LECTURE NOTES 16

THE MAGNETIC VECTOR POTENTIAL $A(\vec{r})$

We saw in electro<u>statics</u> that $\boxed{\vec{\nabla}\times\vec{E}=0}$ {always} (due to intrinsic / microscopic nature of the electrostatic field) permitted us to introduce a <u>scalar</u> potential $V(\vec{r})$ such that:

$\boxed{\vec{E}(\vec{r})\equiv-\vec{\nabla}V(\vec{r})}$ {n.b. $V(\vec{r})$ is uniquely defined, up to an (arbitrary) constant.}

Analogously, in magneto<u>statics</u>, the $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r})=0}$ (always) {$\Rightarrow\exists$ no magnetic charges / no magnetic <u>monopoles</u>} permits us to introduce a <u>magnetic</u> <u>vector</u> <u>potential</u> $\vec{A}(\vec{r})$ such that:

$$\boxed{\underbrace{\vec{B}}_{\text{Teslas}}(\vec{r})\equiv\underbrace{\vec{\nabla}}_{1/m}\times\underbrace{\vec{A}(\vec{r})}_{\text{Tesla-Meters}}}$$ $\Rightarrow$ S.I. units of the magnetic vector potential $\vec{A}(\vec{r})$ = Tesla-meters

<u>Then</u>: $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r})=\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)=0 \quad \{\text{always}\}}$

The divergence of a curl of a vector field $\vec{F}(\vec{r})$ is <u>always</u> zero

<u>Ampere's Law</u>:

In differential form: $\boxed{\vec{\nabla}\times\vec{B}(\vec{r})=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}(\vec{r})\right)-\nabla^2\vec{A}(\vec{r})=\mu_o\vec{J}_{free}(\vec{r})}$

Now, just as in the case of electrostatics, where $V(\vec{r})$ was uniquely defined up to an arbitrary constant (V_o), then let: $\boxed{V'(\vec{r})\equiv V(\vec{r})+V_o}$

then: $\vec{E}(\vec{r})=-\vec{\nabla}V'(\vec{r})=-\vec{\nabla}\left(V(\vec{r})+V_o\right)=-\vec{\nabla}V(\vec{r})-\overbrace{\vec{\nabla}V_o}^{=0}=-\vec{\nabla}V(\vec{r})$

i.e. $\boxed{\vec{E}(\vec{r})=-\vec{\nabla}V'(\vec{r})=-\vec{\nabla}V(\vec{r})}$

An analogous thing occurs in magneto<u>statics</u> - we can add / we have the freedom to add to the magnetic vector potential $\vec{A}(\vec{r})$ the gradient of any scalar function $\boxed{\vec{\mathbb{A}}(\vec{r})\equiv\vec{\nabla}\Phi_m(\vec{r})}$ where $\Phi_m(\vec{r})\equiv$ <u>magnetic</u> <u>scalar</u> <u>potential</u> $\boxed{\text{SI Units of magnetic scalar potential } \Phi_m(\vec{r})=\text{Tesla-m}^2}$

<u>Then</u>: $\boxed{\vec{A}'(\vec{r})\equiv\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})=\vec{A}(\vec{r})+\vec{\nabla}\Phi_m(\vec{r})}$ $\Leftarrow$ Formally known as a <u>Gauge</u> <u>Transformation</u>

The <u>curl of the gradient of a scalar field</u> ($\Phi_m(\vec{r})$ here) <u>automatically/always</u> <u>vanishes</u>, i.e.:

$$\boxed{\begin{aligned}\vec{B}(\vec{r})&=\vec{\nabla}\times\vec{A}'(\vec{r})=\vec{\nabla}\times\left(\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})\right)=\vec{\nabla}\times\vec{A}(\vec{r})+\vec{\nabla}\times\vec{\mathbb{A}}(\vec{r})=\vec{\nabla}\times\vec{A}(\vec{r})+\overbrace{\vec{\nabla}\times\vec{\nabla}\Phi_m(\vec{r})}^{\equiv 0\, Always!!!}\\&=\vec{\nabla}\times\vec{A}(\vec{r})\end{aligned}}$$

Note that the magnetic scalar <u>potential</u> $\Phi_m(\vec{r})$ has <u>same</u> physical units as magnetic <u>flux</u> Φ_m :

Tesla-m^2 = Weber (Magnetic <u>flux</u>, $\Phi_m = \int_S \vec{B}(\vec{r})\bullet d\vec{A}$!!) <u>*eeek*</u>!!!!

$\Rightarrow$ Please do <u>*NOT*</u> confuse the magnetic scalar <u>potential</u> $\Phi_m(\vec{r})$ (= a <u>scalar</u> <u>*point*</u> <u>function</u>, whose value can <u>change</u> at each/every point in space, $\vec{r}$) with the magnetic <u>flux</u> Φ_m (which is a <u>constant</u> scalar quantity (i.e. a pure number), independent of position) $\boxed{\Phi_m(\vec{r}) \neq \Phi_m}$!!!

Thus, like the scalar potential $V(\vec{r})$, the magnetic vector potential $\vec{A}(\vec{r})$ is (also) uniquely defined, but only up to an (arbitrary) vector function $\boxed{\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\Phi_m(\vec{r})}$.

$$\boxed{\vec{A}'(\vec{r}) \equiv \vec{A}(\vec{r}) + \vec{\mathbb{A}}(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})}$$

The definition $\boxed{\vec{B}(\vec{r}) \equiv \vec{\nabla}\times\vec{A}(\vec{r})}$ specifies the <u>curl</u> of $\vec{A}(\vec{r})$, but in order to <u>fully</u> <u>specify</u> the vector field $\vec{A}(\vec{r})$, we <u>additionally</u> need to specify the <u>divergence</u> of $\vec{A}(\vec{r})$, $\boxed{\vec{\nabla}\bullet\vec{A}(\vec{r})}$.

We can <u>exploit</u> the freedom of the <u>definition</u> of $\vec{A}(\vec{r})$ to <u>eliminate</u> the divergence of $\vec{A}(\vec{r})$ - i.e. a <u>specific</u> choice of $\vec{A}(\vec{r})$ will make $\vec{A}(\vec{r})$ <u>divergenceless</u>: $\boxed{\vec{\nabla}\bullet\vec{A}(\vec{r}) = 0}$ $\Leftarrow$ <u>Coulomb Gauge</u>

<u>If</u>: $\boxed{\vec{A}'(\vec{r}) \equiv \vec{A}(\vec{r}) + \vec{\mathbb{A}}(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})}$

<u>Then</u>: $\boxed{\vec{\nabla}\bullet\vec{A}'(\vec{r}) = \vec{\nabla}\bullet\vec{A}(\vec{r}) + \vec{\nabla}\bullet\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\bullet\vec{A}(\vec{r}) + \vec{\nabla}\bullet\vec{\nabla}\Phi_m(\vec{r}) = \vec{\nabla}\bullet\vec{A}(\vec{r}) + \nabla^2\Phi_m(\vec{r})}$

While the <u>original</u> magnetic vector potential, $\vec{A}(\vec{r})$ is not/may not be divergenceless, we can make $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\mathbb{A}}(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})$ divergenceless, i.e. $\boxed{\vec{\nabla}\bullet\vec{A}'(\vec{r}) = 0}$ <u>*if*</u> we <u>chose</u> $\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\Phi_m(\vec{r})$ such that $\boxed{\vec{\nabla}\bullet\vec{\mathbb{A}}(\vec{r}) = \nabla^2\Phi_m(\vec{r}) = -\vec{\nabla}\bullet\vec{A}(\vec{r})}$ $\Leftarrow$ <u>Coulomb Gauge</u>

<u>A Simple Illustrative Example</u>:

Suppose $\exists$ a region of space that has a <u>uniform/constant</u> magnetic field, e.g. $\vec{B}(\vec{r}) = B_o\hat{z}$.

<u>Then</u>: $\vec{B}(\vec{r}) = B_o\hat{z} = \vec{\nabla}\times\vec{A}(\vec{r}) = \left(\dfrac{\partial A_y(\vec{r})}{\partial x} - \dfrac{\partial A_x(\vec{r})}{\partial y}\right)\hat{z}$.

<u>Thus (here)</u>: $\vec{A}(\vec{r}) = A_x(\vec{r})\hat{x} + A_y(\vec{r})\hat{y} + \underbrace{\cancel{A_z(\vec{r})}}_{=0}\hat{z} = A_x(\vec{r})\hat{x} + A_y(\vec{r})\hat{y}$

If $A_x(\vec{r}) = -\tfrac{1}{2}B_o y$ and $A_y(\vec{r}) = \tfrac{1}{2}B_o x$, then $\vec{A}(\vec{r}) = -\tfrac{1}{2}B_o y\hat{x} + \tfrac{1}{2}B_o x\hat{y}$, and thus we see that this choice of magnetic vector potential indeed gives us the correct $\vec{B}$ -field:

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \left(\frac{\partial A_y(\vec{r})}{\partial x} - \frac{\partial A_x(\vec{r})}{\partial y}\right)\hat{z} = \tfrac{1}{2}B_o + \tfrac{1}{2}B_o = B_o\hat{z}$$

Is $\vec{\nabla}\cdot\vec{A}=0$ satisfied? $\vec{\nabla}\cdot\vec{A}=\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\left(\cancel{\frac{\partial\left(-\frac{1}{2}B_o y\right)}{\partial x}}+\cancel{\frac{\partial\left(\frac{1}{2}B_o x\right)}{\partial y}}+\cancel{\frac{\partial(0)}{\partial z}}\right)=0$ <u>Yes</u>!!!

Note that we could also have <u>instead</u> chosen/used a <u>different</u> magnetic vector potential: $\vec{A}'(\vec{r})=\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})=\vec{A}(\vec{r})+\vec{\nabla}\Phi_m(\vec{r})$ where e.g. $\vec{\mathbb{A}}(\vec{r})=\vec{\nabla}\Phi_m(\vec{r})=\vec{A}_o$, i.e. where $\vec{A}_o$ is any (arbitrary) <u>constant</u> vector, $\vec{A}_o=A_{ox}\hat{x}+A_{oy}\hat{y}+A_{oz}\hat{z}$. Since (here) $\vec{\mathbb{A}}(\vec{r})=\vec{\nabla}\Phi_m(\vec{r})=\vec{A}_o$, then $\vec{\mathbb{A}}(\vec{r})=\vec{\nabla}\Phi_m(\vec{r})=\vec{A}_o=A_{ox}\hat{x}+A_{oy}\hat{y}+A_{oz}\hat{z}$ means that the <u>gradient</u> of the magnetic scalar potential (here) is: $\vec{\nabla}\Phi_m(\vec{r})=\frac{\partial(A_{ox}x)}{\partial x}\hat{x}+\frac{\partial(A_{oy}y)}{\partial y}\hat{y}+\frac{\partial(A_{oz}z)}{\partial z}\hat{z}=A_{ox}\hat{x}+A_{oy}\hat{y}+A_{oz}\hat{z}=\vec{A}_o=\vec{\mathbb{A}}(\vec{r})$ and thus the magnetic scalar potential itself (here) is: $\Phi_m(\vec{r})=A_{ox}x\hat{x}+A_{oy}y\hat{y}+A_{oz}z\hat{z}$.

Thus, here for the case of a constant/uniform magnetic field $\vec{B}(\vec{r})=B_o\hat{z}$ we see that there is in fact a continuum of allowed magnetic vector potentials $\vec{A}'(\vec{r})=\vec{A}(\vec{r})+\vec{A}_o=\vec{A}(\vec{r})+\vec{\nabla}\Phi_m(\vec{r})$ that <u>simultaneously</u> satisfy $\vec{B}(\vec{r})=B_o\hat{z}=\vec{\nabla}\times\vec{A}'(\vec{r})$ and $\vec{\nabla}\cdot\vec{A}'(\vec{r})=0$ with the addition of an (arbitrary) <u>constant</u> magnetic vector potential $\vec{A}_o=A_{ox}\hat{x}+A_{oy}\hat{y}+A_{oz}\hat{z}$ contribution with corresponding magnetic scalar potential $\Phi_m(\vec{r})=A_{ox}x\hat{x}+A_{oy}y\hat{y}+A_{oz}z\hat{z}$. Note that this is <u>exactly</u> analogous to the situation in electrostatics where the scalar electric potential $V(\vec{r})$ is unique, up to an arbitrary constant, V_o because there exists <u>*no*</u> <u>absolute</u> voltage reference in our universe – i.e. <u>absolute</u> measurements of the scalar electric potential are <u>meaningless</u> - <u>*only*</u> potential <u>*differences*</u> have physical significance!!!

We used this simple example of the constant/uniform magnetic field $\vec{B}(\vec{r})=B_o\hat{z}$ to elucidate this particular aspect of the magnetic vector potential $\vec{A}(\vec{r})$. Here in this particular example, we found that the addition of an arbitrary <u>constant</u> vector $\vec{\mathbb{A}}(\vec{r})=\vec{A}_o=A_{ox}\hat{x}+A_{oy}\hat{y}+A_{oz}\hat{z}=\vec{\nabla}\Phi_m(\vec{r})$ to the magnetic vector potential $\vec{A}(\vec{r})$ was allowed, i.e. $\vec{A}'(\vec{r})=\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})=\vec{A}(\vec{r})+\vec{A}_o$, which leaves the magnetic field $\vec{B}(\vec{r})$ unchanged. In general there are many instances involving more complicated physics situations, where $\vec{B}(\vec{r})\neq$ constant vector field, where indeed $\vec{B}(\vec{r})=\vec{\nabla}\times\vec{A}'(\vec{r})$ and $\vec{\nabla}\cdot\vec{A}'(\vec{r})=0$ <u>are</u> simultaneously satisfied for $\vec{A}'(\vec{r})=\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})$, because it is possible to determine/find a corresponding magnetic scalar potential $\Phi_m(\vec{r})$ for the problem satisfying $\nabla^2\Phi_m(\vec{r})=-\vec{\nabla}\cdot\vec{A}(\vec{r})$, but it is (very) important to understand that, in general, the allowed $\vec{\mathbb{A}}(\vec{r})=\vec{\nabla}\Phi_m(\vec{r})$ (very likely) may <u>*not*</u> be simply a <u>constant</u> vector field, but indeed one which <u>varies</u> in space (i.e. with position vector, $\vec{r}$)! Here again, however, the new $\vec{A}'(\vec{r})=\vec{A}(\vec{r})+\vec{\mathbb{A}}(\vec{r})$ will also be such that $\vec{B}(\vec{r})=\vec{\nabla}\times\vec{A}'(\vec{r})$ will be unchanged, exactly analogous to $V'(\vec{r})=V(\vec{r})+V_o$ leaving $\vec{E}(\vec{r})=-\vec{\nabla}V'(\vec{r})$ unchanged.

So we see that *if* $\boxed{\nabla^2\Phi_m(\vec{r}) = -\vec{\nabla}\cdot\vec{A}(\vec{r})}$ then yes, $\boxed{\vec{\nabla}\cdot\vec{A}'(\vec{r}) = 0}$.

It is *always* possible to find an $\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\Phi_m(\vec{r})$ in order to make $\boxed{\vec{\nabla}\cdot\vec{A}'(\vec{r}) = 0}$.

Note however that this situation is then formally mathematically identical to Poisson's Equation, for the magnetic scalar potential $\Phi_m(\vec{r})$ because:

$$\boxed{\nabla^2\Phi_m(\vec{r}) = -\rho_m(\vec{r})} \Leftarrow \text{Analogous to } \boxed{\nabla^2 V(\vec{r}) = -\frac{\rho_{Tot}(\vec{r})}{\varepsilon_0}} \text{ in electrostatics!!!}$$

Equivalent ***magnetic*** volume charge density $\Leftarrow$ Physically, $\rho_m(\vec{r})$ could e.g. be due to ***bound*** effective magnetic charges associated with a magnetic material…

If we assume that the equivalent magnetic volume charge density, $\rho_m(\vec{r}) \neq 0$ and we want $\vec{\nabla}\cdot\vec{A}'(\vec{r}) = 0$

Then: $\vec{\nabla}\cdot\vec{A}(\vec{r}) + \vec{\nabla}\cdot\vec{\mathbb{A}}(\vec{r}) = \vec{\nabla}\cdot\vec{A}(\vec{r}) + \nabla^2\Phi_m(\vec{r}) = 0$

Or: $\vec{\nabla}\cdot\vec{A}(\vec{r}) - \rho_m(\vec{r}) = 0 \;\Rightarrow\; \boxed{\vec{\nabla}\cdot\vec{A}(\vec{r}) = \rho_m(\vec{r})}$

Then, the solution to Poisson's equation for the magnetic scalar potential $\Phi_m(\vec{r})$ is of the form:

$$\boxed{\Phi_m(\vec{r}) = \frac{1}{4\pi}\int_{v'}\frac{\rho_m(\vec{r}')}{\mathscr{r}}d\tau'} \Leftarrow \text{Analogous to } \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{v'}\frac{\rho_{Tot}(\vec{r}')}{\mathscr{r}}d\tau'} \text{ in electrostatics}$$

with $\mathscr{r} = |\vec{\mathscr{r}} - \vec{\mathscr{r}}'|$

(n.b. these two relations are both valid *assuming* that $\rho_m(\vec{r}')$ and $\rho_{ToT}(\vec{r}')$ *vanish* when $r' \to \infty$!)

So then if $\boxed{\rho_m(\vec{r}') = \vec{\nabla}\cdot\vec{A}(\vec{r}')}$, and $\rho_m(\vec{r}') = \vec{\nabla}\cdot\vec{A}(\vec{r}')$ vanishes as $r' \to \infty$, then the magnetic scalar potential $\Phi_m(\vec{r})$ is given by:

$$\boxed{\Phi_m(\vec{r}) = \frac{1}{4\pi}\int_{v'}\frac{\rho_m(\vec{r}')}{\mathscr{r}}d\tau' = \frac{1}{4\pi}\int_{v'}\frac{\vec{\nabla}\cdot\vec{A}(\vec{r}')}{\mathscr{r}}d\tau'}$$

{Note that if $\vec{\nabla}\cdot\vec{A}(\vec{r}') = \rho_m(\vec{r}')$ does not go to zero at infinity, then we'll have to use some other means in order to obtain an appropriate $\Phi_m(\vec{r})$, e.g. in an analogous manner to that which we've had to do for the (electric) scalar potential $V(\vec{r})$ associated with problems that have electric charge distributions extending out to infinity.}

Thus, this choice of $\Phi_m(\vec{r})$ *ensures* that indeed we *can* always make the magnetic vector potential $\vec{A}'(\vec{r})$ divergenceless, i.e. the condition $\vec{\nabla}\cdot\vec{A}'(\vec{r}) = 0$ (Coulomb Gauge) can always be met, for the case of magnetostatics. Note that if $\rho_m(\vec{r}') = 0$ then $\rho_m(\vec{r}') = \vec{\nabla}\cdot\vec{A}(\vec{r}') = 0$.

With the choice of the magnetic scalar potential:

$$\boxed{\Phi_m(\vec{r}) = \frac{1}{4\pi}\int_{v'} \frac{\rho_m(\vec{r}')}{\mathrm{r}} d\tau'} \text{ and } \boxed{\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{A}_o = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})}, \text{ and } \boxed{\vec{\nabla}\cdot\vec{A}'(\vec{r}) = 0}$$

Then Ampere's Law (in differential form) becomes:

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}'(\vec{r})\right) = \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A}'(\vec{r})\right)}_{=0} - \nabla^2\vec{A}'(\vec{r}) = \mu_0\vec{J}_{free}(\vec{r})$$

Which gives: $\boxed{\nabla^2\vec{A}'(\vec{r}) = -\mu_0\vec{J}_{free}(\vec{r})}$ ⇐ Vector form of Poisson's equation for magnetostatics.

i.e.: $\begin{Bmatrix} \nabla^2 A'_x(\vec{r}) = -\mu_o J_{x\,free}(\vec{r}) \\ \nabla^2 A'_y(\vec{r}) = -\mu_o J_{y\,free}(\vec{r}) \\ \nabla^2 A'_z(\vec{r}) = -\mu_o J_{z\,free}(\vec{r}) \end{Bmatrix}$ ⇐ The three separate/independent scalar forms of Poisson's equation are connected by: $\vec{J}_{free}(\vec{r}) = J_{x\,free}(\vec{r})\hat{x} + J_{y\,free}(\vec{r})\hat{y} + J_{z\,free}(\vec{r})\hat{z}$

n.b. in Cartesian coordinates: $\nabla^2\vec{A}'(\vec{r}) = \left(\nabla^2 A'_x(\vec{r})\right)\hat{x} + \left(\nabla^2 A'_y(\vec{r})\right)\hat{y} + \left(\nabla^2 A'_z(\vec{r})\right)\hat{z}$

However, in curvilinear coordinates (i.e. spherical-polar or cylindrical coordinates)

e.g. spherical-polar coordinates: $\begin{Bmatrix} \hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z} \\ \hat{\theta} = \cos\theta\cos\varphi\hat{x} + \cos\theta\sin\varphi\hat{y} - \sin\theta\hat{z} \\ \hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y} \end{Bmatrix}$

Note that the unit vectors $\hat{r}, \hat{\theta} + \hat{\varphi}$ for spherical-polar coordinates are in fact ***explicit*** functions of the vector position, $\vec{r}$ i.e. $\hat{r} = \hat{r}(\vec{r})$, $\hat{\theta} = \hat{\theta}(\vec{r})$ and $\hat{\varphi} = \hat{\varphi}(\vec{r})$ and therefore $\hat{r}, \hat{\theta} + \hat{\varphi}$ ***must*** also be ***explicitly*** differentiated in calculating the Laplacian ∇^2 of a vector function (here, $\vec{A}'(\vec{r})$) in curvilinear (i.e. either spherical-polar and/or cylindrical) coordinates!!! This is extremely important to keep in mind, for the future…

⇒ The ***safest*** way to calculate the Laplacian of a ***vector*** function $\nabla^2\vec{A}(\vec{r})$ in terms of curvilinear coordinates, is to ***NOT* use curvilinear coordinates**!!! Failing that, then one should use:

$$\nabla^2\vec{A}(\vec{r}) = \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A}(\vec{r})\right)}_{=0\text{ in the Coulomb Gauge}} - \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}(\vec{r})\right) = -\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)$$

If $\rho_m(\vec{r}') = 0$ then (automatically) $\rho_m(\vec{r}') = \vec{\nabla}\cdot\vec{A}(\vec{r}') = 0$ and we can use $\vec{A}(\vec{r})$ directly.

Hence, *if* $\boxed{\nabla^2\vec{A}(\vec{r}) = -\mu_0\vec{J}_{free}(\vec{r})}$ (vector Poisson equation for magnetostatics),

then *if* $\vec{J}_{free}(\vec{r}) \to 0$ as $r \to \infty$, then $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{free}(\vec{r})}{\mathrm{r}} d\tau'}$ where $\mathrm{r} = |\vec{\mathrm{r}}| = |\vec{\mathrm{r}} - \vec{\mathrm{r}}'|$

Generalizing this for a moving point charge as well as for line, surface and volume current densities (with $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$), we summarize these results in the following table:

$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} q_{free} \frac{\vec{v}(\vec{r}')}{\text{r}}$	$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} q_{free} \frac{\vec{v}(\vec{r}') \times \hat{\text{r}}}{\text{r}^2}$
$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{C'} \frac{\vec{I}_{free}(\vec{r}')}{\text{r}} d\ell' = \frac{\mu_o}{4\pi} I_{free} \int_{C'} \frac{d\vec{\ell}'}{\text{r}}$	$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{C'} \frac{\left(\vec{I}_{free}(\vec{r}') d\ell' \times \hat{\text{r}}\right)}{\text{r}^2} = \frac{\mu_o}{4\pi} I_{free} \int_{C'} \frac{\left(d\vec{\ell}'(\vec{r}') \times \hat{\text{r}}\right)}{\text{r}^2}$
$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{S'} \frac{\vec{K}_{free}(\vec{r}')}{\text{r}} da'$	$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{S'} \frac{\left(\vec{K}_{free}(\vec{r}') \times \hat{\text{r}}\right)}{\text{r}^2} da'$
$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}_{free}(\vec{r}')}{\text{r}} d\tau'$	$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\left(\vec{J}_{free}(\vec{r}') \times \hat{\text{r}}\right)}{\text{r}^2} d\tau'$

Note that: $\vec{A} \parallel \vec{v}, \vec{I}, d\vec{\ell}, \vec{K}, \vec{J}$ i.e. $\vec{A}$ is always parallel to the direction of motion of current, with relative velocity $\vec{v}$, whereas $\vec{B} = \left(\vec{\nabla} \times \vec{A}\right) \perp \vec{v}, \vec{I}, d\vec{\ell}, \vec{K}, \vec{J}$.

Note also that $\vec{B}$ and $\vec{A}$ both vanish when $\vec{v} \to 0$
(e.g. in the rest frame of a current (e.g. a proton or an electron beam)).

A Tale of Two Reference Frames:

For a pure point electric charge/point electric monopole moment, q we know that if it is moving in the lab frame with speed $v << c$ {c = speed of light in vacuum} that the magnetic field $\vec{B}_q(\vec{r})$ observed in the lab frame is:

$$\vec{B}_q(\vec{r}) = \vec{\nabla} \times \vec{A}_q(\vec{r}) = \frac{1}{c^2}\left(\vec{v} \times \vec{E}_q(\vec{r})\right) = \frac{q}{4\pi\varepsilon_0 c^2}\left(\vec{v} \times \frac{\hat{r}}{r^2}\right) = \frac{\mu_o}{4\pi}\left(q\vec{v} \times \frac{\hat{r}}{r^2}\right)$$

Thus in the lab frame where this charged particle is moving, the magnetic vector potential $\vec{A}_q(\vec{r})$ associated with this moving charged particle (as observed in the lab frame) has a non-zero curl.

Contrast this with the situation in the rest frame of this pure point electric charge particle, where the magnetic field vanishes, i.e. the magnetic vector potential $\vec{A}_q(\vec{r})$ associated with this charged particle has no curl!!!

We will find out (next semester, in P436) that: $\boxed{\vec{\nabla} \bullet \vec{A}'(\vec{r},t) = -\frac{1}{c^2}\frac{\partial V(r,t)}{\partial t}}$ in electrodynamics.

$\Rightarrow \exists$ connection between the $\vec{A}(\vec{r},t)$ – field and electric scalar potential $V(\vec{r},t)$ - they are in fact the 3 spatial & 1 temporal components of the relativistic 4-potential in electrodynamics !!!

Uses of the Magnetic Scalar Potential $\Phi_m(\vec{r})$**:**

In certain (limited) circumstances for magnetostatics, it _is_ _actually_ possible to have the magnetic field $\vec{B}(\vec{r})$ _directly_ related to the (negative) gradient of a magnetic scalar potential $\Phi_m(\vec{r})$, i.e. $\boxed{\vec{B}(\vec{r}) = -\mu_o \vec{\nabla}\Phi_m(\vec{r})}$, in direct analogy to that for electrostatics $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$.

However, while $\vec{\nabla}\cdot\vec{B}(\vec{r}) = -\mu_o \vec{\nabla}\cdot\vec{\nabla}\Phi_m(\vec{r}) = -\mu_o \nabla^2\Phi_m(\vec{r}) = 0$ _is_ satisfied, i.e. $\nabla^2\Phi_m(\vec{r}) = 0$ is Laplace's equation for the magnetic scalar potential, $\Phi_m(\vec{r})$ (n.b. implying that $\rho_m(\vec{r}) \equiv 0$), Ampere's law $\vec{\nabla}\times\vec{B}(\vec{r}) = -\mu_o \underbrace{\vec{\nabla}\times\vec{\nabla}\Phi_m(\vec{r})}_{\equiv 0\, Always!!!} = \mu_o \vec{J}(\vec{r})$ is _not_ satisfied/is _violated_ (!!!) _unless_ $\vec{J}(\vec{r}) \equiv 0$ _everywhere_ in the region(s) of interest. These current-free regions _must_ _also_ be _simply-connected_. {A region *D* (e.g. in a plane) is _connected_ if any two points in the region can be connected by a _piecewise smooth curve lying entirely within D_. A region *D* is a _simply connected_ region if _every closed curve in D encloses only points that are in D_.}

The use of $\vec{B}(\vec{r}) = -\vec{\nabla}\Phi_m(\vec{r})$ is in fact helpful for determining the magnetic fields associated with e.g. current-carrying filamentary wires, current loops/magnetic dipoles, and e.g. the magnetic fields associated with magnetized materials/magnetized objects.

The SI Units of the magnetic vector potential $\vec{A}$ are Tesla-meters (= magnetic field strength per unit length), which is also equal to _Newtons/Ampere_ (force per unit current) = kg-meter/Ampere-sec^2 = kg-meter/Coulomb-sec = (kg-meter/sec)/Coulomb = momentum per unit charge, since (kg-meter/sec) are the physical units associated with momentum $p = "mv"$.

Thus, for the $\vec{A}$-field:

$$\boxed{1 \text{ Tesla-meter} = 1 \text{ unit of } \frac{\text{force}}{\text{Ampere of current}} = 1 \text{ unit of } \frac{\text{momentum}}{\text{Coulomb of charge}}}$$

and for the $\vec{B}$-field, from $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$:

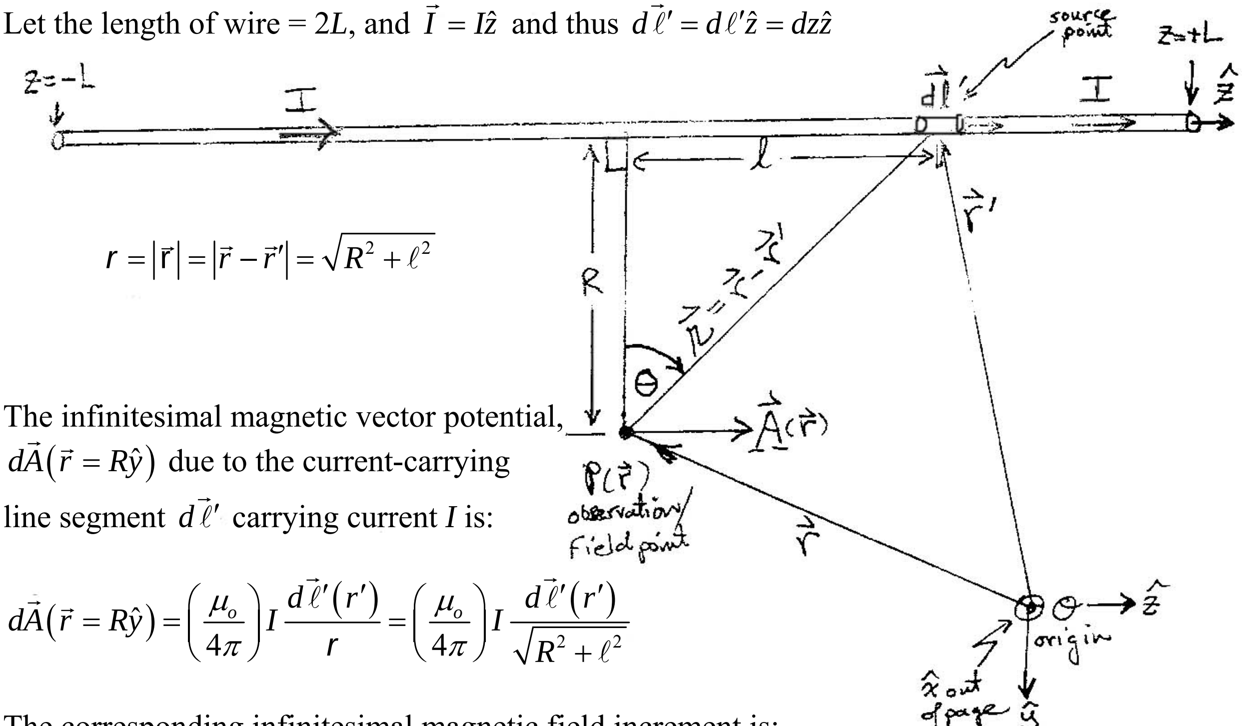
$$\boxed{1 \text{ Tesla} = 1 \text{ unit of } \frac{\text{force}}{\text{Ampere of current}} \Big/ \text{meter} = \frac{N}{A-m} = \frac{\text{momentum}}{\text{Coulomb of charge}} \Big/ \text{meter}}$$

Physically, the $\vec{A}$-field has units of force per Ampere of current (or momentum per Coulomb of electric charge), and thus physically, the magnetic field $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ is the _curl_ of the force per unit current (or momentum per unit charge) field. Note also that force, $F = dP/dt$ and current, $I = dQ/dt$ such that the magnetic vector potential $\vec{A}$ physically also has units of $\boxed{\left(\frac{\text{Force}}{\text{Current}}\right) = \frac{F}{I} = \frac{dp/dt}{dq/dt} = \frac{dp/\cancel{dt}}{dq/\cancel{dt}} = \frac{\Delta p}{\Delta q}}$ and thus $\vec{B}$ is the curl of this physical quantity.

The Magnetic Vector Potential of a Long Straight Wire Carrying a Steady Current

For a filamentary wire carrying steady current I, the magnetic vector potential and magnetic field are: $\boxed{\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \int_{C'} \frac{d\vec{\ell}'}{r}}$ and $\boxed{\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \int_{C'} \frac{\left(d\vec{\ell}' \times \hat{r}\right)}{r^2}}$

Let the length of wire = $2L$, and $\vec{I} = I\hat{z}$ and thus $d\vec{\ell}' = d\ell'\hat{z} = dz\hat{z}$



$$r = |\vec{r}| = |\vec{r} - \vec{r}'| = \sqrt{R^2 + \ell^2}$$

The infinitesimal magnetic vector potential, $d\vec{A}(\vec{r} = R\hat{y})$ due to the current-carrying line segment $d\vec{\ell}'$ carrying current I is:

$$d\vec{A}(\vec{r} = R\hat{y}) = \left(\frac{\mu_o}{4\pi}\right) I \frac{d\vec{\ell}'(r')}{r} = \left(\frac{\mu_o}{4\pi}\right) I \frac{d\vec{\ell}'(r')}{\sqrt{R^2 + \ell^2}}$$

The corresponding infinitesimal magnetic field increment is:

$$d\vec{B}(\vec{r} = R\hat{y}) = \vec{\nabla} \times d\vec{A}(\vec{r} = R\hat{y}) = \left(\frac{\mu_o}{4\pi}\right) I \vec{\nabla} \times \frac{d\vec{\ell}'(r')}{r} = \left(\frac{\mu_o}{4\pi}\right) I \left(\vec{\nabla} \times \frac{dz\hat{z}}{\sqrt{R^2 + \ell^2}}\right)$$

<u>Now</u>:

$$\vec{A}(\vec{r} = R\hat{y}) = \int_{C'} d\vec{A}(\vec{r} = R\hat{y}) = \left(\frac{\mu_o}{4\pi}\right) I \int_{z=-L}^{z=+L} \frac{dz\hat{z}}{\sqrt{R^2 + \ell^2}} = 2\left(\frac{\mu_o}{4\pi}\right) \int_0^L \frac{d\ell}{\sqrt{R^2 + \ell^2}}$$

$$= 2\left(\frac{\mu_o}{4\pi}\right) I \ln\left[\ell + \sqrt{R^2 + \ell^2}\right]\Big|_0^L \hat{z} = 2\left(\frac{\mu_o}{4\pi}\right) I \left\{\ln\left[L + \sqrt{R^2 + L^2}\right] - \ln R\right\}$$

More generally, for a $\perp$ distance R away from a long straight wire of length $2L$ carrying a steady current I:

$$\boxed{\vec{A}(\vec{r} = R\hat{y}) = 2\left(\frac{\mu_o}{4\pi}\right) I \ln\left[\left(\frac{L}{R}\right)\left\{1 + \sqrt{1 + \left(R/L\right)^2}\right\}\right]\hat{z}}$$

If $L >> R$, then: $\boxed{\vec{A}(\vec{r} = R\hat{y}) \approx 2\left(\frac{\mu_o}{4\pi}\right) I \ln\left(\frac{2L}{R}\right)\hat{z}}$ {Since $\sqrt{1+\varepsilon} \approx 1$ for $\varepsilon \ll 1$.}

Note that if $L \to \infty$ (or $R \to 0$), then $\vec{A}(\vec{r} = R\hat{y})$ <u>diverges</u> (logarithmically)!

This is OK (unphysical anyway!) because even if $\vec{A} \to \infty$, $\vec{B} = \vec{\nabla} \times \vec{A} \neq \infty$!!! (necessarily!!!)

So, for a $\perp$ distance R away from a long straight wire of length $2L$ carrying a steady current I:

$$\boxed{\vec{A}(\vec{r}=R\hat{y}) = 2\left(\frac{\mu_o}{4\pi}\right) I \ln\left[\left(\frac{L}{R}\right)\left\{1+\sqrt{1+\left(R/L\right)^2}\right\}\right]\hat{z} = A_z(\vec{r}=R\hat{y})\,\hat{z}}$$

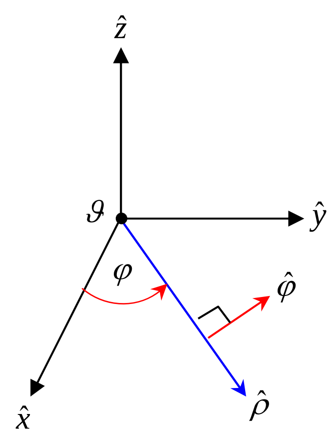
Then $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ Let's do this in cylindrical coordinates: (note: $\rho = \sqrt{x^2+y^2} = R$ here):

$$\boxed{\vec{B}(\rho=R) = \left(\frac{1}{\rho}\cancelto{=0}{\frac{\partial A_z}{\partial\varphi}} - \cancelto{=0}{\frac{\partial A_\varphi}{\partial z}}\right)\hat{\rho}\Bigg|_{\rho=R} + \left(\cancelto{=0}{\frac{\partial A_\rho}{\partial z}} - \frac{\partial A_z}{\partial\rho}\right)\hat{\varphi}\Bigg|_{\rho=R} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\cancelto{=0}{\left(\rho A_\varphi\right)} - \cancelto{=0}{\frac{\partial A_\rho}{\partial\varphi}}\right)\hat{z}\Bigg|_{\rho=R} = -\frac{\partial Az}{\partial\rho}\hat{\varphi}\Bigg|_{\rho=R}}$$

<u>Or</u>:

$$\boxed{\vec{B}(\rho=R) = \frac{1}{R}\begin{vmatrix}\hat{\rho} & \hat{\varphi} & \hat{z}\\ \frac{\partial}{\partial\rho} & 0 & \frac{\partial}{\partial z}\\ 0 & 0 & A_z\end{vmatrix}_{\rho=R} = -\frac{\partial A_z}{\partial\rho}\hat{\varphi}\Bigg|_{\rho=R}}$$

<u>In Cylindrical Coordinates</u>:



<u>Thus</u>:

$$\boxed{\begin{aligned} B_\rho &= 0\\ B_\varphi &= \frac{1}{\cancel{R}}\bullet\cancel{R}\frac{\partial Az}{\partial\rho}\hat{\varphi}\Bigg|_{\rho=R} = -\frac{\partial Az}{\partial\rho}\hat{\varphi}\Bigg|_{\rho=R}\\ B_z &= 0\end{aligned}}$$

Then:

$$\boxed{\vec{B}(\rho=R) = -\frac{\partial Az}{\partial\rho}\hat{\varphi}\Big|_{\rho=R} = -2\left(\frac{\mu_o}{4\pi}\right) I \frac{\partial}{\partial R}\left[\ln\left(L+\sqrt{L^2+R^2}\right) - \ln R\right]\hat{\varphi}}$$

<u>Now if</u>: $U(R) \equiv \left(L+\sqrt{L^2+R^2}\right)$

<u>Then</u>: $$\frac{\partial}{\partial R}\ln\left(U(R)\right) = \frac{1}{U(R)}\left[\frac{dU(R)}{dR}\right] = \frac{R}{\left(L+\sqrt{L^2+R^2}\right)\left(\sqrt{L^2+R^2}\right)}$$

<u>Since</u>: $\dfrac{dU(R)}{dR} = \dfrac{R}{\sqrt{L^2+R^2}}$ <u>and</u>: $\dfrac{\partial}{\partial R}\left(\ln(R)\right) = \dfrac{1}{R}$

<u>Then finally</u>:

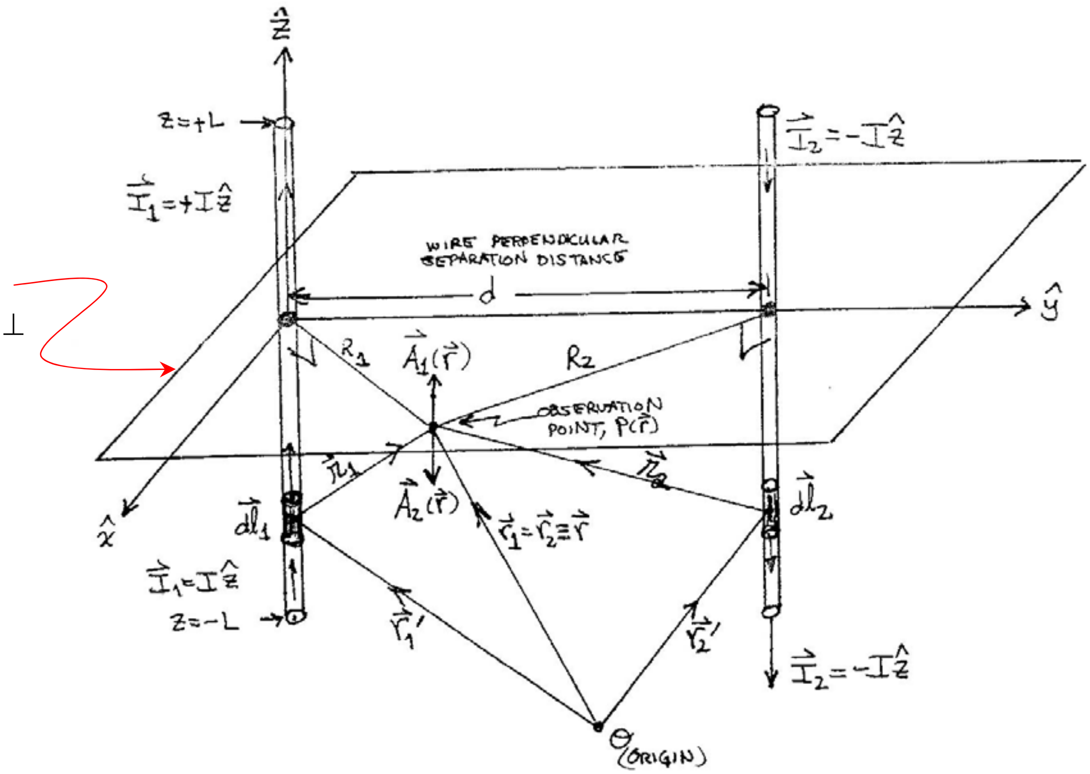
$$\boxed{\vec{B}(\rho=R) = -\left(\frac{\mu_o}{2\pi}\right) I \left\{\frac{R}{L^2\sqrt{1+\left(R/L\right)^2}\left(1+\sqrt{1+\left(R/L\right)^2}\right)} - \frac{1}{R}\right\}\hat{\varphi}} \Leftarrow$$

B-field associated with filamentary wire of length 2***L*** carrying steady current ***I***.

Note that as $L\to\infty$, then $\boxed{\vec{B}(\rho=R) = -\left(\frac{\mu_o}{2\pi}\right)\frac{I}{R}\hat{\varphi}}$ $\Leftarrow$ *i.e.* exactly the same as we obtained for ∞-long straight filamentary wire carrying steady current I (see previous P435 Lecture Notes)!!!

The Magnetic Vector Potential $\vec{A}(\vec{r})$ and Magnetic Field $\vec{B}(\vec{r})=\vec{\nabla}\times\vec{A}(\vec{r})$ Associated with a Pair of Long, Parallel Wires Carrying Steady Currents $I_1=+I\hat{z}$ and $I_2=-I\hat{z}$, Separated by a Perpendicular Separation Distance, *d*

Plane ⊥ to wires and containing ⊥ separation distance *d*



$$d\vec{\ell}_1=+d\ell\hat{z}=+dz\hat{z} \qquad \vec{r}_1=\vec{r}_1-\vec{r}_1'=\vec{r}-\vec{r}_1'$$

$$d\vec{\ell}_2=-d\ell\hat{z}=-dz\hat{z} \qquad \vec{r}_2=\vec{r}_2-\vec{r}_2'=\vec{r}-\vec{r}_2' \quad (\vec{r}_1=\vec{r}_2=\vec{r})$$

For simplicity's sake, assume $L >> R_1, R_2$.

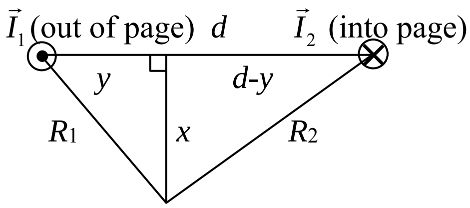
<u>Then</u>: $\boxed{\vec{A}_1(R_1)\approx+\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{R_1}\right)\hat{z}}$ and $\boxed{\vec{A}_2(R_2)\approx-\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{R_1}\right)\hat{z}}$

Then, using the principle of linear superposition:

$$\vec{A}_{TOT}(\vec{r})=\vec{A}_1(\vec{r})+\vec{A}_2(\vec{r})\approx+\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{R_1}\right)\hat{z}-\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{2L}{R_2}\right)\hat{z}$$

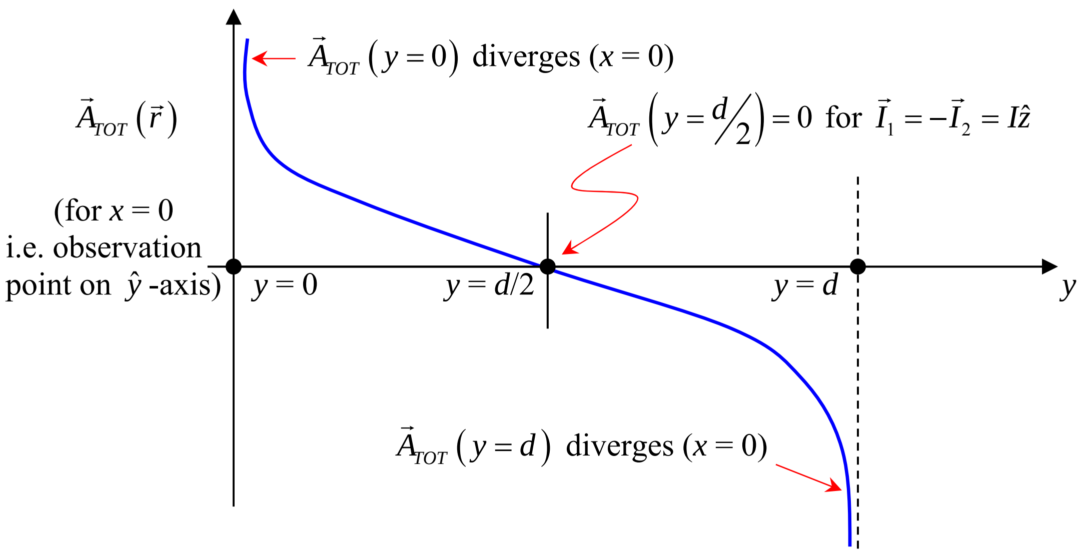
<u>Or</u>: $\boxed{\vec{A}_{TOT}(\vec{r})\approx\left(\frac{\mu_o}{2\pi}\right)I\ln\left(\frac{R_2}{R_1}\right)\hat{z}=\left(\frac{\mu_o}{4\pi}\right)I\ln\left(\frac{R_2}{R_1}\right)^2\hat{z}}$ for $L >> R_1, R_2$.

Now let us re-locate the local origin to be at the LHS wire, where it intersects the $\perp$-plane:



$$R_1^2 = x^2 + y^2$$
$$R_2^2 = x^2 + \left(d^2 - y^2\right)$$

Then: $$\boxed{\vec{A}_{TOT}(\vec{r}) \simeq \left(\frac{\mu_o}{4\pi}\right) I \ln\left(\frac{R_2}{R_1}\right)^2 \hat{z} = \left(\frac{\mu_o}{4\pi}\right) I \ln\left[\frac{x^2 + (d-y)^2}{x^2 + y^2}\right]\hat{z}}$$ for $L >> R_1, R_2$.



Note that $\vec{A}_{TOT} = A_z\hat{z}$ i.e. $A_x^{TOT} = A_y^{TOT} = 0$ (since currents only in $\pm\hat{z}$-direction) !!!

Note also that $\vec{A}_{TOT} = A_z\hat{z}$ changes sign – its direction is parallel to the <u>closest</u> current!!!

Then: $\boxed{\vec{B}_{TOT}(\vec{r}) = \vec{\nabla}\times\vec{A}_{TOT}(\vec{r})}$, in Cartesian $(\hat{x}-\hat{y}-\hat{z})$ coordinates:

$$\boxed{B_x^{TOT} = +\frac{\partial A_z^{TOT}}{\partial y} = -\left(\frac{\mu_o}{2\pi}\right) I\left[\frac{(d-y)}{R_2^2} + \frac{y}{R_1^2}\right]} \qquad R_1^2 = x^2 + y^2$$

$$\boxed{B_y^{TOT} = +\frac{\partial A_z^{TOT}}{\partial x} = -\left(\frac{\mu_o}{2\pi}\right) I\left[\frac{x}{R_2^2} - \frac{x}{R_1^2}\right]} \qquad R_2^2 = x^2 + \left(d^2 - y^2\right)$$

$$\boxed{B_z^{TOT} = 0}$$

At the point $x = 0$ and $y = d/2$ (i.e. at the center point, midway between the two conductors): {where $R_1 = R_2 = R = d/2$}:

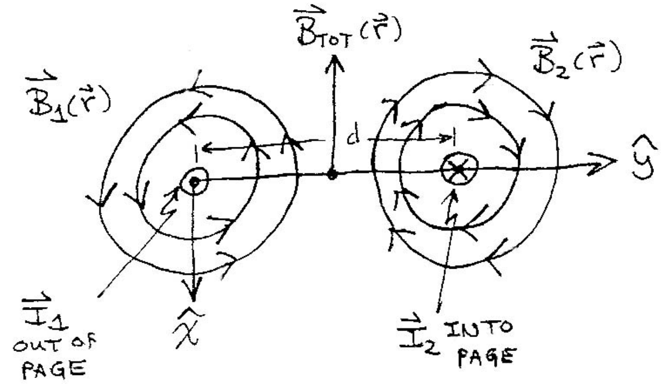
$$B_x^{TOT} = -\left(\frac{\mu_o}{2\pi}\right) I \left[\frac{d/2}{\left(d/2\right)^2} + \frac{d/2}{\left(d/2\right)^2}\right] = -\left(\frac{\mu_o}{2\pi}\right) I \left[\frac{1}{d/2} + \frac{1}{d/2}\right] = -\left(\frac{\mu_o}{2\pi}\right) I \frac{4}{d} = -\left(\frac{2\mu_o}{\pi}\right)\frac{I}{d}$$

$$B_y^{TOT} = 0$$

$$B_z^{TOT} = 0$$

$$\vec{B}_{TOT}\left(x = 0, y = d/2, z = 0\right) = -\frac{2\mu_o}{\pi}\left(\frac{I}{d}\right)\hat{x}$$

Top View:



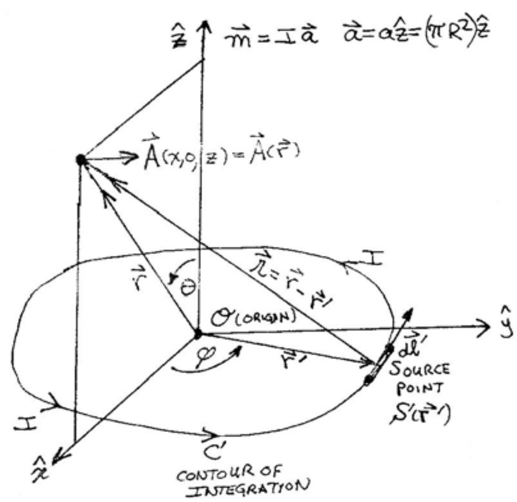
The Magnetic Vector Potential $\vec{A}(\vec{r})$ and Magnetic Field $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$

Associated with a Magnetic Dipole Loop

(For Large Source-Observer Separation Distances)

For simplicity's sake, let us choose the observation / field point $P(\vec{r})$ to lie in the *x*-*z* plane:

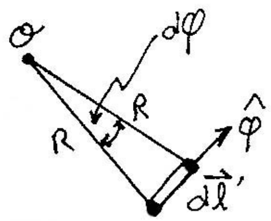
Observation / field Point $P(\vec{r}) = P(x,0,z)$

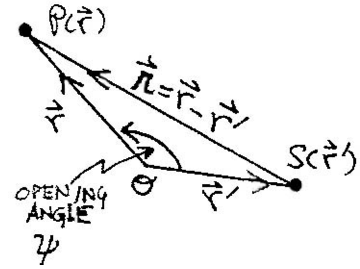


Magnetic Dipole Loop has radius R

$R = |\vec{r}'|$

$\vec{I} = I\hat{\varphi}$





$d\vec{\ell}' = Rd\varphi\hat{\varphi}$

$\left(d\ell' = Rd\varphi\right)$

{from the arc length formula "$S = R\theta$"}

$\cos\Psi = \hat{r}\bullet\hat{r}'$ = opening angle between $\hat{r}$ and $\hat{r}'$

$\cos\Psi = \dfrac{\vec{r}\bullet\vec{r}'}{|\vec{r}|\bullet|\vec{r}'|} = \dfrac{\vec{r}\bullet\vec{r}'}{r\bullet r'} = \dfrac{\vec{r}\bullet\vec{r}'}{rR}$ where: $r' = |\vec{r}'| = R$

Now: $\vec{A}(\vec{r}) = \left(\dfrac{\mu_o}{4\pi}\right)\oint_{C'}\dfrac{Id\vec{\ell}'(\vec{r}')}{\mathscr{r}} = \left(\dfrac{\mu_o}{4\pi}\right)I\oint_{C'}\dfrac{d\vec{\ell}'(\vec{r}')}{\mathscr{r}}$

$\Rightarrow \vec{A}(\vec{r})$ (here) is a function of $\hat{x}$ and $\hat{y}$ only (actually only $\hat{\varphi}$) and not $\hat{z}$ since $\vec{I}d\ell' = Id\vec{\ell}'$ lies in the *x*-*y* plane and $\boxed{I = I\hat{\varphi}}$ (here).

Since we evaluate $\vec{A}$ in the *x*-*z* plane, the only component of $d\vec{\ell}'$ that will contribute to $\vec{A}$ (there) will be in the $\hat{y}$ direction (n.b. $\vec{A}$ is parallel to the closest current from the observation point *P*(*r*)).

$\Rightarrow$ We only want the component of $d\vec{\ell}'(\vec{r}')$ along the $\hat{y}$-axis, $d\vec{\ell}(\vec{r}')\cos\varphi$ (Note: If we wanted to evaluate $\vec{A}$ e.g. in the *y*-*z* plane, then we would want only the component of $d\vec{\ell}'(\vec{r}')$ along the $\hat{x}$-axis, $d\vec{\ell}(\vec{r}')\sin\varphi$)

Then: $\vec{A}(\vec{r}) = \vec{A}(x,0,z) = \left(\dfrac{\mu_o}{4\pi}\right)I\int_0^{2\pi}\dfrac{(Rd\varphi)\cos\varphi}{\mathscr{r}}\hat{\varphi} \leftarrow \hat{\varphi} = \hat{y}$ in the *x*-*z* plane for $\vec{r} = (x,0,z)$.

Now: $\mathscr{r}^2 = r^2 + R^2 - 2rR\cos\Psi$ (from the Law of Cosines) and $\mathscr{r} = |\vec{r} - \vec{r}'|$

And: $\dfrac{\mathscr{r}}{r} = \left(1 - \dfrac{R^2}{r^2} + \dfrac{2rR}{r^2}\cos\Psi\right)^{1/2} \approx 1 - \dfrac{1}{2}\left(\dfrac{R}{r}\right)^2 + \left(\dfrac{R}{r}\right)\cos\Psi$ if $R \ll r, \mathscr{r}$ and $r \approx \mathscr{r}$.

And: $\vec{r}\bullet\vec{r}' = rr'\cos\Psi = (x\hat{x} + z\hat{z})\bullet(R\cos\varphi\hat{x} + R\sin\varphi\hat{y}) = xR\cos\varphi$

(if the observation / field point $P(\vec{r}) = P(x,0,z)$ lies in the *x*-*z* plane)

Thus: $\dfrac{\mathscr{r}}{r} \simeq 1 - \dfrac{1}{2}\left(\dfrac{R}{r}\right)^2 + \dfrac{xR\cos\varphi}{r^2}$ for $r >> R$ and $r \simeq \mathscr{r} = |\vec{r} - \vec{r}'|$

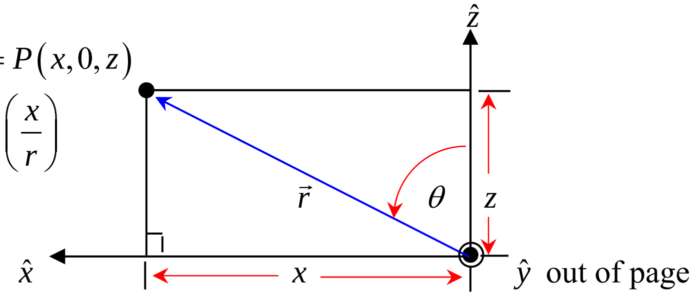
Then: $\vec{A}(\vec{r}) = \vec{A}(x,0,z) \simeq \left(\dfrac{\mu_o}{4\pi}\right)\dfrac{I}{r}\int_{\varphi=0}^{\varphi=2\pi}R\cos\varphi\left(1 - \dfrac{1}{2}\left(\dfrac{R}{r}\right)^2 + \dfrac{xR}{r^2}\cos\varphi\right)d\varphi\hat{y}$

Or: $\vec{A}(\vec{r}) = \vec{A}(x,0,z) \simeq \left(\dfrac{\mu_o}{4\pi}\right)\dfrac{I\pi R^2}{r^3}x\hat{y}$ for $r >> R$ and $r \simeq \mathscr{r} = |\vec{r} - \vec{r}'|$

But notice that in the *x*-*z* plane:

$P(\vec{r}) = P(x,0,z)$

$\sin\theta = \left(\frac{x}{r}\right)$



$\therefore$ $\boxed{\vec{A}(\vec{r}) = \vec{A}(x,0,z) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{I\pi R^2}{r^2}\sin\theta\hat{y}}$ for $r >> R$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$

But in the *x*-*z* plane, $\hat{\varphi} = \hat{y}$, and since the direction of $\vec{A}(\vec{r})$ is always parallel to the current:

$\therefore$ $\boxed{\vec{A}(\vec{r}) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{I\pi R^2}{r^2}\sin\theta\hat{\varphi}}$ for $r >> R$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$ {n.b. $\vec{A}(\vec{r}) \parallel \vec{I} \parallel \hat{\varphi}$ }

The magnetic dipole moment associated with this current-carrying loop is $\vec{m} = m\hat{z}$

$\vec{m} = I\vec{a} = Ia\hat{z}$ (for this planar loop) where (here): $\vec{a} = \pi R^2\hat{z}$ (by the right-hand rule)

$\vec{m} = I\pi R^2\hat{z}$ and: $m = |\vec{m}| = Ia = I\pi R^2 = \pi R^2 I$

<u>Note that</u>: $\boxed{\hat{z}\times\hat{r} = \left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\times\hat{r} = -\sin\theta\left(\hat{\theta}\times\hat{r}\right) = -\sin\theta\left(-\hat{\varphi}\right) = +\sin\theta\hat{\varphi}}$

Thus, the quantity: $\boxed{\vec{m}\times\hat{r} = Ia\hat{z}\times\hat{r} = I\pi R^2\left(\hat{z}\times\hat{r}\right) = I\pi R^2\sin\theta\hat{\varphi}}$

$\therefore$ $\boxed{\vec{A}(\vec{r}) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\vec{m}\times\hat{r}}{r^2} = \left(\frac{\mu_o}{4\pi}\right)\frac{\vec{m}\times\vec{r}}{r^3}}$ for $r >> R$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$

Now $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ in spherical coordinates for the magnetic dipole (with magnetic dipole moment $\vec{m} = I\vec{a} = I\pi R^2\hat{z}$) is:

$$\boxed{B_r(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{2m}{r^3}\cos\theta}$$

$$\boxed{B_\theta(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\sin\theta}$$

$$\boxed{B_\varphi(\vec{r}) = 0}$$

$$\boxed{\vec{B}(\vec{r}) = B_r(\vec{r})\hat{r} + B_\theta(\vec{r})\hat{\theta} + B_\varphi(\vec{r})\hat{\varphi}}$$

valid for $r >> R$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$

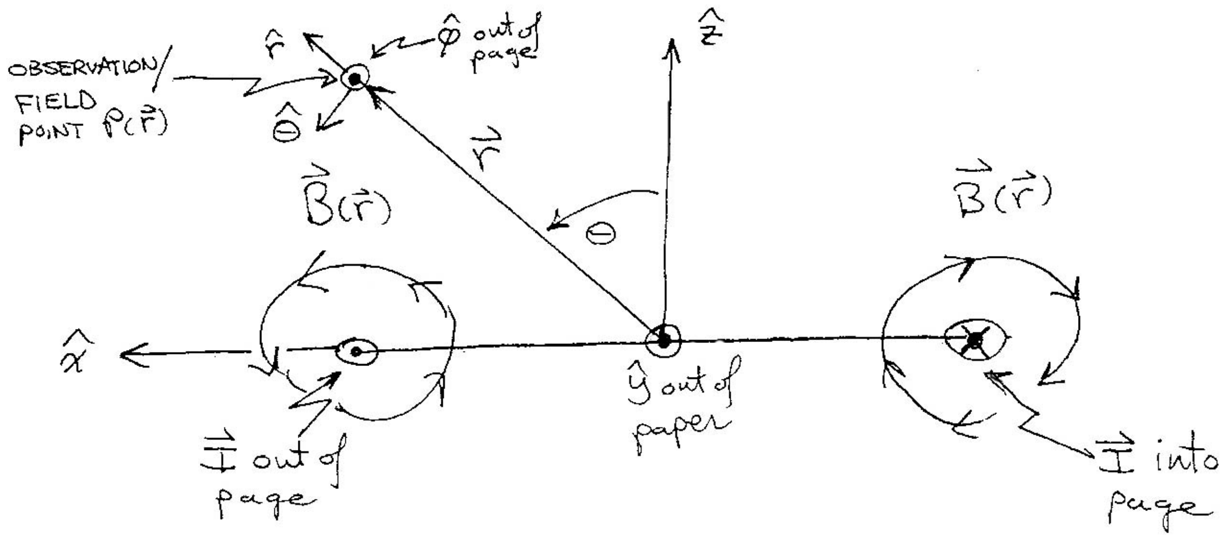
<u>Thus</u>: $\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{m}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)}$ for $r >> R$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$ and $\boxed{\vec{m} = I\vec{a} = I\pi R^2\hat{z}}$.

cf w/ the $\vec{E}$-field associated with a physical electric dipole with dipole moment $\boxed{\vec{p} = q\vec{d}}$:

$\boxed{\vec{E}(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{p}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)}$ for $r >> d$ and $\mathscr{r} \simeq r = |\vec{r} - \vec{r}'|$ and $\boxed{\vec{p} = q\vec{d}}$.

We already know what this $\vec{B}(\vec{r})$ looks like it – it is "solenoidal" around the current loop:

Cross-Sectional View of a Magnetic Dipole Loop:



Thus if $\boxed{\vec{m} = I\vec{a} = I\pi R^2\hat{z}}$ = a constant vector, $\vec{A}(\vec{r}) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\vec{m}\times\vec{r}}{r^3}$ for $r >> R$ and $r \simeq \mathbf{r} = |\vec{r}-\vec{r}'|$

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\nabla\times\frac{\vec{m}\times\vec{r}}{r^3} \quad \text{for } r >> R \text{ and } r \simeq \mathbf{r} = |\vec{r}-\vec{r}'|$$

$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left[-\left(\vec{m}\bullet\vec{\nabla}\right)\frac{\vec{r}}{r^3} + \vec{m}\underbrace{\left(\vec{\nabla}\bullet\frac{\vec{r}}{|\vec{r}|^3}\right)}_{=0}\right] \Leftarrow \left(\vec{\nabla}\bullet\frac{\vec{r}}{|\vec{r}|^3} = 0\right)$$

$$\text{e.g. } m_x\frac{\partial}{\partial x}\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = \frac{m_x\hat{x}}{|\vec{r}|^3} - 3m_x x\frac{\vec{r}}{|\vec{r}|^5} \quad \therefore \left(\vec{m}\bullet\vec{\nabla}\right)\frac{\vec{r}}{|\vec{r}|^3} = \frac{\vec{m}}{|\vec{r}|^3} - \frac{3(\vec{m}\bullet\vec{r})\vec{r}}{|\vec{r}|^5} = -\frac{3(\vec{m}\bullet\hat{r})\hat{r}-\vec{m}}{|\vec{r}|^3}$$

$$\text{and: } \vec{\nabla}\bullet\frac{\vec{r}}{|\vec{r}|^3} = \frac{3}{|\vec{r}|^3} - \vec{r}\bullet\frac{3\vec{r}}{|\vec{r}|^5} = \frac{3}{|\vec{r}|^3} - \frac{3\vec{r}\bullet\vec{r}}{|\vec{r}|^5} = \frac{3}{|\vec{r}|^3} - \frac{3|\vec{r}|^2}{|\vec{r}|^5} = \frac{3}{|\vec{r}|^3} - \frac{3}{|\vec{r}|^3} = 0$$

$$\therefore \boxed{\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\nabla\times\frac{\vec{m}\times\vec{r}}{|\vec{r}|^3} = \left(\frac{\mu_o}{4\pi}\right)\left[\frac{3(\vec{m}\bullet\hat{r})\hat{r}-\vec{m}}{|\vec{r}|^3}\right]} \Leftarrow \boxed{\text{Magnetic Field of a Magnetic Dipole Loop}}$$

for $r >> R$ (far away) and $r \simeq \mathbf{r} = |\vec{r}-\vec{r}'|$

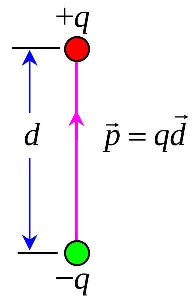
⇒ The magnetic field of a distant circuit ($r >> R$) does not depend on its detailed geometry, but only its magnetic ***dipole*** moment, $\vec{m}$!!! ⇐ Important (conceptual) result!!!

Compare this result for $\vec{B}(\vec{r})$ for the magnetic dipole loop, with magnetic dipole moment $\vec{m} = I\vec{a}$, with result for the electric dipole field $\vec{E}(\vec{r})$ associated with a physical electric dipole moment $\vec{p} = q\vec{d}$:

$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left[\frac{3(\vec{m}\bullet\hat{r})\hat{r} - \vec{m}}{|\vec{r}|^3}\right]} \quad \boxed{\vec{m} = I\vec{a}}, \text{ for } |\vec{r}| \gg R \text{ (far away) and } r \simeq r = |\vec{r} - \vec{r}'|$$

$$\boxed{\vec{E}(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\left[\frac{3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}}{|\vec{r}|^3}\right]} \quad \boxed{\vec{p} = q\vec{d}}, \text{ for } |\vec{r}| \gg |\vec{d}| \text{ (far away) and } r \simeq r = |\vec{r} - \vec{r}'|$$

When $r \approx R$ (or less) for magnetic dipole loop
Or $r \approx d$ (or less) for electric dipole, then will be able to "see" / observe / detect higher-order moments - e.g. quadrupole, octupole, sextupole, etc. . .moments of the $\vec{B}(\vec{E})$ fields.



The statement that the magnetic field of a distant circuit ($r >> R$) does not depend on its detailed geometry, but only its magnetic ***dipole*** moment, $\vec{m} = I\vec{a}$ (n.b. this is also true for the electrostatic case, with $\vec{p} = q\vec{d}$) are very useful!!!

If one can compute $\vec{m} = I\vec{a}$ then one can obtain $\vec{A}(\vec{r})$ and hence $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ (or if have $\vec{p} = q\vec{d}$ then can obtain $\vec{E}(\vec{r})$) for $r >> R$ (or d) and $r \simeq r = |\vec{r} - \vec{r}'|$. EASY!!!

Magnetic Flux Conservation

If $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ then $\vec{\nabla}\bullet\vec{B}(\vec{r}) = \vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{A}(\vec{r})\right) = 0$ is automatically satisfied everywhere $(\forall\ \vec{r})$

If $\vec{\nabla}\bullet\vec{B}(\vec{r}) \equiv 0$ for each/every point, $\vec{r}$ in a volume v bounded by its surface S

Then: $\int_v \vec{\nabla}\bullet\vec{B}(\vec{r})\,d\tau = \oint_S \vec{B}(\vec{r})\bullet\hat{n}dA$ (by the Divergence Theorem)

What is $\oint_S \vec{B}(\vec{r})\bullet\hat{n}dA$?? $\oint_S \vec{B}(\vec{r})\bullet\hat{n}dA \equiv 0$

Recall Gauss' Law for $\vec{E}$ (and/or $\vec{D}$) were:

$\boxed{\Phi_D \equiv \oint_S \vec{D}\bullet\hat{n}dA = Q_{free}^{enclosed}}$ = net electric displacement flux through closed surface S.

$\boxed{\Phi_E \equiv \oint_S \vec{E}(\vec{r})\bullet\hat{n}dA = Q_{Tot}^{enclosed}/\varepsilon_o}$ = net electric flux through closed surface S.

Thus: $\boxed{\Phi_m \equiv \oint_S \vec{B}(\vec{r})\bullet\hat{n}dA = 0}$ = net magnetic flux through closed surface $S \equiv 0$!!!

$\Rightarrow$ Magnetic flux is conserved $\rightarrow$ magnetic field lines have no beginning / no end points (because $\exists$ no magnetic charge(s)!)

The SI units of magnetic flux Φ_m are Tesla-m^2 = Webers

Again: Do not confuse magnetic flux, Φ_m with the magnetic scalar potential $\Phi_m(\vec{r})$
(they even have the same units!!! Webers / Tesla-m[2]) WAA-*HEE* !!!

They are not the same thing!!!

$$\Phi_m = \int_S \vec{B}(\vec{r})\bullet\hat{n}dA$$

$$\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Phi_m(\vec{r})$$

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}'(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) + \vec{\nabla}\times\vec{\nabla}\Phi_m(\vec{r})$$

Magnetic Flux — Area element — Magnetic Vector Potential — Magnetic Scalar Potential

Don't confuse these either!!!

$$\vec{\nabla}\times\vec{A}(\vec{r}) = \nabla^2\Phi_m(\vec{r}) = -\rho_m(\vec{r}) \qquad \Phi_m(\vec{r}) = \frac{1}{4\pi}\int_v \frac{\vec{\nabla}\bullet\vec{A}(\vec{r})}{r}d\tau \longrightarrow \text{gives } \vec{\nabla}\bullet\vec{A}'(\vec{r}) = 0$$

The magnetic flux through a surface *S* (not necessarily closed!!!):

$$\boxed{\Phi_m = \int_S \vec{B}(\vec{r})\bullet\hat{n}dS = \int_S\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\bullet\hat{n}dS = \oint_C \vec{A}(\vec{r})\bullet d\vec{\ell}}$$ by Stoke's Theorem

n.b. not a closed surface!

Magnetic flux enclosed by contour C: $\boxed{\Phi_m = \oint_C \vec{A}(\vec{r})\bullet d\vec{\ell}}$

n.b. This $d\vec{\ell}$ is NOT a line segment associated with a line current *I* !!!

$$\Phi_m = \int_S \vec{B}(\vec{r})\bullet\hat{n}dS = \int_S\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\bullet\hat{n}dS = \oint_C \vec{A}(\vec{r})\bullet d\vec{\ell} = \oint_{C'} \vec{\nabla}\Phi_m(\vec{r})\bullet d\vec{\ell} = \Phi_m$$

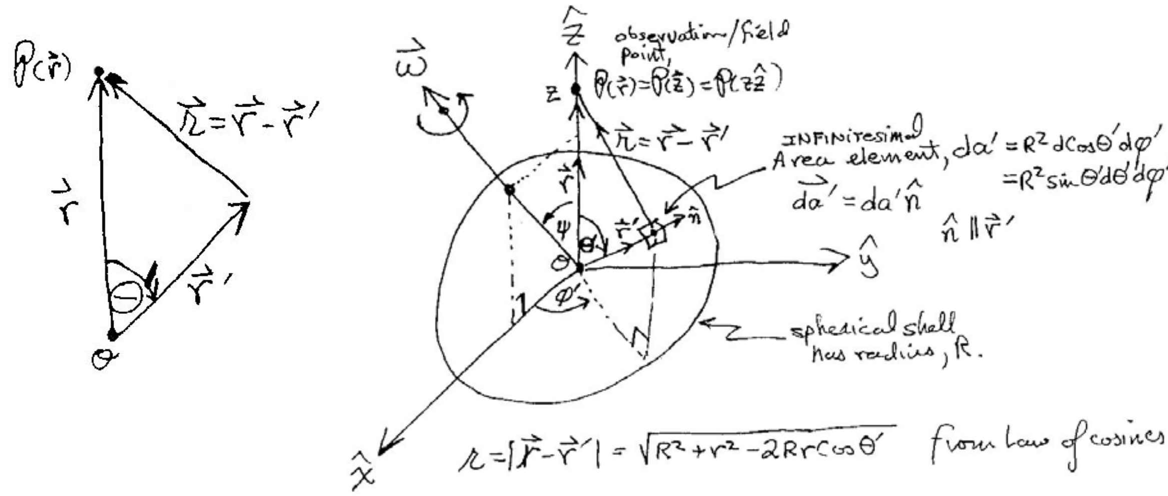
Griffiths Example 5.11:

A spherical shell of radius R carries a uniform surface charge density σ and rotates with constant angular velocity $\vec{\omega}$, Determine the magnetic vector potential it produces at point $\vec{r}$.

A rotating surface charge density σ produces a surface/sheet current density $\vec{K}(\vec{r}') = \sigma\vec{v}(\vec{r}')$

The magnetic vector potential is thus: $\boxed{\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{S'} \frac{\vec{K}(\vec{r}')}{r} da'}$

For ease of integration, choose the observation/field point $P(\vec{r}) = P(z\hat{z})$ (i.e. $r = z\hat{z}$) along the $+\hat{z}$ -axis and $\vec{\omega}$ to lie in the *x*-*z* plane. Choose the origin $\mathcal{O}$ to be at the center of the sphere, as shown in the figure below:



$r = |\vec{r} - \vec{r}'| = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$ from the law of cosines

$\vec{K}(\vec{r}') = \sigma\vec{v}(\vec{r}')$ and: $\vec{v}(\vec{r}') = \vec{\omega} \times \vec{r}'$

$\vec{\omega} = \omega\sin\psi\,\hat{x} + \omega\cos\psi\,\hat{z}$ (here)

$\vec{r}' = R\sin\theta'\cos\varphi'\hat{x} + R\sin\theta'\sin\varphi'\hat{y} + R\cos\theta'\hat{z}$

$\vec{v}(\vec{r}') = \vec{\omega} \times \vec{r}'$

$$\vec{v}(\vec{r}') = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta' \end{vmatrix}$$

$$\vec{v}(\vec{r}') = R\omega\left[-(\cos\psi\sin\theta'\sin\varphi')\hat{x} + (\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta')\hat{y} + (\sin\psi\sin\theta'\sin\varphi')\hat{z}\right]$$

Now since $\int_0^{2\pi} \sin\varphi' d\varphi' = \int_0^{2\pi} \cos\varphi' d\varphi' = 0$

Then terms involving <u>only</u> $\sin\varphi'$ or $\cos\varphi'$ in the integral for $\vec{A}(\vec{r})$ contribute <u>nothing</u>.

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{S'}\frac{\vec{K}(\vec{r}')}{r}da' = \frac{\mu_o}{4\pi}\int_{S'}\frac{\sigma(\vec{\omega}\times\vec{r}')}{r}da'}$$

with $\vec{K}(\vec{r}') = \sigma(\vec{\omega}\times\vec{r}')$ and $da' = R^2\sin\theta' d\theta' d\varphi'$ and $r = \sqrt{R^2 + r^2 - 2Rr\cos\varphi'}$

<u>Then</u>: $\vec{A}(\vec{r}) = \frac{\mu_o R^3\sigma\omega\sin\psi}{2}\left(\int_0^\pi \frac{\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}d\theta'\right)\hat{y}$

<u>Let</u>: $u \equiv \cos\theta'$ $\quad\theta' = 0 \Rightarrow u = +1$

$du = \sin\theta' d\theta'$ $\quad\theta' = \pi \Rightarrow u = -1$

$$\int_0^\pi \frac{\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}d\theta' = \int_{-1}^{+1}\frac{udu}{\sqrt{R^2+r^2-2Rru}}$$

$$= -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\Bigg|_{u=-1}^{u=+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

If $|\vec{r}| < R$ (i.e. inside sphere) then this integral $= \left(\frac{2r}{3R^2}\right)$

If $|\vec{r}| > R$ (i.e. outside sphere) then this integral $= \left(\frac{2R}{3r^2}\right)$

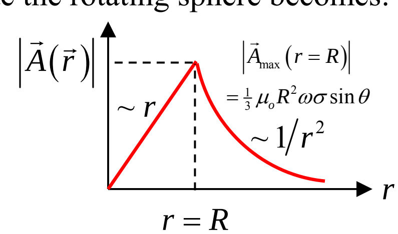
<u>Now</u>: $\boxed{\vec{\omega}\times\vec{r} = -\omega r\sin\psi\,\hat{y}}$

<u>Then</u>:

$$\boxed{\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_o R\sigma}{3}(\vec{\omega}\times\vec{r}) \text{ for } |\vec{r}| < R \text{ (inside sphere)}\\ \vec{A}(\vec{r}) &= \frac{\mu_o R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) \text{ for } |\vec{r}| > R \text{ (outside sphere)}\end{aligned}}$$

If we now <u>rotate</u> the problem so that $\vec{\omega} = \omega\hat{z}$ and $\vec{r} = (r, \vartheta, \varphi)$ then $\boxed{\vec{\omega}\times\vec{r} = -\omega r\sin\psi\,\hat{y}} \Rightarrow \boxed{\omega r\sin\theta\hat{\varphi}}$, thus with $\vec{\omega}$ rotated to $\vec{\omega} = \omega\hat{z}$ and field point now located at $\vec{r} = (r, \vartheta, \varphi)$, the magnetic vector potential $\vec{A}(\vec{r})$ inside/outside the rotating sphere becomes:

$$\boxed{\begin{aligned}\vec{A}(r,\vartheta,\varphi) &= \frac{\mu_o R\omega\sigma}{3}r\sin\theta\hat{\varphi} \quad (|\vec{r}| < R\text{, inside sphere})\\ \vec{A}(r,\vartheta,\varphi) &= \frac{\mu_o R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\varphi} \quad (|\vec{r}| > R\text{, outside sphere})\end{aligned}}$$



<u>Then</u>: $\boxed{\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \frac{2\mu_o R\omega\sigma}{3}\overbrace{(\cos\theta\hat{r} - \sin\theta\hat{\theta})}^{=\hat{z}} = \frac{2}{3}\mu_o R\omega\hat{z} = \frac{2}{3}\mu_o R\vec{\omega}\text{!!! } (\vec{\omega} = \omega\hat{z})}$

Griffiths Example 5.12:

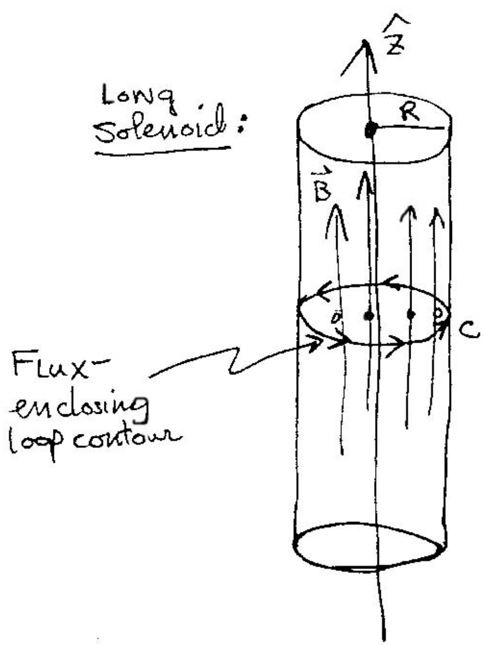
Determine the magnetic vector potential $\vec{A}(\vec{r})$ of an infinitely long solenoid with *n* turns / unit length, radius *R* and steady current *I*

$\Rightarrow$ n.b. The current extends to infinity, so we *cannot* use $\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{C'} \frac{I}{r} d\vec{\ell}'$ because it diverges!

But we *do* know that:

Magnetic flux $\Phi_m = \int_C \vec{A}(\vec{r}) \cdot d\vec{\ell} = \int_{S'} \left(\vec{\nabla} \times \vec{A}(\vec{r})\right) \cdot d\vec{a} = \int_{S'} \vec{B} \cdot d\vec{a}$

Flux-enclosing loop / contour



$\Phi_m = \int_S \vec{B}(\vec{r}) \cdot d\vec{a}$

But we know from Ampere's Circuital Law that:

$\vec{B}_{inside}(r \le R) = \mu_o n I \hat{z}$ = uniform & constant

$\therefore \Phi_m^{inside} = (\mu_o n I) * (\pi R^2) = \mu_o n I \pi R^2$

But: $\Phi_m^{inside} = \int_C \vec{A}(r = R) \cdot d\vec{\ell}$ where $d\vec{\ell} = R d\varphi \hat{\varphi}$

$\therefore \Phi_m^{inside} = A(r = R) 2\pi R$

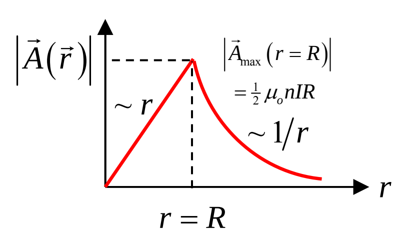
Now $\vec{A}_{solenoid}$ must be parallel to $\vec{I} = I\hat{\varphi}$ for the "ideal solenoid" (i.e. no pitch angle)

$\Rightarrow \vec{A}(\vec{r}) = A(\vec{r})\hat{\varphi}$

Then: $\vec{A}(r = R) = \frac{\Phi_m^{inside}}{2\pi R} = \frac{\mu_o n I \pi R^2}{2\pi R}\hat{\varphi} = \frac{1}{2}\mu_o n I R \hat{\varphi}$

If $r > R$, then more generally, we have: $\vec{A}_{outside}(r > R) = \frac{1}{2}\mu_o n I \left(\frac{R^2}{r}\right)\hat{\varphi}$

For $r < R$, then: $\vec{A}_{inside}(r < R) = \frac{1}{2}\mu_o n I r \hat{\varphi}$



Note that: $\vec{A}(\vec{r}) = A_\varphi(\vec{r})\hat{\varphi}$ (only) for the infinitely long ideal solenoid.

Does $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$?

$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi(\vec{r})\right)\hat{z}$ in cylindrical coordinates

$$\vec{B}_{outside}(r>R) = \frac{1}{2}\mu_o nI\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{R^2}{r}\right)\hat{z} = \frac{1}{2}\mu_o nI\frac{1}{r}\overbrace{\frac{\partial}{\partial r}\left(R^2\right)}^{=0}\hat{z} \equiv 0$$

$$\vec{B}_{inside}(r<R) = \frac{1}{2}\mu_o nI\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r^2\right)\hat{z} = \frac{1}{2}\mu_o nI\left(\frac{1}{r}\right)(2r)\hat{z} = \mu_o nI\hat{z}$$

Does $\vec{\nabla}\bullet\vec{A}(\vec{r}) = 0$?? (Coulomb Gauge) $\vec{A}(\vec{r}) = A_\varphi(\vec{r})\hat{\varphi}$

In Cylindrical Coordinates:

$$\vec{\nabla}\bullet\vec{A}(\vec{r}) = \frac{1}{r}\frac{\partial A_\varphi(\vec{r})}{\partial\varphi} = 0$$ because $\vec{A}(\vec{r})$ has <u>NO</u> <u>*explicit*</u> φ-dependence!

$$\vec{A}_{outside}(r>R) = \frac{1}{2}\mu_o nI\left(\frac{R^2}{r}\right)\hat{\varphi}$$

$$\vec{A}_{inside}(r<R) = \frac{1}{2}\mu_o nIr\hat{\varphi}$$

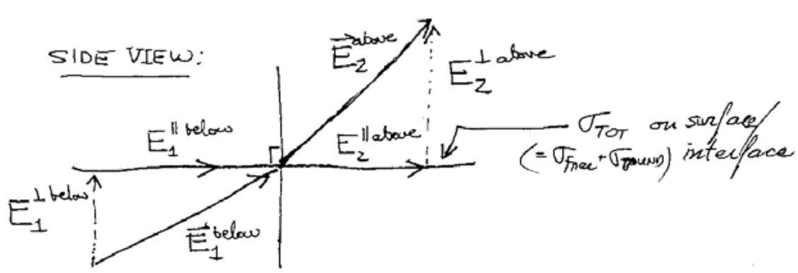
<u>Magnetostatic Boundary Conditions</u>

In the case of electrostatics, we learned (via use of Gauss' Law - $\Phi_E = \oint_S \vec{E}(\vec{r})\bullet\hat{n}da = Q_{Tot}^{enclosed}/\varepsilon_o$) that the normal component of $\vec{E}(\vec{r})$ suffers a discontinuity whenever there is a surface charge density (<u>free</u> or <u>bound</u>) present on a surface / interface:

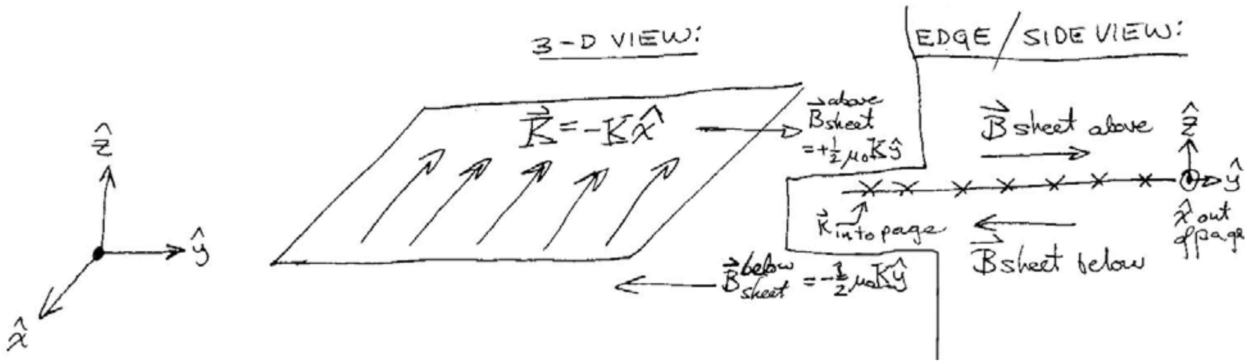
$$E_2^{\perp\,above} - E_1^{\perp\,below} = \frac{\sigma_{Tot}}{\varepsilon_o} = \frac{\left(\sigma_{free}+\sigma_{bound}\right)}{\varepsilon_o} = \left.\frac{\partial V_2^{above}}{\partial n}\right|_{surface} - \left.\frac{\partial V_1^{below}}{\partial n}\right|_{surface}$$

$\left(E_2^{\parallel above} = E_1^{\parallel above}\right)$ ($E^{\parallel}$ is continuous across interface)

n.b. $\perp$ = perpendicular component relative to surface, $\parallel$ = parallel component relative to surface:



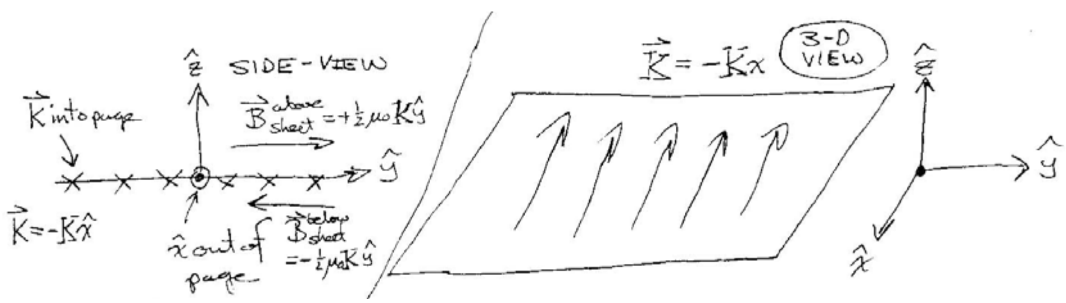
Consider a thin conducting sheet of material carrying a surface current density of $\vec{K} = -K\hat{x} = K\left(-\hat{x}\right)$ Amperes/meter



Now imagine that this current sheet $\vec{K} = -K\hat{x} = K\left(-\hat{x}\right)$ is "placed" in an external magnetic field, e.g. created / emanating from some other current-carrying circuit <u>below</u> this current sheet.

Call this external magnetic field that is below the original current sheet $\vec{B}_{1_{ext}}^{below}$.

What we discover is that the magnetic field <u>above</u> the current sheet $\vec{B}_{2_{ext}}^{above}$ is not parallel to $\vec{B}_{1_{ext}}^{below}$ - it has been <u>refracted</u> by the current sheet (in the <u>tangential</u> direction - with respect to the surface)!



The physical origin for this is simple to understand. <u>Below</u> the current sheet, the current sheet itself <u>adds</u> to the tangential component of B_{ext}^{below} a component $\vec{B}_{sheet}^{below} = -\frac{1}{2}\mu_o K\hat{y}$ (for $\vec{K} = -K\hat{x}$), however, <u>above</u> the current sheet, the current sheet adds to the tangential component of B_{ext}^{above} a component $\vec{B}_{sheet}^{above} = +\frac{1}{2}\mu_o K\hat{y}$ (for $\vec{K} = -K\hat{x}$).

<u>So if</u>: $\boxed{\vec{B}_{ext} = B_{ext_x}^{\parallel}\hat{x} + B_{ext_y}^{\parallel}\hat{y} + B_{ext_z}^{\perp}\hat{z}}$

<u>Then</u>: $\boxed{\vec{B}_{ext}^{below} = B_{ext_x}^{\parallel below}\hat{x} + B_{ext_y}^{\parallel below}\hat{y} + B_{ext_z}^{\perp below}\hat{z}}$ $\parallel$ = parallel to surface

<u>And</u>: $\boxed{\vec{B}_{ext}^{above} = B_{ext_x}^{\parallel above}\hat{x} + B_{ext_y}^{\parallel above}\hat{y} + B_{ext_z}^{\perp above}\hat{z}}$ $\perp$ = perpendicular to surface

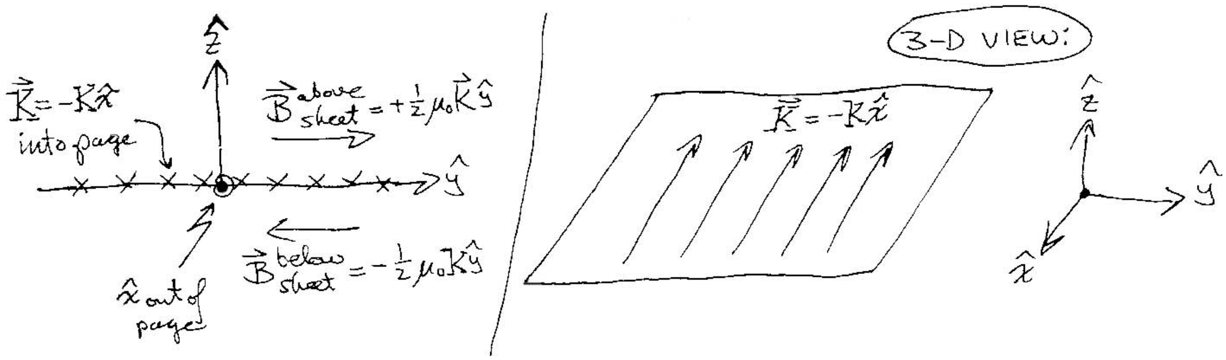
Then by the principle of linear superposition, $\vec{B}_{Tot} = \vec{B}_{ext} + \vec{B}_{sheet}$.

Hence, <u>below</u> the current sheet ($\vec{K} = -K\hat{x}$):

$$\vec{B}_{TOT}^{below} = B_{ext_x}^{\|below}\hat{x} + \left(B_{ext_y}^{\|below} \overbrace{-\frac{1}{2}\mu_o K}^{=\vec{B}_{sheet}^{below}} \right)\hat{y} + B_{ext_z}^{\perp below}\hat{z} = B_{TOT_x}^{below}\hat{x} + B_{TOT_y}^{below}\hat{y} + B_{TOT_z}^{below}\hat{z}$$

And <u>above</u> the current sheet ($\vec{K} = -K\hat{x}$):

$$\vec{B}_{TOT}^{above} = B_{ext_x}^{\|above}\hat{x} + \left(B_{ext_y}^{\|above} \overbrace{+\frac{1}{2}\mu_o K}^{=\vec{B}_{sheet}^{above}} \right)\hat{y} + B_{ext_z}^{\perp above}\hat{z} = B_{TOT_x}^{above}\hat{x} + B_{TOT_y}^{above}\hat{y} + B_{TOT_z}^{above}\hat{z}$$



Thus, (comparing $\vec{B}_{TOT}^{above}$ vs. $\vec{B}_{TOT}^{below}$ component-by-component), we see that:

1)	$B_{TOT_x}^{\|below} = B_{TOT_x}^{\|above}$ $B_{ext_x}^{\|below} = B_{ext_x}^{\|above}$	Tangential (to sheet / surface) component of $\vec{B}_{TOT}$ <u>parallel</u> to sheet current $\vec{K} = -K\hat{x}$ is <u>continuous</u>.
2)	$B_{TOT_y}^{\|below} \neq B_{TOT_y}^{\|above}$ $B_{TOT_y}^{\|above} - B_{TOT_y}^{\|below} = \mu_o K$	Tangential (to sheet/surface) component of $\vec{B}_{TOT}$ <u>perpendicular</u> to sheet current $\vec{K} = -K\hat{x}$ is <u>discontinuous</u> by an amount $\mu_o K$ across sheet / surface.
3)	$B_{TOT_z}^{\perp below} = B_{TOT_z}^{\perp above}$ $B_{ext_z}^{\perp below} = B_{ext_z}^{\perp above}$	Normal (to sheet/surface) component of $\vec{B}_{TOT}$ is continuous across sheet / surface.

Mathematically, these 3 statements can be compactly combined into a single expression:

$\boxed{\vec{B}_{TOT}^{above} - \vec{B}_{TOT}^{below} = \mu_o \vec{K} \times \hat{n}}$ where the unit normal to the surface, $\hat{n} = \hat{z}$ (here, as drawn above).

As we found in electrostatics, that the scalar electric potential $V(\vec{r})$ was continuous across any boundary $\boxed{V_{above}(\vec{r}) = V_{below}(\vec{r})}$, likewise, the magnetic vector potential $\vec{A}(\vec{r})$ is <u>also</u> continuous across any boundary, i.e. $\boxed{\vec{A}_{above}(\vec{r}) = \vec{A}_{below}(\vec{r})}$ <u>provided that</u>: $\boxed{\vec{\nabla} \bullet \vec{A}(\vec{r}) = 0}$, which guarantees that $\boxed{A_{\perp}^{above}(\vec{r}) = A_{\perp}^{below}(\vec{r})}$ and *<u>also</u>* <u>provided that</u>: $\boxed{\vec{\nabla} \times \vec{A}(\vec{r}) \ \left(= \vec{B}(\vec{r})\right)}$, which, in integral form, i.e. $\boxed{\oint_C \vec{A}(\vec{r}) \bullet d\vec{\ell} = \oint_S \vec{B}(\vec{r}) \bullet d\vec{a} = \Phi_m}$ guarantees that $\boxed{A_{\parallel}^{above}(\vec{r}) = A_{\parallel}^{below}(\vec{r})}$.

However, note that the <u>normal derivative</u> of $\vec{A}(\vec{r})$, since $\vec{A}(\vec{r}) \parallel \vec{K}(\vec{r})$ then $\vec{A}(\vec{r})$ also "inherits" the discontinuity associated with $\vec{B}(\vec{r})$: $\boxed{B_{TOT_y}^{\parallel above} - B_{TOT_y}^{\parallel below} = \mu_o K}$ (see #2 on previous page), and since $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$, thus we have a discontinuity in the (normal) slope(s) of $\vec{A}(\vec{r})$ on either side of the boundary/current sheet.

We can understand the origin of this condition on the normal derivative(s) of $\vec{A}(\vec{r})$ taken just above/below an "interface" e.g. for the specific case of the current sheet $\vec{K} = -K\hat{x}$. From $\boxed{B_{TOT_y}^{\parallel above} - B_{TOT_y}^{\parallel below} = \mu_o K}$ we know that the discontinuity in the $\vec{B}$-field is in the $\hat{y}$-direction, whereas since the magnetic vector potential associated with the current sheet $\vec{A}(\vec{r})$ is always parallel to the current, and since $\vec{K} = -K\hat{x}$ we know that the component of $\vec{A}_{TOT}(\vec{r})$ that we are concerned with here is in the $\hat{y}$-direction. But from: $\boxed{\vec{B}_{TOT} = \vec{\nabla} \times \vec{A}_{TOT}}$, then: $\boxed{B_{TOT_y} = \left(\vec{\nabla} \times \vec{A}_{TOT}\right)_y}$ thus we need to worry only about the $\hat{y}$-component of the curl of $\vec{A}_{TOT}(\vec{r})$, which is:

$$\boxed{B_{TOT_y} = \left(\vec{\nabla} \times \vec{A}_{TOT}\right)_y = \left(\frac{\partial A_{TOT_x}}{\partial z} - \frac{\partial A_{TOT_z}}{\partial x}\right)}$$

Then, noting that the $\hat{z}$-direction is perpendicular (i.e. normal) to the plane of the current sheet:

$$B_{TOT_y}^{\parallel above} - B_{TOT_y}^{\parallel below} = \mu_o K = \left(\frac{\partial A_{TOT_x}^{\parallel above}}{\partial z} - \frac{\partial A_{TOT_z}^{\perp above}}{\partial x}\right)\Bigg|_{surface} - \left(\frac{\partial A_{TOT_x}^{\parallel below}}{\partial z} - \frac{\partial A_{TOT_z}^{\perp below}}{\partial x}\right)\Bigg|_{surface}$$

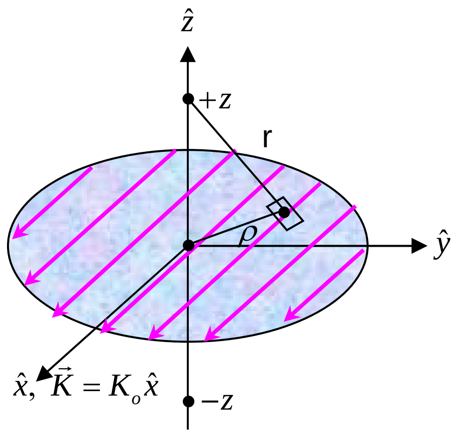
$$= \left(\frac{\partial A_{TOT_x}^{\parallel above}}{\partial z} - \frac{\partial A_{TOT_x}^{\parallel below}}{\partial z}\right)\Bigg|_{surface} - \underbrace{\left(\frac{\partial A_{TOT_z}^{\perp above}}{\partial x} - \frac{\partial A_{TOT_z}^{\perp below}}{\partial x}\right)\Bigg|_{surface}}_{\substack{=0 \\ A_{TOT_z}^{\perp}\ suffers\ \text{no discontinuity}}} = \left(\frac{\partial A_{TOT_x}^{\parallel above}}{\partial n} - \frac{\partial A_{TOT_x}^{\parallel below}}{\partial n}\right)\Bigg|_{surface}$$

Neither $A_{TOT_z}^{\perp}$ nor $A_{TOT_y}^{\parallel}$ suffer discontinuities in their slopes at the current sheet – only $A_{TOT_x}^{\parallel above}$ does - in the normal (i.e. $\hat{z}$) direction. Therefore, we can most generally write this condition on the discontinuity in the normal derivative on $\vec{A}(\vec{r})$ as:

$$\boxed{\frac{\partial \vec{A}^{above}(\vec{r})}{\partial n}\Bigg|_{surface} - \frac{\partial \vec{A}^{below}(\vec{r})}{\partial n}\Bigg|_{surface} = -\mu_o \vec{K}}$$

The Magnetic Vector Potential $\vec{A}(\vec{r})$ Associated with a Finite Circular Disk Sheet Current

We wish to delve a bit deeper into the nature of the magnetic vector potential, $\vec{A}(\vec{r})$ and also $\vec{B}(\vec{r})=\vec{\nabla}\times\vec{A}(\vec{r})$ associated with current sheets. Consider a sheet current $\vec{K}=K_o\hat{x}$ flowing on the surface of a finite circular disk of radius *R*, lying in the *x-y* plane as shown in the figure below:



To keep it simple, we'll just calculate $\vec{A}(\vec{r})$ at an arbitrary point along the $\hat{z}$ -axis above and below the *x-y* plane. The magnetic vector potential $\vec{A}(\vec{r})$ associated with a sheet current is:

$$\vec{A}(\vec{r})=\frac{\mu_o}{4\pi}\int_{S'}\frac{\vec{K}(\vec{r}')}{\mathrm{r}}da'=\frac{\mu_o K_o\hat{x}}{4\pi}\int_{S'}\frac{da'}{\mathrm{r}}$$

We deliberately chose a sheet current flowing on a finite circular disk of radius *R* so that we could easily carry out the integration. The area element da' on the circular disk (in cylindrical coordinates) is $da'=d\rho(\rho d\varphi)=\rho d\rho d\varphi$, and from the figure above, we see that: $\mathrm{r}=\sqrt{\rho^2+z^2}$.

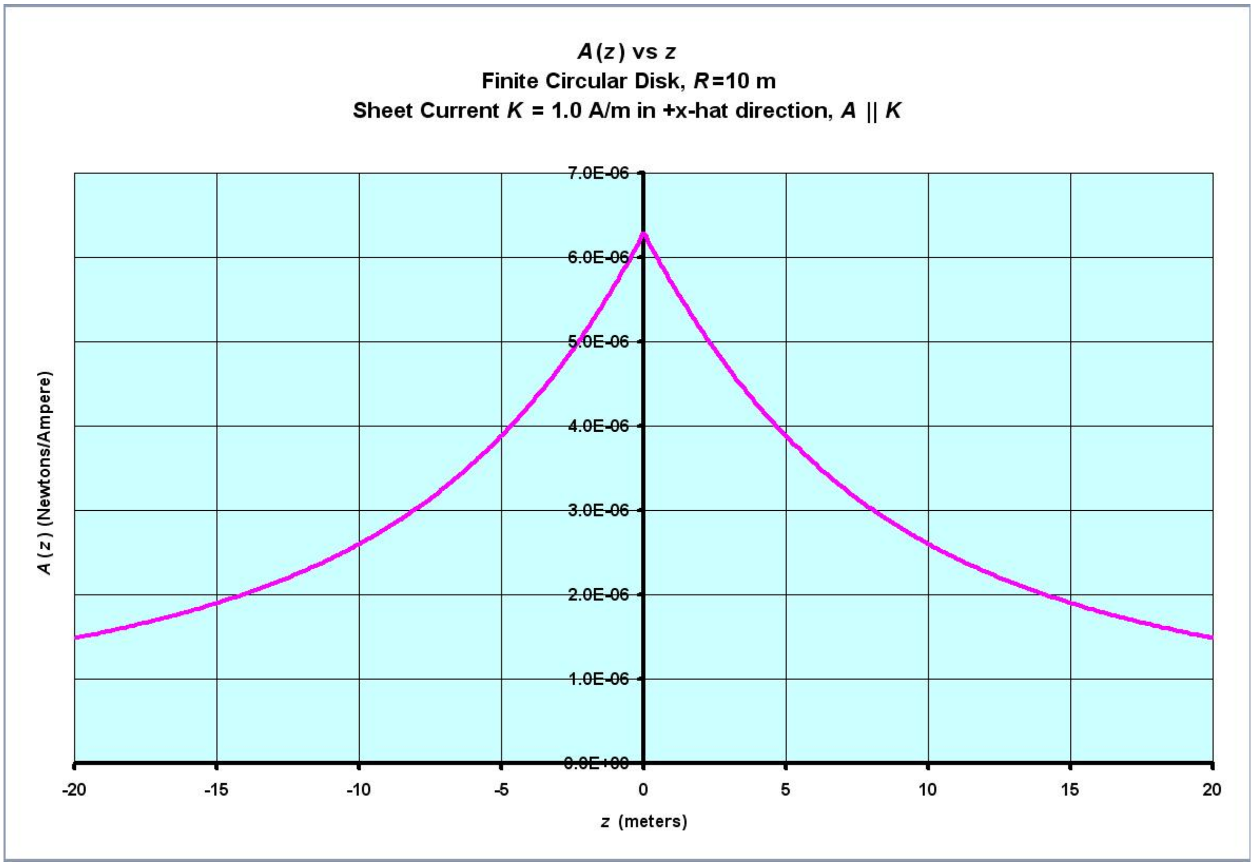
<u>Thus</u>:

$$\vec{A}(z)=\frac{\mu_o K_o\hat{x}}{4\pi}\int_{\rho=0}^{\rho=R}\int_{\varphi=0}^{\varphi=2\pi}\frac{\rho\, d\rho d\varphi}{\sqrt{\rho^2+z^2}}=\frac{\cancel{2\pi}\mu_o K_o\hat{x}}{\cancel{4\pi}}\int_{\rho=0}^{\rho=R}\frac{\rho\, d\rho}{\sqrt{\rho^2+z^2}}=\frac{1}{2}\mu_o K_o\hat{x}\left[\sqrt{\rho^2+z^2}\right]_{\rho=0}^{\rho=R}$$
$$=\frac{1}{2}\mu_o K_o\hat{x}\left[\sqrt{R^2+z^2}-\sqrt{z^2}\right]=\frac{1}{2}\mu_o K_o\left[\sqrt{R^2+z^2}-\sqrt{z^2}\right]\hat{x}$$

Now there is a subtlety here that we need to notice before proceeding further – since we are interested in knowing $\vec{A}(z)$ at an arbitrary point along the $\hat{z}$ -axis - above <u>and/or</u> below the *x-y* plane, thus *z* can be either positive or negative. Note that both the $\sqrt{R^2+z^2}$ and $\sqrt{z^2}$ terms are <u>*always*</u> ≥ 0 for <u>both</u> positive <u>and/or</u> negative *z* (in particular: $\sqrt{z^2}=|z|\neq z$!). Thus, in order to <u>preserve</u> this fact, we <u>explicitly</u> keep expression for the magnetic vector potential $\vec{A}(z)$ as:

$$\vec{A}(z)=\frac{1}{2}\mu_o K_o\left(\sqrt{R^2+z^2}-\sqrt{z^2}\right)\hat{x}$$

A plot of the magnetic vector potential $\vec{A}(z)$ vs. z is shown in the figure below for a circular disk of radius R = 10 m and sheet current $\vec{K} = K_o\hat{x} = 1.0\hat{x}$ Amperes/meter.



Note that $\vec{A}(z)$ is a maximum when z = 0, right on the sheet current. Note also the discontinuity in the slope(s) of $\vec{A}(z)$ on either side of z = 0, which arises due to the presence of the sheet current in the x-y plane, since:

$$\left.\frac{\partial \vec{A}^{above}(\vec{r})}{\partial n}\right|_{surface} - \left.\frac{\partial \vec{A}^{below}(\vec{r})}{\partial n}\right|_{surface} = -\mu_o\vec{K}$$

or:

$$\left.\frac{\partial \vec{A}^{above}(z \geq 0)}{\partial z}\right|_{z=0} - \left.\frac{\partial \vec{A}^{below}(z \leq 0)}{\partial z}\right|_{z=0} = -\mu_o\vec{K}$$

Care/thought must ***also*** be taken when carrying out the normal derivatives (slopes) above and below the x-y plane – look carefully at the slopes for $z > 0$ and $z < 0$ in the above figure, and compare this information to what we calculate:

$$\frac{\partial \vec{A}^{above}(z \geq 0)}{\partial z} = \tfrac{1}{2}\mu_o K_o \frac{\partial}{\partial z}\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)\hat{x} = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{z^2}}\right)\hat{x} = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\hat{x}$$

$$\frac{\partial \vec{A}^{below}(z \leq 0)}{\partial z} = \tfrac{1}{2}\mu_o K_o \frac{\partial}{\partial z}\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)\hat{x} = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{z^2}}\right)\hat{x} = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{R^2+z^2}} + 1\right)\hat{x}$$

Thus we see that indeed:

$$\left.\frac{\partial \vec{A}^{above}(z \geq 0)}{\partial z}\right|_{z=0} - \left.\frac{\partial \vec{A}^{below}(z \leq 0)}{\partial z}\right|_{z=0} = -\tfrac{1}{2}\mu_o K_o \hat{x} - \tfrac{1}{2}\mu_o K_o \hat{x} = -\mu_o K_o \hat{x} = -\mu_o \vec{K}$$

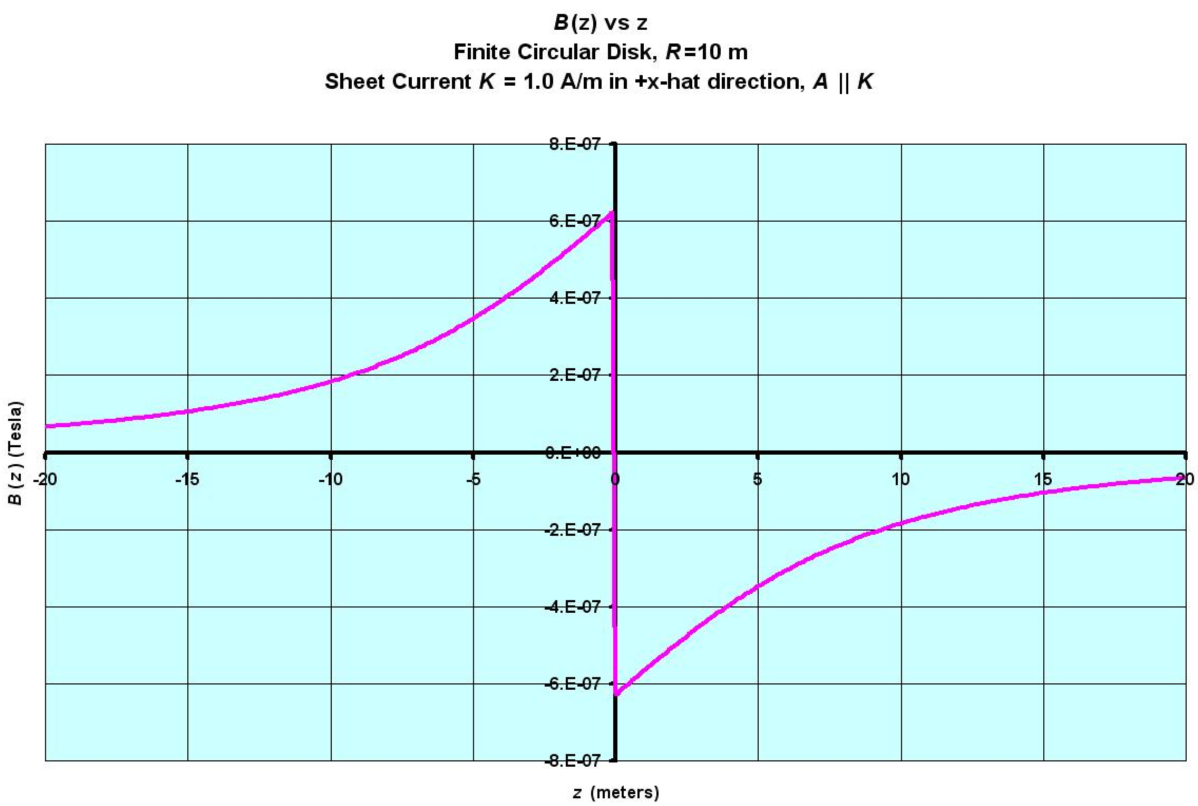
The magnetic field $\vec{B}(z)$ at an arbitrary point along the along the $\hat{z}$-axis – either above and/or below the *x*-*y* plane is calculated using $\vec{B}(z) = \vec{\nabla} \times \vec{A}(z)$ in Cartesian coordinates. Since $\vec{A}(z) = A_x(z)\hat{x}$ (only), then: $\vec{B}(z) = \vec{\nabla} \times \vec{A}(z) = \vec{\nabla} \times A_x(z)\hat{x} = \frac{\partial A_x(z)}{\partial z}\hat{y}$

Thus:

$$\vec{B}^{above}(z \geq 0) = \vec{\nabla} \times \vec{A}^{above}(z \geq 0) = \vec{\nabla} \times A_x^{above}(z \geq 0)\hat{x} = \frac{\partial A_x^{above}(z \geq 0)}{\partial z}\hat{y} = \tfrac{1}{2}\mu_o K_o \left(\frac{z}{\sqrt{R^2 + z^2}} - 1\right)\hat{y}$$

$$\vec{B}^{below}(z \leq 0) = \vec{\nabla} \times \vec{A}^{below}(z \leq 0) = \vec{\nabla} \times A_x^{below}(z \leq 0)\hat{x} = \frac{\partial A_x^{below}(z \leq 0)}{\partial z}\hat{y} = \tfrac{1}{2}\mu_o K_o \left(\frac{z}{\sqrt{R^2 + z^2}} + 1\right)\hat{y}$$

The figure below shows the magnetic field $\vec{B}(z)$ vs. *z* along the $\hat{z}$-axis with a sheet current $\vec{K} = K_o \hat{x}$ flowing on the surface of the finite disk of radius *R*, lying in the *x*-*y* plane:



We now investigate what happens in the limit that the radius of the sheet current-carrying circular disc, $R \to \infty$, i.e. it becomes an <u>infinite</u> planar sheet current. We discover that the magnetic vector potential $\vec{A}(\vec{r})$ associated with the sheet current $\vec{K} = K_o\hat{x}$ becomes <u>infinite</u> (i.e. $\vec{A}(\vec{r})$ diverges):

$$\lim_{R\to\infty}\left(\vec{A}(z)\right) = \frac{1}{2}\mu_o K_o\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)\hat{x} \to \frac{1}{2}\mu_o K_o\left(\sqrt{\infty^2+z^2} - \sqrt{z^2}\right)\hat{x}$$

whereas the boundary condition on the discontinuity in the normal derivative of $\vec{A}(\vec{r})$ across the sheet current lying in the *x*-*y* plane at *z* = 0 still exists, and is well-behaved (i.e. finite):

$$\left.\frac{\partial \vec{A}^{above}(z \ge 0)}{\partial z}\right|_{z=0} - \left.\frac{\partial \vec{A}^{below}(z \le 0)}{\partial z}\right|_{z=0} = -\tfrac{1}{2}\mu_o K_o\hat{x} - \tfrac{1}{2}\mu_o K_o\hat{x} = -\mu_o K_o\hat{x} = -\mu_o\vec{K}$$

We also discover that the magnetic field $\vec{B}(\vec{r})$ is <u>also</u> well-behaved (i.e. finite) – and <u>constant</u> – independent of the height/depth *z* above/below the *x*-*y* plane (!!):

$$\lim_{R\to\infty}\left(\vec{B}^{above}(z \ge 0)\right) = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{\infty^2+z^2}} - 1\right)\hat{y} = -\tfrac{1}{2}\mu_o K_o\hat{y}$$

$$\lim_{R\to\infty}\left(\vec{B}^{below}(z \le 0)\right) = \tfrac{1}{2}\mu_o K_o\left(\frac{z}{\sqrt{\infty^2+z^2}} + 1\right)\hat{y} = +\tfrac{1}{2}\mu_o K_o\hat{y}$$