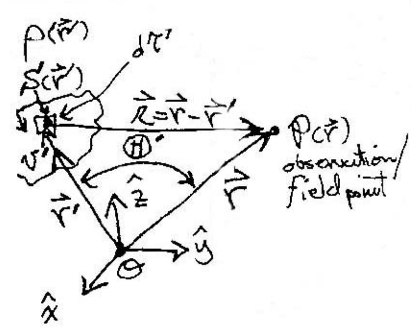
LECTURE NOTES 17

MULTIPOLE EXPANSION OF THE MAGNETIC VECTOR POTENTIAL $\vec{A}(\vec{r})$

As we saw in the case of electrostatics, we carried out a multipole expansion of the scalar electrostatic potential $V(\vec{r}) = \sum_{n=0}^{\infty} V_n(\vec{r})$ that was valid for distant observation points (field points) $P(\vec{r})$ far from a localized electrostatic source charge density distribution $\rho_{TOT}(\vec{r}\,')$, which in turn enabled us to a corresponding solution for $\vec{E}(\vec{r}) = \sum_{n=0}^{\infty} \vec{E}_n(\vec{r})$ via $\boxed{\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})}$.

$$\boxed{V(\vec{r}) = \sum_{n=0}^{\infty} V_n(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_{v'} (r')^n P_n(\cos\Theta')\, \rho(\underline{r}')\, d\tau'}$$

with: $\boxed{\cos\Theta' = \hat{r} \bullet \hat{r}'}$ and $\boxed{r = |\vec{r}| \gg r' = |\vec{r}\,'|}$

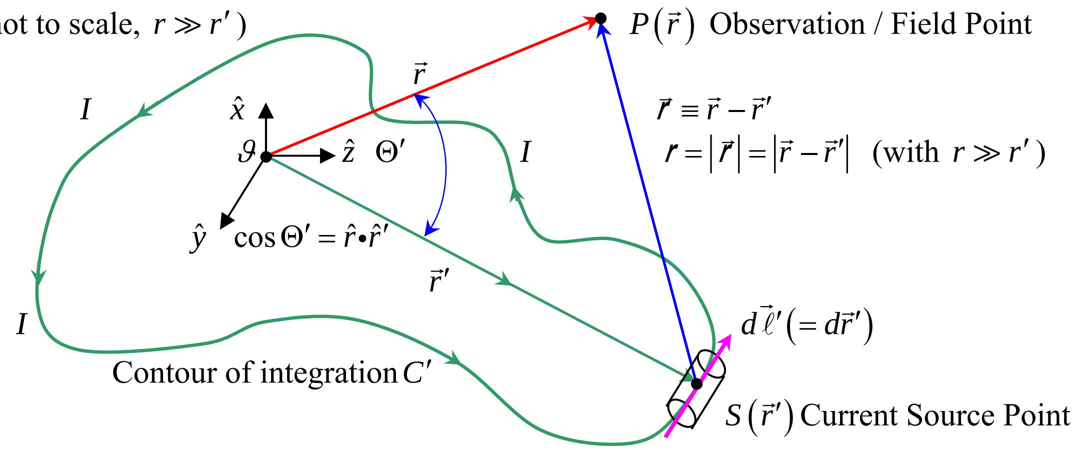


Likewise, we can similarly/analogously carry out the same kind of multipole expansion for the magnetic vector potential $\vec{A}(\vec{r}) = \sum_{n=0}^{\infty} \vec{A}_n(\vec{r})$**,** obtaining an expression for the magnetic vector potential that is valid for distant observation / field points $P(\vec{r})$ far from a localized magnetostatic source current density distribution – e.g. a filamenary/line current $I(\vec{r}\,')$, a surface current density $\vec{K}(\vec{r}\,')$, or a volume current density $\vec{J}(\vec{r}\,')$, which Obtaining a solution for $\vec{A}(\vec{r})$ then enables us to obtain a corresponding solution for the magnetic field $\vec{B}(\vec{r}) = \sum_{n=0}^{\infty} \vec{B}_n(\vec{r})$ via $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$.

Thus, we carry out a power series / Taylor series / binomial expansion in r'/r with $r' \ll r$ for $\vec{A}(\vec{r})$ {as we did in the electrostatics case for $V(\vec{r})$} where r $(\gg r')$ is the distance from the origin (located near to the charge / current source distribution). For $r \gg r'$, the multipole moment expansion will be dominated by the lowest-order non-vanishing multipole; higher-order terms in the expansion can be neglected/ignored.

Suppose we have a filamentary/line current loop, as shown in the figure below:

(Drawing not to scale, $r \gg r'$)



As we found before for the case of electrostatics, we can write a power-series expansion of $1/\mathscr{r}$ (for $r \gg r'$) as:

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\Theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\Theta')}_{\substack{Ordinary\ Legendre' \\ polynomial\ of\ 1st \\ kind,\ of\ order\ n}}} \text{ with } \boxed{\cos\Theta' = \hat{r}\bullet\hat{r}'}$$

Then, for a filamentary/line current source distribution with steady current I:

$$\boxed{\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\oint_{C'}\frac{\vec{I}(\vec{r}')d\ell'}{\mathscr{r}} = \left(\frac{\mu_o}{4\pi}\right)I\oint_{C'}\frac{d\vec{\ell}'(\vec{r}')}{\mathscr{r}}} \quad \text{(for } \left|\vec{I}(\vec{r}')\right| = I = \text{constant } \forall\ \vec{r}'\text{)}$$

$$\boxed{\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint_{C'}(r')^n P_n(\cos\Theta')\,d\vec{\ell}'(\vec{r})} \quad \text{with } \boxed{\cos\Theta' = \hat{r}\bullet\hat{r}'}$$

$$\boxed{\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\left\{\frac{1}{r}\oint_{C'}d\vec{\ell}'(\vec{r}') + \frac{1}{r^2}\oint_{C'}r'(\cos\Theta')\,d\vec{\ell}'(\vec{r}') + \frac{1}{r^3}\oint_{C'}(r')^2\left(\frac{3}{2}\cos^2\Theta' - \frac{1}{2}\right)d\vec{\ell}'(\vec{r}') + \ldots\right\}}$$

The first term ($\sim 1/r$) in the expansion is the magnetic monopole term, the 2nd term ($\sim 1/r^2$) is the magnetic dipole term, the 3rd term ($\sim 1/r^3$) is the magnetic quadrupole term, etc. for the multipole expansion of the magnetic vector potential $\vec{A}(\vec{r})$.

Thus, we see that: $\boxed{\vec{A}(\vec{r}) = \sum_{n=0}^{\infty}\vec{A}_n(\vec{r})}$ where n = order of the magnetic multipole, and:

$$\boxed{\vec{A}_n(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{1}{r^{n+1}}\oint_{C'}(r')^n P_n(\cos\Theta')\,\vec{I}(\vec{r}')\,d\ell'} \quad \text{for filamentary/line currents } \vec{I}(\vec{r}')$$

$$\boxed{\vec{A}_n(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{1}{r^{n+1}}\oint_{S'}(r')^n P_n(\cos\Theta')\,\vec{K}'(\vec{r}')\,da'_\perp} \quad \text{for surface/sheet current densities } \vec{K}(\vec{r}')$$

$$\boxed{\vec{A}_n(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{1}{r^{n+1}}\oint_{v'}(r')^n P_n(\cos\Theta')\,\vec{J}(\vec{r}')\,d\tau'} \quad \text{for volume current densities } \vec{J}(\vec{r}')$$

The reader of these lecture notes may have already realized that since (empirically) there are no (N/S) magnetic charges / no magnetic monopoles have been (conclusively / convincingly) ever observed in our universe, i.e. all magnetic field phenomena arises from (relative) motional effects of electric charges that the $n = 0$ term in the multipole expansion of the magnetic vector potential $\vec{A}(\vec{r})$ does not exist in nature. Mathematically we can <u>also</u> see this for the $n = 0$ term:

$\boxed{\vec{A}_0(\vec{r}) \equiv 0}$ because e.g. $\boxed{\oint_{C'} d\vec{\ell}'(\vec{r}') \equiv 0}$ around a <u>closed</u> contour of integration

This is a consequence of Maxwell's equation $\vec{\nabla}\bullet\vec{B}(\vec{r}) = 0$

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = \sum_{n=1}^{\infty} \vec{A}_n(\vec{r})}$$

Thus, the dominant term for magnetostatics is the ($n = 1$) magnetic dipole term, e.g. for a filamentary/line current $\vec{I}(\vec{r}')$:

$$\boxed{\begin{aligned} \vec{A}_1(\vec{r}) = \vec{A}_{dipole}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r^2}\oint_{C'} r'\cos\Theta' d\vec{\ell}'(\vec{r}') \quad \text{with} \quad \cos\Theta' = \hat{r}\bullet\hat{r}' \quad \text{and} \quad \vec{r} = r\hat{r}, \;\; \vec{r}' = r'\hat{r}' \\ &= \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r^2}\oint_{C'} (\hat{r}\bullet\vec{r}')\, d\vec{\ell}'(\vec{r}') \\ &= \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r^3}\oint_{C'} (\vec{r}\bullet\vec{r}')\, d\vec{\ell}'(\vec{r}') \end{aligned}}$$

Now if $\vec{C}$ = any <u>constant</u> vector, then (see Griffiths 1.106, 7 & 8 p. 57):

$$\boxed{\oint_{C'} (\vec{C}\bullet\vec{r}')\, d\vec{\ell}' = \vec{a}\times\vec{C} = -\vec{C}\times\vec{a}}$$

<u>Where</u>: $\boxed{\vec{a} \equiv \int_{S'} d\vec{a} = \frac{1}{2}\oint_{C'} \vec{r}'\times d\vec{\ell}'(\vec{r}')}$ = vector area of the contour loop

<u>And</u>: $\boxed{\vec{a} = a\hat{n}}$ where the unit normal $\hat{n}$ associated with the vector area enclosed by the contour loop is defined by the right hand rule.

<u>Thus (here)</u>: $\vec{r} = \vec{C}$ because the observation / field-point $P(\vec{r})$ (<u>by definition</u>) <u>*is*</u> a <u>constant</u> vector, pointing from the defined origin ϑ to the observation / field point $P(\vec{r})$.

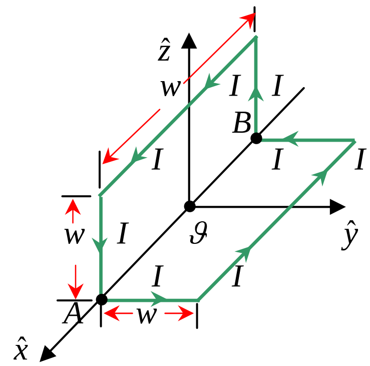
<u>Then</u>: $\boxed{\oint_{C'} (\hat{r}\bullet\vec{r}')\, d\vec{\ell}' = \int_{S'} d\vec{a}'\times\hat{r} = -\hat{r}\times\int_{S'} d\vec{a}'}$

<u>and thus</u>: $\boxed{\vec{A}_{dipole}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{\vec{m}\times\hat{r}}{r^2}}$ <u>where</u>: $\boxed{\vec{m} \equiv I\int_{S'} d\vec{a}' = I\vec{a}}$ = magnetic dipole moment of loop.

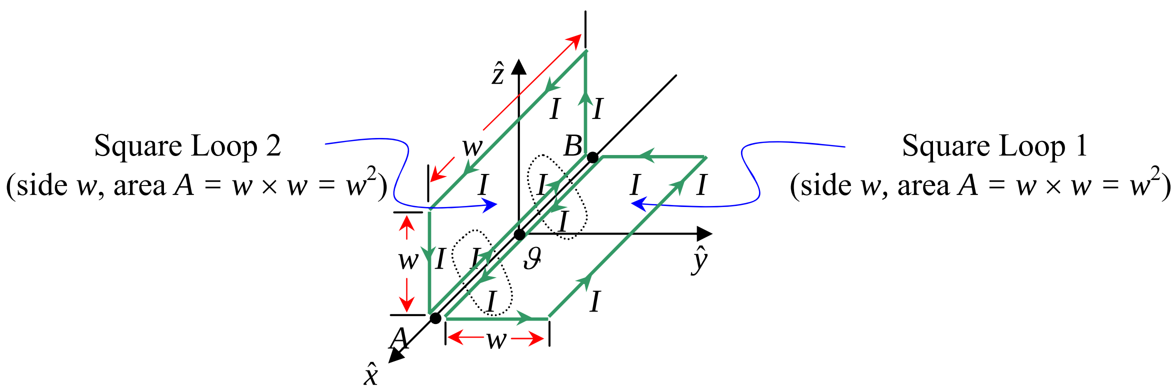
(SI units of $|\vec{m}|$ = Amp – m^2)

<u>Griffiths Example 5.13</u>:

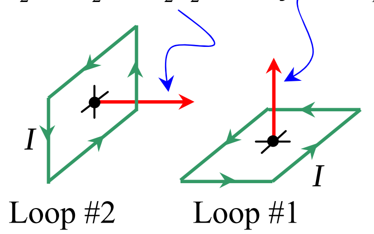
Determine the magnetic dipole moment $\vec{m}$ associated with a "book-end" shaped loop carrying steady current I as shown in the figure below:



<u>Use the principle of linear superposition</u>: Superpose <u>two</u> square current loops – one in the *x-y* plane (of square side *w*) and another one in the *x-z* plane (also of square side *w*). The side in common (line segment $\overline{AB}$) to both square loops have currents I flowing in *opposite* directions, hence the total current along line segment $\overline{AB}$ vanishes!



$\vec{m}_2 = I\vec{a}_2 = Ia_2\hat{n}_2 = Iw^2\hat{y}$ $\vec{m}_1 = I\vec{a}_1 = Ia_1\hat{n}_1 = Iw^2\hat{z}$



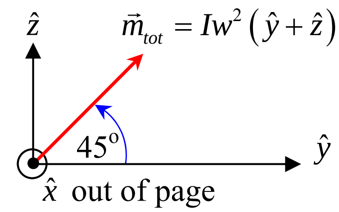
By the principle of linear superposition, the <u>total</u> magnetic dipole moment is:

$$\boxed{\vec{m}_{tot} = \vec{m}_1 + \vec{m}_2 = I\vec{a}_1 + I\vec{a}_2}$$

$$\boxed{\vec{m}_{tot} = Iw^2\hat{z} + Iw^2\hat{y} = Iw^2\left(\hat{y} + \hat{z}\right)}$$

$$\boxed{m_{tot} = \left|\vec{m}_{tot}\right| = \sqrt{m_1^2 + m_2^2} = \sqrt{2}Iw^2}$$

$$\left.\begin{array}{l} \boxed{\vec{a}_1 = Ia_1\hat{n}_1 = w^2\hat{z}} \\ \boxed{\vec{a}_2 = Ia_2\hat{n}_2 = w^2\hat{y}} \end{array}\right\} \text{by the right-hand rule}$$



Note that (here) the magnetic dipole moments $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_{tot}$ are <u>independent</u> of the choice of origin because the magnetic <u>monopole</u> moment of this magnetic charge distribution is <u>zero</u>.

Recall that the electric dipole moment $\vec{p}$ associated with an electric charge distribution is also independent of the choice of origin, but <u>ONLY</u> when the electric monopole moment (i.e. the net electric charge) associated with that electric charge distribution is <u>zero</u>.

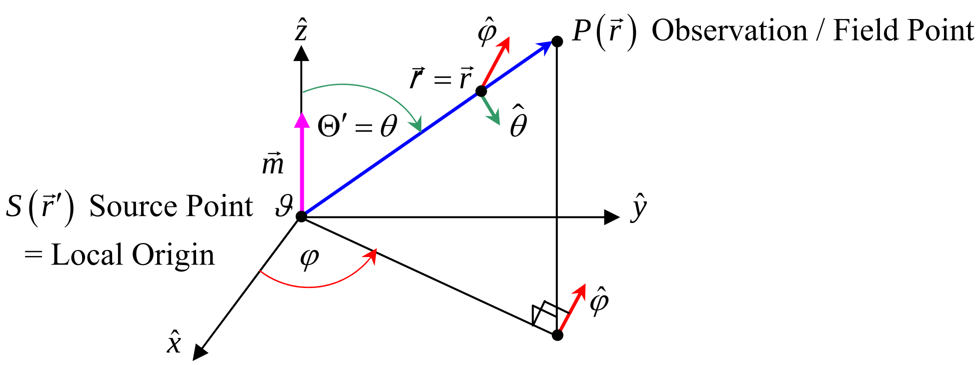
The magnetic dipole moments discussed thus far are obviously for a <u>physical</u> magnetic dipole – i.e. one with finite spatial extent. A pure / ideal magnetic dipole moment has <u>NO</u> spatial extent – its area $\vec{a} \to 0$ while its current $I \to \infty$, keeping the product $\vec{m} = I\vec{a} =$ constant.

For $r \gg r'$, we asymptotically realize the case for an ideal / pure / point magnetic dipole, e.g. magnetic moments of atoms, molecules, etc. have $r' \simeq$ few Ångstroms (~ few x 10^{-10} m) whereas $r \sim 1$ – few cm typically.

<u>The Magnetic Field Associated with a Magnetic Dipole Moment</u>

It is easiest to first calculate the magnetic vector potential $\vec{A}(\vec{r})$ and then calculate the corresponding magnetic field $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ associated with a magnetic dipole moment $\vec{m}$ by choosing (without any loss of generality) to have the origin ϑ at the location of the magnetic dipole, i.e. place $\vec{m}$ at $\vec{r}' = 0$ and also orient the magnetic dipole moment such that $\vec{m} = m\hat{z}$ (i.e. align $\vec{m} \parallel$ to the $\hat{z}$-axis).

<u>Then</u>: $\boxed{\cos\Theta' = \hat{r} \cdot \hat{r}' = \cos\theta}$ (i.e. θ = the usual polar angle) and: $\boxed{\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}' = \vec{r} - 0 = \vec{r}}$, $\boxed{r \gg r'}$



SI units of $|\vec{m}|$ = Amp-m^2 <u>Note</u>: $\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$

$\vec{m} = I\vec{a}$

From the multipole moment expansion of the magnetic vector potential $\vec{A}(\vec{r})$ we have:

$$\boxed{\vec{A}_{\text{dipole}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{\vec{m}\times\hat{r}}{r^2}}$$

$$\boxed{\vec{m} = m\hat{z} = m\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)}$$

$$\boxed{\hat{z}\times\hat{r} = \left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\times\hat{r}}$$

And: $\boxed{\left|\vec{m}\times\hat{r}\right| = m\sin\Theta}$ where $\Theta =$ opening angle between $\hat{z}$ and $\hat{r}$

$\hat{r}\times\hat{\theta} = \hat{\varphi}$	$\hat{\theta}\times\hat{r} = -\hat{\varphi}$	$\hat{r}\times\hat{r} = 0$
$\hat{\theta}\times\hat{\varphi} = \hat{r}$	$\hat{\varphi}\times\hat{\theta} = -\hat{r}$	$\hat{\theta}\times\hat{\theta} = 0$
$\hat{\varphi}\times\hat{r} = \hat{\theta}$	$\hat{r}\times\hat{\varphi} = -\hat{\theta}$	$\hat{\varphi}\times\hat{\varphi} = 0$

$\Leftarrow$ Very Useful Table # 2

But $\Theta = \theta$ here, and thus $\boxed{\left|\vec{m}\times\hat{r}\right| = m\sin\theta}$.

Note that $\vec{m}\times\hat{r}$ points in the $+\hat{\varphi}$ direction, because $\boxed{\vec{m}\times\hat{r} = m\hat{z}\times\hat{r} = +m\sin\theta\hat{\varphi}}$.

$\therefore$ $\boxed{\vec{A}_{\text{dipole}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{m\sin\theta}{r^2}\hat{\varphi}}$ i.e. $\boxed{\vec{A}_{\text{dipole}} = fcn(r,\theta)\hat{\varphi}}$ valid for $r \gg$ characteristic size of $\vec{m}$, i.e. $r \gg \sqrt{a}$.

Then: $\boxed{\vec{B}_{\text{dipole}}(\vec{r}) = \vec{\nabla}\times\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_o}{4\pi}\left(\frac{m}{r^3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)}$ valid for $r \gg$ characteristic size of $\vec{m}$, i.e. $r \gg \sqrt{a}$.

Compare this result to the electric field of an electric dipole with electric dipole moment $\vec{p} = q\vec{d}$:

$\boxed{\vec{E}_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{p}{r^3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)}$ valid for $r \gg$ characteristic size of $\vec{p} = q\vec{d}$, i.e. $r \gg d$.

They have the same form!!

We can also write $\vec{B}_{\text{dipole}}(\vec{r})$ in coordinate-free form by using:

$$\vec{m} = m\hat{z} = m\left[\cos\theta\hat{r} - m\sin\theta\hat{\theta}\right] = (\vec{m}\bullet\hat{r})\hat{r} + \left(\vec{m}\bullet\hat{\theta}\right)\hat{\theta} \quad \{\cos\theta = \hat{z}\bullet\hat{r} \text{ and } \sin\theta = \hat{z}\bullet\hat{\theta}\}$$

Then:
$$3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m} = 3m\cos\theta\hat{r} + m\sin\theta\hat{\theta} - m\cos\theta\hat{r}$$
$$= 2m\cos\theta\hat{r} + m\sin\theta\hat{\theta}$$

Then, in coordinate-free form the magnetic field associated with a physical magnetic dipole moment, $\vec{m} = I\vec{a}$ is:

$\boxed{\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_o}{4\pi}\left(\frac{1}{r^3}\right)\left[3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m}\right]}$ valid for $r \gg$ characteristic size of $\vec{m}$, i.e. $r \gg \sqrt{a}$.

Compare this result to the coordinate-free form of the electric field associated with a physical electric dipole with dipole moment, $\vec{p}$:

$\boxed{\vec{E}_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right]}$ valid for $r \gg$ characteristic size of $\vec{p} = q\vec{d}$, i.e. $r \gg d$.

For completeness' sake, we give the coordinate-free form of the magnetic field associated with a point magnetic dipole moment, $\vec{m}$:

$$\boxed{\vec{B}^{\text{point}}_{\text{dipole}}(\vec{r}) = \frac{\mu_o}{4\pi}\left(\frac{1}{r^3}\right)\left[3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m}\right] + \frac{8\mu_o}{3}\vec{m}\delta^3(\vec{r})} \quad \boxed{\text{n.b. valid for all } r.}$$

Note that the δ-function term compensates for the singularity at $r = 0$ associated with the first term, and arises from calculating the average magnetic field over an infinitesimally small sphere of infinitesimal radius ε that entirely contains the current density associated with the magnetic dipole moment $\vec{m}$ (See Griffiths Problem 5.59, p. 254). In quantum mechanics, this δ-function term is responsible for hyperfine splitting of bound electron energy levels in atoms!

Compare this result to the coordinate-free form of the electric field associated with a point electric dipole moment, $\vec{p}$ (See P435 Lect. Notes 8, p. 8, and/or Griffiths Problem 3.42, p.157):

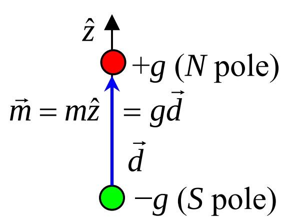
$$\boxed{\vec{E}^{\text{point}}_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right] - \frac{1}{3\varepsilon_o}\vec{p}\delta^3(\vec{r})} \quad \boxed{\text{n.b. valid for all } r.}$$

where again the δ-function term compensates for the singularity at $r = 0$ associated with the first term, and arises from calculating the average electric field over an infinitesimally small sphere of infinitesimal radius ε that entirely contains the charge densities associated with the electric dipole moment $\vec{p}$.

Creation of a Magnetic Dipole Moment from *N* & *S* Magnetic Charges:

We can create a magnetic dipole moment $\vec{m}$ (at least conceptually) in a manner completely analogous to that associated with making an electric dipole moment $\vec{p} = q\vec{d}$ from two opposite electric charges $+q$ and $-q$, but instead using *N* and *S* magnetic charges $\pm g$ for $\boxed{\vec{m} = g\vec{d}}$:

n.b. SI units of: Ampere-m^2 $\Rightarrow$ SI units of magnetic charge, g: Ampere-meters



We summarize below the magnetic dipole moments associated with filamentary/line, surface/sheet and volume current densities:

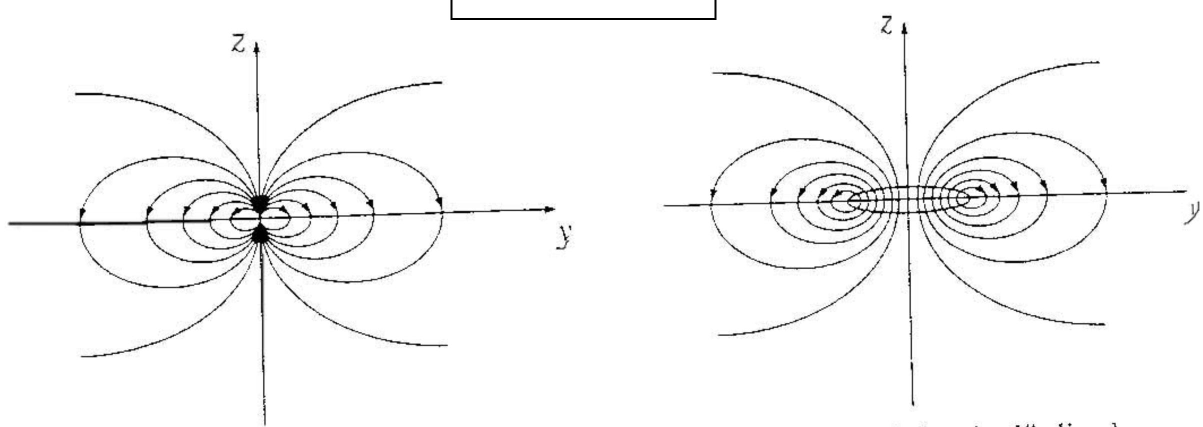
$$\boxed{\vec{m} = \frac{1}{2}\oint_{C'}\vec{r}'\times\vec{I}(\vec{r}')\,d\ell' = \frac{1}{2}I\oint_{C'}\vec{r}'\times d\vec{\ell}'} \qquad \text{If } \left|\vec{I}\right| = I = \text{constant} \ \ \forall\ \vec{r}'$$

$$\boxed{\vec{m} = \frac{1}{2}\oint_{S'}\vec{r}'\times\vec{K}(\vec{r}')\,da'_\perp = \frac{1}{2}K\oint_{S'}\vec{r}'\times\hat{K}\,da'_\perp} \qquad \text{If } \left|\vec{K}\right| = K = \text{constant} \ \ \forall\ \vec{r}'$$

$$\boxed{\vec{m} = \frac{1}{2}\oint_{v'}\vec{r}'\times\vec{J}(\vec{r}')\,d\tau'}$$

A comparison of the magnetic dipole fields associated with a pure / ideal / point magnetic dipole moment, $\vec{m}_{point}$ versus a physical / finite spatial extent magnetic dipole moment $\vec{m}_{phys} = I\vec{a} = I\pi R^2\hat{z}$ (e.g. $a = |\vec{a}| = \pi R^2$ for a magnetic dipole loop of radius R)

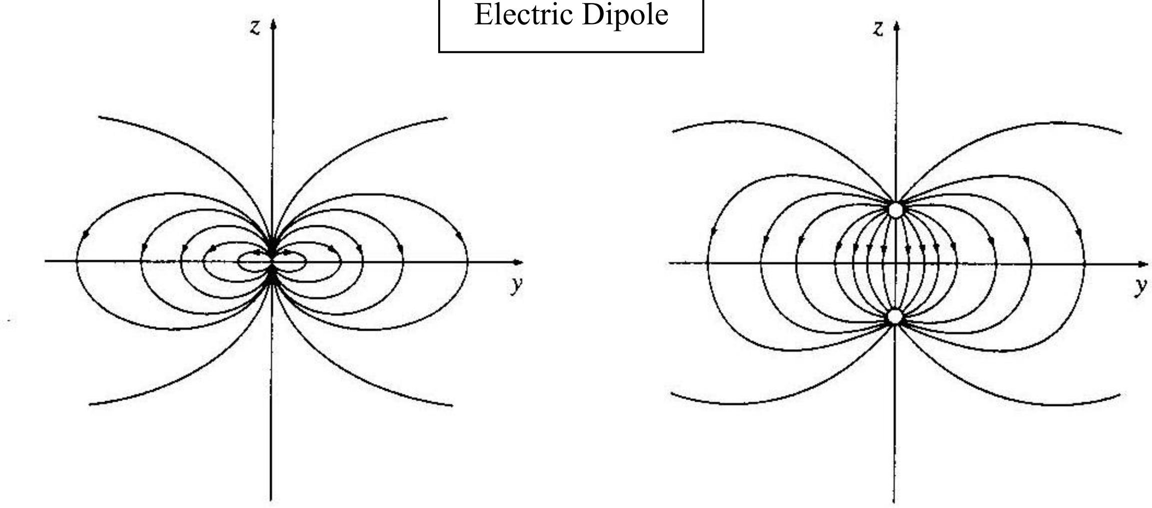
Pure vs. Physical
Magnetic Dipole



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

A comparison of the electric dipole fields associated with a pure / ideal / point electric dipole moment, $\vec{p}_{point}$ versus a physical / finite spatial extent electric dipole moment $\vec{p}_{phys} = q\vec{d}$

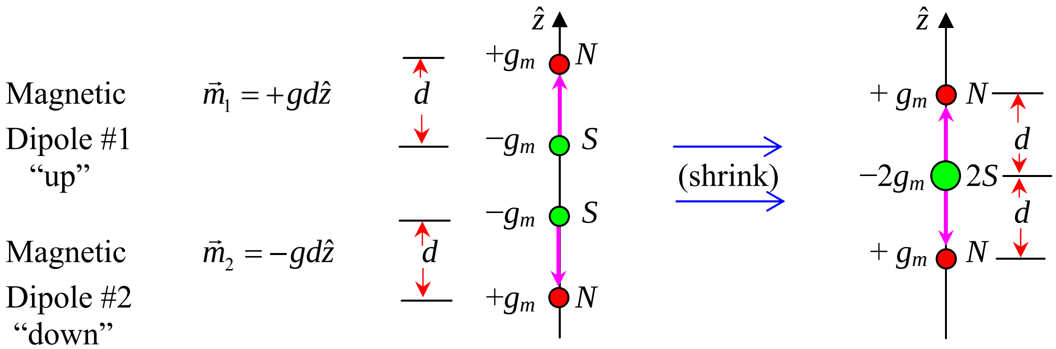
Pure vs. Physical
Electric Dipole



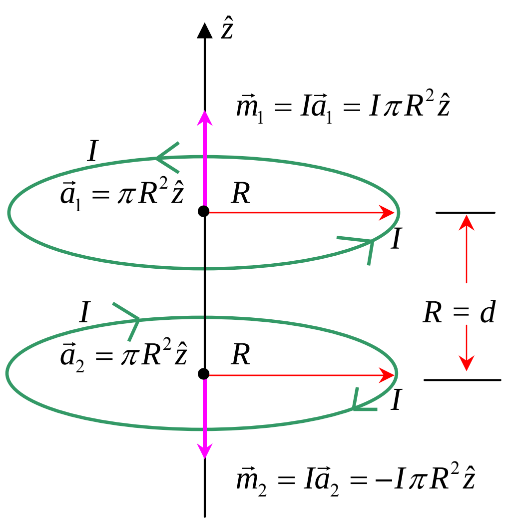
(a) Field of a "pure" dipole (b) Field of a "physical" dipole

The Magnetic Vector Potential $\vec{A}_{quad}(\vec{r})$ and Magnetic Field $\vec{B}_{quad}(\vec{r})$ Associated with a Magnetic Quadrupole Moment Q_m

We can create / build a magnetic quadrupole moment using two back-to-back magnetic dipoles in analogy to how an electric quadrupole was generated from two back-to-back electric dipoles - i.e. use the principle of linear superposition, e.g. using magnetic charges $+g_m = N$, $-g_m = S$ poles, or using two identical current loops back-to-back to make a linear magnetic quadrupole:



SI units of magnetic charge: g_m = Ampere-meters



Two identical magnetic dipole loops carrying opposing equal currents I, each of radius R and separation distance $d = R$.

Then (for $r \gg r'$): $\vec{A}_{quad}(\vec{r}) = \left(\dfrac{\mu_o}{4\pi}\right)\dfrac{I}{r^3}\oint_{C'}(r')^2 P_2(\cos\Theta')\, d\vec{\ell}'(\vec{r}')$ for line currents,

$$= \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r^3}\oint_{C'}(r')^2\left(\frac{3}{2}\cos^2\Theta' - \frac{1}{2}\right)d\vec{\ell}'(\vec{r}') \quad \text{with } \cos\Theta' = \hat{r}\cdot\hat{r}'$$

The Magnetic Quadrupole Moment Tensor (in terms of discrete magnetic charges):

discrete magnetic charges

$$\boxed{\overleftrightarrow{Q}_m \equiv \frac{1}{2}\sum_{i=1}^{n=3}\left(3\vec{r}_i\vec{r}_i - \overleftrightarrow{1}r_i^2\right)g_{m_i}}$$

Unit Dyadic: $\overleftrightarrow{1} = \begin{pmatrix} \hat{x}\hat{x} & 0 & 0 \\ 0 & \hat{y}\hat{y} & 0 \\ 0 & 0 & \hat{z}\hat{z} \end{pmatrix}$ in Cartesian coordinates

$\vec{r}_1 = +d\hat{z}$ $\quad g_{m_1} = +g_m$ $\quad r_i^2 = \vec{r}_i \bullet \vec{r}_i \quad (i = 1,2,3)$

$\vec{r}_2 = 0\hat{z}$ $\quad g_{m_2} = -2g_m$

$\vec{r}_3 = -d\hat{z}$ $\quad g_{m_3} = +g_m$

Thus: $$\boxed{\overleftrightarrow{Q}_m = \underbrace{\frac{1}{2}g_m\left(3d^2\hat{z}\hat{z} - d^2\overleftrightarrow{1}\right)}_{\substack{\text{charge \#1 } +g_m \\ \text{at } \vec{r}_1 = +d\hat{z}}} - \underbrace{\frac{2}{2}g_m\left(\overset{=0}{\cancel{3\bullet 0}}\,\hat{z}\hat{z} - \overset{=0}{\cancel{0}}\overleftrightarrow{1}\right)}_{\substack{\text{charge \#2 } -2g_m \\ \text{at } \vec{r}_2 = 0\hat{z}}} + \underbrace{\frac{1}{2}g_m\left(3d^2\hat{z}\hat{z} - d^2\overleftrightarrow{1}\right)}_{\substack{\text{charge \#3 } +g_m \\ \text{at } \vec{r}_3 = -d\hat{z}}} = 2g_m d^2\frac{\left(3\hat{z}\hat{z} - \overleftrightarrow{1}\right)}{2}}$$

Then for $r \gg r'$:

$$\boxed{\vec{A}_{quad}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)2g_m d^2\left(\frac{1}{r^3}\right)\frac{\left(3\cos^2\Theta' - 1\right)}{2} = \left(\frac{\mu_o}{4\pi}\right)2g_m d^2\left(\frac{1}{r^3}\right)P_2\left(\cos\Theta'\right)} \quad \boxed{\cos\Theta' = \hat{r}\bullet\hat{r}'}$$

$\boxed{Q_m = \left|\overleftrightarrow{Q}_m\right| = 2g_m d^2}$ Amp-meters*meters^2 = Amp-meters^3

In terms of current loops: $\boxed{Q_m = 2md = 2Iad = 2\pi R^2 dI = 2\pi R^3 I}$ $(R = d)$ Amp-meters^3

$\boxed{Q_m = 2g_m d^2}$ or: $\boxed{Q_m = 2md = 2Iad = 2I\left(\pi R^2\right)d = 2I\left(\pi R^3\right)}$ $\{d = R \text{ here}\}$.

Magnetic dipole current loop separation distance, d

Thus for $r \gg r'$: $\boxed{\vec{A}_{quad}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{Q_m}{r^3}P_2\left(\cos\Theta'\right)}$ where $\boxed{\cos\Theta' = \hat{r}\bullet\hat{r}'}$

We can also write this in coordinate-free form {valid for $r \gg r'$, d $(R = d$ here$)$} as:

$$\boxed{\vec{A}_{quad}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{Q_m}{r^3}\left\{\hat{r}\times\left[\frac{3\hat{z}\hat{z} - \overleftrightarrow{1}}{2}\right]\times\hat{r}\right\} = \left(\frac{\mu_o}{4\pi}\right)\frac{Q_m}{r^3}\left[\frac{3\cos^2\Theta' - 1}{2}\right] = \left(\frac{\mu_o}{4\pi}\right)\frac{Q_m}{r^3}P_2\left(\cos\Theta'\right)}$$

We can obtain $\vec{B}_{quad}(\vec{r})$ from: $\vec{B}_{quad}(\vec{r}) = \vec{\nabla}\times\vec{A}_{quad}(\vec{r})$ (…an exercise for the energetic student...)

Thus, we can write the multipole expansion of the magnetic vector potential $\vec{A}(\vec{r})$ as:

$$\boxed{\vec{A}(\vec{r}) = \sum_{n=1}^{\infty} \vec{A}_n(\vec{r}) = \underbrace{\left(\frac{\mu_o}{4\pi}\right)\frac{\hat{m}\times\hat{r}}{r^2}}_{\substack{\text{magnetic dipole} \\ \text{term } n=1}} + \underbrace{\left(\frac{\mu_o}{4\pi}\right)\frac{\hat{r}\times\overleftrightarrow{Q}_m\times\hat{r}}{r^3}}_{\substack{\text{magnetic quadrupole} \\ \text{term } n=2}} + \ldots.}$$

Once $\vec{A}(\vec{r})$ is determined, we can obtain $\vec{B}(\vec{r})$ from $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$.

We can write the magnetic quadrupole tensor $\overleftrightarrow{Q}_m$ as:

$$\boxed{\overleftrightarrow{Q}_m = \begin{pmatrix} Q_{m_{xx}} & Q_{m_{yx}} & Q_{m_{zx}} \\ Q_{m_{xy}} & Q_{m_{yy}} & Q_{m_{zy}} \\ Q_{m_{xz}} & Q_{m_{yz}} & Q_{m_{zz}} \end{pmatrix}}$$

Note that (as for the electric quadrupole moment tensor $\overleftrightarrow{Q}_e$) the magnetic quadrupole moment tensor $\overleftrightarrow{Q}_m$ has only <u>six</u> independent components because $\boxed{Q_{m_{ij}} = Q_{m_{ji}}}$ and also note that $\boxed{Q_{m_{xx}} + Q_{m_{yy}} + Q_{m_{zz}} = 0}$ i.e. $\overleftrightarrow{Q}_m$ (like $\overleftrightarrow{Q}_e$) is traceless.

<u>For a linear magnetic quadrupole (oriented along the $\hat{z}$-axis</u>:

$\boxed{Q_{m_{xx}} = Q_{m_{yy}}}$ and thus: $\boxed{Q_{m_{zz}} = -2Q_{m_{xx}} = -2Q_{m_{yy}} = 2g_m d^2}$

Thus, the magnetic quadrupole tensor for a linear magnetic quadrupole is of the form:

$$\boxed{\overleftrightarrow{Q}_m^{linear} = 2g_m d^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \Leftarrow$$

For a linear magnetic quadrupole oriented along the $\hat{z}$-axis consisting of magnetic charges g_m.

or:

$$\boxed{\overleftrightarrow{Q}_m^{linear} = 2I\left(\pi R^3\right) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \Leftarrow$$

For two back-to-back magnetic dipole loops carrying steady current I separated by a vertical distance d.

Compare these results for the magnetic quadrupole to that for the electric quadrupole (P435 Lecture Notes 8, p. 13-15)

Another Kind of Magnetic Quadrupole Using Four Bar Magnets: