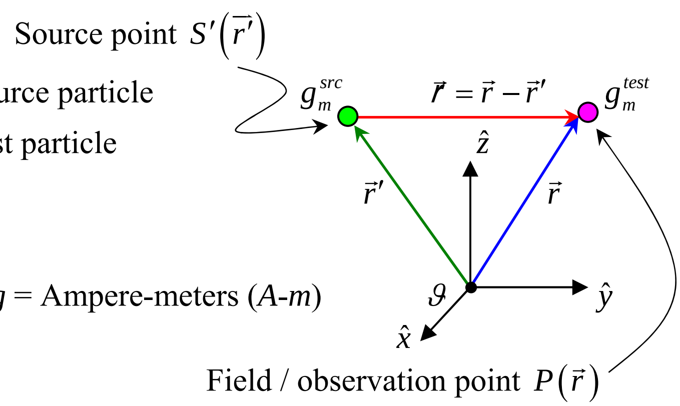
LECTURE NOTES 18

MAGNETIC MONOPOLES – FUNDAMENTAL / POINTLIKE MAGNETIC CHARGES

No fundamental, point-like isolated separate North or South magnetic poles – i.e. *N* or *S* magnetic charges have ever been conclusively / reproducibly observed. In principle, there is no physical law, or theory, that forbids their existence. So we may well ask, why did “nature” not “choose” to have magnetic monopoles in our universe – or, if so, why are they so extremely rare, given that many people (including myself) have looked for / tried to detect their existence…

If magnetic monopoles / fundamental point magnetic charges did exist in nature, they would obey a Coulomb-type force law (just as electric charges do):

$$\boxed{\vec{F}_m(\vec{r}) = g_m^{test}\vec{B}_m(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{g_m^{test}g_m^{src}}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}}$$



Where:
- g_m^{src} = magnetic charge of source particle
- g_m^{test} = magnetic charge of test particle
- $g_m \equiv +g\ (\equiv North\ pole)$
- $g_m \equiv -g\ (\equiv South\ pole)$
- SI units of magnetic charge g = Ampere-meters (*A-m*)

Units check: $F_m(Newtons) = \left(\frac{\mu_o}{4\pi}\right)\frac{g_m^2}{\boldsymbol{\mathscr{r}}^2}$ $\quad \mu_o = 4\pi\times10^{-7}\ Newtons/Ampere^2\ \left(N/A^2\right)$

$$\text{Newton} = N = \left(\frac{N}{A^2}\right)\frac{(A-m)^2}{m^2} = \left(\frac{N}{\cancel{A^2}}\right)\frac{\cancel{A^2}-\cancel{m^2}}{\cancel{m^2}} = N$$

Then (the radial) $\vec{B}$-field of a magnetic monopole is:

$$\boxed{\vec{B}_m(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{g_m^{src}}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}} \quad (\text{SI units = Tesla} = \frac{N}{A-m})}$$

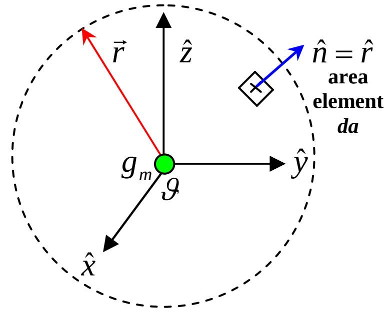
Units check: $Tesla = \dfrac{N}{A-m} = \left(\dfrac{N}{\cancel{A^2}}\right)\dfrac{\cancel{A^2}-\cancel{m}}{\cancel{m^2}} = \dfrac{N}{A-m}$

$1\ \text{Tesla} \equiv 1\dfrac{Newton}{Ampere-meter}$ $1\ \text{T} = 1\ \dfrac{N}{A-m}$
g_m = Ampere-meters (*A-m*)

The magnetic flux associated with a magnetic monopole with magnetic charge g_m^{src} is:

$$\Phi_m \equiv \oint_S \vec{B}_m(\vec{r}) \cdot \hat{n} da = \left(\frac{\mu_o}{\cancel{4\pi}}\right)\frac{g_m^{src}}{\cancel{r^2}}\left(\cancel{4\pi}\,\cancel{r^2}\right) = \mu_o g_m^{src} \quad \text{or:} \quad \boxed{\Phi_m = \mu_o g_m^{src}} \quad \text{(Webers, or Tesla-m}^2\text{)}$$

$\hat{\mathscr{r}} = \vec{r}$, and $\mathscr{r} = r$ because the source charge g_m^{src} is located at the origin ϑ:



SI units of magnetic flux:
Tesla-meters^2 $(\Phi_m = B \cdot A)$

1 (T–m^2) = $1\left(\frac{N-m}{A}\right) = 1$ Weber (Wb)

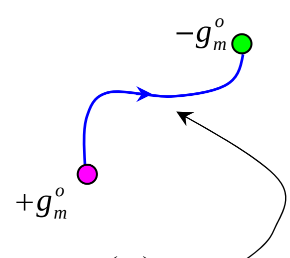
Units check:

$$T - m^2 \left\{ = \frac{N}{A-m} \cdot m^2 = \frac{N-m}{A} \right\} = \frac{N}{A^{\cancel{2}}} \cdot \cancel{A} - m = \frac{N-m}{A}$$

We know that electric charge is quantized in units of e = 1.602x 10^{-19} Coulombs. Similarly we would expect magnetic charges to also be quantized (if they do indeed exist in nature).

We know that magnetic flux Φ_m is quantized - the quantum (i.e. the smallest unit) of magnetic flux is one flux quantum:

$$\boxed{\Phi_m^o \equiv \left(\frac{h}{e}\right) = \frac{6.626\times10^{-34}\,Joule-sec}{1.602\times10^{-19}\,coulombs} = 4.136\times10^{-15}\,Webers}$$



$$1\Phi_m^o = \left(\frac{h}{e}\right) = \mu_o g_m^o$$

Where: h = Planck's constant = 6.626 x 10^{-34} Joule-sec

Units check: $\Phi_m^o = Webers = Tesla - m^2 = \left(\frac{N}{A-\cancel{m}}\right) \cdot m^{\cancel{2}} = \left(\frac{N-m}{A}\right)$

And: $\left(\frac{h}{e}\right) = \frac{Joule-sec}{coulombs} = \frac{Joules}{Amp} = \frac{Newton-meters}{Amp} = \left(\frac{N-m}{A}\right)$

Then the smallest integer unit of quantized magnetic charge g_m^o is:

$$\boxed{\Phi_m^o = \mu_o g_m^o} \quad \text{or:} \quad \boxed{g_m^o = \frac{1}{\mu_o}\Phi_m^o = \frac{1}{\mu_o}\left(\frac{h}{e}\right) = \frac{4.136\times10^{-15}\,Wb}{4\pi\times10^{-7}\,N/A^2} = 3.2914\times10^{-9}} \text{ Ampere-meters } (A\text{-}m)$$

Now, it's possible that magnetic monopoles could exist with integer multiples of this smallest / quantized amount of magnetic charge g_m^o, i.e. $g_m^n = n g_m^o$ where n = integer = ±1, ±2, ±3, ... and: $+g_m^o = N\ (North\ Pole)$ and $-g_m^o = S\ (South\ Pole)$.

Then if $\boxed{\Phi_m^o = \mu_o g_m^o = \left(\frac{h}{e}\right)}$, then $\boxed{\Phi_m^n = \mu_o n g_m^o = \mu_o g_m^n = n\left(\frac{h}{e}\right)}$ i.e. $\boxed{\Phi_m^n = n\Phi_m^o}$.

<u>And</u>: $\boxed{\mu_o g_m^n = n\left(\frac{h}{e}\right)} \Rightarrow \boxed{e g_m^n = n\left(\frac{h}{\mu_o}\right)}$ but: $\boxed{\frac{1}{\mu_o} = \varepsilon_o c^2}$ where $c = 3 \times 10^8$ m/s (speed of light)

Defining $\boxed{\hbar \equiv \left(\frac{h}{2\pi}\right)}$ then $g_m^n = \frac{n\varepsilon_o h c^2}{e} = \frac{n}{2} 4\pi\varepsilon_o \left(\frac{h}{2\pi}\right) c^2 = \frac{n}{2}\underbrace{\left(\frac{4\pi\varepsilon_o \hbar c}{e^2}\right)}_{\equiv 1/\alpha_{em}} ec$

i.e. $\boxed{g_m^n = \frac{n}{2}\left(\frac{1}{\alpha_{em}}\right) ec}$ (A-m) and the fine-structure constant $\boxed{\alpha_{em} \equiv \frac{e^2}{4\pi\varepsilon_o \hbar c} = \frac{1}{137.036...}}$

(dimensionless quantity) and numerically, $2\alpha_{em} = \frac{2}{137.036} \simeq \frac{1}{68.5}$

Then: $\boxed{g_m^n = 68.5 n (ec) \quad A\text{-}m \qquad (n = \pm 1, \pm 2, \pm 3, \ldots)}$

Thus, we see that the relative strength of magnetic monopole (e.g. North-South pole) attraction is huge in comparison to that associated with electric monopole (e.g. e^+-e^-) attraction:

$$\frac{F_m^{(n)}}{F_e} = \frac{\left|\vec{F}_m^{(n)}\right|}{\left|\vec{F}_e\right|} = \frac{\left(\frac{\mu_o}{4\pi}\right) g_m^{n\,2} \big/ r^2}{\left(\frac{1}{4\pi\varepsilon_o}\right) e^2 \big/ r^2} = \varepsilon_o \mu_o \left(\frac{g_m^n}{e}\right)^2 = \frac{1}{\cancel{c^2}} 68.5^2 n^2 \cancel{c^2} = (68.5n)^2 \simeq 4700 n^2$$

$(n = \pm 1, \pm 2, \pm 3, \ldots)$

Force of magnetic attraction between N-S monopoles $\gg$ Force of attraction between two (opposite) electric charges.

If $\boxed{\Phi_m^o = \left(\frac{h}{e}\right)}$ magnetic flux quantum (= 4.136 x 10^{-15} Wb)

<u>Then</u>: $\boxed{\frac{\Phi_m^o}{\Phi_E^o} = \frac{\oint_S \vec{B}\cdot\hat{n}da}{\oint_S \vec{E}\cdot\hat{n}da} = \frac{\mu_o g_m^o}{e/\varepsilon_o} = \varepsilon_o\mu_o\left(\frac{g_m^o}{e}\right) = \frac{1}{c^2}\left(\frac{g_m^o}{e}\right)}$

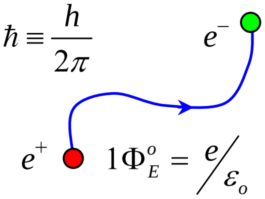
But: $\boxed{g_m^o = 68.5ec}$ or: $\boxed{\left(\frac{g_m^o}{e}\right) = 68.5c = \left(\frac{1}{2\alpha_{em}}\right)c}$ where: $\boxed{\alpha_{em} = \frac{e^2}{4\pi\varepsilon_o\hbar c} = \frac{1}{137.036...}}$

$\therefore \quad \boxed{\frac{\Phi_m^o}{\Phi_E^o} = \frac{1}{c^2}\left(\frac{g_m^o}{e}\right) = \frac{1}{c^2}\left(\frac{1}{2\alpha_{em}}\right)c = \frac{1}{c}\left(\frac{1}{2\alpha_{em}}\right) = \frac{1}{2\alpha_{em}c} = 68.5\big/c}$

We can rearrange this latter relation to obtain the electric flux quantum:

$$\Phi_E^o = 2\alpha_{em} c \Phi_m^o = 2\alpha_{em} c\left(\frac{h}{e}\right) = 2\left(\frac{e^{\cancel{2}}}{4\pi\varepsilon_o \hbar \cancel{c}}\right)\cancel{c}\left(\frac{h}{\cancel{e}}\right) = 2\left(\frac{eh}{4\pi\varepsilon_o \hbar}\right) \quad \text{where } \hbar \equiv \frac{h}{2\pi}$$

$$\Phi_E^o = 2\left(\frac{eh}{4\pi\varepsilon_o \left(h/2\pi\right)}\right) = \left(\frac{\cancel{4\pi}\, e\, \cancel{h}}{\cancel{4\pi}\, \varepsilon_o\, \cancel{h}}\right) = \left(e/\varepsilon_o\right)$$



<u>Numerically</u>:

$$\boxed{\Phi_E^o = \left(e/\varepsilon_o\right)} = \text{Electric Flux Quantum} = \frac{1.602\times10^{-19}\,Coulombs}{8.85\times10^{-12}\,\frac{Coulombs^2}{N-m^2}} = 1.810\times10^{-8}\,\frac{N-m^2}{Coulomb}$$

$$\boxed{\Phi_m^o = \left(\mu_o g_m^o\right) = \left(h/e\right)} = \text{Magnetic Flux Quantum} = 4.136 \times 10^{-15}\ Wb\ (T\text{–}m^2)$$

$$\boxed{\frac{\Phi_m^o}{\Phi_E^o} = \frac{1}{2\alpha_{em} c}} \text{ and } \boxed{g_m^n = \frac{n}{2\alpha_{em}} ec} \text{ Ampere-meters where } n = \pm1,\ \pm2,\ \pm3$$

<u>Gauss' Law</u>:

$$\Phi_m^o = \mu_o g_m^o = \frac{h}{e} = 4.136\times10^{-15} Wb\left(= T - m^2 = \left(\frac{N}{A-m}\right) - m^2\right) = \left(\frac{F_m}{g_m}\right)\bullet Area = B_m \bullet Area$$

$$\Phi_E^o = \frac{e}{\varepsilon_o} = 1.810\times10^{-8}\left(\left(\frac{N}{c}\right) - m^2\right) = \left(\frac{F_E}{e}\right)\bullet Area = E \bullet Area$$

The Dirac Quantization Condition

In 1931, Paul Adrian Maurice Dirac showed (see P.A.M. Dirac, Proc. Roy. Soc., London, Ser. A133, 60, (1931)) that quantization of electric charge (i.e. why e is e could be explained if magnetic monopoles existed, because then:

$e * \mu_o g_m = \dfrac{e g_m}{\varepsilon_o c^2} = nh$ (SI units)	Dirac Quantization Condition

e = electric charge = 1.602 x 10^{-19} Coulombs
g_m = magnetic charge (SI units of Ampere-meters)
$\mu_o = 4\pi \times 10^{-7}\, Newtons / Ampere^2$ (magnetic permeability of free space)
$\varepsilon_o = 8.85 \times 10^{-12}\, F/m \left(\dfrac{Coulombs^2}{Newton - meter^2} \right)$ (electric permittivity of free space)
$c = 1/\sqrt{\varepsilon_o \mu_o}$ = speed of light (in free space/vacuum) = 3 × 10^8 meters / second
n = integer ($\neq 0$) $n = \pm 1, \pm 2, \pm 3, \ldots$
h = Planck's Constant = 6.626 x 10^{-34} Joule-sec = (N-m-s)

Dirac originally obtained this relation by considering the motion of an electron circling a magnetic monopole of magnetic charge g_m, with radial magnetic field

$$\vec{B}_m(\vec{r}) = \left(\frac{\mu_o}{4\pi} \right) \frac{g_m}{r^2} \hat{r} \quad \text{(SI units: Tesla = } N / A\text{-}m\text{)}$$

Quantum mechanically, the wave function $\psi_e(\vec{r})$ of the electron circling the magnetic monopole (assumed to be infinitely heavy) must be single-valued in φ, i.e. $\boxed{\psi_e(\varphi = 2\pi n) = \psi_e(\varphi = 0)}$ in analogy e.g. to the Bohr model of the Hydrogen atom (e^- bound to proton, p)

In other words for the electron-monopole system, Dirac demanded:

$$\boxed{\psi_e(\vec{r}) \to \psi'_e(\vec{r}) = \psi_e(r,\theta) e^{i\varphi} = \psi_e(r,\theta) e^{i(2\pi n)}}$$

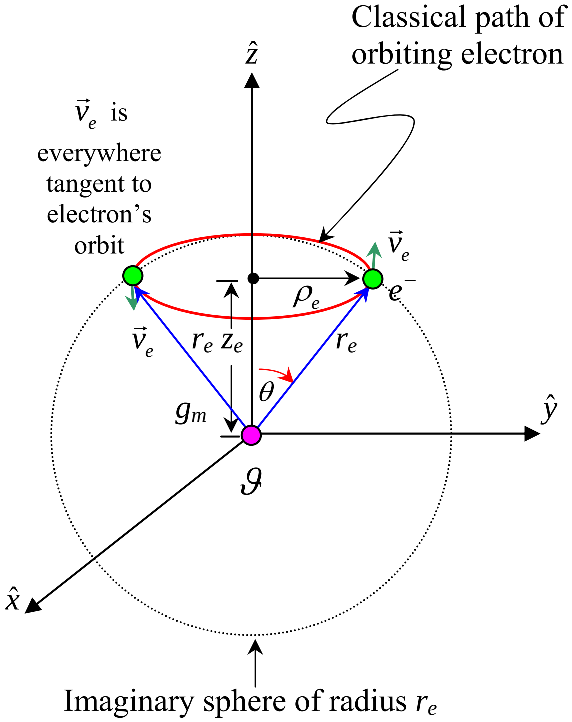
The quantum physics of the $e^- g_m$ system gives $2\pi n = 2\pi (e \mu_o g_m)/h$ where $n = \pm 1, \pm 2, \pm 3, \ldots$

Or: $\boxed{n = \dfrac{e \mu_o g_m}{h}}$ or: $\boxed{e \mu_o g_m = nh}$ ⇐ Dirac Quantization Condition (in SI units)

Using: $\boxed{c^2 = \dfrac{1}{\varepsilon_o \mu_o}}$ and: $\boxed{\alpha_{em} \equiv \dfrac{e^2}{4\pi \varepsilon_o \hbar c}}$ = fine structure constant (dimensionless) and $\hbar \equiv \dfrac{h}{2\pi}$

This relation can be rewritten as: $\boxed{g_m = \dfrac{n}{2\alpha_{em}} ec}$ or: $\boxed{\dfrac{g_m}{e} = \dfrac{n}{2\alpha_m} c \simeq 68.5 nc}$

Classically, the motion of an electron circling a magnetic monopole is shown below, at a height z_c above it (at the origin).



The "Lorentz" force acting on the electron is:

$$\vec{F}_e^m(\vec{r}_e) = -e\vec{v}_e \times \vec{B}_m(\vec{r}_e)$$

$$= -e\vec{v}_e \times \left(\frac{\mu_o}{4\pi}\right)\frac{g_m}{r_e^2}\hat{r}$$

n.b. assume $g_m =$ <u>North</u> magnet pole i.e. $g_m > 0$

$$= -\left(\frac{e\mu_o g_m}{4\pi}\right)\frac{\vec{v}_e \times \hat{r}}{r_e^2}$$

$$= -\left(\frac{nh}{4\pi}\right)\frac{\vec{v}_e \times \hat{r}}{r_e^2}$$

$$= -\left(\frac{nh}{4\pi}\right)\frac{m_e\vec{v}_e \times \hat{r}}{r_e^2}\left(\frac{c^2}{m_e c^2}\right)$$

$$= -\left(\frac{nh}{4\pi}\right)\frac{\vec{p}_e \times \vec{r}_e}{r_e^3}\left(\frac{c^2}{m_e c^2}\right)$$

The angular momentum of the electron is $\vec{L}_e = \vec{r}_e \times \vec{p}_e$:

$$\therefore \boxed{\vec{F}_e^m(\vec{r}_e) = +\left(\frac{nh}{4\pi}\right)\frac{\vec{L}_e c^2}{m_e c_2 r_e^3}}$$

(n.b. the <u>direction</u> of $\vec{L}_e$ is not constant here..)

The magnetic force acting on the electron bends it around in the orbit as shown in above figure.

We can also view this from a different perspective: the electron circling the magnetic monopole in this manner creates an electric current I

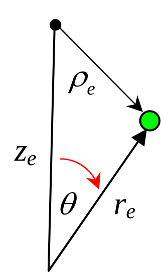
$$I = \frac{e}{\tau_{orbit}} = \frac{e}{(C/v_e)} \qquad \tau_{orbit} = \frac{C}{v_e} = \frac{2\pi\rho_e}{v_e} = \frac{2\pi r_e \sin\theta}{v_e}$$

Which in turn creates an orbital magnetic dipole moment:

$$\vec{m}_e = I\vec{a} = I\pi\rho_e^2(-\hat{z}) = \frac{ev_e}{2\pi\rho_e}\pi\rho_e^2(-\hat{z}) \text{ (SI units Amp-m}^2\text{)}$$

$$\vec{m}_e = \tfrac{1}{2}ev_e\rho_e(-\hat{z}) \quad \text{where} \quad \rho_e = r_e\sin\theta$$

$$\boxed{\vec{m}_e = -\tfrac{1}{2}ev_e\rho_e\hat{z} = -\tfrac{1}{2}e\vec{v}_e r_e\sin\theta\hat{z}}$$



<u>Reminder</u>: $I =$ <u>conventional current</u>, which for a circulating e^- <u>electron</u>, flows in the direction <u>opposite</u> to the electron's orbital motion (see above figure).

The orbital magnetic moment of the electron then interacts with the magnetic monopole.

Quantum mechanically, the e^- behaves as a wave, not a point particle and thus the wavefunction $\psi_e(\vec{r}_e)$ of the electron spreads out along its orbit as a periodic wave in such a way that an integer number of deBroglie wavelengths fit around the classical circumferential path, i.e.

$$n\lambda_n = C = 2\pi\rho_e \text{ with } n = 1, 2, 3, \ldots.$$

Note also that the electron is *not* actually bound to the magnetic monopole – its orbit is stable. $\exists$ no binding energy between these two particles; given an initial electron velocity, e.g. $\vec{v}_e = -v_e\hat{x}$, with the electron initially at height z_e above the magnetic monopole, the radial magnetic field of the magnetic monopole will bend the electron's path into orbit shown. Recall also that magnetic forces do no work…

The Duality Transformation for Electromagnetism

Because of the intimate connection between $\vec{E}$ and $\vec{B}$ at the microscopic / fundamental / elementary particle physics level, there (obviously) exists an intimate connection between $\vec{E}$ and $\vec{B}$ at the macroscopic level.

A duality transformation is a simultaneous rotation in an abstract mathematical space by an angle φ of all electric and magnetic phenomena, which leaves all of the laws associated with the physics of electromagnetism unchanged – it's a "knob" that allows us to rotate space $\rightleftarrows$ time!!!

Electromagnetism is invariant under a duality transformation.

By carrying out a duality transformation, we simultaneously rotate all electric and magnetic phenomena by an angle φ in this abstract mathematical space, thus we can change electric fields $\rightleftarrows$ magnetic fields, electric charges $\rightleftarrows$ magnetic charges and we would never know the difference!

Note: In carrying out duality transformations on all electric and magnetic phenomena, in order for all of these quantities to transform properly, each duality transform *pair* must have the same physical units.

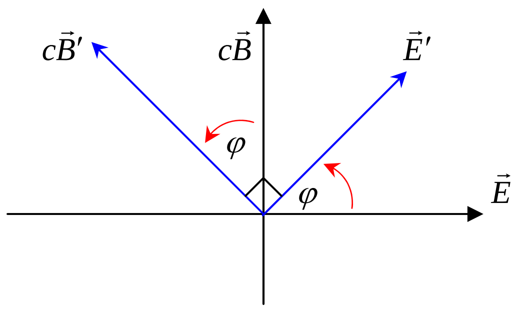
e.g. $\left(\vec{E} \rightleftarrows c\vec{B}\right)$, $\left(ec \rightleftarrows g_m\right)$, $\left(\varepsilon_o \rightleftarrows \dfrac{1}{\mu_o c^2}\right)$ n.b. c^2 = invariant under a duality transform

Other duality transform pairs are electric and magnetic currents and/or charge densities:

$$\left\{\begin{array}{l} \vec{J}_e \rightleftarrows \frac{1}{c}\vec{J}_m \\ \vec{K}_e \rightleftarrows \frac{1}{c}\vec{K}_m \\ \vec{I}_e \rightleftarrows \frac{1}{c}\vec{I}_m \end{array}\right\} \qquad \left\{\begin{array}{l} \rho_e \rightleftarrows \frac{1}{c}\rho_m \\ \sigma_e \rightleftarrows \frac{1}{c}\sigma_m \\ \lambda_e \rightleftarrows \frac{1}{c}\lambda_m \end{array}\right\}$$

$$I_m \equiv \frac{dQ_m}{dt}$$

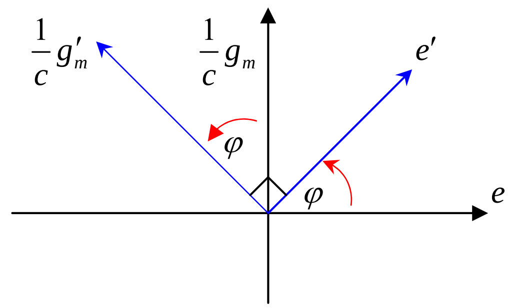
Duality Transform for $\vec{E} \rightleftarrows c\vec{B}$



$$\vec{E}' = \vec{E}\cos\varphi + c\vec{B}\sin\varphi$$

$$c\vec{B}' = c\vec{B}\cos\theta - \vec{E}\sin\varphi$$

Duality Transform for e and $\frac{1}{c}g_m$



$$e' = e\cos\varphi + \frac{1}{c}g_m\sin\varphi$$

$$\frac{1}{c}g_m' = \frac{1}{c}g_m\cos\varphi - e\sin\varphi$$

Duality Transforms for:

$$\vec{J}_e \rightleftarrows \frac{1}{c}\vec{J}_m$$

$$\vec{K}_e \rightleftarrows \frac{1}{c}\vec{K}_m$$

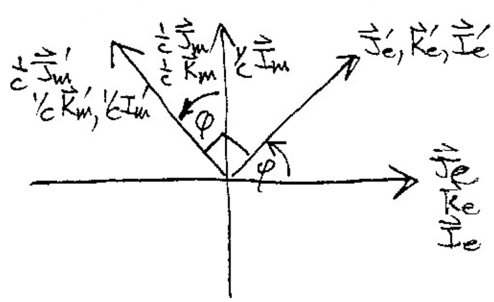
$$\vec{I}_e \rightleftarrows \frac{1}{c}\vec{I}_m$$

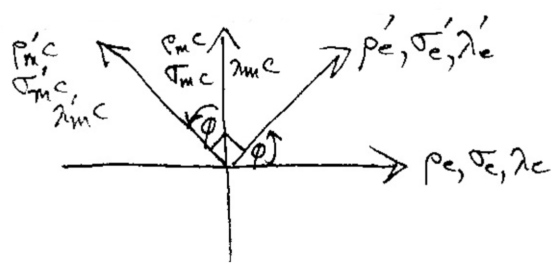
Duality Transforms for:

$$\rho_e \rightleftarrows \frac{1}{c}\rho_m$$

$$\sigma_e \rightleftarrows \frac{1}{c}\sigma_m$$

$$\lambda_e \rightleftarrows \frac{1}{c}\lambda_m$$





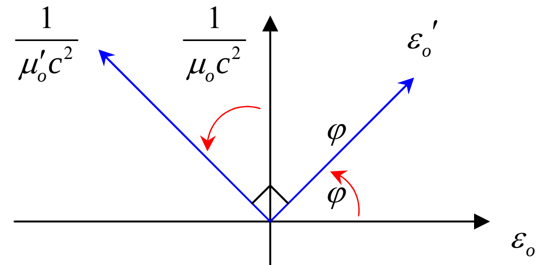
$$\begin{pmatrix}\vec{J}_e' \\ \vec{K}_e' \\ \vec{I}_e'\end{pmatrix} = \begin{pmatrix}\vec{J}_e \\ \vec{K}_e \\ \vec{I}_e\end{pmatrix}\cos\varphi + \frac{1}{c}\begin{pmatrix}\vec{J}_m \\ \vec{K}_m \\ \vec{I}_m\end{pmatrix}\sin\varphi$$

$$\frac{1}{c}\begin{pmatrix}\vec{J}_m' \\ \vec{K}_m' \\ \vec{I}_m'\end{pmatrix} = \frac{1}{c}\begin{pmatrix}\vec{J}_m \\ \vec{K}_m \\ \vec{I}_m\end{pmatrix}\cos\varphi - \begin{pmatrix}\vec{J}_e \\ \vec{K}_e \\ \vec{I}_e\end{pmatrix}\sin\varphi$$

$$\begin{pmatrix}\rho_e' \\ \sigma_e' \\ \lambda_e'\end{pmatrix} = \begin{pmatrix}\rho_e \\ \sigma_e \\ \lambda_e\end{pmatrix}\cos\varphi + \frac{1}{c}\begin{pmatrix}\rho_m \\ \sigma_m \\ \lambda_m\end{pmatrix}\sin\varphi$$

$$\frac{1}{c}\begin{pmatrix}\rho_m' \\ \sigma_m' \\ \lambda_m'\end{pmatrix} = \frac{1}{c}\begin{pmatrix}\rho_m \\ \sigma_m \\ \lambda_m\end{pmatrix}\cos\varphi - \begin{pmatrix}\rho_e \\ \sigma_e \\ \lambda_e\end{pmatrix}\sin\varphi$$

Duality Transform for ε_o and $\dfrac{1}{\mu_o c^2}$



$$\varepsilon_o' = \varepsilon_o \cos\varphi + \frac{1}{\mu_o c^2}\sin\varphi \quad \text{and} \quad \frac{1}{\mu_o' c^2} = \frac{1}{\mu_o c^2}\cos\varphi - \varepsilon_o \sin\varphi$$

We define the 2 × 2 duality transform rotation matrices as: $R(\varphi) \equiv \begin{pmatrix} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ and its inverse $R^{-1}(\varphi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{pmatrix}$ Then $RR^{-1} = R^{-1}R = \overleftrightarrow{1} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{unit matrix}}$

<u>Then</u>: $\begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix} = R(\varphi)\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}$ or: $\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} = R^{-1}(\varphi)\begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix}$

$$\begin{pmatrix} \varepsilon_o' \\ \frac{1}{\mu_o' c^2} \end{pmatrix} = R(\varphi)\begin{pmatrix} \varepsilon_o \\ \frac{1}{\mu_o c^2} \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \varepsilon_o \\ \frac{1}{\mu_o c^2} \end{pmatrix} = R^{-1}(\varphi)\begin{pmatrix} \varepsilon_o' \\ \frac{1}{\mu_o' c^2} \end{pmatrix}$$

$$\begin{pmatrix} e' \\ g_m'/c \end{pmatrix} = R(\varphi)\begin{pmatrix} e \\ g_m/c \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} e \\ g_m/c \end{pmatrix} = R^{-1}(\varphi)\begin{pmatrix} e' \\ g_m'/c \end{pmatrix} \quad \text{etc....}$$

<u>An Example of the Use / Application of the Duality Transform</u> $\begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix} = R(\varphi)\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}$

Convert the solenoidal magnetic field associated with the motion of an electric charge $(v_e \ll c)$ into the solenoidal electric field associated with the motion of a magnetic charge $(v_{g_m} \ll c)$!

Start with $\boxed{\vec{B}_e(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) q \frac{\vec{v}_e \times \hat{r}}{r^2}}$

choose: $\varphi = 90^o$ then $R(90^o) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\vec{E}' = c\vec{B}$ $\quad$ $c\vec{B}' = -E'$

$\varepsilon_o' = \dfrac{1}{\mu_o c^2}$ $\quad$ $e' = \dfrac{1}{c} g_m$

$\dfrac{1}{\mu_o' c^2} = -\varepsilon_o$ $\quad$ $\dfrac{1}{c} g_m' = -e$

Multiply both sides by *c*: $\quad c\vec{B} = \left(\dfrac{\mu_o}{4\pi}\right) cq \dfrac{\vec{v} \times \hat{r}}{r^2}$

Change $cB \to E'$: $\quad E' = -\left(\dfrac{\mu_o c}{4\pi}\right)(-e)\dfrac{\vec{v} \times \hat{r}}{r^2}$

Change $-e \to \dfrac{1}{c} g_m'$: $\quad \vec{E}' = -\left(\dfrac{\mu_o c}{4\pi}\right)\left(\dfrac{1}{c} g_m'\right)\dfrac{\vec{v} \times \hat{r}}{r^2} = -\left(\dfrac{\mu_o}{4\pi}\right) g_m' \dfrac{\vec{v} \times \hat{r}}{r^2}$

Change $\mu_o \to \dfrac{1}{\varepsilon_o' c^2}$: $\quad \vec{E}' = -\dfrac{1}{c^2}\left(\dfrac{1}{4\pi\varepsilon_o}\right) g_m' \dfrac{\vec{v} \times \hat{r}}{r^2}$

All *EM* quantities - everything electromagnetic - now duality-transformed
Now, drop primes everywhere (i.e. can't tell the difference after φ-rotation!!)

$$\boxed{\vec{E}_m(\vec{r}) = -\frac{1}{c^2}\left(\frac{1}{4\pi\varepsilon_o}\right) g_m \frac{\vec{v}_m \times \hat{r}}{r^2}} \rightleftarrows \boxed{B_e(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) q \frac{\vec{v}_e \times \hat{r}}{r^2}}$$

This is where / how the minus sign arises!!