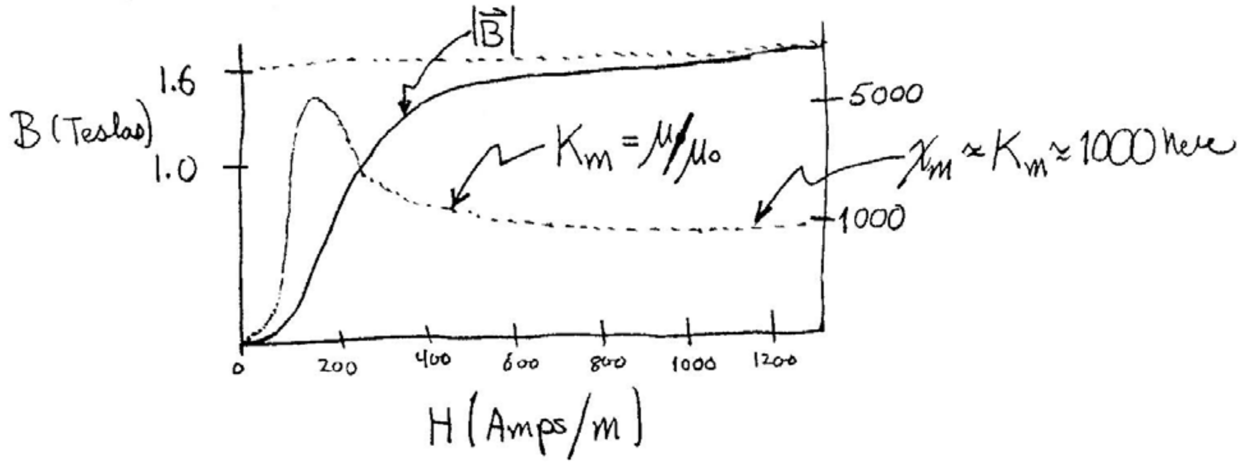
A plot of the Langevín Function $\left[ \coth(u) - \frac{1}{u} \right] = \frac{\mathrm{M}}{n_{mol} m_{mol}}$ versus $u = \frac{m_{mol} \langle B_{loc} \rangle}{k_B T}$ is shown below:



## Ferromagnetic Materials

Ferromagnetic materials are "hard" magnetic materials – i.e. magnetic materials with <u>permanent</u> magnetic polarization $\vec{M}(\vec{r})$, analogous to the permanent electric polarization $\vec{P}(\vec{r})$ associated e.g. with bar electrets. Like bar electrets, where the constitutive relation $\boxed{\vec{D}(\vec{r})=\varepsilon_o\vec{E}(\vec{r})+\vec{P}(\vec{r})}$ is valid, and where the relation $\boxed{\vec{D}(\vec{r})=\varepsilon\vec{E}(\vec{r})}$ ($\varepsilon = constant$) is <u>*not*</u> valid (it is valid <u>*only*</u> for linear/Class-A dielectric materials), for "hard" magnetic materials (such as ferro-magnets), the constitutive relation $\boxed{\vec{H}(\vec{r})=\vec{B}(\vec{r})/\mu_o-\vec{M}(\vec{r})}$ is valid, however the relation $\boxed{\vec{H}(\vec{r})=\vec{B}(\vec{r})/\mu}$ ($\mu = constant$) is <u>*not*</u> valid – it is valid <u>*only*</u> for "soft"(i.e. <u>*linear*</u>) magnetic dielectric materials, because $\mu=\mu(H)$ and is dependent on the past history of the ferromagnetic material! The relations $\boxed{\vec{D}(\vec{r})=\varepsilon\vec{E}(\vec{r})}$ and $\boxed{\vec{H}(\vec{r})=\vec{B}(\vec{r})/\mu}$ <u>can</u> be used for permanently polarized materials, however one must realize that $\varepsilon$ and $\mu$ are <u>*not*</u> constants in such materials.

**<u>The Magnetization Curve and Relative Magnetic Permeability</u>** $K_m \equiv \mu/\mu_o$ **<u>of Annealed Iron</u>:**

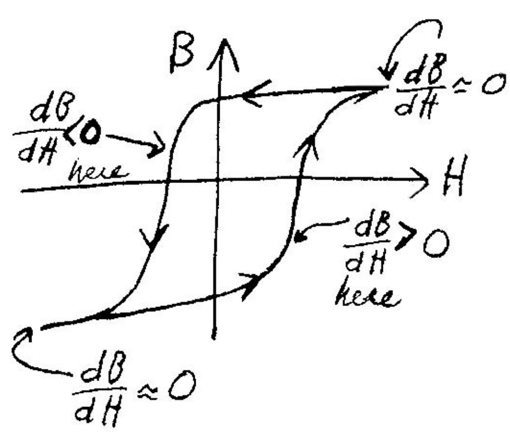



$K_m$ = relative (*a.k.a.* <u>differential</u>) magnetic permeability: $\boxed{K_m(H)\equiv\mu(H)/\mu_o=\frac{1}{\mu_o}\frac{dB}{dH}}$

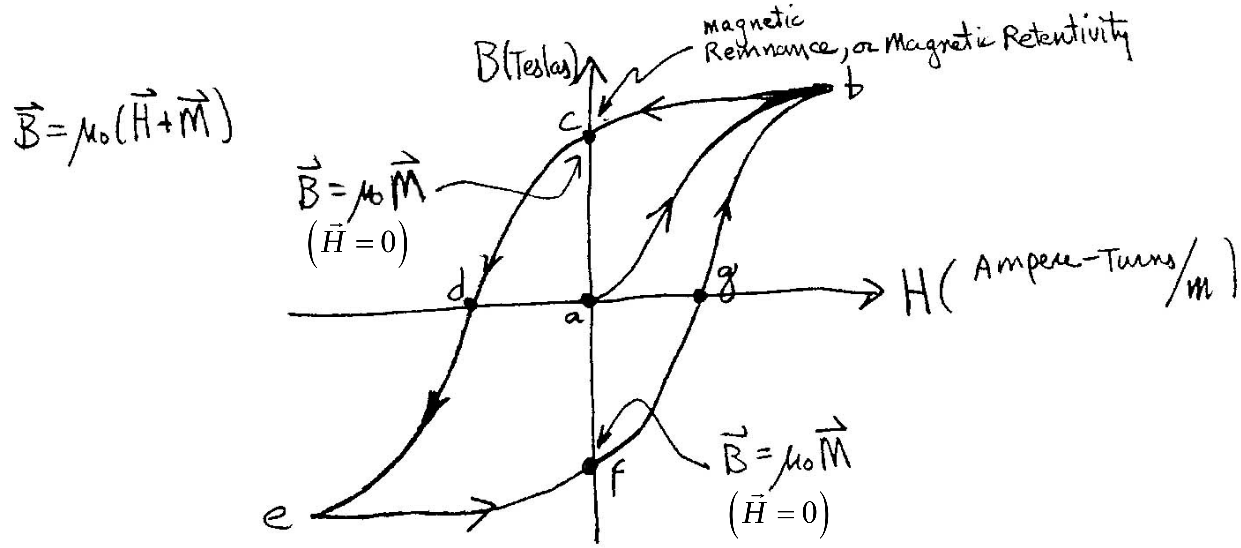
$K_m$ is also known as the <u>differential</u> magnetic permeability because (here) $\boxed{B=\mu(H)H}$ i.e. the magnetic permeability $\mu(H)$ is <u>no</u>t a constant: $\boxed{\mu(H)=K_m(H)\mu_o}$, and thus $K_m(H)$ is also not a constant because:

$$\boxed{\frac{dB(H)}{dH}=\mu(H)=K_m(H)\mu_o \Rightarrow K_m(H)=\frac{1}{\mu_o}\frac{dB(H)}{dH}}$$

$\frac{dB(H)}{dH}$ is the (local) slope of the *B*-*H* hysteresis curve at a given point on the *B*-*H* curve.



**Magnetic Hysteresis of Annealed Iron (Analog of Electro-Hysteresis for Bar Electret):**



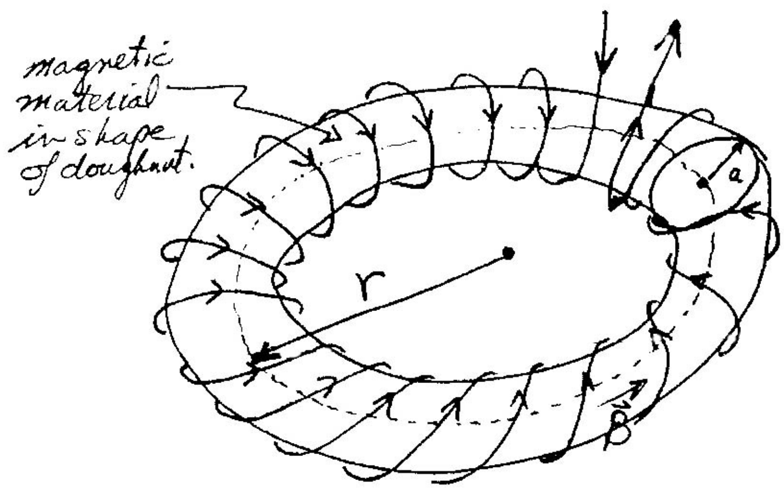

For a ferromagnetic material with a non-linear *B*-*H* hysteresis relation as shown in the above graph of *B* vs. *H* for the material, the present state of its magnetization (magnetic dipole moment per unit volume) $\vec{M}$ depends on the past history of the material's magnetization.

The *B*-*H* curve for a ferromagnetic material can be mapped out directly by simultaneously measuring *B* and *H* over one complete hysteresis cycle, making a series of infinitesimal increments in $dH$ (which by Ampere's law (in integral form) is proportional to the free current $I_{free}$ flowing e.g. in a solenoid coil in which the sample of ferromagnetic material is placed), and measuring the resulting *B*-field of the ferromagnetic material at that value of the applied *H*-field.

One can also compute the resulting infinitesimal increment in the magnetic field $dB$ for each infinitesimal increment in $dH$ . Then at each such measurement, the differential magnetic permeability $\boxed{K_m \equiv \mu/\mu_o = \frac{1}{\mu_o}\frac{dB}{dH}}$ (= local slope of the *B*-*H* curve at a given point $(H, B)$ on the *B*-*H* curve) can be calculated. Note that in order to know where one is at any point on the *B*-*H* curve for a ferromagnetic material that is being mapped out in this manner, (at a minimum, at least) the initial/starting values of *B* and *H* must be known: $(H_{start}, B_{start})$. If the sample of material has never been magnetized before, then $(H_{start}, B_{start}) = (0,0)$ - this is point *a* on the above graph. If the sample of ferromagnetic material has instead e.g. been previously fully magnetically-charged, then e.g. if $H_{start} = 0$, the sample of ferromagnetic material will either be at point *c* or point *f* on the above *B*-*H* curve. Thus, a measurement of $B_{start}$ with $H_{start} = 0$ will unambiguously determine which one of these two points the sample of ferromagnetic material is presently at. Thus from the starting point $(H_{start}, B_{start})$ the *B*-field at any desired ending point $(H_{end}, B_{end})$ on the *B* vs. *H* curve then can be computed from:

$$\boxed{B_{end} = B_{start} + \int_{start}^{end}\left(\frac{dB(H)}{dH}\right)dH = B_{start} + \int_{start}^{end}\mu(H)dH = B_{start} + \mu_o\int_{start}^{end}K_m(H)dH}$$

**The Rowland Ring (= Toroid with Core of Ferromagnetic Material):**




Winding #1 (Primary) has $N_1$ turns, carrying current $I_1$

Winding #2 (secondary) not shown.

Ampere's Circuital Law for $H$: $\boxed{\oint_C \vec{H}(\vec{r})\cdot d\vec{\ell} = I_{free}^{encl}}$

Inside the ring: $H = \dfrac{N_1 I_1}{2\pi r} = \left(\dfrac{N_1}{2\pi r}\right) I_1$

$B = \Phi_m / A$ where $A$ = cross-sectional area of core = $\pi a^2$

Magnetic flux through Rowland Ring $\Phi_m = BA$

**Energy Dissipated in a Hysteresis Cycle**

The extra power provided by a source (i.e. a battery) in causing the magnetic material of the Rowland Ring's core to (slowly) go around the hysteresis loop (e.g. for one cycle) is:

$$\text{Power, } P = \frac{dW}{dt} = I_1\left(N_1 \frac{d\Phi_m}{dt}\right) = I_1 N_1 A \frac{dB}{dt} \quad \text{where } A = \pi a^2 = \text{cross-sectional area}$$

$$= \frac{N_1 I_1 A(2\pi r)}{(2\pi r)} \frac{dB}{dt} \quad \text{(mean) circumference of Roland Ring: } = 2\pi r$$

$$= \frac{N_1 I_1 V_{RR}}{(2\pi r)} \frac{dB}{dt} \quad \text{the volume of the Roland Ring: } V_{RR} = 2\pi r A = 2\pi r\left(\pi a^2\right)$$

$$= HV \frac{dB}{dt} \quad \text{since: } H = \frac{N_1 I_1}{2\pi r}$$

The instantaneous power dissipated is $\boxed{P = \dfrac{dW}{dt} = HV_{RR}\dfrac{dB}{dt}}$ and thus the infinitesimal work/energy increment at a given point on the hysteresis loop is: $\boxed{dW = HV_{RR} dB = (HdB)V_{RR}}$ and therefore the work done (= energy dissipated in the ferromagnetic core of the Roland Ring) in one hysteresis cycle is $\boxed{W = \oint_{cycle} dW = V_{RR}\oint_{cycle} HdB}$

Then: $W = V_{RR}\oint_{cycle} HdB = V_{RR}\oint_{cycle} \vec{H}\bullet d\vec{B}$

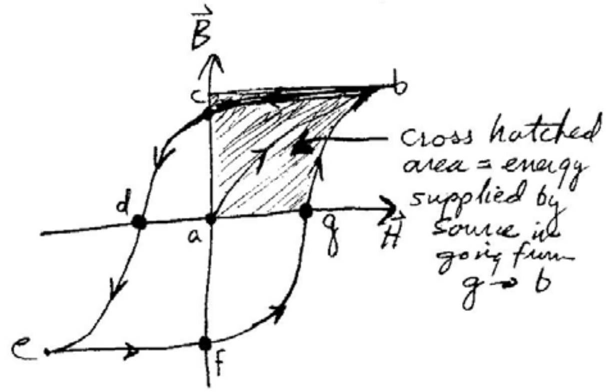
$W = W_{b\to c} + W_{c\to d} + W_{d\to e} + W_{e\to f} + W_{f\to g} + W_{g\to b}$

$$W = V_{RR}\left[\begin{array}{l}\int_b^c \vec{H}\bullet d\vec{B} + \int_c^d \vec{H}\bullet d\vec{B} + \int_d^e \vec{H}\bullet d\vec{B} \\ +\int_e^f \vec{H}\bullet d\vec{B} + \int_f^g \vec{H}\bullet d\vec{B} + \int_g^b \vec{H}\bullet d\vec{B}\end{array}\right]$$

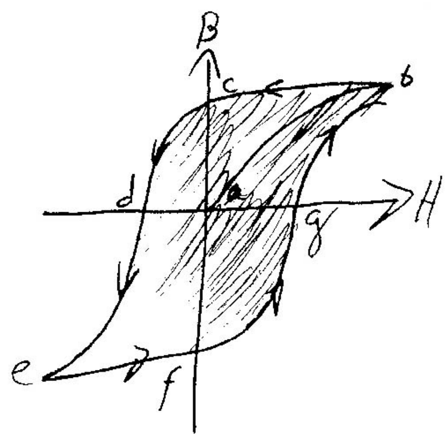
$W = V_{RR} \times$(Area enclosed by the hysteresis loop in the *B*-*H* plane)

$W$ = energy dissipated / hysteresis cycle

Going from point *g* to point *b* in the *B*-*H* plane:



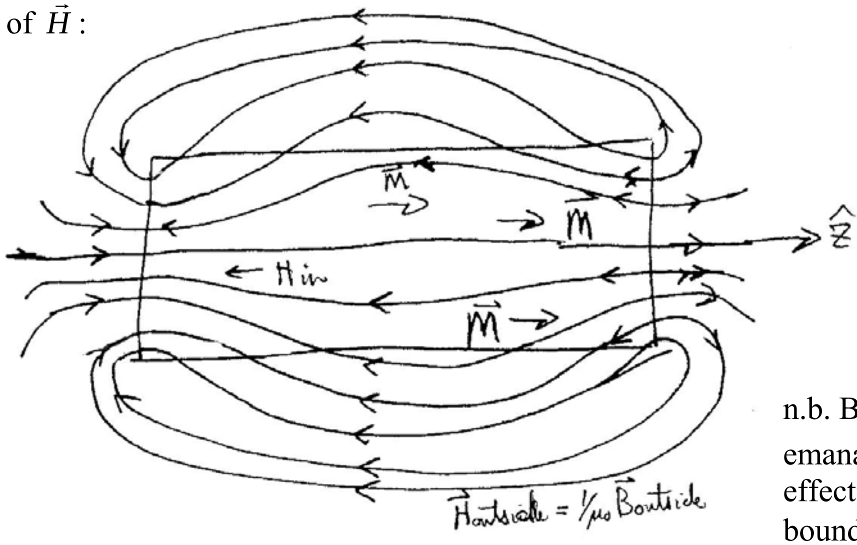

Going around entire hysteresis loop for 1 cycle:




Please see/read the UIUC Physics 401 Lab Handout on the Rowland Ring for more information
It is available at the following URL: http://online.physics.uiuc.edu/courses/phys401/

## $\vec{B}$ and $\vec{H}$ Fields of a Bar Magnet (Permanent Magnet) with Uniform Magnetization $\vec{M}$

Lines of $\vec{H}$:



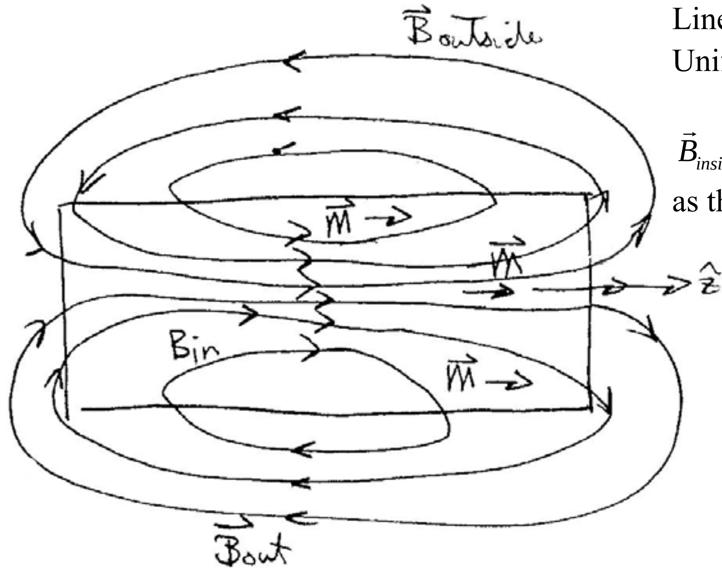

$$\boxed{\vec{B} = \mu_o \vec{H} + \mu_o \vec{M}}$$

$$\boxed{\vec{M} = M_o \hat{z}}$$

$\vec{H}_{inside}$ is in opposite direction to $\vec{M}$ !!!

n.b. Both $\vec{M}$ and $\vec{H}_{inside}$ emanates from/terminate on effective surface magnetic bound charge $\sigma_m$ on the ends the bar magnet!

Lines of $\vec{B}$:




Lines of $\vec{M}$ are constant: Uniform $\vec{M} = M_o \hat{z}$ inside

$\vec{B}_{inside}$ is in the same direction as the magnetization $\vec{M}$

Compare these pictures for bar magnet to that for bar electret (P435 Lecture Notes 10, page 33)

Outside the magnet: $\vec{M} = 0$

$\therefore \quad \vec{B}_{outside} = \mu_o \vec{H}_{outside}$

$\vec{B}_{outside}$ is in same direction as $\vec{H}_{outside}$

Inside the magnet: $\vec{H}_{inside} = \frac{1}{\mu_o}\vec{B}_{inside} - \vec{M}$

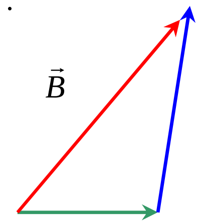
Near the center of bar magnet, $\vec{H}^{center}_{inside} \approx 0$

$\therefore \quad \vec{B}^{center}_{inside} \approx \mu_o \vec{M}$

<u>Note</u>: $\vec{B} = \mu \vec{H}$ where $\mu$ = constant is valid *<u>only</u>* in "soft" magnet materials that are linear and isotropic (i.e. Class "A")

Vector relationship for $\vec{B}$, $\vec{H}$ and $\vec{M}$:

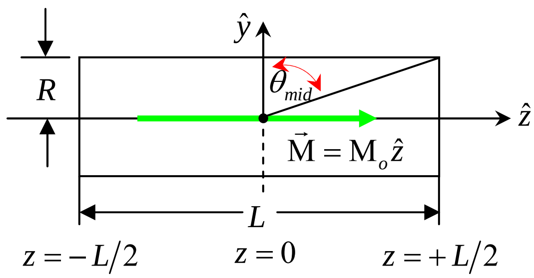
$$\vec{B} = \mu_o \vec{H} + \mu_o \vec{M}$$

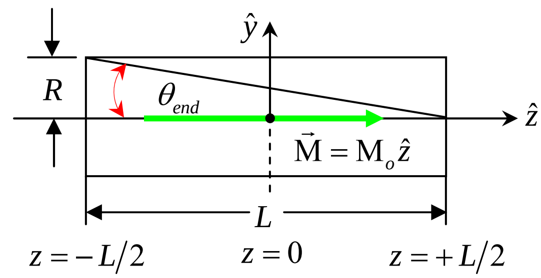



Inside a bar magnet, $\vec{B}_{inside}$ and $\vec{H}_{inside}$ point in ≈ opposite directions.

Magnetic field of a cylindrical bar magnet with $\vec{M} = M_o \hat{z}$ is analogous to that for short solenoid!

Arbitrary point on the z-axis: $\boxed{B_z^{\substack{short\\solenoid}}(z_o) = \left(\frac{\mu_o}{2}\right) nI\left(\sin\theta_2' - \sin\theta_1'\right)}$ Just replace: $\boxed{nI \to M}$ !!!




On the symmetry axis:

At the center of the cylindrical bar magnet:

$$\boxed{\vec{B}_{\substack{short\\solenoid}}(z=0) = \mu_o nI \sin\theta_m' \hat{z}}$$

$$\boxed{\vec{B}_{CBM}(z=0) = \mu_o \vec{M} \sin\theta_{mid}}$$

$$\boxed{\vec{H}_{CBM}(z=0) = -\vec{M}\left(1 - \sin\theta_{mid}\right)}$$

At one end of the cylindrical bar magnet:

$$\boxed{\vec{B}_{\substack{short\\solenoid}}\left(z = +\frac{L}{2}\right) = \left(\frac{\mu_o}{2}\right) nI \sin\theta_{end} \hat{z}}$$

$$\boxed{\vec{B}_{CBM}\left(z = +\frac{L}{2}\right) = \mu_o \vec{M} \frac{\sin\theta_{end}}{2}}$$

$$\boxed{\vec{H}_{CBM}\left(z = +\frac{L}{2}\right) = -\vec{M}\left(1 - \frac{\sin\theta_{end}}{2}\right)}$$

$$\vec{B}(\vec{r}) = \mu_o\left(\vec{H}(\vec{r}) + \vec{M}(\vec{r})\right)$$

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_o\left(\underbrace{\vec{\nabla}\times\vec{H}(\vec{r})}_{=0} + \underbrace{\vec{\nabla}\times\vec{M}(\vec{r})}_{=0}\right)$$

Since $J_f = 0$ (Ampere's Law) Because $\vec{M} = M_o \hat{z}$ is uniform

∴ Lines of $\vec{H}$ are not refracted (i.e. bent) at the cylindrical surface of magnet.

## Example(s) of Calculating $\vec{B}$, $\vec{M}$ and $\vec{H}$ Inside & Outside Magnetized Materials

Consider a long cylinder of length $L$ and radius $R$ such that $L >> R$ (analogous to long solenoid) made of magnetizable material (not yet specified if diamagnetic, paramagnetic or ferromagnetic). However, we will assume that the type of material (and/or its magnetic history, in the case of ferromagnetic material) is known; thus the magnetic permeability $\mu$ of the material in known, or equivalently the magnetic susceptibility $\chi_m$ of the material is known.

$$\mu \equiv \mu_o\left(1+\chi_m\right) \qquad K_m \equiv \frac{\mu}{\mu_o} = \left(1+\chi_m\right)$$

The following table lists the magnetic susceptibilities of various types of linear diamagnetic and linear paramagnetic materials (most at ≈ room temperature, $T = 20^o$ C):

| **Material** | **Susceptibility** | **Material** | **Susceptibility** |
|---|---|---|---|
| *Diamagnetic:* | | *Paramagnetic:* | |
| Bismuth | $-1.6 \times 10^{-4}$ | Oxygen | $1.9 \times 10^{-6}$ |
| Gold | $-3.4 \times 10^{-5}$ | Sodium | $8.5 \times 10^{-6}$ |
| Silver | $-2.4 \times 10^{-5}$ | Aluminum | $2.1 \times 10^{-5}$ |
| Copper | $-9.7 \times 10^{-6}$ | Tungsten | $7.8 \times 10^{-5}$ |
| Water | $-9.0 \times 10^{-6}$ | Platinum | $2.8 \times 10^{-4}$ |
| Carbon Dioxide | $-1.2 \times 10^{-8}$ | Liquid Oxygen ($-200^o$ C) | $3.9 \times 10^{-3}$ |
| Hydrogen | $-2.2 \times 10^{-9}$ | Gadolinium | $4.8 \times 10^{-1}$ |

Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, $20^o$ C). Source: Handbook of Chemistry and Physics, 67th ed. (Boca Raton: CRC Press, Inc., 1986)

For these linear magnetic materials, the magnetization (magnetic dipole moment per unit volume) $\vec{M}$ is related to the $\vec{H}$ field via: $\boxed{\vec{M} = \chi_m \vec{H}_{inside}}$.

But we also have: $\boxed{\vec{H}_{inside} = \frac{1}{\mu_o}\vec{B}_{inside} - \vec{M}}$

then:
$$\vec{M} = \chi_m \vec{H}_{inside} = \chi_m\left(\frac{1}{\mu_o}\vec{B}_{inside} - \vec{M}\right) = \frac{\chi_m}{\mu_o}\vec{B}_{inside} - \chi_m\vec{M}$$

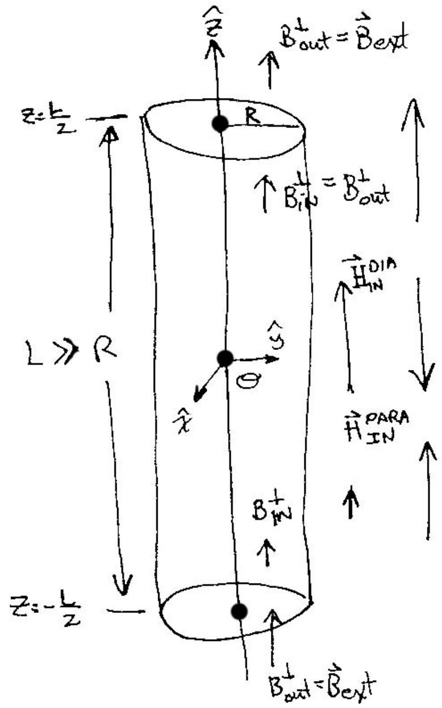
or:
$$\vec{M} + \chi_m\vec{M} = \frac{\chi_m}{\mu_o}\vec{B}_{inside} \quad \text{or:} \quad \left(1+\chi_m\right)\vec{M} = \frac{\chi_m}{\mu_o}\vec{B}_{inside}$$

or:
$$\boxed{\vec{M} = \frac{\chi_m}{\mu_o\left(1+\chi_m\right)}\vec{B}_{inside} = \frac{\chi_m}{\mu}\vec{B}_{inside}} \text{ where } \boxed{\mu = \mu_o\left(1+\chi_m\right)}$$

Suppose we place a sample of magnetic material in a uniform external applied magnetic field $\vec{B}_{ext} = B_o\hat{z} = 1\text{ Tesla }\hat{z}$

Then if $L >> R$, the resulting magnetization is (ideally) uniform:

$\boxed{\vec{M} \simeq \pm M_o\hat{z}}$ (Simplifying assumption!)




$\vec{B}_{ext} = B_o\hat{z} = 1\text{ Tesla }\hat{z}$

For diamagnetic materials:

$$\vec{M}_{dia} = -\frac{|\chi_m^{dia}|}{\mu_{dia}}\vec{B}_{inside}$$

$\chi_m^{dia} < 0$

For paramagnetic materials:

$$\vec{M}_{para} = +\frac{|\chi_m^{para}|}{\mu_{para}}\vec{B}_{inside}$$

$\chi_m^{para} > 0$

Use boundary condition on interface: $B_2^\perp = B_1^\perp \Rightarrow B_{in}^\perp = B_{out}^\perp = \vec{B}_{ext} = B_o\hat{z}$

1) If the magnetic material is diamagnetic $\left(\chi_m^{dia} < 0\right)$ and $|\chi_m^{dia}| \ll 1$ then: $\boxed{\vec{M}_{dia} = -\frac{|\chi_m^{dia}|}{\mu_{dia}}\vec{B}_{inside}}$

But since $\chi_m^{dia} < 0$ and $|\chi_m^{dia}| \ll 1$ then: $\boxed{\vec{M}_{dia} \simeq -\frac{|\chi_m^{dia}|}{\mu_o}\vec{B}_{ext}}$

$\vec{M}_{dia}$ is antiparallel to $\vec{B}_{ext}$, i.e. $\vec{M}_{dia} = -M_o\hat{z}$ for $\vec{B}_{ext} = B_o\hat{z}$, then $|M_o^{dia}| \simeq \left(|\chi_m^{dia}|/\mu^{dia}\right)B_o$

Inside a diamagnetic material $\boxed{\vec{H}_{inside}^{dia} \parallel \vec{B}_{ext}}$

$$\boxed{\vec{H}_{inside}^{dia} \simeq \frac{1}{\mu_o}\vec{B}_{ext} - \vec{M}_{dia} = \frac{1}{\mu_o}\vec{B}_{ext} + \frac{|\chi_m^{dia}|}{\mu_o}\vec{B}_{ext} = \frac{1+|\chi_m^{dia}|}{\mu_o}\vec{B}_{ext}}$$

2) If the magnetic material is paramagnetic $\left(\chi_m^{para} > 0\right)$ and $|\chi_m^{para}| \ll 1$ then: $\boxed{\vec{M}_{para} = \frac{|\chi_m^{para}|}{\mu_{para}}\vec{B}_{inside}}$

But since $\chi_m^{para} < 0$ and $|\chi_m^{para}| \ll 1$ then: $\boxed{\vec{M}_{para} \simeq +\frac{|\chi_m^{para}|}{\mu_o}\vec{B}_{ext}}$

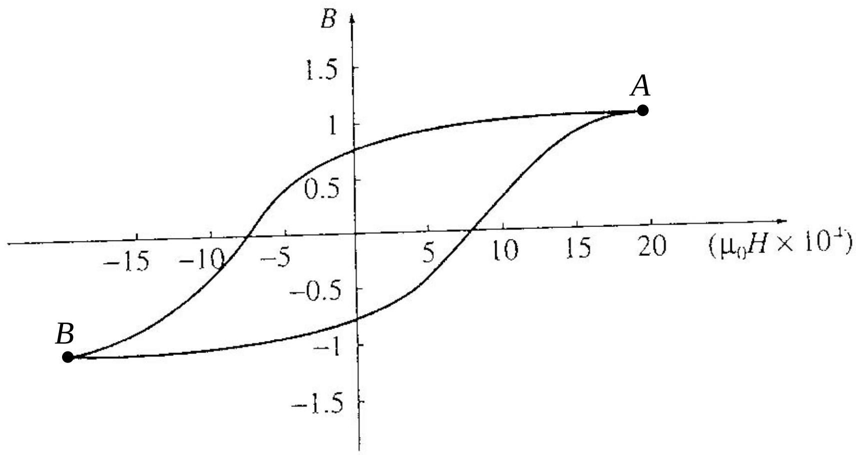
$\vec{M}_{para}$ is parallel to $\vec{B}_{ext}$, i.e. $\vec{M}_{para} = +M_o\hat{z}$ for $\vec{B}_{ext} = B_o\hat{z}$, then $|M_o^{para}| \simeq \left(|\chi_m^{para}|/\mu^{para}\right)B_o$

Inside a paramagnetic material $\boxed{\vec{H}_{inside}^{para} \parallel \vec{B}_{ext}}$

$$\boxed{\vec{H}_{inside}^{para} \simeq \frac{1}{\mu_o}\vec{B}_{ext} - \vec{M}_{para} = \frac{1}{\mu_o}\vec{B}_{ext} - \frac{|\chi_m^{para}|}{\mu_o}\vec{B}_{ext} = \frac{1-|\chi_m^{para}|}{\mu_o}\vec{B}_{ext}}$$

If the magnetic material is ferromagnetic then ∃ a non-linear relation between the magnetization $\vec{\mathrm{M}}$ and $\vec{H}_{inside}$, and also between $\vec{B}_{inside}$ and $\vec{H}_{inside}$

→ need to know $\vec{B}_{inside}$ vs. $\mu_o \vec{H}_{inside}$ (or $\vec{\mathrm{M}}$ vs. $\vec{H}_{inside}$) hysteresis curve, e.g. for iron:




For $\vec{B}_{ext} = B_o \hat{z} = 1$ Tesla $\hat{z}$ (point $A$ or $B$ in figure above), this hysteresis curve says two things:

1) $\boxed{\vec{H}_{inside} \parallel \vec{B}_{ext}}$ and

2) $\mu_o \vec{H}^{ferro}_{inside} \approx 10 \times 10^{-4} = 10^{-3}$ Tesla $\hat{z}$ ( $\mu_o \vec{H}$ has same units (Tesla) as $\vec{B}$ )

<u>But</u>: $\mu_o = 4\pi \times 10^{-7}\ Newtons/Ampere^2$ $\quad$ $1 \text{ Tesla} = 1 \dfrac{Newton}{Amp - meter}$

$$\rightarrow \quad \vec{H}^{ferro}_{inside} \simeq \frac{1}{\mu_o} 10^{-3} T = \frac{1}{4\pi \times 10^{-7}\ N/A^2} 10^{-3} \hat{z} \frac{N}{A-m} = \frac{1}{4\pi} 10^4 \hat{z}\ Amps/meter$$

(Actually, SI Units of $\vec{H}$ are Ampere-<u>*turns*</u>/meter – e.g. for long solenoid $H = nI = NI/L$)

| | |
|---|---|
| $\vec{H}^{ferro}_{inside} \simeq 795.8\ Amps/meter \approx 800\ Amps/meter\ \hat{z}$ | very small! |

<u>Then</u>: $\vec{\mathrm{M}}^{ferro} = \chi^{ferro}_m \vec{H}^{ferro}_{inside} = \dfrac{1}{\mu_o} \vec{B}_{ext} - \vec{H}^{ferro}_{inside}$

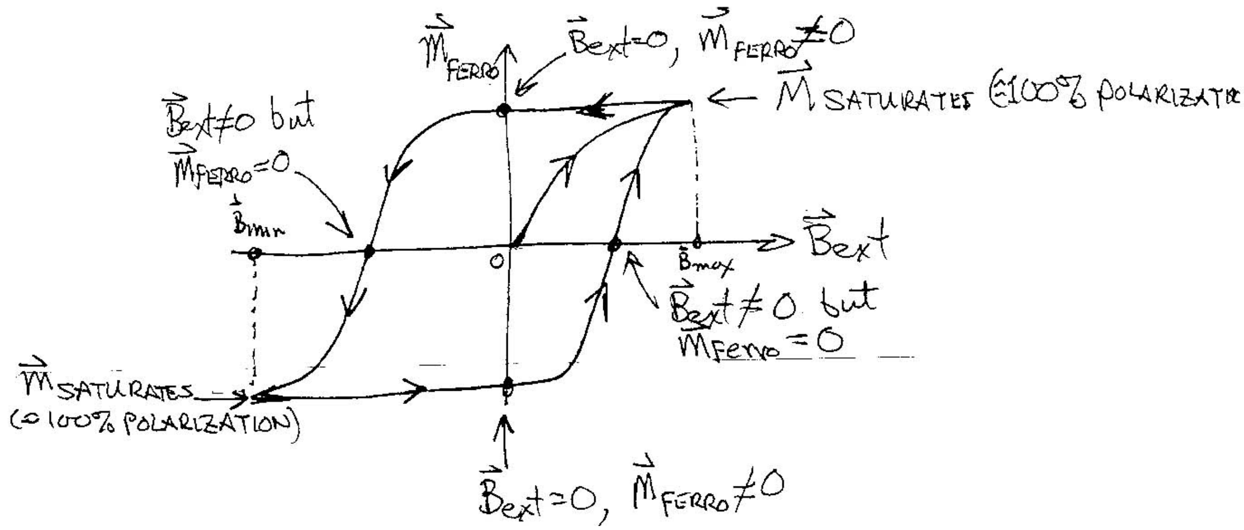
$$\vec{\mathrm{M}}^{ferro} \simeq \frac{1}{4\pi \times 10^{-7} \left( N/A^2 \right)} 1 \left( \frac{N}{A-m} \right) \hat{z} - 800 \left( A/m \right) \hat{z} \simeq \frac{1}{4\pi} \times 10^1 \hat{z} \left( A/m \right) - 800 \left( A/m \right) \hat{z}$$

$$\approx 800{,}000 - 800 \left( A/m \right) z \qquad\qquad \chi^{ferro}_m = \frac{\left| \vec{\mathrm{M}}^{ferro} \right|}{\left| \vec{H}^{ferro}_{inside} \right|} \simeq 1000!!!$$

| | | | |
|---|---|---|---|
| $\vec{\mathrm{M}}^{ferro} \simeq 8 \times 10^5 \hat{z} \left( A/m \right)$ | $\vec{H}^{ferro}_{inside} \parallel \vec{B}_{ext}$ | $\vec{\mathrm{M}}^{ferro} \parallel \vec{B}_{ext}$ | $\vec{\mathrm{M}}^{ferro} \parallel \vec{H}^{ferro}_{inside}$ |

After the magnetizing field $\vec{B}_{ext}$ is turned off / removed, then if $\vec{\mathrm{M}}_{ferro}\left(\vec{B}_{ext}=\text{off}\right)\simeq\vec{\mathrm{M}}_{ferro}\left(\vec{B}_{ext}=\text{on}\right)$ (i.e. the magnetization changes very little when the ferromagnetic material is fully-charged):

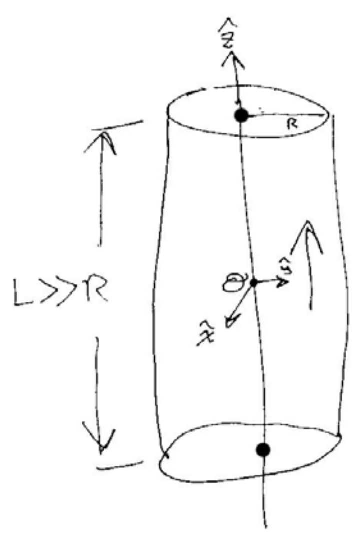



Then: $\vec{B}^{inside}=\left(\dfrac{\mu_{ferro}}{\chi_m^{ferro}}\right)\vec{\mathrm{M}}_{ferro}$ where $\mu_{ferro}=\mu_o\left(1+\chi_m^{ferro}\right)$ and $\chi_m^{ferro}\approx 1000\gg 1$

$$\vec{B}^{inside}=\frac{\mu_o\left(1+\chi_m^{ferro}\right)}{\chi_m^{ferro}}\vec{\mathrm{M}}_{ferro}\simeq\mu_o\vec{\mathrm{M}}_{ferro}\text{ but }\vec{\mathrm{M}}_{ferro}\approx 8\times10^5\,\hat{z}\left(A/m\right)$$

Thus: $$\boxed{\vec{B}^{inside}\approx 4\pi\times10^{-7}\left(N/A^2\right)\bullet 8\times10^5\left(A/m\right)\hat{z}\simeq 1\left(N/A-m\right)\hat{z}=1\text{ Tesla }\hat{z}}$$

Since $B_{inside}^{\perp}=B_{outside}^{\perp}$ at the interface/boundary of a ferromagnetic material, a measurement of the normal component of the magnetic field at the center of one of the end surfaces of the magnetic material (e.g. using a {transverse} Hall Probe) will give $B_{outside}^{surface}=B_{outside}^{\perp}=B_{inside}^{\perp}\simeq 1$ Tesla $\hat{z}$ for a fully-charged ferromagnetic material.




$$\vec{\mathrm{M}}_{ferro}\simeq 9\times10^5\,\hat{z}\left(A/m\right)$$

$$\Rightarrow\vec{B}_{inside}=\vec{B}_{outside}\left(z=\pm\tfrac{L}{2}\right)\simeq 1\text{ Tesla }\hat{z}$$

(for $L >> R$, "ideal" magnet assumed)

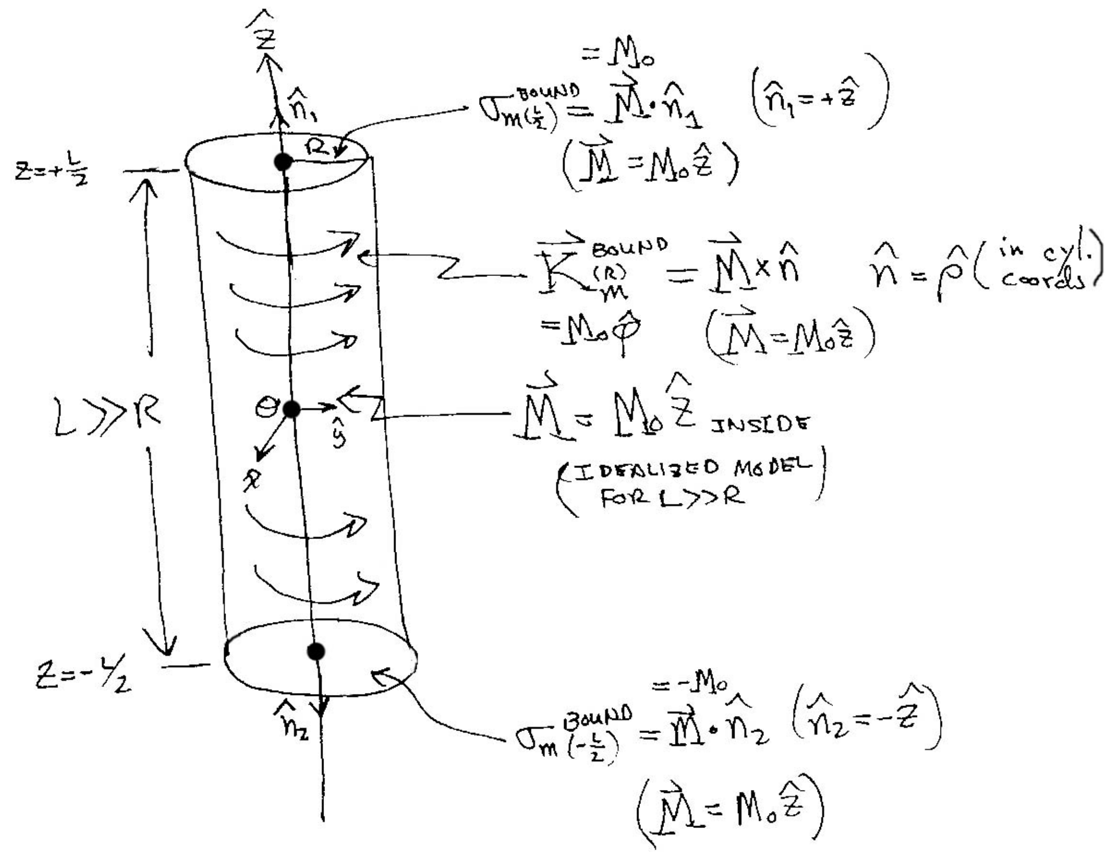
Then the $H$-field inside the magnetic material is:

$$\vec{H}_{inside}=\frac{1}{\mu_o}\vec{B}_{inside}-\vec{\mathrm{M}}_{ferro}=\frac{\cancel{\mu_o}}{\cancel{\mu_o}}\left(\frac{\left(1+\chi_m^{ferro}\right)}{\chi_m^{ferro}}\right)\vec{\mathrm{M}}_{ferro}-\vec{\mathrm{M}}_{ferro}$$

$$\vec{H}_{inside}=\frac{1}{\chi_m^{ferro}}\vec{\mathrm{M}}_{ferro}\simeq\frac{1}{1000}8\times10^5\,\hat{z}\left(A/m\right)\simeq 800\left(A/m\right)\hat{z}$$

Once the magnetization, $\vec{\mathrm{M}}$ of the ferromagnetic material has been determined, then e.g. the <u>equivalent bound current densities</u>: $\vec{K}_m^{bound} = \vec{\mathrm{M}} \times \hat{n}\big|_{surface} \left(= \mathrm{M}_o \hat{z} \times \hat{n} \text{ here}\right)$ and $\vec{J}_m^{bound} = \vec{\nabla} \times \vec{\mathrm{M}}$ $\left(= 0 \text{ here in our idealized } \vec{\mathrm{M}} = \mathrm{M}_o \hat{z} \text{ model}\right)$ can be determined.

Alternatively, the equivalent pole densities of magnetic strength can be determined from $\sigma_m^{bound}(\vec{r}) = \vec{\mathrm{M}}(\vec{r}) \cdot \hat{n}\big|_{surface} \left(= \mathrm{M}_o \hat{z} \times \hat{n} \text{ here}\right)$ and $\rho_m^{bound}(\vec{r}) = -\vec{\nabla} \times \vec{\mathrm{M}}(\vec{r})$ (= 0 here in our idealized $\vec{\mathrm{M}} = \mathrm{M}_o \hat{z}$ model).




We can then determine the magnetic field $\vec{B}(\vec{r})$, valid for any observation/field point $P(\vec{r})$ via two equivalent methods:

**<u>Method #1</u>:** Equivalent surface and volume current densities $\boxed{\vec{K}_m^{bound} = \vec{\mathrm{M}} \times \hat{n}\big|_{surface}}$, $\boxed{\vec{J}_m^{bound} = \vec{\nabla} \times \vec{\mathrm{M}}}$

First, determine the magnetic vector potential $\vec{A}(\vec{r})$ and then the magnetic field $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$:

$$\boxed{\vec{A}_K(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{S'} \frac{\vec{K}_m^{bound}(\vec{r}\,')}{\mathscr{r}}\, da'} \qquad \boxed{\vec{B}_K(\vec{r}) = \vec{\nabla} \times \vec{A}_K(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{v'} \frac{\vec{K}_m^{bound}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'}$$

$$\boxed{\vec{A}_J(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{v''} \frac{\vec{J}_m^{bound}(\vec{r}\,')}{\mathscr{r}}\, d\tau'} \qquad \boxed{\vec{B}_J(\vec{r}) = \vec{\nabla} \times \vec{A}_J(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \int_{v'} \frac{\vec{J}_m^{bound}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'}$$

$$\boxed{\vec{A}_{Tot}(\vec{r}) = \vec{A}_K(\vec{r}) + \vec{A}_J(\vec{r})} \qquad \boxed{\vec{B}_{Tot}(\vec{r}) = \vec{B}_K(\vec{r}) + \vec{B}_J(\vec{r})}$$

**Method #2:** Equivalent pole densities of magnetic strength $\boxed{\sigma_m^{Bound} \equiv \vec{M} \bullet \hat{n}\big|_{surface}}$, $\boxed{\rho_m^{Bound} \equiv -\vec{\nabla} \bullet \vec{M}}$

First, determine the magnetic scalar potential $V_m(\vec{r})$ and then the magnetic field $\vec{B}(\vec{r})$ using the relations $\boxed{\vec{B}_{inside}(\vec{r}) = \mu_o \vec{M}(\vec{r}) - \vec{\nabla} V_m(\vec{r})}$ and $\boxed{\vec{B}_{outside}(\vec{r}) = -\vec{\nabla} V_m(\vec{r})}$:

$$\boxed{V_m^{\sigma_m^{Bound}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{S'} \frac{\sigma_m^{Bound}(\vec{r}')}{\mathscr{r}} da'}$$

$$\boxed{V_m^{\rho_m^{Bound}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \frac{\rho_m^{Bound}(\vec{r}')}{\mathscr{r}} d\tau'}$$

$$\boxed{V_m^{Tot}(\vec{r}) = V_m^{\sigma_m^{Bound}}(\vec{r}) + V_m^{\rho_m^{Bound}}(\vec{r})}$$

$$\boxed{\vec{B}(\vec{r}) \equiv \mu_o \vec{M}(\vec{r}) - \vec{\nabla} V_m(\vec{r})}$$

$$\boxed{\vec{B}(\vec{r}) = \mu_o \vec{M}(\vec{r}) + \left(\frac{\mu_o}{4\pi}\right)\int_{S'} \frac{\sigma_m^{Bound}(\vec{r}')\,\hat{\mathscr{r}}}{\mathscr{r}^2} da' + \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \frac{\rho_m^{Bound}(\vec{r}')\,\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'}$$

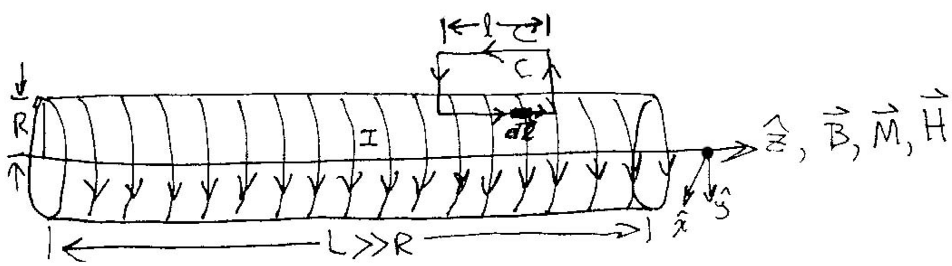
Once $\vec{B}_{inside}(\vec{r})$ and $\vec{B}_{outside}(\vec{r})$ have been determined (using either method) then $\vec{H}_{inside}(\vec{r})$ and $\vec{H}_{outside}(\vec{r})$ can be obtained from:

$$\boxed{\vec{H}_{inside}(\vec{r}) = \frac{1}{\mu_o}\vec{B}_{inside}(\vec{r}) - \vec{M}(\vec{r})} \quad \text{and} \quad \boxed{\vec{H}_{inside}(\vec{r}) = \frac{1}{\mu_o}\vec{B}_{inside}(\vec{r})} \quad (\text{outside, } \vec{M}(\vec{r}) \equiv 0)$$

**Comparison of Magnetic Field Strengths $|\vec{B}|$ for a Long Solenoid**
**With and Without a Magnetic Core**

$$n \equiv \frac{\#turns}{meter} \equiv \frac{N}{L}$$

$N$ = total # turns




Consider a long "ideal" solenoid of radius $R$ and length $L >> R$ has $n$ turns/meter, carrying a steady current $\vec{I} = I\hat{\varphi}$ (i.e. ideal solenoid – for simplicity's sake neglect pitch angle here).
If solenoid has "soft" annealed iron magnetic core of magnetic permeability $\mu = \mu_o(1+\chi_m) \approx 1001\mu_o$ i.e. a magnetic susceptibility $\chi_m = 1000$, then $\vec{H}_{inside} = \frac{1}{\mu_o}\vec{B}_{inside} - \vec{M}$ or $\vec{H}_{inside} = \frac{1}{\mu}\vec{B}_{inside}$ with $\vec{M} = \chi_m \vec{H}_{inside}$.

Ampere's Circuital Law for $\vec{H}$ (integral form) $\oint_C \vec{H}(\vec{r})\bullet d\vec{\ell} = I_{free}^{enclosed}$ with contour of integration, $C$ indicated in the picture above.

$\oint_C \vec{H}(\vec{r})\bullet d\vec{\ell} = I_{free}^{enclosed} \Rightarrow H_{inside} \parallel \hat{z}$ by axial rotational invariance/axial symmetry of solenoid.

Shrink contour $C$ to $\varepsilon$ above/below windings, then get: $H_{inside}\ell = n\ell I \Rightarrow \boxed{\vec{H}_{inside}(\vec{r}) = nI\hat{z}}$.

<u>Then</u>: $\boxed{\vec{B}_{inside}(\vec{r}) = \mu\vec{H}_{inside}(\vec{r}) = \mu nI\hat{z}}$ and $\boxed{\vec{M}(\vec{r}) = \chi_m\vec{H}_{inside}(\vec{r}) = 1000nI\hat{z}}$

Note that $\vec{B}_{inside}(\vec{r}) \parallel \vec{H}_{inside}(\vec{r}) \parallel \vec{M}(\vec{r}) \parallel \hat{z}$ for long "ideal" solenoid with "soft" annealed iron core.

Now compare these results to that for a long ideal solenoid <u>without</u> a magnetic core (for the same steady current $I$):

$$\boxed{\vec{H}_{inside}^{no\ core}(\vec{r}) = \vec{H}_{inside}^{w/\ core}(\vec{r}) = nI\hat{z}}$$

$$\boxed{\vec{B}_{inside}^{no\ core}(\vec{r}) = \mu_o\vec{H}_{inside}^{no\ core}(\vec{r}) = \mu_o nI\hat{z}}$$

<u>Thus</u>: $$\boxed{\frac{\left|\vec{B}_{inside}^{w/\ core}(\vec{r})\right|}{\left|\vec{B}_{inside}^{no\ core}(\vec{r})\right|} = \frac{\mu\left|\vec{H}_{inside}^{w/\ core}(\vec{r})\right|}{\mu_o\left|\vec{H}_{inside}^{no\ core}(\vec{r})\right|} = \frac{\mu}{\mu_o} = 1001}$$

Another way to look at / view this: If one wants $\left|\vec{B}_{inside}^{w/core}\right| = \left|\vec{B}_{inside}^{no\ core}\right|$, then one can reduce the current in the solenoid that has e.g. a "soft" annealed iron magnetic core by a factor of $\mu_o/\mu \simeq 1/1001$ for same $\vec{B}$-field!!!