

## LECTURE NOTES 19

### MAGNETIC FIELDS IN MATTER THE MACROSCOPIC MAGNETIZATION, $\vec{M}$

There exist many types of materials which, when placed in an external magnetic field  $\vec{B}_{ext}(\vec{r})$  become magnetized — i.e. at the microscopic level  $\exists$  internal atomic/molecular magnetic dipole moments  $\vec{m}_{atom}$  or  $\vec{m}_{molecular}$ , which, in the presence of the external aligning magnetic field  $\vec{B}_{ext}(\vec{r})$  produce magnetic torques  $\vec{\tau}(\vec{r}) = \vec{m}(\vec{r}) \times \vec{B}_{ext}(\vec{r})$  which act on the individual atomic/molecular dipole moments, thereby causing a net alignment of the atomic/molecular magnetic dipole moments  $\vec{m}_{atom}/\vec{m}_{molecular}$  which in turn results in a net, macroscopic magnetic polarization, also known as the magnetization,  $\vec{M}(\vec{r})$ . This is analogous to the situation associated with dielectric materials where electrostatic torques  $\vec{\tau}(\vec{r}) = \vec{p}(\vec{r}) \times \vec{E}_{ext}(\vec{r})$  act on individual atomic/molecular electric dipole moments  $\vec{p}_{atom}/\vec{p}_{molecular}$  in an external electric field  $\vec{E}_{ext}(\vec{r})$  resulting in a net, macroscopic electric polarization,  $\vec{P}(\vec{r})$ .

In the absence of an external applied magnetic field (i.e.  $\vec{B}_{ext}(\vec{r}) = 0$ ) the macroscopic alignment of the atomic/molecular magnetic dipole moments  $\vec{m}_{atom}/\vec{m}_{molecular}$  (in many, but not all magnetic materials) is random, due to fluctuations in the internal thermal energy of the material at finite temperature (e.g. room temperature). Thus, no net macroscopic magnetization  $\vec{M}(\vec{r})$  exists in many such materials for  $\vec{B}_{ext}(\vec{r}) = 0$  at finite (absolute) temperature,  $T$ .

We define the macroscopic magnetic polarization (a.k.a. magnetization)  $\vec{M}(\vec{r})$  of a magnetic material in complete analogy to that associated with the macroscopic electric polarization  $\vec{P}(\vec{r})$  of a dielectric material:

Macroscopic Electric Polarization  $\vec{P}(\vec{r})$ :

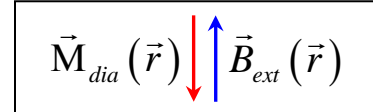
$\vec{P}(\vec{r}) = \left( \frac{\text{electric dipole moment}}{\text{unit volume}} \right) \text{ at point } \vec{r}$	SI Units of $\vec{P}$ : <u>Coulombs/m<sup>2</sup></u>
$\vec{P}(\vec{r}) = n_{mol} \langle \vec{p}_{mol}(\vec{r}) \rangle \equiv \sum_{i=1}^N \vec{p}_{mol_i}(\vec{r}_i) / \text{Volume}, V = \sum_{i=1}^N Q_i \vec{d}_i(\vec{r}_i) / \text{Volume}, V$	
$n_{mol} = \frac{\# \text{ atoms/molecules}}{\text{unit volume}}$	

Macroscopic Magnetic Polarization/Magnetization  $\vec{M}(\vec{r})$ :

$\vec{M}(\vec{r}) = \left( \frac{\text{magnetic dipole moment}}{\text{unit volume}} \right) \text{ at point } \vec{r}$	SI Units of $\vec{M}$ : <u>Amperes/meter</u>
$\vec{M}(\vec{r}) = n_{mol} \langle \vec{m}_{mol}(\vec{r}) \rangle \equiv \sum_{i=1}^N \vec{m}_{mol_i}(\vec{r}_i) / \text{Volume}, V = \sum_{i=1}^N I_i \vec{a}_i(\vec{r}_i) / \text{Volume}, V$	

Note that the magnetization  $\vec{M}(\vec{r})$  has SI units the same as that for a surface current density,  $\vec{K}(\vec{r})$  (Amperes/meter), whereas the electric polarization  $\vec{P}(\vec{r})$  has SI units the same as that for a surface charge density,  $\sigma(\vec{r})$  (Coulombs/m<sup>2</sup>).

There are (at least) four kinds of magnetism:

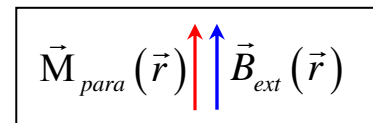


### 1.) DIAMAGNETISM:

The induced macroscopic magnetization  $\vec{M}_{dia}(\vec{r})$  is antiparallel to  $\vec{B}_{ext}(\vec{r})$ . Due to the physics origin of diamagnetism at the microscopic scale – i.e. at the atomic/molecular scale, ALL substances are diamagnetic! However, diamagnetism is very a weak phenomenon – other kinds of magnetism (see below) can “over-ride”/mask out the diamagnetic behavior of a material.

Diamagnetism results from changes induced in the orbits of electrons in the atoms/molecules of a substance, due to the applied/external magnetic field. The direction of the change in orbital motion of the electrons is such that it opposes the change in applied magnetic flux (this is nothing more than Lenz’s Law acting at the microscopic/atomic/molecular scale!).

Superconductors are examples of strong diamagnets – they are in fact perfect diamagnets, completely\* screening out the applied external magnetic field  $\vec{B}_{ext}(\vec{r})$  (\* if no flux-pinning defects are present in the superconducting material). Note that  $\vec{M}_{dia}(\vec{r})$  vanishes when  $\vec{B}_{ext}(\vec{r}) = 0$ .

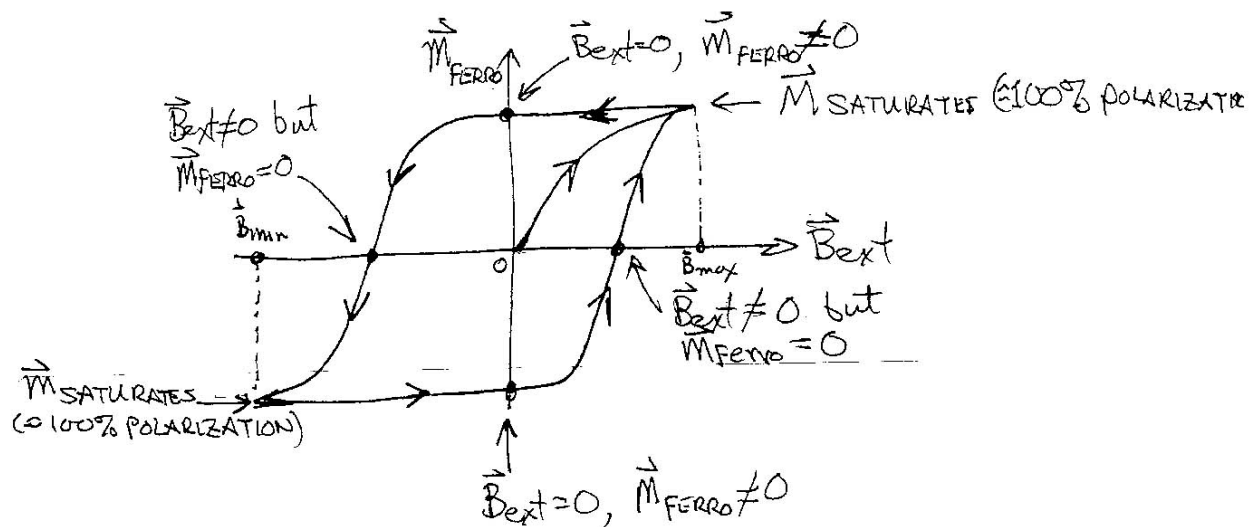


### 2.) PARAMAGNETISM:

The induced macroscopic magnetization,  $\vec{M}_{para}(\vec{r})$  is parallel to  $\vec{B}_{ext}(\vec{r})$ . Atoms or molecules that have a net orbital and/or intrinsic spin magnetic dipole moment  $\vec{m}$  (e.g. atoms/molecules with unpaired electrons – such as *Al, Ba, Ca, Na, Sr, U, ...* and also metals – due to the magnetic dipole moments  $\vec{m}$  associated with intrinsic spins of the conduction electrons) are paramagnetic materials. The external applied magnetic field  $\vec{B}_{ext}(\vec{r})$  exerts a torque on these atomic/molecular magnetic dipole moments  $\vec{m}$  which tends to (partially) align them, giving rise to a net  $\vec{M}_{para}(\vec{r})$  which is parallel to  $\vec{B}_{ext}(\vec{r})$ . The energy of alignment  $U_M(\vec{r}) = -\vec{m}(\vec{r}) \cdot \vec{B}_{ext}(\vec{r})$  is a minimum when  $\vec{m}$  is parallel to  $\vec{B}_{ext}(\vec{r})$ . This is analogous to the net induced electric polarization  $\vec{P}(\vec{r})$  which is parallel to  $\vec{E}_{ext}(\vec{r})$  in dielectric materials, the energy of alignment  $U_E(\vec{r}) = -\vec{p}(\vec{r}) \cdot \vec{E}_{ext}(\vec{r})$  when  $\vec{p}$  is parallel to  $\vec{E}_{ext}(\vec{r})$ . Note that  $\vec{M}_{para}(\vec{r})$  also vanishes when  $\vec{B}_{ext}(\vec{r}) = 0$ .

3.) FERROMAGNETISM: The macroscopic magnetization,  $\vec{M}_{ferro}(\vec{r})$  depends on the (entire) past history of exposure to  $\vec{B}_{ext}(\vec{r})$  !! There exists a non-linear hysteresis-type relation between  $\vec{M}_{ferro}(\vec{r})$  and  $\vec{B}_{ext}(\vec{r})$ . Iron and other ferromagnetic materials have a “macroscopic” crystalline domain structure (a typical scale length involves many thousands of atoms), within a domain (nearly) all of the atomic/molecular magnetic dipole moments  $\vec{m}$  are aligned parallel to each other  $\Rightarrow \vec{M}_{domain}(\vec{r})$  can be very large. However, the orientation of  $\vec{M}_{domain}$  over many domains is  $\approx$  random, unless  $\vec{B}_{ext}(\vec{r}) \neq 0$ . However, ferromagnetic materials have a critical temperature (known as the Curie Temperature  $T_c$ ) below which the domains can spontaneously align – a phase transition occurs in the material at this temperature! In the presence of an external applied magnetic field  $\vec{B}_{ext}$  the alignment of ferromagnetic domains tends to be parallel to  $\vec{B}_{ext}$ , but it is in fact (more) complicated than this, because it is history dependent!!! The alignment arises from quantum mechanics – intrinsic spin and the Pauli exclusion principle. Thus,  $\vec{M}_{ferro}(\vec{r})$  does not vanish when  $\vec{B}_{ext}(\vec{r}) = 0$ !!!

History-Dependence / Hysteresis Relation Between  $\vec{M}_{ferro}$  and  $\vec{B}_{ext}$  for Ferromagnetic Materials for  $T < T_c$  (= Curie Temperature):



Ferromagnetic behavior vanishes for  $T > T_c$ . The material then becomes paramagnetic.

The arrows indicate the path taken for  $\vec{M}_{ferro}$ :  $\vec{B}_{ext}$  starts at  $\vec{B}_{ext} = 0$ , then goes to  $\vec{B}_{max}$ , then through 0, going to  $B_{min}$ , then through 0 again and then going to  $B_{max}$ , etc....

#### 4.) ANTI-FERROMAGNETISM (a.k.a. FERRIMAGNETISM)

In some magnetically-ordered materials  $\exists$  an anti-parallel alignment of intrinsic spins, due to two (or more) inter-penetrating crystalline structures, such that no spontaneous magnetization in the bulk material occurs. Ferrimagnetism/antiferromagnetism occurs for temperatures  $T < T_{Ne'el}$ . Materials exhibiting antiferromagnetic properties are relatively uncommon – e.g.  $URu_2Si_2$ .

## FORCES & TORQUES ON MAGNETIC DIPOLES

When a magnetic dipole with magnetic dipole moment  $\vec{m}$  is placed in an external magnetic field  $\vec{B}_{ext}$  a torque on the magnetic dipole  $\vec{\tau}_M = \vec{m} \times \vec{B}_{ext}$  will occur, just as we saw for the case of an electric dipole with electric dipole moment  $\vec{p}$  when it is placed in an external electric field  $\vec{E}_{ext}$  giving rise to a torque on the electric dipole  $\vec{\tau}_E = \vec{p} \times \vec{E}_{ext}$ .

As we also learned for the case of an electric dipole in a uniform external electric field, similarly, for a magnetic dipole placed in a uniform external magnetic field, there is no net force acting on the magnetic dipole.

For a magnetic dipole with magnetic dipole moment  $\vec{m}$  (e.g. arising from a current loop) placed in an uniform external magnetic field  $\vec{B}_{ext}$  the net force on the is zero:

$$\vec{F}_m^{net} = I \oint_{C'} d\vec{\ell}'(\vec{r}') \times \vec{B}_{ext}(\vec{r}') = I \underbrace{\left( \oint_{C'} d\vec{\ell}'(\vec{r}') \right)}_{=0} \times \vec{B}_{ext}(\vec{r}') = 0$$

cf w/ that for an electric dipole placed in a uniform external electric field  $\vec{E}_{ext}$ :

$$\vec{F}_p^{net} = \vec{F}_+(\vec{r}_+) + \vec{F}_-(\vec{r}_-) = q\vec{E}_{ext}(\vec{r}_+) - q\vec{E}_{ext}(\vec{r}_-) = q(\vec{E}_{ext}(\vec{r}_+) - \vec{E}_{ext}(\vec{r}_-)) = 0$$

The nature of the magnetic (electric) torque  $\vec{\tau}_M = \vec{m} \times \vec{B}_{ext}$  ( $\vec{\tau}_E = \vec{p} \times \vec{E}_{ext}$ ) is such that it tends to align  $\vec{m}$  ( $\vec{p}$ ) with (i.e. parallel to) the applied/external  $\vec{B}_{ext}$  ( $\vec{E}_{ext}$ ) respectively.

⇒ The effect(s) of magnetic torque explains paramagnetism, with  $\vec{M}_{para} \parallel \vec{B}_{ext}$ . One might be tempted to believe that paramagnetism should be a universal phenomenon, common to all materials. However, paramagnetism is connected to the intrinsic magnetic dipole moment of an unpaired electron and/or its orbital magnetic dipole moment. Because of the Pauli exclusion principle (identical fermions, here, electrons) cannot be in the exact same quantum state, hence pairs of electrons can only be in the same quantum state with one of them spin-up, and the other spin down. Thus, torques on paired magnetic dipole moments (or more correctly, the  $\vec{B}$ -fields associated with the paired electron magnetic dipole moments  $\vec{m}_e$ ) cancel.

⇒ Paramagnetism only arises in atoms/molecules with an odd number of electrons – the outermost electron is unpaired ⇒ hence it (alone) is subject to magnetic torque(s).

As we saw in the case for an electric dipole with electric dipole moment  $\vec{p}$  in a non-uniform external electric field  $\vec{E}_{ext}$ , a non-zero force acts on the electric dipole. Similarly, for a magnetic dipole, with magnetic dipole moment  $\vec{m}$  in a non-uniform external magnetic field  $\vec{B}_{ext}$  experiences a non-zero force:

$$\vec{F}_m(\vec{r}) = \vec{\nabla}(\vec{m}(\vec{r}) \cdot \vec{B}_{ext}(\vec{r})) = (\vec{m}(\vec{r}) \cdot \vec{\nabla}) \vec{B}_{ext}(\vec{r}) \quad \{\text{last step valid iff } \vec{m}(\vec{r}) = \text{constant vector}\}$$

$$\vec{F}_p(\vec{r}) = \vec{\nabla}(\vec{p}(\vec{r}) \cdot \vec{E}_{ext}(\vec{r})) = (\vec{p}(\vec{r}) \cdot \vec{\nabla}) \vec{E}_{ext}(\vec{r}) \quad \{\text{last step valid iff } \vec{p}(\vec{r}) = \text{constant vector}\}$$

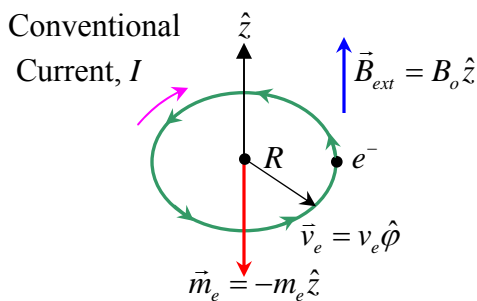
Similarly, work (= potential energy) of a magnetic (electric) dipole moment in an external magnetic (electric) field,  $\vec{B}_{ext}$  ( $\vec{E}_{ext}$ ) are (respectively) given by:

$$\boxed{W_{\vec{m}} = P.E._{\vec{m}} = -\vec{m} \cdot \vec{B}_{ext}} \text{ vs. } \boxed{W_{\vec{p}} = P.E._{\vec{p}} = -\vec{p} \cdot \vec{E}_{ext}}$$

### The Physics of Diamagnetism

Atomic electrons orbit/revolve around the nucleus of the atom at some mean / average / characteristic radius,  $R$ . Atomic electrons bound to the nucleus of an atom no longer behave like point-like particles, but as quantum-mechanical matter waves. However, an orbiting atomic electron “wave” still constitutes a circulating current:

$$\boxed{I_{QM} \sim \frac{ev_e}{\lambda_e} = \frac{ev_e}{C}} \Big|_{\text{Gnd State}} = \frac{ev_e}{2\pi R} \Big|_{\text{Gnd State}} \quad \boxed{\lambda_e \approx C = 2\pi R} \text{ for ground state}$$



Classically, a circulating point electric charge has:

$$\boxed{I_{Class} = \frac{e}{\tau_{orbit}}} \quad \text{with} \quad \boxed{\tau_{orbit} = \frac{C}{v_e} = \frac{2\pi R}{v_e}} \quad \Rightarrow \quad \boxed{I_{Class} = \frac{ev_e}{2\pi R} = I_{QM}}$$

Then: 
$$\boxed{\vec{m} = I\vec{a} = -\left(\frac{ev_e}{2\cancel{\pi}R}\right) \cancel{R}^2 \hat{z} = -\frac{1}{2}(ev_e R) \hat{z}}$$

↑  
due to  $e^-$  charge

With no external magnetic field applied  $\vec{B}_{ext} = 0$ , thus the forces acting on the atomic electron are:

$$\vec{F}_{electrostatic} = \vec{F}_{centripetal}$$

$$-\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{R^2} \hat{r} = -m_e \vec{a}_{centripetal} = -m_e \frac{v_e^2}{R} \hat{r} \quad \Rightarrow \quad \text{Equation A: } \boxed{\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{R^2} \hat{r} = m_e \frac{v_e^2}{R} \hat{r}}$$

$m_e$  = mass of electron

$Z$  = nuclear electric charge # { $+Ze$  = nuclear charge}

With an external magnetic field present  $\vec{B}_{ext} \neq 0$ , thus the forces acting on the atomic electron are:

$$\vec{F}_{EM}^{net} = \vec{F}_{electrostatic} + \vec{F}_B = \vec{F}'_{centripetal}$$

$$\vec{F}_{EM}^{net} = \vec{F}_{electrostatic} + \vec{F}_B = -\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{R^2} \hat{r} - e(\vec{v}_e \times \vec{B}_{ext})$$

Suppose  $\vec{B}_{ext} = B_0 \hat{z}$  and  $\vec{v}_e = v_e \hat{\phi}$  (as shown in above pix)

Then:  $|\hat{\phi} \times \hat{z}| = \hat{\phi} \times (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \quad \theta = 90^\circ$

$$= -\hat{\phi} \times \hat{\theta} = -(-\hat{r}) = \underline{\underline{+\hat{r}}} \quad \sin \theta = \sin 90^\circ = 1$$

$$\cos \theta = \cos 90^\circ = 0$$

$\hat{r} \times \hat{\theta} = \hat{\phi}$	$\hat{\theta} \times \hat{r} = -\hat{\phi}$
$\hat{\theta} \times \hat{\phi} = \hat{r}$	$\hat{\phi} \times \hat{\theta} = -\hat{r}$
$\hat{\phi} \times \hat{r} = \hat{\theta}$	$\hat{r} \times \hat{\phi} = -\hat{\theta}$

Then:  $\vec{F}_{EM}^{net} = -\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{R^2} - ev'_e B_0 \hat{r} = \vec{F}'_{centripetal} = -m_e \frac{v_e'^2}{R} \hat{r}$

Then for  $\vec{B}_{ext} \neq 0$  we have Equation B:  $\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{R^2} + ev'_e B_0 = m_e \frac{v_e'^2}{R}$

Note that since we have an additional term on LHS of Equation B, then we see that:

$$v_e' (\vec{B}_{ext} \neq 0) \neq v_e (\vec{B}_{ext} = 0).$$

Subtract Equation A from Equation B:

$$\underbrace{ev'_e B_0}_{>0} = \frac{m_e}{R} \underbrace{(v_e'^2 - v_e^2)}_{>0} \Rightarrow v_e' > v_e \text{ for } \vec{B}_{ext} = +B_0 \hat{z} \quad (v_e' < v_e \text{ for } \vec{B}_{ext} = -B_0 \hat{z})$$

If the change in  $v_e$ ,  $\Delta v_e \equiv (v_e' - v_e)$  is small, then:  $v_e'^2 - v_e^2 = (v_e' - v_e)(v_e' + v_e) = \Delta v_e (v_e' + v_e)$

But:  $v_e' = v_e + \Delta v_e$  (since  $\Delta v_e \equiv (v_e' - v_e)$ )

$$\begin{aligned} \therefore v_e'^2 - v_e^2 &= \Delta v_e ((v_e + \Delta v_e) + v_e) = \Delta v_e (v_e + \Delta v_e + v_e) = \Delta v_e (2v_e + \Delta v_e) \\ &= 2v_e \Delta v_e + \underbrace{\Delta v_e^2}_{neglect} \approx 2v_e \Delta v_e \end{aligned}$$

$$\therefore ev'_e B_0 = e(v_e + \Delta v_e) B_0 \approx \frac{m_e}{R} (2v_e \Delta v_e)$$

$$\approx e \cancel{v_e} B_0 \approx \frac{2 \cancel{v_e} m_e}{R} \Delta v_e$$

or:  $\Delta v_e \approx \frac{eB_0 R}{2m_e}$

But if:  $I = \frac{ev_e}{2\pi R}$  and  $I' = \frac{ev'_e}{2\pi R}$  and  $v'_e > v_e$

Then:  $\Delta I = I' - I = \frac{e(v'_e - v_e)}{2\pi R} = \frac{e\Delta v_e}{2\pi R}$  but:  $\Delta v_e \approx \frac{eB_o R}{2m_e}$

$\therefore \Delta I = \frac{e^2 B_o R}{4\pi m_e} = \frac{e^2 B_o}{4\pi m_e}$


$$\Delta I = \frac{e^2 B_o}{4\pi m_e}$$

Then:  $m = Ia = I\pi R^2$  and  $m' = I'a = I'\pi R^2$  ( $a = \pi R^2$ )

Thus:  $\Delta m = m' - m = (I' - I)a = \Delta I a = \Delta I \pi R^2$

$\therefore \Delta m = \Delta I \pi R^2 = \left( \frac{e^2 B_o}{4\pi m_e} \right) \pi R^2 = \frac{e^2 B_o R^2}{4m_e}$

But recall that  $\vec{m} = -m\hat{z}$

 i.e.  $\vec{m}$  points down.

Therefore:  $\Delta \vec{m} = -\frac{e^2 B_o R^2}{4m_e} \hat{z}$ ,  $\vec{B}_{ext} = B_o \hat{z}$

Or:  $\Delta \vec{m} = -\left( \frac{e^2 R^2}{4m_e} \right) \vec{B}_{ext}$

The point is, that for diamagnetic materials, the change in the magnetic dipole moment  $\vec{m}$ ,  $\Delta \vec{m}$  is opposite to the direction of  $\vec{B}_{ext}$  - i.e. if  $\vec{B}_{ext} = B_o \hat{z}$  increases, then  $\vec{m}$  also increases, but in the opposite direction to try to cancel/buck the external/applied magnetic field,  $\vec{B}_{ext}$ . This is a simply a manifestation of Lenz's Law at the atomic scale!!!

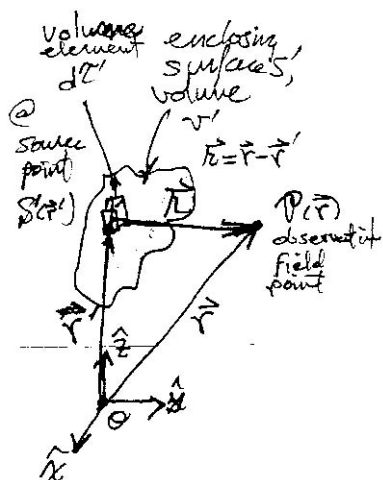
This is what phenomenon of diamagnetism is due to, at least from a  $\approx$  semi-classical perspective.

The induced dipole moments in diamagnetic materials (essentially every material) point in the direction opposite to the applied magnetic field. The macroscopic magnetization  $\vec{M}$  resulting from diamagnetism is relatively speaking very small. Diamagnetism (except in superconductors) is extremely weak.

THE MAGNETIC VECTOR POTENTIAL  $\vec{A}(\vec{r})$ , THE MAGNETIC FIELD  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$   
 OF A MAGNETIZED OBJECT WITH MAGNETIZATION  $\vec{M}(\vec{r})$ 

Recall that the magnetic vector potential  $\vec{A}(\vec{r})$  of a magnetic dipole with magnetic dipole moment  $\vec{m}$  (Amp-m<sup>2</sup>) is:

$$\vec{A}_{\text{dipole}}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \frac{\vec{m} \times \hat{r}}{r^2} \quad \{\text{SI Units: Tesla-meters} = \text{Newtons/Ampere} = F/I \text{ !!!}\}$$



Thus, in a magnetized object with macroscopic magnetization (magnetic dipole moment per unit volume)  $\vec{M}(\vec{r}')$ , each volume element  $d\tau'$  within the volume  $v'$  has a magnetic dipole moment associated with it of:  $\vec{m}(\vec{r}') = \vec{M}(\vec{r}') d\tau'$ .

Thus, the infinitesimal contribution to the magnetic vector potential  $\vec{A}(\vec{r})$  due to the magnetic dipole moment  $\vec{m}(\vec{r}')$  associated with the macroscopic magnetization  $\vec{M}(\vec{r}')$  in the infinitesimal volume element  $d\tau'$  is:

$$d\vec{A}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \frac{\vec{m}(\vec{r}') \times \hat{r}}{r^2} = \left( \frac{\mu_0}{4\pi} \right) \frac{\vec{M}(\vec{r}') d\tau' \times \hat{r}}{r^2} \quad \text{with } \boxed{\vec{r} = \vec{r} - \vec{r}'}$$

Then the total magnetic vector potential  $\vec{A}(\vec{r})$  is obtained by integrating this expression over the entire volume  $v'$  of the magnetized material:

$$\vec{A}(\vec{r}) = \int_{v'} d\vec{A}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \int_{v'} \frac{\vec{M}(\vec{r}') d\tau' \times \hat{r}}{r^2}$$

Now again: 
$$\vec{\nabla}' \left( \frac{1}{r} \right) = \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} = \frac{\hat{r}}{r^2}$$

Thus: 
$$\vec{A}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \int_{v'} \left[ \vec{M}(\vec{r}') \times \left( \vec{\nabla}' \frac{1}{r} \right) \right] d\tau'$$

Integrating by parts, and using  $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$ :

Then: 
$$\vec{A}(\vec{r}) = \left( \frac{\mu_0}{4\pi} \right) \left\{ \int_{v'} \frac{1}{r} \left[ \vec{\nabla}' \times \vec{M}(\vec{r}') \right] d\tau' - \int_{v'} \vec{\nabla}' \times \left[ \frac{\vec{M}(\vec{r}')}{r} \right] d\tau' \right\}$$

Then using:  $\int_v \vec{\nabla} \times \vec{V}(\vec{r}) d\tau = -\oint_s \vec{V}(\vec{r}) \times d\vec{a}$  ( $\vec{V}(\vec{r}) =$  Arbitrary Vector Point Function)  
 (See Griffiths Problem 1.60 (b), page 56)



$$\underline{\text{Thus:}} \quad \vec{A}(\vec{r}) = \left( \frac{\mu_o}{4\pi} \right) \left\{ \int_{v'} \frac{1}{r} \left[ \vec{\nabla}' \times \vec{M}(\vec{r}') \right] d\tau' + \oint_{S'} \frac{1}{r} \left[ \vec{M}(\vec{r}') \times d\vec{a}' \right] \right\}$$

$$\underline{\text{But:}} \quad d\vec{a}' = \hat{n} da'$$

$$\underline{\text{Then:}} \quad \vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{1}{r} \left[ \vec{\nabla}' \times \vec{M}(\vec{r}') \right] d\tau' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{1}{r} \left[ \vec{M}(\vec{r}') \times \hat{n} \right] da' = \vec{A}_{\text{Bound}}(\vec{r})$$

$$\equiv \vec{J}_{\text{Bound}}(\vec{r}') \quad \equiv \vec{K}_{\text{Bound}}(\vec{r}')$$

$$\underline{\text{Or:}} \quad \vec{A}_{\text{Bound}}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}_{\text{Bound}}(\vec{r}')}{r} d\tau' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\vec{K}_{\text{Bound}}(\vec{r}')}{r} da' \quad \text{with } \vec{r} = \vec{r} - \vec{r}'$$

Compare this result to that which we obtained for the magnetic vector potential  $\vec{A}(\vec{r})$  associated with a free volume current density  $\vec{J}_{\text{free}}(\vec{r}')$  and a free surface/sheet current density  $\vec{K}_{\text{free}}(\vec{r}')$  (see P435 Lecture Notes 16, page 6):

$$\vec{A}_{\text{free}}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}_{\text{free}}(\vec{r}')}{r} d\tau' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\vec{K}_{\text{free}}(\vec{r}')}{r} da' \quad \text{with } \vec{r} = \vec{r} - \vec{r}'$$

Thus for a magnetized material with macroscopic magnetization (magnetic dipole moment per unit volume)  $\vec{M}(\vec{r}')$  contained within in the enclosing source volume  $v'$  bounded by the surface  $S'$ , the magnetic vector potential at the field/observation point  $\vec{A}(\vec{r})$  arising from the sum total of the macroscopic magnetization  $\vec{M}(\vec{r}')$  present in the material can be equivalently represented by contributions from an equivalent bound volume current density  $\vec{J}_{\text{Bound}}(\vec{r}') \equiv \vec{\nabla}' \times \vec{M}(\vec{r}')$  and an equivalent bound surface current density  $\vec{K}_{\text{Bound}}(\vec{r}') \equiv \vec{M}(\vec{r}') \times \hat{n}_{\text{surface}}$  where  $\hat{n}$  = outward unit normal at the surface of the magnetized material.

On the interior of the magnetized material:

$$\underbrace{\vec{J}_{\text{Bound}}(\vec{r}')}_{\text{Amps/m}^2} \equiv \underbrace{\vec{\nabla}'}_{1/m} \times \underbrace{\vec{M}(\vec{r}')}_{\text{Amps/m}} = \text{equivalent bound volume current density, SI units} = \underline{\underline{\text{Amps/m}^2}}$$

On the surface(s) of the magnetized material:

$$\underbrace{\vec{K}_{\text{Bound}}(\vec{r}')}_{\text{Amps/m}} \equiv \underbrace{\vec{M}(\vec{r}')}_{\text{Amps/m}} \times \hat{n} = \text{equivalent bound surface current density, SI units} = \underline{\underline{\text{Amps/m}}}$$

$$\underline{\text{Then:}} \quad \vec{A}_{\text{Bound}}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}_{\text{Bound}}(\vec{r}')}{r} d\tau' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\vec{K}_{\text{Bound}}(\vec{r}')}{r} da' \quad \text{with } \vec{r} = \vec{r} - \vec{r}'$$

**MAGNETIC MATERIALS**

$$\vec{J}_{Bound}(\vec{r}') \equiv \vec{\nabla} \times \vec{M}(\vec{r}')$$

$$\vec{K}_{Bound}(\vec{r}') \equiv \vec{M}(\vec{r}') \times \hat{n}_{surface}$$

 $\Leftrightarrow$ 
**DIELECTRIC MATERIALS**

$$\rho_{Bound}(\vec{r}') = -\vec{\nabla} \cdot \vec{P}(\vec{r}')$$

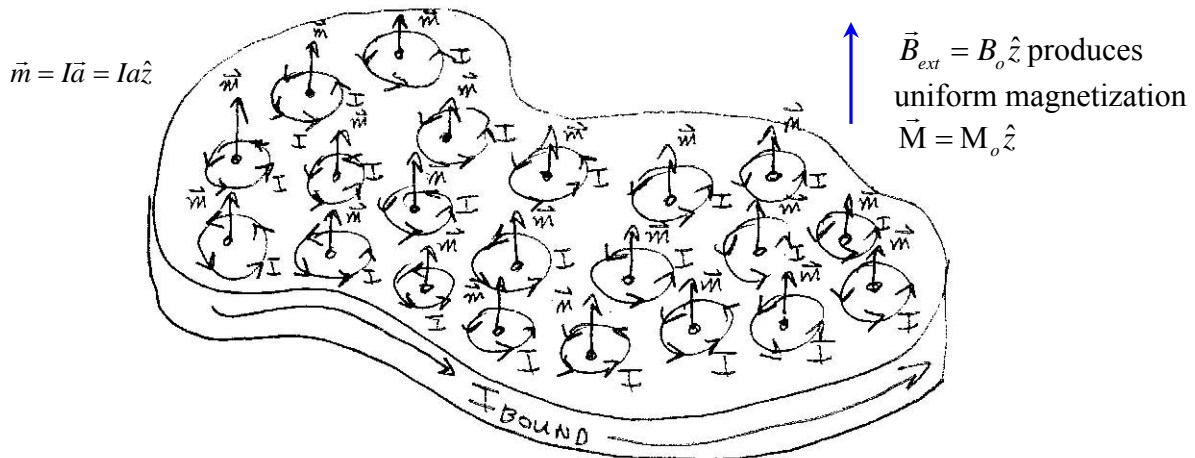
$$\sigma_{Bound}(\vec{r}') = \vec{P}(\vec{r}') \cdot \hat{n}_{surface}$$

So again, instead of integrating over the macroscopic magnetization  $\vec{M}(\vec{r}')$  (or polarization  $\vec{P}(\vec{r}')$ ) arising from the direct contributions from the infinitesimal magnetic (and/or electric) dipoles  $\vec{m}(r')$  (and/or  $\vec{p}(r')$ ), we replace these by macroscopic bound volume and surface current distributions  $\vec{J}_{Bound}(\vec{r}')$  and  $\vec{K}_{Bound}(\vec{r}')$  (and/or  $\rho_{Bound}(\vec{r}')$  and  $\sigma_{Bound}(\vec{r}')$ ); we can then obtain  $\vec{A}(\vec{r})$  (and/or  $V(\vec{r})$ ). Once  $\vec{A}(\vec{r})$  (and/or  $V(\vec{r})$ ) is known, we can then obtain  $\vec{B}(\vec{r})$  from  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$  (and/or  $\vec{E}(\vec{r})$  from  $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ ) !

Note that for a magnetized material with macroscopic magnetization  $\vec{M}(\vec{r}')$  we can also obtain the equivalent bound current,  $I_{Bound}$  from:

$$\vec{I}_{Bound} = \int_{S'_{\perp}} \vec{J}_{Bound}(\vec{r}') da'_{\perp} + \int_{C'_{\perp surface}} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp} \Big|_{surface}$$

Consider the equivalent bound surface current  $\vec{K}_{Bound}$  associated with a thin slab of magnetized material that has been placed in uniform magnetic field  $\vec{B}_{ext} = B_o \hat{z}$ , in turn producing a uniform macroscopic magnetization (magnetic dipole per unit volume)  $\vec{M} = M_o \hat{z}$ . At the microscopic level, atoms and/or molecules will tend to have their induced and/or permanent magnetic dipole moments lined up parallel/anti-parallel to  $\vec{B}_{ext}$  for paramagnetic / diamagnetic materials, respectively. Suppose that the material is paramagnetic, as shown in the figure below:



It can be seen from the above figure that on the interior of the uniformly magnetized material the atomic/molecular microscopic currents will cancel each other (for uniform magnetization,  $\vec{M} = M_o \hat{z}$ ) except on the periphery (i.e. the surface) of the magnetic material.

For uniformly magnetized material(s), e.g.  $\vec{M} = M_o \hat{z}$ :  $\vec{J}_{Bound}(\vec{r}) \equiv \vec{\nabla} \times \vec{M}(\vec{r}) = \vec{\nabla} \times (M_o \hat{z}) = 0$ .

Then:  $\vec{I}_{Bound} = \int_{C_{\perp} surface} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp} |_{surface}$  for uniform magnetization, e.g.  $\vec{M} = M_o \hat{z}$ .

**Example:** Consider a cylindrical rod of radius  $a$  and length  $\ell$  of magnetized material immersed in a uniform  $\vec{B}_{ext} = B_o \hat{z}$  as shown in the figure below. Then the magnetization is uniform, e.g.

$\vec{M} = M_o \hat{z}$ . Thus, no equivalent bound volume current density  $\vec{J}_{Bound}(\vec{r})$  exists, because

$\vec{J}_{Bound}(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}) = \vec{\nabla} \times (M_o \hat{z}) = 0$  for uniform magnetization,  $\vec{M} = M_o \hat{z}$ .

$\vec{M} = n_{mol} \langle \vec{m}_{mol} \rangle$   
 $= M_o \hat{z}$

$\vec{M} = \langle \vec{m}_{Tot} \rangle / Volume$

$\vec{M} = \langle \vec{m}_{Tot} \rangle / \pi a^2 \ell$

$\langle \vec{m}_{Tot} \rangle =$  total dipole moment  
of magnetized  
cylinder

$\vec{B}_{ext} = B_o \hat{z}$  produces uniform  $\vec{M} = M_o \hat{z}$

$$\vec{I}_{Bound} = \oint_{C_{\perp} surface} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp} |_{surface}$$

$$= K_{bound} \ell \hat{\phi}$$

$$\vec{K}_{Bound} = \vec{M} \times \hat{n} |_{surface} \text{ with } \hat{n}_{surface} = \hat{\rho}$$

$$= M_o (\hat{z} \times \hat{\rho}) = M_o \hat{\phi} = K_o \hat{\phi}$$

$$\therefore \vec{I}_{Bound} = K_{bound} \ell \hat{\phi} = M_o \ell \hat{\phi},$$

i.e.  $\vec{K}_{Bound} = M_o \hat{\phi} = K_o \hat{\phi}$

If  $\vec{B}_{ext} \neq$  uniform magnetic field, will result in a non-uniform magnetization, i.e.  $\vec{M} \neq$  uniform, which in turn also implies that the equivalent bound volume current density  $\vec{J}_{Bound} = \vec{\nabla} \times \vec{M} \neq 0$ . This means that at microscopic level the atomic/molecular current loops no longer cancel each other (completely) in the interior region of the magnetized material. Hence for  $\vec{M} \neq$  uniform:

$$\vec{J}_{Bound}(\vec{r}') = \vec{\nabla} \times \vec{M}(\vec{r}') \neq 0 \Rightarrow \vec{I}_{Bound}^{Volume} = \int_{S'_{\perp}} \vec{J}_{Bound}(\vec{r}') da_{\perp}$$

Similarly, we also expect for non-uniform  $\vec{M}$  that  $\vec{K}_{Bound}(\vec{r}') = \vec{M}(\vec{r}') \times \hat{n} |_{surface} \neq 0$  and thus we will also have an equivalent bound surface current:

then  $\vec{I}_{Bound}^{Surface} = \int_{C_{\perp} surface} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp} |_{surface}$  (for magnetized cylinder in above figure:  $d\ell'_{\perp} = dz$ )

Then using the principle of linear superposition:

$$\vec{I}_{Bound}^{Tot} = \vec{I}_{Bound}^{Volume} + \vec{I}_{Bound}^{Surface} = \int_{S'_{\perp}} \vec{J}_{Bound}(\vec{r}') da_{\perp} + \int_{C_{\perp} surface} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp} |_{surface}$$

Note that these equivalent bound currents are flowing in different places in/on the magnetized material – one is flowing inside the material, the other is flowing on the surface of the material.

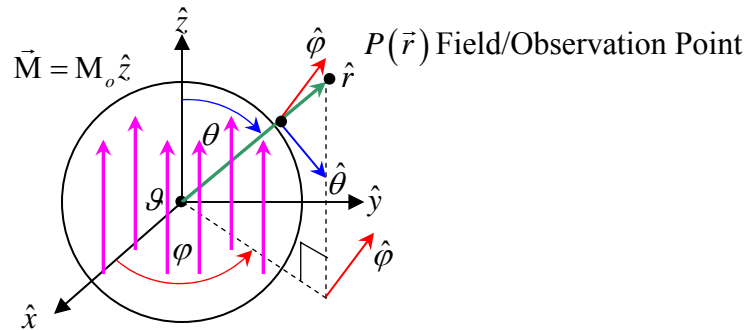
Note also that:  $\vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{M}(\vec{r})) = 0$  *always* in magnetostatics, because  $\vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r})$  is the LHS of the Continuity Equation for equivalent bound currents (i.e. conservation of bound charge):

$$\vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r}, t) = -\frac{\partial \rho_{Bound}(\vec{r}, t)}{\partial t} = 0 \quad \text{if } \rho_{Bound}(\vec{r}, t) \neq \text{fcn}(t).$$

Note also that (here):  $\vec{\nabla} \times (\vec{\nabla} \times \vec{M}(\vec{r})) = \vec{\nabla}(\vec{\nabla} \cdot \vec{M}(\vec{r})) - \nabla^2 \vec{M}(\vec{r}) = 0$  i.e.:  $\vec{\nabla}(\vec{\nabla} \cdot \vec{M}(\vec{r})) = \nabla^2 \vec{M}(\vec{r})$   
 {We will come back to this relation in the near future...}

### Griffiths Example 6.1:

Determine the magnetic field  $\vec{B}(\vec{r})$  associated with a uniformly magnetized sphere of radius  $R$  with uniform magnetization  $\vec{M} = M_o \hat{z}$  as show in the figure below. Choose the local origin  $\mathcal{G}$  to be at the center of the magnetized sphere:



Since the magnetization of the sphere is uniform, then:  $\vec{J}_{Bound}(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}) = \vec{\nabla} \times (M_o \hat{z}) = 0$ .

However:  $\vec{K}_{Bound}(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}|_{surface} = M_o (\hat{z} \times \hat{r}) = M_o \sin \theta \hat{\phi}$  where:  $\hat{n}_{surface} = \hat{r}$

Note:  $\hat{z} \times \hat{r} = (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \times \hat{r} = 0 - \sin \theta (\hat{\theta} \times \hat{r}) = + \sin \theta \hat{\phi}$  since:  $\hat{r} \times \hat{r} = 0, \hat{\theta} \times \hat{r} = -\hat{\phi}$

Now recall that we learned in Griffiths Example 5.11 (p. 236-7)/P435 Lecture Note 16 p. 18-19 (the charged spinning hollow sphere) that:  $\vec{K}_{free} = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{r}' = \sigma \omega R \sin \theta \hat{\phi}$

Uniformly Magnetized Sphere:

$$\vec{K}_{Bound} = M_o \sin \theta \hat{\phi}$$

$$\vec{B}_{inside} (r < R) = \frac{2}{3} \mu_o M_o \hat{z} = \frac{2}{3} \mu_o \vec{M}$$

$$\vec{B}_{outside} (r > R) = \left( \frac{\mu_o}{4\pi} \right) \frac{m}{r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$

$$\vec{m} = \frac{4}{3} \pi R^3 \vec{M} = \frac{4}{3} \pi R^3 M_o \hat{z}$$

Charged Spinning Hollow Sphere:

$$\vec{K}_{free} = \sigma \omega R \sin \theta \hat{\phi} \Rightarrow M_o = \sigma \omega R$$

$$\vec{B}_{inside} (r < R) = \frac{2}{3} \mu_o (\sigma \omega R) \hat{z}$$

$$\vec{B}_{outside} (r > R) = \left( \frac{\mu_o}{4\pi} \right) \frac{m}{r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$

$$\vec{m} = \frac{4}{3} \pi R^3 (\sigma \omega R) \hat{z} = \frac{4}{3} \pi R^4 \sigma \omega \hat{z}$$

For magnetic media, we have obtained the following relations:

Bound current continuity equation: 
$$\vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r}, t) = -\frac{\partial \rho_{Bound}(\vec{r}, t)}{\partial t}$$

Equivalent bound volume current density  $\vec{J}_{Bound}(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r})$ , where  $\vec{M}(\vec{r}) =$  magnetization (*a.k.a.* magnetic dipole moment per unit volume)  $\vec{M}(\vec{r}) = n_{mol} \langle \vec{m}_{mol}(\vec{r}) \rangle = \langle \vec{m}_{Tot}(\vec{r}) \rangle / \text{volume}$  and the equivalent bound surface current density  $\vec{K}_{Bound}(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}|_{surface} \neq 0$  with corresponding relations

$$\vec{I}_{Bound}^{Volume} = \int_{S_{\perp}} \vec{J}_{Bound}(\vec{r}) da_{\perp} \quad \text{and} \quad \vec{I}_{Bound}^{Surface} = \int_{C_{\perp} \text{ surface}} \vec{K}_{Bound}(\vec{r}) d\ell_{\perp} |_{surface}$$

Using the principle of linear superposition: total current density = free current + bound current density:

$$\begin{aligned} \vec{J}_{tot}(\vec{r}) &= \vec{J}_{free}(\vec{r}) + \vec{J}_{Bound}(\vec{r}) \\ \vec{K}_{tot}(\vec{r}) &= \vec{K}_{free}(\vec{r}) + \vec{K}_{Bound}(\vec{r}) \end{aligned}$$

Ampere's Circuital Law becomes (in differential form) for the magnetic field  $\vec{B}(\vec{r})$ :

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_o \vec{J}_{Tot}(\vec{r}) = \mu_o \vec{J}_{free}(\vec{r}) + \mu_o \vec{J}_{Bound}(\vec{r})$$

Note that this is the analog of Gauss' Law (in differential form) for the electric field  $\vec{E}(\vec{r})$ :

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_o} \rho_{Tot}(\vec{r}) = \frac{1}{\epsilon_o} (\rho_{free}(\vec{r}) + \rho_{Bound}(\vec{r}))$$

Now:  $\vec{J}_{Bound}(\vec{r}) \equiv \vec{\nabla} \times \vec{M}(\vec{r}) \quad \therefore \quad \vec{\nabla} \times \vec{B}(\vec{r}) = \mu_o \vec{J}_{free}(\vec{r}) + \mu_o (\vec{\nabla} \times \vec{M}(\vec{r}))$

or:  $\vec{\nabla} \times \vec{B}(\vec{r}) - \mu_o (\vec{\nabla} \times \vec{M}(\vec{r})) = \mu_o \vec{J}_{free}(\vec{r})$

or:  $\frac{1}{\mu_o} \vec{\nabla} \times \vec{B}(\vec{r}) - \vec{\nabla} \times \vec{M}(\vec{r}) = \vec{J}_{free}(\vec{r})$

or:  $\vec{\nabla} \times \left\{ \frac{1}{\mu_o} \vec{B}(\vec{r}) - \vec{M}(\vec{r}) \right\} = \vec{J}_{free}(\vec{r})$

We now define the auxiliary field: 
$$\vec{H}(\vec{r}) \equiv \frac{1}{\mu_o} \vec{B}(\vec{r}) - \vec{M}(\vec{r})$$
 SI Units of  $\vec{H}$  = Amperes/meter  
– the same as that for  $\vec{M}$  !!!

We could call  $\vec{H}(\vec{r})$  the magnetic displacement, in analogy to the electric displacement:

$$\vec{D}(\vec{r}) = \epsilon_o \vec{E}(\vec{r}) + \vec{P}(\vec{r})$$

But usually we just call  $\vec{H}$  “the  $\vec{H}$ -field”.

Ampere's Law for the  $\vec{H}$ -field (in differential form) then becomes:

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}_{free}(\vec{r}) \quad \text{where:} \quad \vec{H}(\vec{r}) \equiv \frac{1}{\mu_0} \vec{B}(\vec{r}) - \vec{M}(\vec{r})$$

Ampere's Law for the  $\vec{H}$ -field is the analog of Gauss' Law for the  $\vec{D}$ -field, in differential form:

$$\vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho_{free}(\vec{r}) \quad \text{where:} \quad \vec{D}(\vec{r}) \equiv \epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r})$$

n.b. both  $\vec{H}(\vec{r})$  and  $\vec{D}(\vec{r})$  are auxiliary fields,  $\vec{B}(\vec{r})$  and  $\vec{E}(\vec{r})$  are fundamental fields.

In integral form, these relations become:

$$\oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{free}^{enclosed} \quad \frac{1}{\mu_0} \oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = I_{Tot}^{enclosed} = I_{free}^{enclosed} + I_{Bound}^{enclosed}$$

$$\oint_S \vec{D}(\vec{r}) \cdot d\vec{a} = Q_{free}^{enclosed} \quad \epsilon_0 \oint_S \vec{E}(\vec{r}) \cdot d\vec{a} = Q_{Tot}^{enclosed} = Q_{free}^{enclosed} + Q_{Bound}^{enclosed}$$

We also have the relations:

$\vec{J}_{Bound}(\vec{r}') = \vec{\nabla} \times \vec{M}(\vec{r}')$	$\rho_{Bound}(\vec{r}') = -\vec{\nabla} \cdot \vec{P}(\vec{r}')$
$\vec{K}_{Bound}(\vec{r}') = \vec{M}(\vec{r}') \times \hat{n} _{surface}$	$\sigma_{Bound}(\vec{r}') = \vec{P}(\vec{r}') \cdot \hat{n} _{surface}$
$\vec{I}_{Bound}^{Volume} = \int_{S_{\perp}} \vec{J}_{Bound}(\vec{r}') da'_{\perp}$	$Q_{Bound}^{Volume} = \int_{V'} \rho_{Bound}(\vec{r}') d\tau'$
$\vec{I}_{Bound}^{Surface} = \int_{C_{\perp}} \vec{K}_{Bound}(\vec{r}') d\ell'_{\perp}$	$Q_{Bound}^{Surface} = \int_{S'} \sigma_{Bound}(\vec{r}') da'$
$\vec{I}_{free}^{Volume} = \int_{S_{\perp}} \vec{J}_{free}(\vec{r}') da'_{\perp}$	$Q_{free}^{Volume} = \int_{V'} \rho_{free}(\vec{r}') d\tau'$
$\vec{I}_{free}^{Surface} = \int_{C_{\perp}} \vec{K}_{free}(\vec{r}') d\ell'_{\perp}$	$Q_{free}^{Surface} = \int_{S'} \sigma_{free}(\vec{r}') da'$

And the-time dependent Continuity Equations – separate conservation of bound and free charge:

$$\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} \quad \Leftrightarrow \quad \text{Free charge is conserved.}$$

$$\vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r}, t) = -\frac{\partial \rho_{Bound}(\vec{r}, t)}{\partial t} \quad \Leftrightarrow \quad \text{Bound charge is conserved.}$$

n.b. There are actually two separate bound charge continuity equations here, because we have bound charges in dielectric media and effective bound currents in magnetic media!

Then using the principle of linear superposition:

$$\vec{J}_{Tot}(\vec{r}, t) = \vec{J}_{free}(\vec{r}, t) + \vec{J}_{Bound}(\vec{r}, t)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{J}_{Tot}(\vec{r}, t) = \vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) + \vec{\nabla} \cdot \vec{J}_{Bound}(\vec{r}, t) = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} - \frac{\partial \rho_{Bound}(\vec{r}, t)}{\partial t} = -\frac{\partial \rho_{Tot}(\vec{r}, t)}{\partial t}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{J}_{Tot}(\vec{r}, t) = -\frac{\partial \rho_{Tot}(\vec{r}, t)}{\partial t} \quad \Leftrightarrow \quad \text{Total charge is conserved.}$$

**Griffiths Example 6.2:** A long copper rod of radius  $R$  carries a steady, uniformly-distributed free current  $\vec{I}_{free} = I_{free} \hat{z}$  with  $\vec{J}_{free} = J_{o,free} \hat{z}$  as shown in the figure below. Determine  $\vec{H}(\rho)$  inside and outside the copper rod. Note that copper is weakly diamagnetic, so at the microscopic level the magnetic dipoles of the copper atoms will align opposite/antiparallel to the magnetic field  $\vec{B} \sim \hat{\phi}$ , resulting in a bound volume current  $\vec{I}_{Bound}^{Volume}$  running antiparallel to the free current  $\vec{I}_{free}^{Volume}$ . All currents are longitudinal (*i.e.* in the  $\pm \hat{z}$  direction).

$$J_{o,free} = I_{free} / \pi R^2 \text{ with } I_{free}^{enclosed}(\rho \leq R) = I_{free} \left( \frac{\rho}{R} \right)^2 \text{ where } \rho = \sqrt{x^2 + y^2} \text{ (in cylindrical coords.)}$$

Use Ampere's Circuital Law for the  $\vec{H}$ -field:

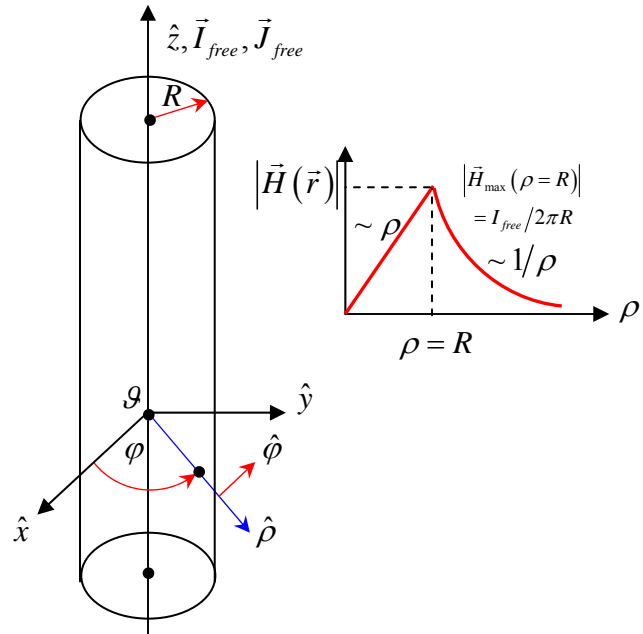
$$\oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{free}^{enclosed}$$

$$\vec{H}^{inside}(\rho \leq R) = \frac{1}{2\pi} \frac{\rho}{R^2} I_{free} \hat{\phi}$$

$$\vec{H}^{outside}(\rho \geq R) = \frac{1}{2\pi\rho} I_{free} \hat{\phi}$$

Note that:

$$\vec{H}^{inside}(\rho = R) = \vec{H}^{outside}(\rho = R)$$



Now:  $\vec{H}(\vec{r}) \equiv \frac{1}{\mu_o} \vec{B}(\vec{r}) - \vec{M}(\vec{r})$  thus:  $\vec{B}(\vec{r}) = \mu_o \vec{H}(\vec{r}) + \vec{M}(\vec{r})$

Then:  $\vec{B}^{outside}(\rho > R) = \mu_o \vec{H}^{outside}(\rho > R) = \frac{\mu_o}{2\pi\rho} I_{free} \hat{\phi}$  (Because  $\vec{M}^{outside}(\rho > R) \equiv 0$ )  
 = same as  $\vec{B}^{outside}$  for non-magnetized wire!

What is  $\vec{B}^{inside}(\rho \leq R)$ ?

$$\vec{B}^{inside}(\rho \leq R) = \mu_o \vec{H}^{inside}(\rho \leq R) + \mu_o \vec{M}(\rho \leq R) = \mu_o (\vec{H}^{inside}(\rho \leq R) + \vec{M}(\rho \leq R))$$

We don't (yet) have the "tools" in hand to know/determine  $\vec{M}(\rho \leq R)$  - but we will, shortly....  
 when we have these, we can then determine  $\vec{B}^{inside}(\rho \leq R)$ .

Note that from Ampere's Circuital Law for the  $\vec{H}$ -field:  $\oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{free}^{enclosed}$

This relation says that if we measure  $I_{free}^{enclosed}$  then we can compute  $\vec{H}$ . This makes  $\vec{H}$  more useful e.g. than  $\vec{D}$  (the electric displacement).

In the "old" days (e.g. 1800's), it was easier to reliably measure a free current  $I$  (in Amps) than voltage  $V$  (in Volts). Reliably measuring a current required the use of galvanometer (an early type of ammeter – which is a very low input impedance device – ideally zero Ohms), whereas reliably measuring the voltage  $V$  (with respect to a local ground) required the use of a voltmeter with a very high input impedance (ideally infinite Ohms), which was very difficult to achieve back then! In the "old" days, a good galvanometer was easy to make a good ammeter, but a good voltmeter was very difficult to make. These days, garden-variety/"vanilla" digital voltmeters typically have input impedances of  $\sim 10$  Meg-Ohms.

Measuring a current thus enabled the monitoring of the  $\vec{H}$ -field, e.g. for an electro-magnet (big fields  $\Rightarrow$  big magnet coils  $\Rightarrow$  lots of current!)

### The Magnetic Permeability $\mu$ and Magnetic Susceptibility $\chi_m$ of Linear Magnetic Materials

Recall that for linear dielectric materials in electrostatics, that:

$$\begin{aligned} \boxed{\vec{D} = \epsilon_o \vec{E} + \vec{P} = \epsilon \vec{E}} \text{ where } \epsilon \text{ is the electric permittivity of the material and } \boxed{\epsilon = \epsilon_o (1 + \chi_e)}. \\ = \epsilon_o (1 + \chi_e) \vec{E} \text{ where } \chi_e \text{ is the electric susceptibility of the dielectric material.} \\ = \epsilon_o \vec{E} + \epsilon_o \chi_e \vec{E} \Rightarrow \boxed{\vec{P} = \epsilon_o \chi_e \vec{E}}. \end{aligned}$$

It would seem reasonable/logical/rational for linear magnetic materials in magnetostatics, that we could define a magnetic permeability  $\mu$  and related magnetic susceptibility  $\chi_m$  in a manner similar to that for how  $\epsilon$  and  $\chi_e$  were defined for linear dielectric materials in electrostatics, i.e.:

$$\boxed{\vec{H} \equiv \frac{1}{\mu_o} \vec{B} - \vec{M} = \frac{1}{\mu} \vec{B}} \text{ with } \boxed{\mu = \mu_o (1 + \chi_m)}.$$

However, the  $1/\mu$  factor really messes things up!!! For if  $\vec{H} = \vec{B}/\mu$  and we want to have  $\mu = \mu_o (1 + \chi_m)$  then  $\vec{H} = \frac{1}{\mu_o (1 + \chi_m)} \vec{B}$  and mathematically there is no rigorous way to separate the RHS of this relation into two separate pieces that would enable us to relate the magnetization  $\vec{M}$  directly to the magnetic field  $\vec{B}$ , analogous to obtaining the relation  $\vec{P} = \epsilon_o \chi_e \vec{E}$  for linear dielectric media.

$$\text{If } \boxed{|\chi_m| \ll 1} \text{ then: } \boxed{\frac{1}{1 + \chi_m} \approx 1 - \chi_m} \text{ and then: } \boxed{\vec{H} \approx \frac{1}{\mu_o} (1 - \chi_m) \vec{B} = \frac{1}{\mu_o} \vec{B} - \frac{1}{\mu_o} \chi_m \vec{B} = \frac{1}{\mu_o} \vec{B} - \vec{M}}$$

Thus, we see that for  $\boxed{|\chi_m| \ll 1}$ , that  $\boxed{\vec{M} \approx \frac{1}{\mu_o} \chi_m \vec{B}}$  in analogy to  $\boxed{\vec{P} = \epsilon_o \chi_e \vec{E}}$ .



However, there are many linear magnetic materials where  $\mu/\mu_o = (1 + \chi_m) \gg 1$  i.e.  $\chi_m \gg 1$  so therefore we cannot use this approximation!!! We must do “something” else!!!

We could e.g. re-define the magnetic permeability of free space,  $\mu_o = 4\pi \times 10^{-7}$  Henrys/meter (=N/A<sup>2</sup>) in terms of its inverse, e.g. define:  $\xi_o \equiv \frac{1}{\mu_o} = \frac{1}{4\pi} \times 10^7$  meters/Henry (= A<sup>2</sup>/N),

Then  $\vec{H} \equiv \frac{1}{\mu_o} \vec{B} - \vec{M}$   $\Rightarrow$   $\vec{H} \equiv \xi_o \vec{B} - \vec{M} = \xi \vec{B}$ , where  $\xi$  is a “new” inverse magnetic permeability

defined such that  $\xi = \xi_o (1 - \chi_m^*)$  with “new” magnetic susceptibility  $\chi_m^*$  such that

$\vec{H} \equiv \xi_o \vec{B} - \vec{M} = \xi \vec{B} = \xi_o (1 - \chi_m^*) \vec{B} = \xi_o \vec{B} - \xi_o \chi_m^* \vec{B}$  and thus  $\vec{M} = \xi_o \chi_m^* \vec{B}$  in analogy to  $\vec{P} = \epsilon_o \chi_e \vec{E}$

for linear dielectrics. But note that since  $\xi \sim 1/\mu$ , then as (old)  $\mu$  increases, the (new)  $\xi$  decreases (and vice-versa)! So this approach has some troubles also... see Appendix at end of this lecture note for a bit more info on this....

However, we shouldn't get too hung-up on this, because e.g. we have already seen that there is a vast difference between the nature of the electric field  $\vec{E}$  vs. the nature of the magnetic field  $\vec{B}$  in terms of how they are specified by their respective divergences and curls, and thus we have absolutely every reason to believe that there is also a vast difference between the nature of the two auxiliary fields  $\vec{D}$  and  $\vec{H}$  in terms of how they are specified by their respective divergences and curls. Hence insisting on (or wanting) “symmetry” between relations associated with  $\vec{E}$  vs. those for  $\vec{B}$  is illusory. In fact, only the macroscopic matter fields  $\vec{P}$  (the electric polarization / electric dipole moment per unit volume) and  $\vec{M}$  (the magnetic polarization/magnetic dipole moment per unit volume) are analogous/similar fields (by deliberate construction on our part)!

What people (Maxwell, et al.) actually did was to start with the auxiliary relation:  $\vec{H} \equiv \frac{1}{\mu_o} \vec{B} - \vec{M}$

Multiply both sides by  $\mu_o$ :  $\mu_o \vec{H} = \vec{B} - \mu_o \vec{M}$ , then rearrange:  $\vec{B} = \mu_o (\vec{H} + \vec{M})$

n.b. This latter relation erroneously causes people to (wrongly) think that the  $\vec{H}$ -field is the fundamental field and therefore that the  $\vec{B}$ -field is the auxiliary field.  $\Rightarrow$  **WRONG !!!**  $\Leftarrow$

For linear magnetic materials, the magnetic permeability  $\mu$  can then be defined such that  $\mu$  connects  $\vec{H}$  to  $\vec{B}$  via the relation  $\vec{H} = \vec{B}/\mu$  (the magnetic analog of  $\vec{D} = \epsilon \vec{E}$ ).

The magnetic susceptibility can then be defined as:  $\mu \equiv \mu_o (1 + \chi_m)$  paralleling that done for linear dielectrics:  $\epsilon \equiv \epsilon_o (1 + \chi_e)$

Then we see that:  $\vec{B} = \mu \vec{H} = \mu_o (1 + \chi_m) \vec{H} = \mu_o \vec{H} + \mu_o \chi_m \vec{H}$  but:  $\vec{B} = \mu_o (\vec{H} + \vec{M}) = \mu_o \vec{H} + \mu_o \vec{M}$

Then “viola”:  $\vec{M} = \chi_m \vec{H}$ , which is not analogous to:  $\vec{P} = \epsilon_o \chi_e \vec{E}$  because we don't have a direct relationship between  $\vec{M}$  and (the fundamental field)  $\vec{B}$ .

However, since  $\vec{H} = \vec{B}/\mu$  then  $\vec{M} = \chi_m \vec{H} = \chi_m \vec{B}/\mu = \left[ \chi_m / (1 + \chi_m) \right] \vec{B} / \mu_o$ .

Now if  $|\chi_m| \ll 1$ , then the factor  $\left[ \chi_m / (1 + \chi_m) \right] \approx \chi_m$ , and  $\mu = \mu_o (1 + \chi_m) \approx \mu_o$ .

Thus:  $\vec{M} = \chi_m \vec{H} \approx \chi_m \vec{B} / \mu_o$  for  $|\chi_m| \ll 1$ .

The following table lists the magnetic susceptibilities for a few typical types of diamagnetic and paramagnetic materials. Note that systematically,  $|\chi_m| \ll 1$  for both types of magnetic materials (except for gadolinium).

Material	Susceptibility	Material	Susceptibility
<i>Diamagnetic:</i>		<i>Paramagnetic:</i>	
Bismuth	$-1.6 \times 10^{-4}$	Oxygen	$1.9 \times 10^{-6}$
Gold	$-3.4 \times 10^{-5}$	Sodium	$8.5 \times 10^{-6}$
Silver	$-2.4 \times 10^{-5}$	Aluminum	$2.1 \times 10^{-5}$
Copper	$-9.7 \times 10^{-6}$	Tungsten	$7.8 \times 10^{-5}$
Water	$-9.0 \times 10^{-6}$	Platinum	$2.8 \times 10^{-4}$
Carbon Dioxide	$-1.2 \times 10^{-8}$	Liquid Oxygen ( $-200^\circ \text{C}$ )	$3.9 \times 10^{-3}$
Hydrogen	$-2.2 \times 10^{-9}$	Gadolinium	$4.8 \times 10^{-1}$

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm,  $20^\circ \text{C}$ ). Source: *Handbook of Chemistry and Physics*, 67th ed. (Boca Raton: CRC Press, Inc., 1986).

In this course, we want to “hang on to” the following:

$\vec{E}(\vec{r})$  and  $\vec{B}(\vec{r})$  are fundamental fields.

$\vec{D}(\vec{r})$  and  $\vec{H}(\vec{r})$  are auxiliary fields associated with the  $E$  &  $M$  properties of matter.

For *linear* dielectrics and *linear* magnetic materials:  $\left\{ \begin{array}{l} \vec{D}(\vec{r}) = \epsilon \vec{E}(\vec{r}) \\ \vec{H}(\vec{r}) = \frac{1}{\mu} \vec{B}(\vec{r}) \end{array} \right\}$

$\epsilon =$  electric permittivity of matter  $= K_e \epsilon_o$

$$K_e = \epsilon_{rel} \equiv \frac{\epsilon}{\epsilon_o} = (1 + \chi_e)$$

dielectric “constant”  
(a.k.a. relative electric permittivity)

electric susceptibility

$\mu =$  magnetic permeability of matter  $= K_m \mu_o$

$$K_m = \mu_{rel} \equiv \frac{\mu}{\mu_o} = (1 + \chi_m)$$

relative magnetic permeability

magnetic susceptibility

### For Diamagnetic Materials:

$$\boxed{\chi_m^{dia} < 0} \Rightarrow \mu^{dia} = \mu_o (1 + \chi_m^{dia}) < \mu_o, \quad K_m^{dia} = \frac{\mu^{dia}}{\mu_o} = (1 + \chi_m^{dia}) < 1$$

### For Paramagnetic Materials:

$$\boxed{\chi_\mu^{para} > 0} \Rightarrow \mu^{para} = \mu_o (1 + \chi_m^{para}) > \mu_o, \quad K_m^{para} = \frac{\mu^{para}}{\mu_o} = (1 + \chi_m^{para}) > 1$$

### For Ferromagnetic Materials:

$\chi_m^{ferro} \gg 0$  but is in fact dependent on past magnetic history of material !!!

A non-linear hysteresis-type relation exists between  $\vec{M}$  vs.  $\vec{H}$  (and/or  $\vec{M}$  vs.  $\vec{B}$ ) for ferromagnetic materials.

Magnetic materials that obey the relation  $\boxed{\vec{B} = \mu \vec{H} = \mu_o (1 + \chi_m) \vec{H} = \mu_o \vec{H} + \mu_o \vec{M}} \Rightarrow \boxed{\vec{M} = \chi_m \vec{H}}$  are known as linear magnetic materials, i.e.  $\mu$  = the magnetic permeability of the magnetic material and  $\mu$  = constant of proportionality between  $\vec{B}$  and  $\vec{H}$ , whereas  $\chi_m$  = magnetic susceptibility of the magnetic material and  $\chi_m$  = constant of proportionality between  $\vec{M}$  and  $\vec{H}$ .

If  $\vec{B}_{ext}$  becomes extremely large, then the relation between  $\vec{B}$  and  $\vec{H}$ , and  $\vec{M}$  and  $\vec{H}$  can/does become non-linear, e.g.  $\vec{B} = \mu (1 + c_2 \mu + c_3 \mu^2 + \dots) \vec{H}$  and  $\vec{M} = \chi_m (1 + a_2 \chi_m + a_3 \chi_m^2 + \dots) \vec{H}$

Note that various crystalline magnetic materials are anisotropic, hence:  $\vec{B} = \vec{\mu} \vec{H}$  and  $\vec{M} = \vec{\chi}_m \vec{H}$

$$\vec{\mu} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \quad \vec{\chi}_m = \begin{pmatrix} \chi_{xx}^m & \chi_{xy}^m & \chi_{xz}^m \\ \chi_{yx}^m & \chi_{yy}^m & \chi_{yz}^m \\ \chi_{zx}^m & \chi_{zy}^m & \chi_{zz}^m \end{pmatrix} \quad \begin{array}{cc} \uparrow & \uparrow \\ \text{magnetic} & \text{magnetic} \\ \text{permeability} & \text{susceptibility} \\ \text{tensor} & \text{tensor} \end{array}$$

Note also that:  $\mu_{ij} = \mu_{ji}$  and  $\mu_{xx} + \mu_{yy} + \mu_{zz} = 0$  and likewise:  $\chi_{ij}^m = \chi_{ji}^m$  and  $\chi_{xx}^m + \chi_{yy}^m + \chi_{zz}^m = 0$ .

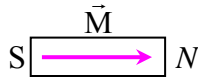
We have the Maxwell relation:  $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$  (no magnetic charges/no magnetic monopoles)

and the constitutive relation:  $\vec{H}(\vec{r}) = \frac{1}{\mu_0} \vec{B}(\vec{r}) - \vec{M}(\vec{r})$ . Then:  $\vec{\nabla} \cdot \vec{H}(\vec{r}) = \frac{1}{\mu_0} \underbrace{\vec{\nabla} \cdot \vec{B}(\vec{r})}_{=0} - \vec{\nabla} \cdot \vec{M}(\vec{r})$

or:  $\vec{\nabla} \cdot \vec{H}(\vec{r}) = -\vec{\nabla} \cdot \vec{M}(\vec{r})$   $\leftarrow$  These divergences do not necessarily vanish!!! Often they don't!  
(especially on the surfaces of magnetized materials)

Only when  $\vec{\nabla} \cdot \vec{M}(\vec{r}) = 0$ , does  $\vec{\nabla} \cdot \vec{H}(\vec{r}) = 0$  (and not vice-versa!!!)

Consider a bar magnet (permanent magnet) with uniform  $\vec{M}(\vec{r}) \neq 0$  inside, thus  $\vec{B}(\vec{r}) \neq 0$  inside or outside:



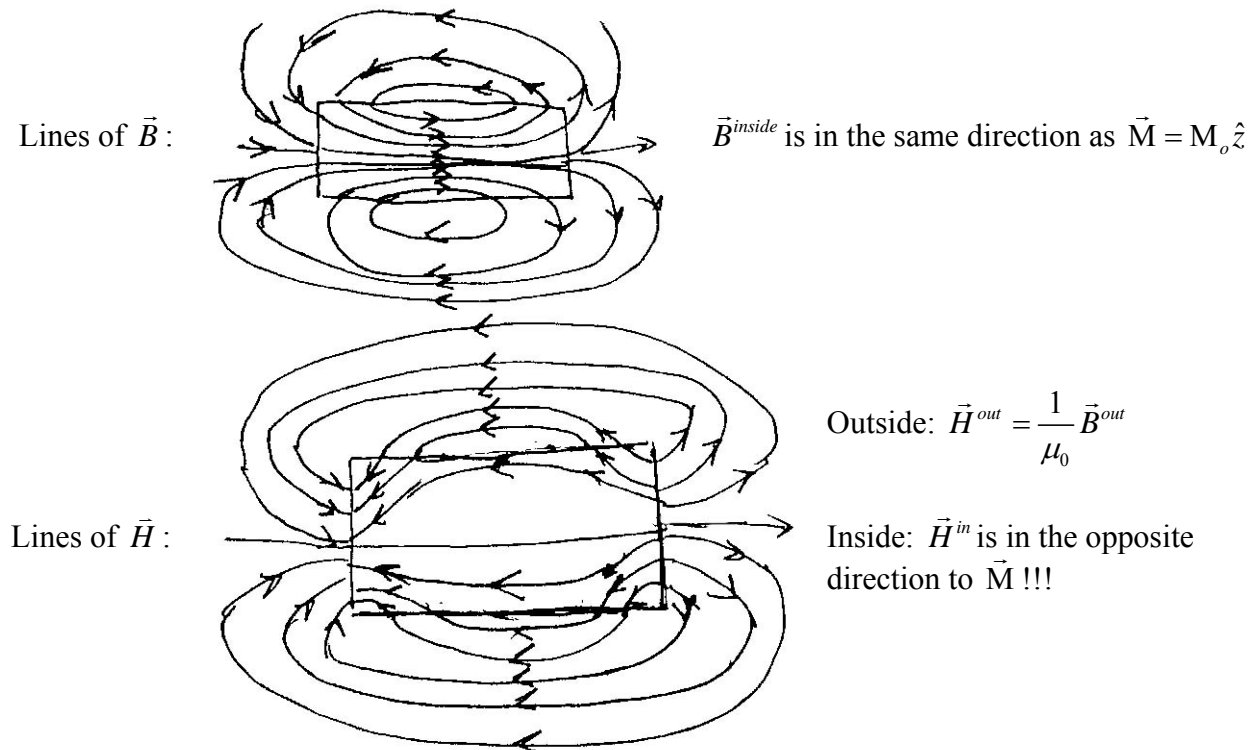
Consider Ampere's Circuital Law for  $\vec{H}$ :  $\oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{free}^{enclosed}$

But:  $\exists$  no free current(s) in a bar magnet – does this mean that  $\vec{H}^{inside}(\vec{r}) = \vec{H}^{outside}(\vec{r}) = 0$ !!!!???

**!!! NONSENSE !!!**

$\vec{M}(\vec{r}) = M_0 \hat{z}$  inside the bar magnet.

$\vec{B}(\vec{r})$  for a cylindrical bar magnet =  $\vec{B}(\vec{r})$  for a short solenoid (w/ no pitch angle).



Compare these pix to that for  $\vec{E}$ ,  $\vec{D}$  and  $\vec{P}$  for the bar electret – see P435 Lecture Notes 10, p. 33.

Note that since  $\vec{J}_{Bound}(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r})$  and  $\vec{M}(\vec{r}) = \chi_m \vec{H}(\vec{r})$  then:  $\vec{J}_{Bound}(\vec{r}) = \chi_m \vec{\nabla} \times \vec{H}(\vec{r})$ ,

However:  $\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}_{free}(\vec{r}) \Rightarrow \vec{J}_{Bound}(\vec{r}) = \chi_m \vec{J}_{free}(\vec{r})$ .

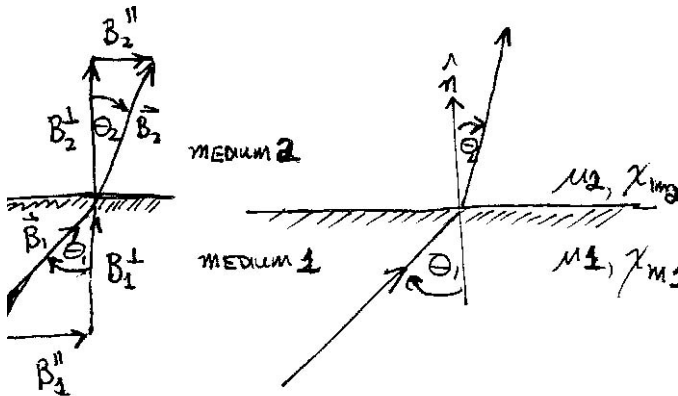
This relation says that *unless* a free current actually flows through a linear magnetic material of susceptibility  $\chi_m$ , with free volume current density  $\vec{J}_{free}(\vec{r})$ , then (and *only* then) will there be / arise a corresponding non-zero equivalent bound volume current density  $\vec{J}_{Bound}(\vec{r})$  which is related to  $\vec{J}_{free}(\vec{r})$  via  $\vec{J}_{Bound}(\vec{r}) = \chi_m \vec{J}_{free}(\vec{r})$ . This is analogous to the relationship that we found

between  $\rho_{Bound}(\vec{r})$  and  $\rho_{free}(\vec{r})$  for linear dielectric materials:  $\rho_{Bound}(\vec{r}) = -\left(1 - \frac{1}{K_e}\right) \rho_{free}(\vec{r})$

{See P435 Lecture Notes 10, page 21}. If  $\vec{J}_{free}(\vec{r}) = 0$  inside a magnetic material, then

$\vec{J}_{bound}(\vec{r}) = 0$  inside the magnetic material, also. In this situation, any/all non-zero effective bound currents can only exist on the surfaces of the magnetic material!!!

### MAGNETOSTATIC BOUNDARY CONDITIONS FOR MAGNETIC MEDIA



$\perp$  = normal (i.e. perpendicular) component relative to plane of interface

$\parallel$  = parallel component relative to plane of interface (tangential component)

From  $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$  (no magnetic charges/no monopoles) in integral form:  $\oint_S \vec{B}(\vec{r}) \cdot \hat{n} da = 0$ .

Use a Gaussian pillbox for the enclosing surface  $S$ , vertically centered on the interface between the two magnetic media. We then shrink the height of pillbox to infinitesimally above/below the interface – then only the top/bottom portions of the surface integral will contribute anything.

We thus obtain a condition on the perpendicular components of  $\vec{B}(\vec{r})$  above/below interface:

$$B_2^{\perp \text{ above}}(\vec{r})|_{\text{surface}} = B_1^{\perp \text{ above}}(\vec{r})|_{\text{surface}}$$

We also have the constitutive relation:  $\vec{B}(\vec{r})/\mu_o = \vec{H}(\vec{r}) + \vec{M}(\vec{r})$  and thus  $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0 \Rightarrow$

$\vec{\nabla} \cdot \vec{H}(\vec{r}) = -\vec{\nabla} \cdot \vec{M}(\vec{r})$ . In integral form this relation becomes:  $\oint_S \vec{H}(\vec{r}) \cdot \hat{n} da = -\oint_S \vec{M}(\vec{r}) \cdot \hat{n} da$ .

Using the same Gaussian pillbox, we obtain the following condition on the perpendicular components of  $\vec{H}(\vec{r})$  and  $\vec{M}(\vec{r})$  above/below interface:

$$\left[ H_2^{\perp \text{above}}(\vec{r}) - H_1^{\perp \text{above}}(\vec{r}) \right]_{\text{surface}} = - \left[ M_2^{\perp \text{above}}(\vec{r}) - M_1^{\perp \text{below}}(\vec{r}) \right]_{\text{surface}}$$

Ampere's Law for  $\vec{B}(\vec{r})$  is:  $\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_o \vec{J}_{\text{Tot}}(\vec{r})$ , with  $\vec{J}_{\text{Tot}}(\vec{r}) = \vec{J}_{\text{free}}(\vec{r}) + \vec{J}_{\text{Bound}}(\vec{r})$  which in integral form becomes:  $\oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = \mu_o I_{\text{Tot}}^{\text{enclosed}}$ , with  $I_{\text{Tot}}^{\text{enclosed}} = I_{\text{free}}^{\text{enclosed}} + I_{\text{Bound}}^{\text{enclosed}}$ . We can take a (rectangular) contour vertically centered above/below the interface between the two magnetic media; we then shrink the height of this contour to be infinitesimally above/below the interface, thus only the tangential portions of the line integral above/below the interface will contribute. We obtain the following condition on the tangential components of  $\vec{B}(\vec{r})$  above / below the interface, where  $\vec{K}_{\text{Tot}}(\vec{r}) = \vec{K}_{\text{free}}(\vec{r}) + \vec{K}_{\text{Bound}}(\vec{r})$  and  $\hat{n}$  is shown in the figure above:

$$\frac{1}{\mu_o} \left[ B_2^{\parallel \text{above}}(\vec{r}) - B_1^{\parallel \text{below}}(\vec{r}) \right]_{\text{surface}} = K_{\text{Tot}}(\vec{r}) \Big|_{\text{surface}}$$

We can write this more compactly/succinctly in vector form as:

$$\frac{1}{\mu_o} \left[ \vec{B}_2^{\text{above}}(\vec{r}) - \vec{B}_1^{\text{below}}(\vec{r}) \right]_{\text{surface}} = \vec{K}_{\text{Tot}}(\vec{r}) \times \hat{n} \Big|_{\text{surface}}$$

Since  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ , this relation can also be equivalently written as:

$$\frac{1}{\mu_o} \left[ \frac{\partial \vec{A}_2^{\text{above}}(\vec{r})}{\partial n} - \frac{\partial \vec{A}_1^{\text{below}}(\vec{r})}{\partial n} \right]_{\text{surface}} = -\vec{K}_{\text{Tot}}(\vec{r}) \Big|_{\text{surface}}$$

Similarly, Ampere's Law for  $\vec{H}(\vec{r})$  is:  $\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}_{\text{free}}(\vec{r})$  which in integral form becomes:

$\oint_C \vec{H}(\vec{r}) \cdot d\vec{\ell} = I_{\text{free}}^{\text{enclosed}}$ . Again, we can take a (rectangular) contour vertically centered above / below the interface between the two magnetic media; we then shrink the height of this contour to be infinitesimally above/below the interface, thus only the tangential portions of the line integral above/below the interface will contribute. We obtain the following condition on the tangential components of  $\vec{H}(\vec{r})$  above/below the interface:  $\left[ H_2^{\parallel \text{above}}(\vec{r}) - H_1^{\parallel \text{below}}(\vec{r}) \right]_{\text{surface}} = K_{\text{free}}(\vec{r}) \Big|_{\text{surface}}$

which can also be written compactly/succinctly in vector form as:

$$\left[ \vec{H}_2^{\text{above}}(\vec{r}) - \vec{H}_1^{\text{below}}(\vec{r}) \right]_{\text{surface}} = \vec{K}_{\text{free}}(\vec{r}) \times \hat{n} \Big|_{\text{surface}}$$

Since  $\vec{B}(\vec{r}) = \mu \vec{H}(\vec{r})$  or:  $\vec{H}(\vec{r}) = \vec{B}(\vec{r}) / \mu$  and  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ , this relation can also be equivalently written as:

$$\left[ \left( \frac{1}{\mu_2} \right) \frac{\partial \vec{A}_2^{\text{above}}(\vec{r})}{\partial n} - \left( \frac{1}{\mu_1} \right) \frac{\partial \vec{A}_1^{\text{below}}(\vec{r})}{\partial n} \right]_{\text{surface}} = -\vec{K}_{\text{free}}(\vec{r}) \Big|_{\text{surface}}$$

### Appendix:

If we wanted to/needed to define the macroscopic magnetization  $\vec{M}$  (auxiliary macroscopic matter field) in terms of the fundamental field  $\vec{B}$ , in analogy to  $\vec{P}(\vec{r}) = \epsilon_o \chi_e \vec{E}(\vec{r})$ . We have seen (above) that given the constitutive relation  $\vec{H}(\vec{r}) = \vec{B}(\vec{r})/\mu_o - \vec{M}(\vec{r})$  we are unable to do so. The problem here actually focuses squarely on  $\mu_o$ , the magnetic permeability of free space:

Note that  $\mu_o$  is actually derived/defined from: 
$$c^2 = \frac{1}{\epsilon_o \mu_o} \Rightarrow \mu_o \equiv \frac{1}{\epsilon_o c^2}$$

The electric permittivity of free space is  $\epsilon_o = 8.85 \times 10^{-12}$  Farads/meter. The Farad is the SI unit of capacitance,  $C$  (in electrostatics) – the ability of something (in this case, the vacuum) to store energy in the electric field of that something. Note that  $\epsilon_o$  has the dimensions of capacitance/unit length – Farads/meter.

The numerical value of the magnetic permeability of free space  $\mu_o$  is defined from the experimental measurement of  $c = 3 \times 10^8$  m/s (speed of light in free space) and the electric permittivity of free space  $\epsilon_o = 8.85 \times 10^{-12}$  Farads/meter, thus:

$$\mu_o \equiv 4\pi \times 10^{-7} \text{ Newtons/Ampere}^2 = (\text{kg-meter/sec}^2)/\text{Ampere}^2 = \text{Henrys/meter}$$

$$\{1 \text{ Newton/ Ampere}^2 = 1 \text{ Henry} = 1 \text{ Tesla-m}^2/\text{Ampere} = 1 \text{ Weber/Ampere}\}$$

The Henry is the SI unit of inductance,  $L$  (in magnetostatics) – the ability of something (in this case, the vacuum) to store energy in the magnetic field of that something. Note that  $\mu_o$  has the dimensions of inductance/unit length – Henrys/meter.

However, if we alternatively define  $\xi_o \equiv 1/\mu_o = \frac{10^7}{4\pi}$  Amperes<sup>2</sup>/Newton = meters/Henry,

Then  $\xi_o$  = inverse magnetic permeability (magnetic “reluctance”??) of free space.

Then: 
$$c^2 = \xi_o / \epsilon_o \quad \text{or:} \quad \xi_o = c^2 \epsilon_o$$

Then the magnetic constitutive relation becomes: 
$$\vec{H} = \frac{1}{\mu_o} \vec{B} - \vec{M} \Rightarrow \vec{H} = \xi_o \vec{B} - \vec{M}$$
 in analogy

to  $\vec{D} = \epsilon_o \vec{E} + \vec{P}$  and we also have (for linear materials)  $\vec{H} = \xi \vec{B}$  in analogy to  $\vec{D} = \epsilon \vec{E}$ .

However, here we will define  $\xi \equiv \xi_o (1 - \chi_m^*)$  in contrast to  $\mu \equiv \mu_o (1 + \chi_m)$ .

Then: 
$$\vec{H} \equiv \xi_o \vec{B} - \vec{M} = \xi \vec{B} = \xi_o (1 - \chi_m^*) \vec{B} = \xi_o \vec{B} - \xi_o \chi_m^* \vec{B}$$
 and thus  $\vec{M} = \xi_o \chi_m^* \vec{B}$  in analogy to  $\vec{P} = \epsilon_o \chi_e \vec{E}$  for linear dielectrics.

Since: 
$$\vec{M} = \chi_m \vec{H} = \chi_m \vec{B} / \mu = \left[ \chi_m / (1 + \chi_m) \right] \vec{B} / \mu_o$$
 we see that  $\xi_o \chi_m^* = \chi_m / (1 + \chi_m) \mu_o$

or: 
$$\chi_m^* = \chi_m / (1 + \chi_m)$$
.