LECTURE NOTES 14

THE MACROSCOPIC MAGNETIC FIELD ASSOCIATED WITH THE RELATIVE MOTION OF AN ELECTRICALLY-CHARGED POINT-LIKE PARTICLE

An electrically charged point-like particle of charge $+q$ (> 0) moving in the lab frame with relative velocity $\vec{v}$ (with respect to a fixed coordinate system, origin ϑ) generates an apparent solenoidal magnetic field $\vec{B}$ in the lab reference frame (cf with particle's own reference frame: $\vec{B}=0$ there!)

For the non-relativistic case (i.e. for $v << c$ {c = speed of light in vacuum}), the strength of $\vec{B}_q(\vec{r})$ from a moving point-charged particle with electric charge q is:

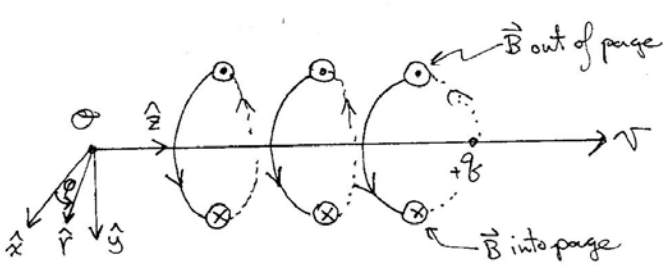
$$\vec{B}_q(\vec{r}) = \frac{1}{c^2}\left(\vec{v}\times\vec{E}_q(\vec{r})\right)$$

where: $\vec{E}_q(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}$ = electrostatic field of charged particle in its own rest frame!

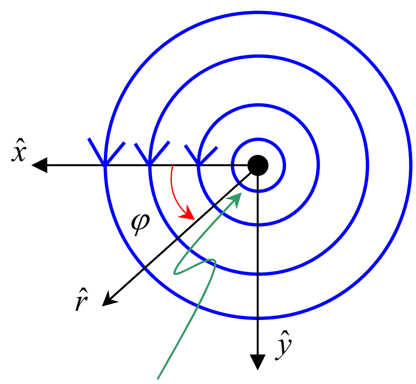
Then: $$\vec{B}_q(\vec{r}) = \frac{q}{4\pi\varepsilon_0 c^2}\left(\vec{v}\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\right) = \frac{\mu_o}{4\pi}\left(q\vec{v}\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\right)$$ (using $c^2 = 1/\varepsilon_o\,\mu_o$)

where: $\vec{\boldsymbol{\mathscr{r}}} = \vec{r}-\vec{r}' = (obs.\,pt - src.\,pt)$ and: $\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} = \frac{\vec{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^3} = \frac{\vec{\boldsymbol{\mathscr{r}}}}{|\vec{\boldsymbol{\mathscr{r}}}|^3}$ since: $\vec{\boldsymbol{\mathscr{r}}} = |\vec{\boldsymbol{\mathscr{r}}}|\,\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}$

Side View: $\vec{v} = v\hat{z}$



Face View: $q > 0$: $\vec{B}_q(\vec{r}) = +B_q(\vec{r})\hat{\varphi}$



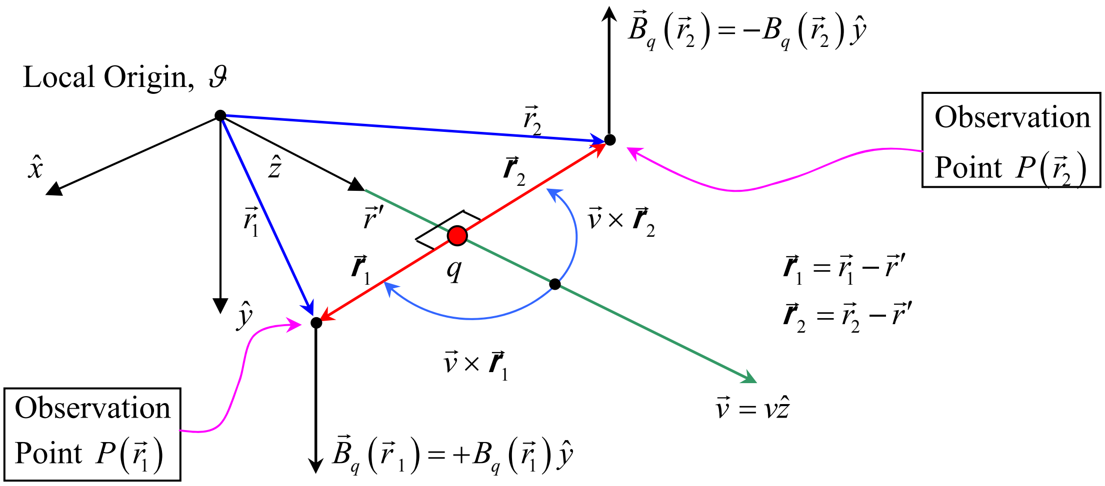
$\vec{v} = v\hat{z},\ \hat{z}$ out of page

For $+q > 0$: $\vec{B}_q(\vec{r}) = +B_q(\vec{r})\hat{\varphi}$

For $-q < 0$: $\vec{B}_q(\vec{r}) = B_q(\vec{r})(-\hat{\varphi}) = -B_q(\vec{r})\hat{\varphi}$

The macroscopic $\vec{B}$-field associated with a moving electrically charged particle is a solenoidal field, i.e. $\vec{B}_q(\vec{r}) = B_q(\vec{r})\hat{\varphi}$!

Take cross-products e.g. $\vec{v}\times\vec{\boldsymbol{r}}_1$ and $\vec{v}\times\vec{\boldsymbol{r}}_2$ to determine the direction of B-field at observation/field points $P(\vec{r}_1)$ and $P(\vec{r}_2)$, as shown in the figure below:



$$\boxed{\vec{B}_q(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}\times\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}\times\frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}^3}\right)\quad Teslas\ \left(= {}^{N}\!/_{Amp-m}\right)}$$

This is the magnetic field observed in the lab frame due to a point-like particle with electric charge q moving with relative velocity $v << c$ in the lab frame.

By deliberate construction (here), $\vec{\boldsymbol{r}}_1 = \vec{r}_1 - \vec{r}\,'$ and $\vec{\boldsymbol{r}}_2 = \vec{r}_2 - \vec{r}\,'$ are (momentarily) $\perp$ to $\vec{v}$
In general: $\vec{v}\times\vec{\boldsymbol{r}} = v\boldsymbol{r}\sin\Theta\,\hat{\varphi}$, where Θ = angle between $\vec{v}$ and $\vec{\boldsymbol{r}}$ (= 90^o here).
Charged particle's velocity vector is $\vec{v} = v\hat{z}$ and vector $\vec{\boldsymbol{r}}$ lies in x-z plane (here).

The above pix shows the situation at the so-called distance of closest approach of the point-like charged particle to the observation/field point $P(\vec{r})$, then $|\vec{\boldsymbol{r}}| = |\vec{r} - \vec{r}\,'|$ = minimum, and $\Theta = 90^0$, $\sin\Theta = \sin(90^0) = 1$. Let's say that this occurs at time $t = 0$.

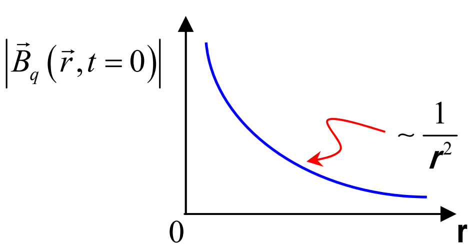
Then (here) at the distance of closest approach: $\boxed{\vec{B}_q(\vec{r}, t=0) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{qv}{\boldsymbol{r}^2}\right)\hat{\varphi}}$

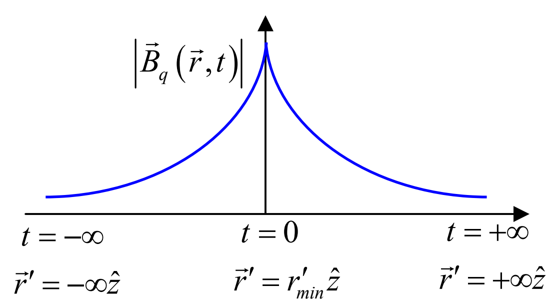
At time $t = -\infty$ the charged particle is infinitely far away, located at $(x,y,z) = (0,0-\infty)$ traveling with $\vec{v} = v\hat{z}$. Then $\vec{B}_q(\vec{r}, t=-\infty) = 0$. Similarly, at time $t = +\infty$ the charged particle is also infinitely far away, located at $(x,y,z) = (0,0+\infty)$ traveling with $\vec{v} = v\hat{z}$. Then $\vec{B}_q(\vec{r}, t=+\infty) = 0$.

B_q-field vs. $\vec{\boldsymbol{r}} = \vec{r} - \vec{r}\,'$:

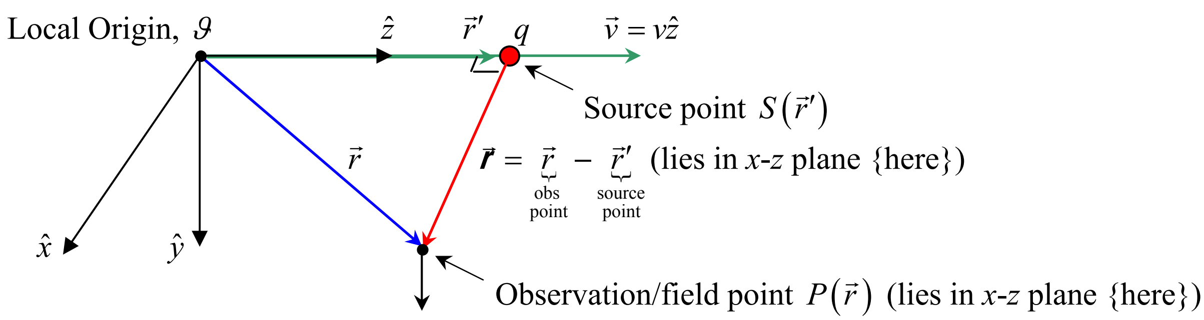
$\boxed{\vec{r}\,'(t) = \vec{r}\,'_{min} + vt\hat{z} = (r'_{min} + vt)\,\hat{z}}$

B_q-field vs. t:





At the distance of closest approach ($t = 0$): $\boldsymbol{r} = |\boldsymbol{\vec{r}}| = |\vec{r} - \vec{r}'| = |\vec{r} - \vec{r}'_{min}| = \boldsymbol{r}_{min}$ = minimum.



$$\boxed{\vec{B}_q(r, t=0) = \frac{\mu_o}{4\pi}\frac{qv}{\boldsymbol{r}^2}\hat{\varphi} = \frac{\mu_o}{4\pi}\frac{qv}{\boldsymbol{r}^2}\hat{y}}$$ at the distance of closest approach ($t = 0$).

Note the similarity between $\vec{E}_q$ and $\vec{B}_q$-fields of a point-like electrically charged particle:

$$\vec{E}_q(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{q\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{q\boldsymbol{\vec{r}}}{\boldsymbol{r}^3}\right)$$

$$\vec{B}_q(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{q\vec{v}\times\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{q\vec{v}\times\boldsymbol{\vec{r}}}{\boldsymbol{r}^3}\right)$$

Both $\vec{E}_q$ and $\vec{B}_q$-fields decrease as $1/\boldsymbol{r}^2$ from point charge, due to $1/\boldsymbol{r}^2$ flux law of virtual photons!!!

The Macroscopic Magnetic Field *B*(*r*) due to a Steady Current *I* Flowing in an Infinitely Long Filamentary Wire

The principle of linear superposition tells us that we can view the macroscopic magnetic field due to a steady current I flowing in a long filamentary wire as the linear superposition of magnetic field contributions associated with each of the individual electric charges q flowing in the filamentary wire that microscopically constitute/make up the macroscopic steady current I:

$$\vec{B}_I(\vec{r},t) = \sum_{i=1}^{N} B_{q_i}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right) q\vec{v} \times \sum_{i=1}^{N}\left(\frac{\hat{\boldsymbol{r}}_i(t)}{\boldsymbol{r}_i^2(t)}\right)$$

Where we have assumed (here) that all charge carriers have the same electric charge q and move along the filamentary wire with the same velocity $\vec{v} = v\hat{z}$.

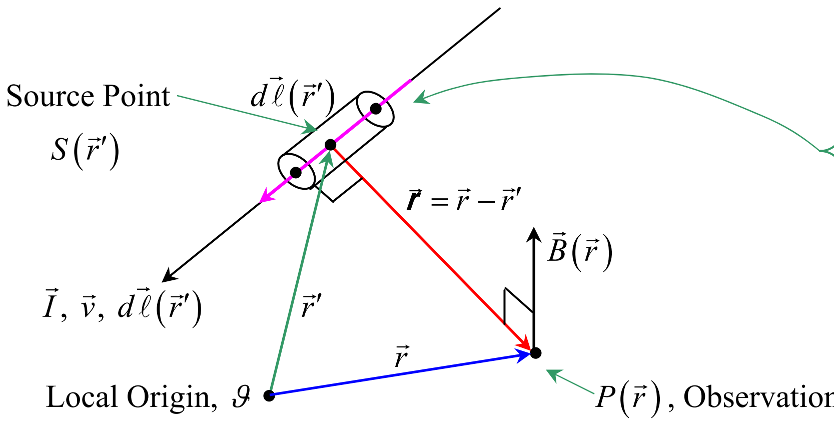
In the limit that the number of electric charges that are microscopically involved in making up the macroscopic current I becomes so large that the spacing between adjacent charge carriers becomes extremely small, e.g. that of atomic dimensions, $\sim$ O(10) Angstroms = 1 nm, then if we are only interested in the net/total macroscopic magnetic field associated with macroscopic distance scales, e.g. $\boldsymbol{r} \sim 1\mu\text{m}$ (and larger), then we can safely replace the summation in the above expression by an integral over a *continuum* of infinitesimal electric charge contributions dq all moving along the filamentary wire with velocity $\vec{v} = v\hat{z}$.

For steady macroscopic currents $I \neq \text{fcn}(t)$ the time dependence drops out/vanishes in the microscopic averaging process! If $N_q \sim \text{O}(10^{24})$ total charge carriers, then the fluctuations are $\sigma_{N_q} = \sqrt{N_q} \sim \text{O}(10^{12})$, thus fractional fluctuations $\sigma_{N_q}/N_q = 1/\sqrt{N_q} \sim \text{O}(10^{-12})$ - negligible!!!

Thus for a macroscopic, steady electric current I flowing in a long (one-dimensional) filamentary wire (of infinitesimal thickness), the infinitesimal contribution $d\vec{B}(\vec{r})$ due to a macroscopic current $I = \lambda_{free} v = dqv/d\ell$ flowing in an infinitesimal length $d\vec{\ell}$ of current-carrying filamentary wire with associated infinitesimal source charge increment $dq = Id\ell/v$ is:

$$\boxed{d\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \left(d\vec{\ell}(\vec{r}\,') \times \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right)} \quad \text{where} \quad \boxed{\boldsymbol{\mathscr{r}} = \vec{r} - \vec{r}\,'}$$

<u>Connection</u>: $I = {qv}/{\ell} \;\Rightarrow\; \boxed{I d\vec{\ell} = dq\vec{v}}$ $\qquad \vec{I} d\ell = I d\vec{\ell}$ (since $\vec{I} \parallel d\vec{\ell}$)

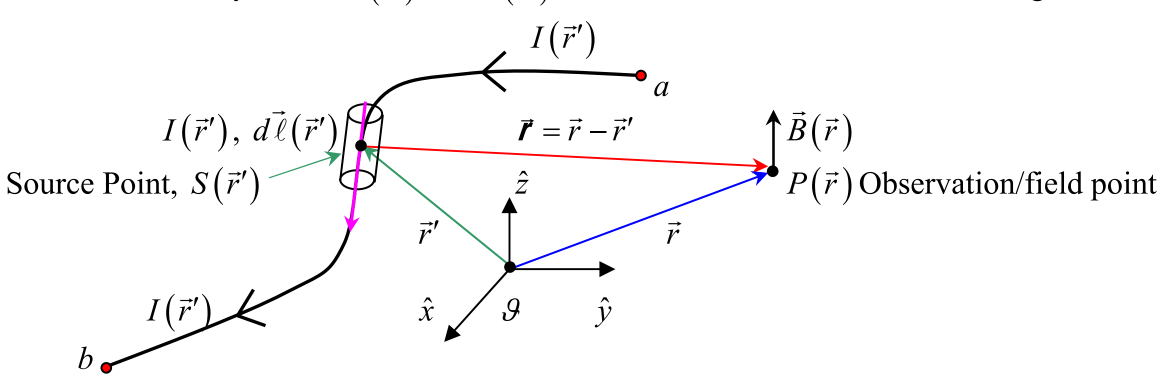


$d\ell = vdt$
$dq = \lambda d\ell = \lambda vdt$
λ line charge (Coulombs/m)
$I = dq/dt = \lambda v$
$I d\vec{\ell} = \lambda d\ell \vec{v} = dq\vec{v}$

n.b. figure drawn for $d\vec{\ell}(\vec{r}\,')$ contribution <u>closest</u> to observation/field point $P(\vec{r})$

<u>Then</u>: $$\vec{B}(\vec{r}) = \int_a^b d\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) \underbrace{I}_{\substack{\text{n.b. assumed}\\ \text{to be constant}\\ \text{everywhere}}} \int_a^b \frac{d\vec{\ell}(\vec{r}\,') \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|^3} = \left(\frac{\mu_o}{4\pi}\right) I \int_a^b \frac{d\vec{\ell}(\vec{r}\,') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

n.b. If $I \neq$ constant everywhere, $\vec{I}(\vec{r}\,') d\ell = I(\vec{r}\,') d\vec{\ell}$ then I <u>must</u> remain <u>inside</u> the integral!



In general this integral can often be difficult to perform analytically, but it can be easy to do on computer, e.g. using numerical integration techniques!

$\vec{I}(\vec{r}')d\ell = I(\vec{r}')d\vec{\ell}$ @ Source point, $\vec{r}'$

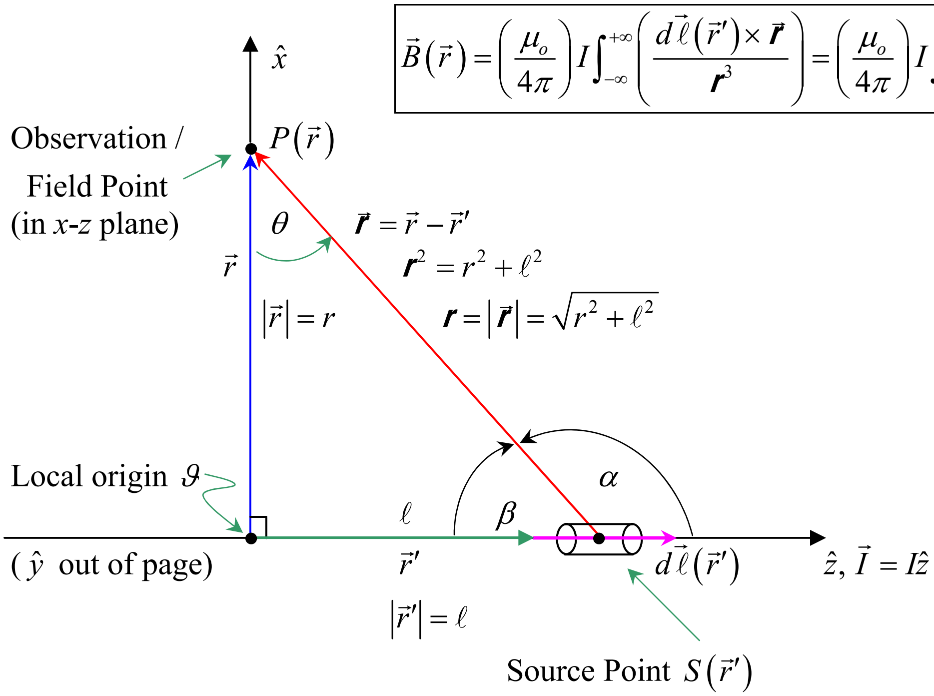
$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_a^b \left(\frac{\vec{I}(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right)d\ell = \left(\frac{\mu_o}{4\pi}\right)\int_a^b \left(\frac{I(\vec{r}')d\vec{\ell}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right) \qquad \boldsymbol{\mathscr{r}} = \vec{r} - \vec{r}'$$

If $I(\vec{r}') =$ constant everywhere, this expression simplifies to: $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\int_a^b \left(\frac{d\vec{\ell}(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right)$

Griffiths Example 5.5 - The Macroscopic Magnetic Field Due to a Steady Current Flowing in an Infinitely Long Wire:

Find the magnetic field $\vec{B}(\vec{r})$ a perpendicular distance *r* away from an infinitely long straight filamentary wire carrying a steady / constant / uniform current *I*.

$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\int_{-\infty}^{+\infty}\left(\frac{d\vec{\ell}(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right) = \left(\frac{\mu_o}{4\pi}\right)I\int_{-\infty}^{+\infty}\left(\frac{d\vec{\ell}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)$$



For this problem, which way does $d\vec{\ell}(\vec{r}')\times\boldsymbol{\mathscr{r}}$ point?

Note that: $d\vec{\ell}(\vec{r}') = d\ell(\vec{r}')\hat{z}$ and that $\boldsymbol{\mathscr{r}}$ lies in the *x*-*z* plane. By the right-hand rule, the cross product: $d\vec{\ell}\times\boldsymbol{\mathscr{r}} = d\ell\hat{z}\times(\mathscr{r}_x\hat{x} + \mathscr{r}_z\hat{z}) = \mathscr{r}_x d\ell\underbrace{(\hat{z}\times\hat{x})}_{=y} + \mathscr{r}_z d\ell\underbrace{(\hat{z}\times\hat{z})}_{=0} = \mathscr{r}_x d\ell\hat{y}$

Thus, $d\vec{\ell}(\vec{r}')\times\boldsymbol{\mathscr{r}} = \mathscr{r}_x d\ell\hat{y}$ points in the $\hat{y}$ direction (i.e. out of the page) here, because the field point $P(\vec{r})$ (as drawn above) lies in the *x*-*z* plane. However as mentioned earlier, $d\vec{\ell}(\vec{r}')\times\boldsymbol{\mathscr{r}}$ actually points in the $\hat{\varphi}$-direction (n.b. $\hat{\varphi} = \hat{y}$ when $\varphi = 0$).

What is the magnitude of $d\vec{\ell}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}$? From the definition of this cross product,

$\left|d\vec{\ell}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}\right| \equiv d\ell\sin\alpha$ where $\alpha =$ opening angle between $d\vec{\ell}(\vec{r}')$ and $\hat{\boldsymbol{\mathscr{r}}}$.

Geometrically: $\alpha = \pi - \beta$ and also: $\pi = \theta + \beta + \frac{\pi}{2}$, hence $\beta = \frac{\pi}{2} - \theta$

$\therefore \quad \alpha = \pi - \beta = \pi - \frac{\pi}{2} + \theta = \frac{\pi}{2} + \theta$, thus:

$$\sin\alpha = \sin\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta + \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}\sin\theta = \cos\theta$$

$\therefore \quad d\vec{\ell} \times \hat{\boldsymbol{r}} = d\ell \sin\alpha\hat{\varphi} = d\ell\cos\theta\hat{\varphi}$

From the above figure, note that: $\cos\theta = \frac{r}{\boldsymbol{r}}$ (or equivalently: $r = \boldsymbol{r}\cos\theta$)

Then: $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \int_{-\infty}^{+\infty} \left(\frac{d\vec{\ell}(\vec{r}\,') \times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right)$ with: $\boldsymbol{r}^2 = r^2 + \ell^2$ and: $d\vec{\ell}(\vec{r}\,') \times \hat{\boldsymbol{r}} = d\ell\cos\theta\hat{\varphi} = d\ell\left(\frac{r}{\boldsymbol{r}}\right)\hat{\varphi}$

Then: $\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I r \int_{-\infty}^{+\infty} \left(\frac{d\ell}{\left(r^2 + \ell^2\right)^{3/2}}\right)\hat{\varphi} = 2\left(\frac{\mu_o}{4\pi}\right)\frac{I \not{r}}{r^{\not 2}}\hat{\varphi} = \frac{\mu_o}{2\pi}\frac{I}{r}\hat{\varphi}$

$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{2\pi}\right)\frac{I}{r}\hat{\varphi}}$$ for an infinitely long filamentary wire carrying a steady current *I*.

<u>An alternative derivation of this result:</u>

Since $r = \boldsymbol{r}\cos\theta$, we can equivalently express $\int_{-\infty}^{\infty} d\ell$ in terms of an integral over θ:

$d\ell\sin\alpha = d\ell\cos\theta$ but <u>note that</u>: $\ell = r\tan\theta$ (or equivalently: $\tan\theta = \frac{\ell}{r}$)

Note also that: $r = \boldsymbol{r}\cos\theta \;\Rightarrow\; \frac{1}{\boldsymbol{r}} = \frac{\cos\theta}{r}$

$d\ell = \frac{r}{\cos^2\theta} d\theta$ is the Jacobian of the transformation from $d\ell \to d\theta$

Now when $\ell = +\infty,\ \theta = \frac{\pi}{2}$ and when $\ell = -\infty,\ \theta = -\frac{\pi}{2}$

$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \int_{-\infty}^{+\infty} \left(\frac{d\vec{\ell}(\vec{r}\,') \times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = \left(\frac{\mu_o}{4\pi}\right) I \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \overbrace{\left(\frac{\cos^2\theta}{r^{\not 2}}\right)}^{1/\boldsymbol{r}^2} \overbrace{\left(\frac{\not{r}}{\cos^2\theta}\right)\cos\theta d\theta}^{d\ell\cos\theta} \hat{\varphi}$$

Then:
$$= \left(\frac{\mu_o}{4\pi}\right) I \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left(\frac{\cos\theta d\theta}{r}\right)\hat{\varphi} = \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\theta d\theta\hat{\varphi}$$

$$= \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]\hat{\varphi} = \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\left[1 - -1\right]\hat{\varphi} = 2\left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\hat{\varphi}$$

$\therefore$ $$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{2\pi}\right)\frac{I}{r}\hat{\varphi}}$$ for an infinitely long filamentary wire carrying steady current *I*.

Note that $\vec{B}(\vec{r})$ falls off as $1/r$ where $r = \perp$ distance from the infinitely long wire.

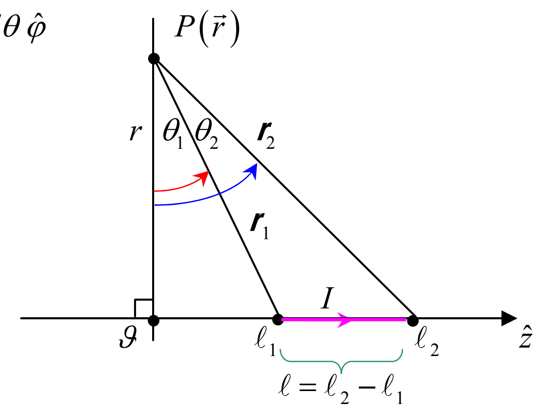
The Macroscopic Magnetic Field due to a Steady Current Flowing in a Finite Length Wire:

What is the $\vec{B}$-field associated with a steady current I flowing down a finite length filamentary wire (of length ℓ)? Same configuration/geometry as before/above for infinite length wire.

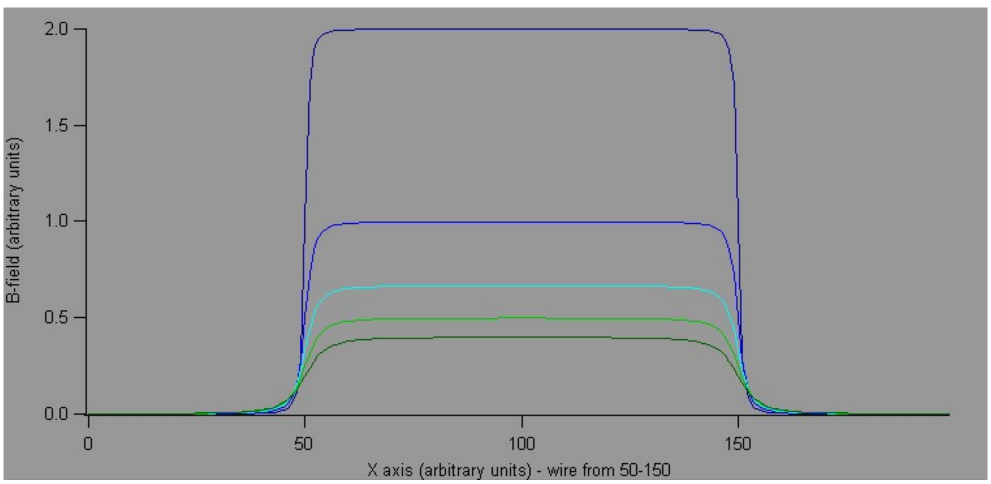
$$\vec{B}_{\substack{finite\\wire}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\int_{\ell_1}^{\ell_2}\frac{d\ell\,\hat{\varphi}}{\left(r^2+\ell^2\right)} = \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta\,\hat{\varphi}$$

$$\boxed{\vec{B}_{\substack{finite\\wire}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\frac{I}{r}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\varphi}}$$

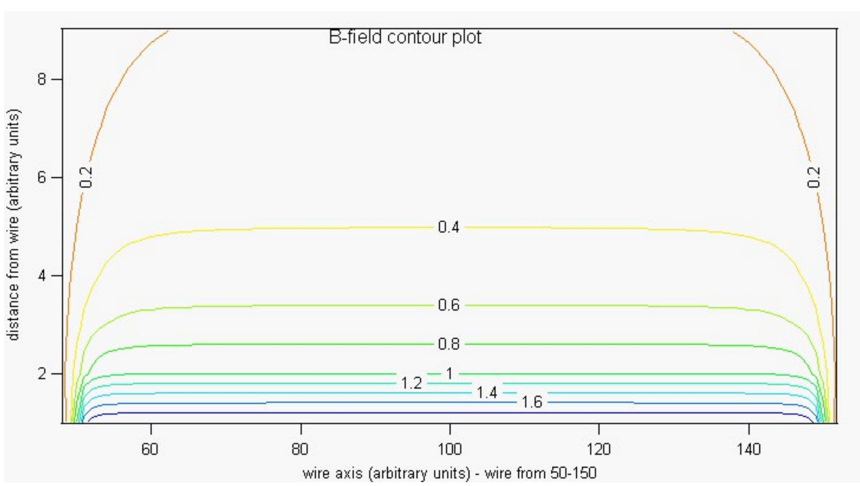
Where: $\boxed{\sin\theta_1 = \frac{\ell_1}{\sqrt{r^2+\ell_1^2}}, \quad \sin\theta_2 = \frac{\ell_2}{\sqrt{r^2+\ell_2^2}}}$



Fall '06 P435 Student – Michael Wiczer made plots of the magnitude of the B-field along finite length wire for 5 different choices of current:

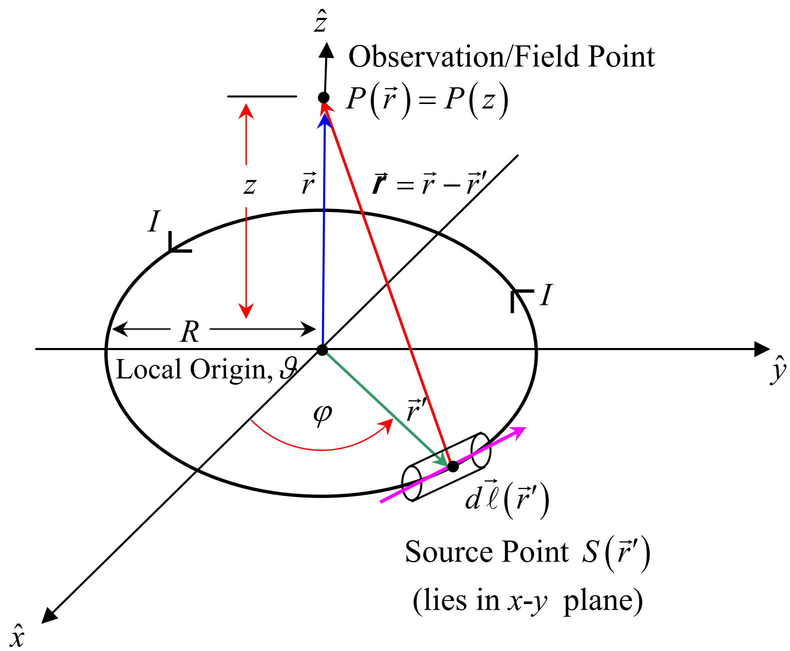


M. Wiczer's contour plot of magnitude of B-field for steady current I flowing down finite-length wire:



Griffiths Example 5.6 - The Macroscopic Magnetic Field Due to Circular Steady Current On the Symmetry Axis of a Current Loop:

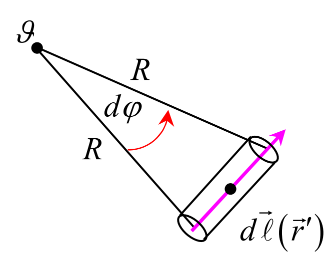
A circular filamentary current loop of radius R lies in x-y plane, with a steady current I circulating anti-clockwise (viewed from above), as shown in the figure below:



$$\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \oint_C \frac{d\vec{\ell}(\vec{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$ or: $$\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \oint_C \frac{d\vec{\ell}(\vec{r}') \times \vec{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^3}$$

What is $d\vec{\ell}(\vec{r}') \times \vec{\boldsymbol{\mathscr{r}}}$ for this particular problem?

By the arc-length formula $S = R\theta$, the infinitesimal segment of current loop $d\ell = \left|d\vec{\ell}(\vec{r}')\right| = Rd\varphi$



and: $$\begin{aligned} d\vec{\ell}(\vec{r}') &= (Rd\varphi)\left(\sin\varphi(-\hat{x}) + \cos\varphi\hat{y}\right) \\ &= Rd\varphi\ \left(-\sin\varphi\hat{x} + \cos\varphi\hat{y}\right) \end{aligned}$$ $\Leftarrow$ can see this easily e.g. when $\varphi = 0$ and $\varphi = \frac{\pi}{2}$

and: $\vec{\boldsymbol{\mathscr{r}}} \equiv \vec{r} - \vec{r}' = \vec{z} - \vec{r}'$ since $\vec{r} = \vec{z} = z\hat{z}$ and $\vec{r}' = R(\cos\varphi\hat{x} + \sin\varphi\hat{y})$

$\therefore$ $\vec{\boldsymbol{\mathscr{r}}} = \vec{z} - \vec{r}' = z\hat{z} - R(\cos\varphi\hat{x} + \sin\varphi\hat{y})$

Thus:

$$\left(d\vec{\ell}(\vec{r}')\times \boldsymbol{r}\right) = d\vec{\ell}(\vec{r}')\times(\vec{r}-\vec{r}') = d\vec{\ell}(\vec{r}')\times(\vec{z}-\vec{r}') = d\vec{\ell}(\vec{r}')\times\left(z\hat{z}-R(\cos\varphi\hat{x}+\sin\varphi\hat{y})\right)$$
$$= Rd\varphi(-\sin\varphi\hat{x}+\cos\varphi\hat{y})\times\left(z\hat{z}-R(\cos\varphi\hat{x}+\sin\varphi\hat{y})\right)$$

Now:

$\hat{x}\times\hat{y}=+\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$	$\hat{x}\times\hat{x}=0$
$\hat{y}\times\hat{z}=+\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$	$\hat{y}\times\hat{y}=0$
$\hat{z}\times\hat{x}=+\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$	$\hat{z}\times\hat{z}=0$

⇐ Very useful table…

Thus:

$$\left(d\vec{\ell}(\vec{r}')\times \boldsymbol{r}\right) = Rd\varphi\Big\{-z\sin\varphi\underbrace{(\hat{x}\times\hat{z})}_{=-\hat{y}} + z\cos\varphi\underbrace{(\hat{y}\times\hat{z})}_{=+\hat{x}}$$
$$+R\sin\varphi\cos\varphi\underbrace{(\hat{x}\times\hat{x})}_{=0} - R\cos^2\varphi\underbrace{(\hat{y}\times\hat{x})}_{=-\hat{z}}$$
$$+R\sin^2\varphi\underbrace{(\hat{x}\times\hat{y})}_{=+\hat{z}} - R\sin\varphi\cos\varphi\underbrace{(\hat{y}\times\hat{y})}_{=0}\Big\}$$

$$\left(d\vec{\ell}(\vec{r}')\times \boldsymbol{r}\right) = Rd\varphi\left\{+z\sin\varphi\hat{y}+z\cos\varphi\hat{x}+R\cos^2\varphi\hat{z}+R\sin^2\varphi\hat{z}\right\}$$
$$= Rd\varphi\left\{+z\sin\varphi\hat{y}+z\cos\varphi\hat{x}+R\left[\cos^2\varphi+\sin^2\varphi\right]\hat{z}\right\}$$
$$= Rd\varphi\left\{+z\sin\varphi\hat{y}+z\cos\varphi\hat{x}+R\hat{z}\right\}$$

Finally:

$$\boxed{\left(d\vec{\ell}(\vec{r}')\times \boldsymbol{r}\right) = Rd\varphi\left\{z(\cos\varphi\hat{x}+\sin\varphi\hat{y})+R\hat{z}\right\}}$$

Now: $\boxed{\boldsymbol{r}=|\boldsymbol{\vec{r}}|=|\vec{r}-\vec{r}'|=\sqrt{z^2+R^2}}$ and $\boxed{\boldsymbol{r}^2=z^2+R^2}$ and also $\boxed{\boldsymbol{r}^3=\left(z^2+R^2\right)^{3/2}}$

Then:

$$\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\oint_C\frac{d\ell(\vec{r}')\times\boldsymbol{r}}{\boldsymbol{r}^3} = \left(\frac{\mu_o}{4\pi}\right)I\int_0^{2\pi}\frac{Rd\varphi\left\{z(\cos\varphi\hat{x}+\sin\varphi\hat{y})+R\hat{z}\right\}}{\left(z^2+R^2\right)^{3/2}}$$
$$= \left(\frac{\mu_o}{4\pi}\right)I\frac{Rz}{\left(z^2+R^2\right)^{3/2}}\left[\int_0^{2\pi}\cos\varphi d\varphi\hat{x}+\int_0^{2\pi}\sin\varphi d\varphi\hat{y}\right]+\left(\frac{\mu_o}{4\pi}\right)\frac{IR^2}{\left(z^2+R^2\right)^{3/2}}\int_0^{2\pi}d\varphi\hat{z}$$

n.b each of these 2 terms will individually vanish / cancel when integrated over all φ - i.e. from $0\le\varphi\le 2\pi$!!

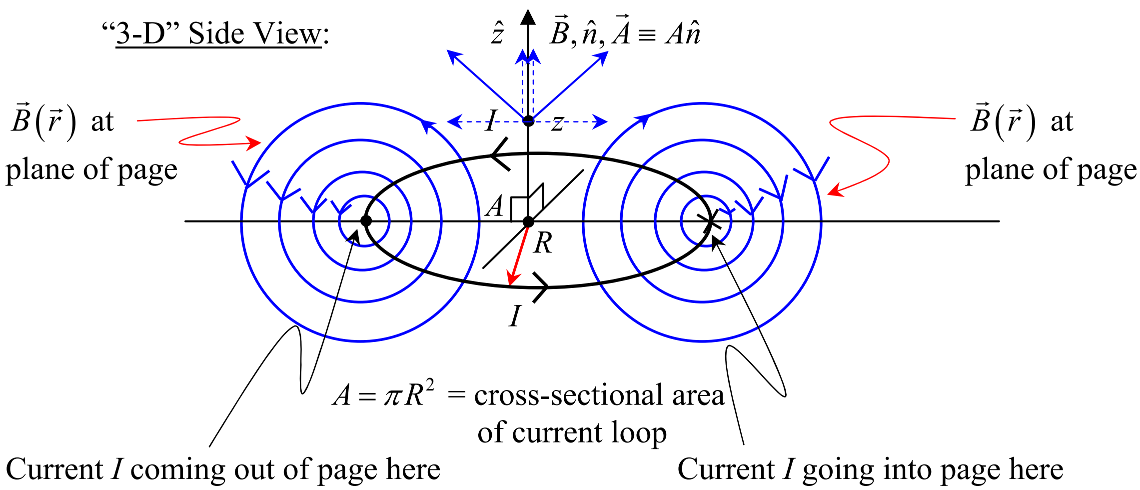
$$\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)I\frac{Rz}{\left(z^2+R^2\right)^{3/2}}\left[+\overset{=0}{\sin\varphi\Big|_0^{2\pi}}\hat{x}-\overset{=0}{\cos\varphi\Big|_0^{2\pi}}\hat{y}\right]+2\pi\left(\frac{\mu_o}{4\pi}\right)\frac{IR^2}{\left(z^2+R^2\right)^{3/2}}\hat{z}$$

Finally:

$$\boxed{\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{2}\right) I \frac{R^2}{\left(z^2+R^2\right)^{3/2}}\hat{z}}$$

The $\vec{B}$-field on symmetry axis of a steady current-carrying loop of radius R points in $\hat{z}$ direction
Note also that $\vec{B}_{loop}(r = z\hat{z})$ falls off as $\sim 1/z^3$ just as a magnetic dipole field should!!
i.e. $\vec{B}_{loop}(r = z\hat{z})$ is the on-axis $\vec{B}$-field of a physical (i.e. spatially-extended) magnetic dipole!
$\Rightarrow$ A loop carrying a current I generates a magnetic dipole field!!!

The $\vec{B}$-field due to a current loop, on the symmetry $(\hat{z})$-axis of the loop:



Note that at the observation point $\vec{r} = z\hat{z}$, the "normal" components of $\vec{B}$ (i.e. those $\perp$ to the $\hat{z}$-axis) cancel, whereas the "parallel" components of $\vec{B}$ (i.e. those || to the $\hat{z}$-axis) add!

The vector area of the current-carrying loop is $\vec{A} = A\hat{n}$ (where $\hat{n}$ is defined by right-hand rule associated with direction of current circulation).

The Magnetic Dipole Moment of a Current-Carrying Loop of Cross-Sectional Area $\vec{A}$

The magnetic dipole moment is defined as: $\boxed{\vec{m} \equiv I\vec{A}}$ $(= \pi R^2 I\hat{z}$ here) (SI units: Ampere-meters^2)
The vector area of the loop is: $\vec{A}_{loop} = \pi R^2 \hat{n}$ ($\hat{n} = \hat{z}$ here), the scalar area of the loop: $A_{loop} = \pi R^2$

The Magnetic Field on the Symmetry Axis of an *N*-Turn Current-Carrying Loop:

Instead of a single current-carrying loop, what would be the $\vec{B}$-field associated with N turns (of very fine wire) for the planar current-carrying loop?

Using the principle of linear superposition: $\boxed{I_{TOT} = NI}$.

Thus on the symmetry axis of an *N*-turn planar loop of radius *R* carrying a steady current *I*:

$$\vec{B}_{\substack{N-\text{turn loop}\\ \text{of radius } R}}(z) = N\vec{B}_{\substack{\text{1-turn loop}\\ \text{of radius } R}}(z) = N\left(\frac{1}{2}\mu_o I \frac{R^2}{\left(z^2+R^2\right)^{3/2}}\hat{z}\right) = \frac{1}{2}\mu_o NI \frac{R^2}{\left(z^2+R^2\right)^{3/2}}\hat{z}$$

The superposition principle also works for / is valid for magnetic phenomena since it is intimately connected to electric phenomena via common / same microscopic physics!

The Macroscopic Magnetic Fields Associated with Line, Surface and Volume Currents:

For line, surface and volume currents, we summarize their corresponding magnetic fields below:

Line Currents: $$\vec{B}_{\substack{line\\ current}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{C'} \frac{I d\vec{\ell}'(\vec{r}')\times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} = \left(\frac{\mu_o}{4\pi}\right)\int_{C'} \frac{\vec{I}(\vec{r}')\times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} d\ell'$$

Surface Currents: $$\vec{B}_{\substack{surface\\ current}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{S'} \frac{\vec{K}(\vec{r}')\times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} dA'$$ where $\vec{\boldsymbol{r}} = \vec{r} - \vec{r}'$, and $\boldsymbol{r} = |\vec{\boldsymbol{r}}| = |\vec{r} - \vec{r}'|$

Volume Currents: $$\vec{B}_{\substack{volume\\ current}}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\int_{v'} \frac{\vec{J}(\vec{r}')\times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} d\tau'$$ and $\hat{\boldsymbol{r}} = \vec{\boldsymbol{r}}/|\vec{\boldsymbol{r}}| = \vec{r} - \vec{r}'/|\vec{r} - \vec{r}'|$

n.b. The primed variables denotes integration over the relevant source current distributions.

The Biot-Savart Law:

The Calculation of Forces on Current-Carrying Conductors Due to Other Current-Carrying Conductors

We have previously derived (in P435 Lecture Notes 13) that the net macroscopic magnetic force $\vec{F}_m$ on a current-carrying wire immersed in an external magnetic field $\vec{B}_{ext}(\vec{r}')$ was:

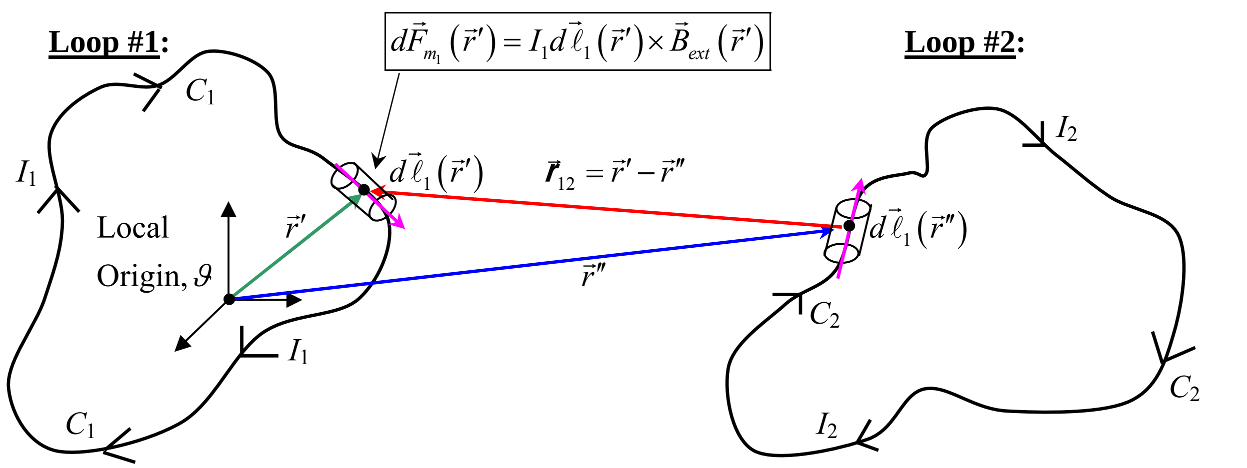
$$\boxed{\vec{F}_m = \int_C d\vec{F}_m(\vec{r}') = I\int_C d\vec{\ell}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')}$$ **iff** I = constant/steady/uniform current!

If $I = I(\vec{r}')$ is not constant/steady/uniform in space, then more generally:

$$\boxed{\vec{F}_m = \int_C \vec{I}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\,d\ell}$$

Now let us consider what $\vec{F}_m$ would be if $\vec{B}_{ext}(\vec{r}')$ is due to a 2[nd] current-carrying loop. For simplicity's sake, we will assume that all currents involved are steady currents.

Two Amperian Current Loops:



The infinitesimal force $d\vec{F}_{m_1}(\vec{r}')$ acting on a line segment $d\vec{\ell}_1(\vec{r}')$ on loop #1 is due to the net macroscopic magnetic field $\vec{B}_{ext}(\vec{r}')$ at the point $\vec{r}'$ that is created by the steady current I_2 flowing in loop #2.

Thus: $\boxed{d\vec{F}_{m_1}(\vec{r}') = I_1 d\vec{\ell}_1(\vec{r}')\times\vec{B}_{ext}(\vec{r}')}$ = infinitesimal force acting on line segment $d\vec{\ell}_1(\vec{r}')$ of loop #1 due to net $\vec{B}$-field from second current loop #2

Then the net force on the original steady current-carrying loop (loop #1) is:

$$\boxed{\vec{F}_{m_1} = \int_{C_1} d\vec{F}_{m_1}(\vec{r}') = I_1\int_{C_1} d\vec{\ell}_1(\vec{r}')\times\underbrace{\vec{B}_{ext}(\vec{r}')}}$$

= magnetic field at source point $\vec{r}'$ (on loop #1) at $d\vec{\ell}_1(\vec{r}')$ due to 2[nd] current I_2 flowing in loop #2

What is the net, macroscopic $\vec{B}$-field at the point $\vec{r}'$ arising from the steady current I_2 flowing in the second current loop #2? <u>It is</u>:

$$\boxed{\vec{B}_{ext}(\vec{r}')=\left(\frac{\mu_o}{4\pi}\right)I_2\oint_{C_2}\frac{d\vec{\ell}_2(\vec{r}'')\times \boldsymbol{\mathscr{r}}_{12}}{|\boldsymbol{\mathscr{r}}_{12}|^3}} \quad \text{where} \quad \boxed{\boldsymbol{\mathscr{r}}_{12}\equiv(\vec{r}'-\vec{r}'')} \quad \text{and} \quad \boxed{|\boldsymbol{\mathscr{r}}_{12}|=|\vec{r}'-\vec{r}''|}$$

<u>Then</u>:

$$\begin{aligned}\vec{F}_{m_1}&=\oint_{C_1}d\vec{F}_{m_1}(\vec{r}')=\oint_{C_1}I_1 d\vec{\ell}_1(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\\ &=\oint_{C_1}I_1 d\vec{\ell}_1(\vec{r}')\times\left[\left(\frac{\mu_o}{4\pi}\right)I_2\oint_{C_2}\frac{d\vec{\ell}_2(\vec{r}'')\times\boldsymbol{\mathscr{r}}_{12}}{|\boldsymbol{\mathscr{r}}_{12}|^3}\right]\\ &=\left(\frac{\mu_o}{4\pi}\right)I_1 I_2\oint_{C_1}d\vec{\ell}_1(\vec{r}')\times\left[\oint_{C_2}\frac{d\vec{\ell}_2(\vec{r}'')\times\boldsymbol{\mathscr{r}}_{12}}{|\boldsymbol{\mathscr{r}}_{12}|^3}\right]\end{aligned}$$

<u>Or</u>:

$$\boxed{\vec{F}_{m_1}=\left(\frac{\mu_o}{4\pi}\right)I_1 I_2\oint_{C_1}\oint_{C_2}\frac{d\vec{\ell}_1(\vec{r}')\times\left(d\vec{\ell}_2(\vec{r}'')\times\boldsymbol{\mathscr{r}}_{12}\right)}{|\boldsymbol{\mathscr{r}}_{12}|^3}} \Leftarrow \text{Biot-Savart Law}$$

Now, Newton's 3[rd] Law of Motion holds here, i.e. that: $\boxed{\vec{F}_{2\to 1}=-\vec{F}_{1\to 2}}$

<u>Thus</u>:

$$\boxed{\vec{F}_{m_2}=\left(\frac{\mu_o}{4\pi}\right)I_2 I_1\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_2(\vec{r}'')\times\left(d\vec{\ell}_1(\vec{r}')\times\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}}$$

where $\boldsymbol{\mathscr{r}}_{21}\equiv(\vec{r}''-\vec{r}')=-(\vec{r}'-\vec{r}'')\equiv-\boldsymbol{\mathscr{r}}_{12}$ and $|\boldsymbol{\mathscr{r}}_{21}|=|\vec{r}''-\vec{r}'|=|\vec{r}'-\vec{r}''|=|\boldsymbol{\mathscr{r}}_{12}|$

This relation can be / is easily obtained by permuting indices $1\rightleftharpoons 2$ and $\vec{r}'\rightleftharpoons\vec{r}''$

<u>Explicit proof</u> - Use the vector triple cross-product identity: $\boxed{\vec{A}\times\vec{B}\times\vec{C}=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)}$

Expand out the integral for $\vec{F}_{m_2}$:

$$\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_2(\vec{r}'')\times\left(d\vec{\ell}_1(\vec{r}')\times\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}=\underbrace{\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1(\vec{r}')\left(d\vec{\ell}_2(\vec{r}'')\bullet\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}}_{\substack{=0\,!!!\\ Why?}}-\oint_{C_2}\oint_{C_1}\frac{\boldsymbol{\mathscr{r}}_{21}\left(d\vec{\ell}_2(\vec{r}'')\bullet d\vec{\ell}_1(\vec{r}')\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}$$

Why does $\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1(\vec{r}')\left(d\vec{\ell}_2(\vec{r}'')\bullet\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}=0\,???$

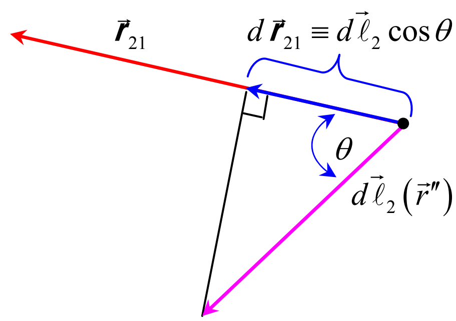
$$\oint_{C_2}\oint_{C_1}\frac{d\vec{\ell}_1(\vec{r}')\left(d\vec{\ell}_2(\vec{r}'')\bullet\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}=\underbrace{\oint_{C_1}\oint_{C_2}}_{\substack{\textit{n.b. Switched order}\\ \textit{of integration}}}\frac{d\vec{\ell}_1(\vec{r}')\left(d\vec{\ell}_2(\vec{r}'')\bullet\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}=\oint_{C_1}d\vec{\ell}_1(\vec{r}')\underbrace{\left[\oint_{C_2}\frac{\left(d\vec{\ell}_2(\vec{r}'')\bullet\boldsymbol{\mathscr{r}}_{21}\right)}{|\boldsymbol{\mathscr{r}}_{21}|^3}\right]}_{\substack{\textit{This integral is carried out}\\ \textit{only over closed contour } C_2\\ \textit{of loop\#2}}}$$

Now: $\dfrac{\left(d\vec{\ell}_2(\vec{r}'')\bullet \boldsymbol{r}_{21}\right)}{|\boldsymbol{r}_{21}|^3} = \dfrac{\left(d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21}\right)}{|\boldsymbol{r}_{21}|^2}$ since $\boldsymbol{r}_{21} = |\boldsymbol{r}_{21}|\,\hat{\boldsymbol{r}}_{21}$

But what is $d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21}$??

$d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21} = d\ell_2\cos\theta$ where θ = opening between $d\vec{\ell}_2(\vec{r}'')$ and $\hat{\boldsymbol{r}}_{21}$

$d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21} = d\ell_2\cos\theta = \left|d\vec{\ell}_2(\vec{r}'')\right|\cos\theta \equiv d\boldsymbol{r}_{21}(\vec{r}'') = \left|d\boldsymbol{r}_{21}(\vec{r}'')\right|$



$\therefore \dfrac{\left(d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21}\right)}{|\boldsymbol{r}_{21}|^2} = \dfrac{d\boldsymbol{r}_{21}}{|\boldsymbol{r}_{21}|^2}$ Then: $\oint_{C_2}\dfrac{\left(d\vec{\ell}_2(\vec{r}'')\bullet \boldsymbol{r}_{21}\right)}{|\boldsymbol{r}_{21}|^3} = \oint_{C_2}\dfrac{\left(d\vec{\ell}_2(\vec{r}'')\bullet \hat{\boldsymbol{r}}_{21}\right)}{|\boldsymbol{r}_{21}|^2} = \oint_{C_2}\dfrac{d\boldsymbol{r}_{21}}{|\boldsymbol{r}_{21}|^2}$

But we know that $\boxed{\oint_C \dfrac{d\boldsymbol{r}}{\boldsymbol{r}^2} \equiv 0}$ around an (arbitrary) closed loop/contour C !!!

e.g. $\boxed{\Delta V = \oint_C \vec{E}_q(\vec{r})\bullet d\vec{\ell} = \dfrac{q}{4\pi\varepsilon_o}\oint_C \dfrac{d\vec{\ell}\bullet\hat{r}}{r^2} \dfrac{q}{4\pi\varepsilon_o}\oint_C \dfrac{dr}{r^2} = 0}$ for a point electric charge!!!

Thus finally:

$$\boxed{\vec{F}_{m_2} = -\left(\frac{\mu_o}{4\pi}\right) I_2 I_1 \oint_{C_2}\oint_{C_1} \frac{\boldsymbol{r}_{21}\left(d\vec{\ell}_2(\vec{r}'')\bullet d\vec{\ell}_1(\vec{r}')\right)}{|\boldsymbol{r}_{21}|^3}}$$

But: $\boldsymbol{r}_{21} \equiv (\vec{r}'' - \vec{r}') = -(\vec{r}' - \vec{r}'') \equiv -\boldsymbol{r}_{12}$

And: $|\boldsymbol{r}_{21}| = |\vec{r}'' - \vec{r}'| = |\vec{r}' - \vec{r}''| = |\boldsymbol{r}_{12}|$

And: $\vec{A}\bullet\vec{B} = \vec{B}\bullet\vec{A}$

Thus:

$$\boxed{\vec{F}_{m_1} = -\left(\frac{\mu_o}{4\pi}\right) I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{\boldsymbol{r}_{12}\left(d\vec{\ell}_1(\vec{r}')\bullet d\vec{\ell}_2(\vec{r}'')\right)}{|\boldsymbol{r}_{12}|^3}}$$

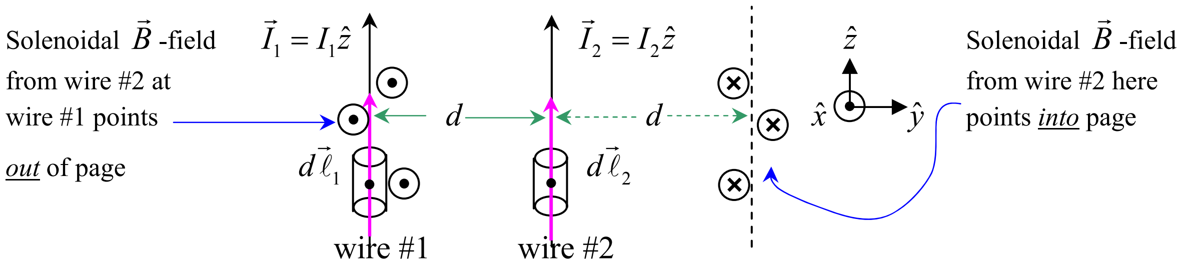
$\therefore \boxed{\vec{F}_{m_2} = -\vec{F}_{m_1}}$

Q.E.D.

Examples of the Use of the Biot-Savart Force Law:

The Net Force Between Two Infinitely Long, Parallel Wires Carrying Steady Currents I_1 and I_2

Referring to the figure below, let's calculate the net force on wire #1 (carrying current $\vec{I}_1 = I_1\hat{z}$) due to the external magnetic field produced by the (parallel) wire #2 (carrying current $\vec{I}_2 = I_2\hat{z}$) located a perpendicular distance $r = d$ away from wire #1. Both wires are infinitely long.



The magnetic field $\vec{B}_{wire\#2}(r = d)$ arising from a steady current $\vec{I}_2 = I_2\hat{z}$ flowing in wire # 2, at a perpendicular distance $r = d$ away from wire # 2 is:

$$\vec{B}_{wire\#2}(r=d) = \left(\frac{\mu_o}{4\pi}\right)\int_{C_2} \frac{\vec{I}_2 d\vec{\ell}_2 \times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} = \left(\frac{\mu_o}{4\pi}\right)\int_{C_2} \frac{I_2 d\vec{\ell}_2 \times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2} = \left(\frac{\mu_o}{2\pi}\right)\frac{I_2}{d}\hat{\varphi}$$

See pages 5-6 of these lecture notes

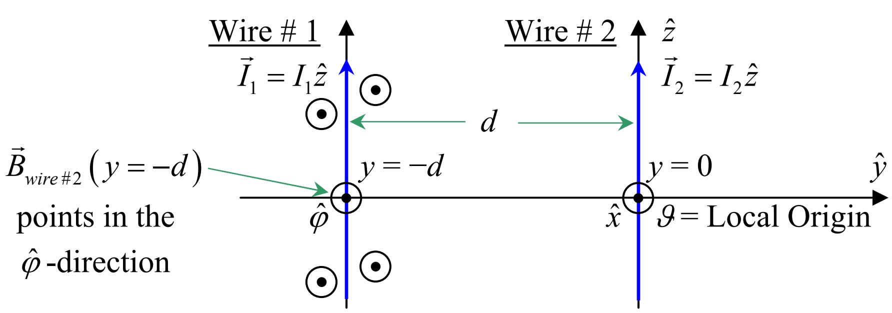
The Biot-Savart Law:
The net magnetic force on current-carrying wire #1 due to *B*-field of current-carrying wire #2 is:

$$\vec{F}_{\substack{m_1 \\ wire\#1}} = \int_{C_1} dF_{m_1}(\vec{r}') = \int_{C_1} \vec{I}_1 d\ell_1(\vec{r}') \times \vec{B}_{wire\#2}(r' = d)$$
$$= \int_{C_1} I_1 d\vec{\ell}_1(\vec{r}') \times \vec{B}_{wire\#2}(r' = d) = I_1\int_{C_1} d\vec{\ell}_1(\vec{r}') \times \vec{B}_{wire\#2}(r' = d)$$

But here: $d\vec{\ell}_1(\vec{r}') = dz\hat{z}$

$$\therefore\ \vec{F}_{\substack{m_1 \\ wire\#1}} = I_1\int_{C_1} dz\hat{z} \times \left(\frac{\mu_o}{2\pi}\right)\frac{I_2}{d}\hat{\varphi} = \left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}\int_{C_1}(\hat{z}\times\hat{\varphi})dz = \left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}(\hat{z}\times\hat{\varphi})\int_{C_1} dz$$

Which way does $(\hat{z}\times\hat{\varphi})$ point? If $\|$-wires are in the *y-z* plane and separated by distance *d*:



Note that $\hat{\varphi}$ at wire # 1 points in the $\hat{x}$ -direction ($\hat{\varphi} \| \hat{x}$ at wire # 1) – i.e. $\hat{\varphi}$ and $\hat{x}$ both point out of the page.

Then: $(\hat{z}\times\hat{\varphi}) = \hat{z}\times\hat{x} = +\hat{y}$ (referring to useful table on p. 8 of these lecture notes).

Then: $$\boxed{\vec{F}_{\substack{m_1\\ wire\#1}}(\vec{r}=-d\hat{y}) = \left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}\,\hat{y}\int_{C_1} dz}$$

But if current-carrying wire #1 is infinitely long, then: $\int_{C_1} dz = \int_{-\infty}^{\infty} dz = \infty + \infty = \infty$!!!

$\Rightarrow$ The net magnetic force $\vec{F}_{\substack{m_1\\ wire\#1}}(r)$ on an infinitely long wire carrying current $\vec{I}_1 = I_1\hat{z}$ due to another infinitely long, parallel wire carrying current $\vec{I}_2 = I_2\hat{z}$ a $\perp$ distance r away is infinite!!!

However, note that the magnetic force per unit length is *finite*: $\boxed{\int_0^{L_1} dz = L_1}$

Define force per unit length as: $$\boxed{\vec{f}_{\substack{m_1\\ wire\#1}}(\vec{r}=-d\hat{y}) \equiv \left(\vec{F}_{\substack{m_1\\ wire\#1}}(r=-d)\Big/L_1\right) = \left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}\,\hat{y}}$$

Then by Newton's 3rd Law, the magnetic force per unit length acting on wire #2 due to the $\vec{B}$-field a perpendicular distance d away from wire #1 is:

$$\boxed{\vec{f}_{\substack{m_2\\ wire\#2}}(\vec{r}=0) \equiv \left(\vec{F}_{\substack{m_2\\ wire\#2}}(r=+d)\Big/L_2\right) = \left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}(-\hat{y}) = -\left(\frac{\mu_o}{2\pi}\right)\frac{I_1 I_2}{d}\,\hat{y}}$$

Thus we see that: $\boxed{\vec{F}_{\substack{m_2\\ wire\#2}} = -\vec{F}_{\substack{m_1\\ wire\#1}}}$ or: $\boxed{\vec{f}_{\substack{m_2\\ wire\#2}} = -\vec{f}_{\substack{m_1\\ wire\#1}}}$ as they must by Newton's 3rd Law

"For every action there is an equal and opposite reaction"

Note that: $\vec{f}_{\substack{m_1\\ wire\#1}} = \underset{\sim\sim\sim}{\_\_}(+\hat{y})$ and $\vec{f}_{\substack{m_2\\ wire\#2}} = \underset{\sim\sim\sim}{\_\_}(-\hat{y})$

i.e. parallel wires carrying currents in the same direction attract each other!

$\Rightarrow$ parallel wires carrying currents in opposite directions repel each other!

Macroscopic Magnetic Forces and Torques on $\left\{\begin{matrix}\textbf{Line}\\ \textbf{Surface}\\ \textbf{Volume}\end{matrix}\right\}$ Current Densities $\left\{\begin{matrix}\vec{I}=\lambda\vec{v}\\ \vec{K}\\ \vec{J}\end{matrix}\right\}$ In an External Magnetic Field $\vec{B}_{ext}(\vec{r})$

1.) Moving point charge q located at $\vec{r}'$ in a externally-applied magnetic field $\vec{B}_{ext}(\vec{r}')$:

$$\boxed{\vec{F}_m = q\vec{v}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')}$$

$$\boxed{\vec{\tau}_m = \vec{r}\times\vec{F}_m = q\vec{r}\times\left[\vec{v}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\right]}$$

2.) Filamentary line current carrying conductor in externally-applied magnetic field $\vec{B}_{ext}(\vec{r}')$:

$$\boxed{\vec{F}_m = \int_{C'}\vec{I}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\,d\ell = \int_{C'} Id\vec{\ell}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')}$$

$$\boxed{\vec{\tau}_m = \int_{C'}\vec{r}'\times d\vec{F}_m(\vec{r}') = \int_{C'}\vec{r}'\times\left[Id\vec{\ell}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\right]}$$

3.) Surface/sheet current densities in externally-applied magnetic field $\vec{B}_{ext}(\vec{r}')$:

$$\boxed{\vec{F}_m = \int_{S'}\vec{K}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\,dA'}$$

$$\boxed{\vec{\tau}_m = \int_{S'}\vec{r}'\times d\vec{F}_m(\vec{r}') = \int_{S'}\vec{r}'\times\left[\vec{K}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\right]dA'}$$

4.) Volume current densities in externally applied magnetic field $\vec{B}_{ext}(\vec{r}')$:

$$\boxed{\vec{F}_m = \int_{v'}\vec{J}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\,d\tau'}$$

$$\boxed{\vec{\tau}_m = \int_{v'}\vec{r}'\times d\vec{F}_m(\vec{r}') = \int_{v'}\vec{r}'\times\left[\vec{J}(\vec{r}')\times\vec{B}_{ext}(\vec{r}')\right]d\tau'}$$

If $\vec{B}_{ext}(\vec{r}')$ is e.g. due to a 2^{nd} current-carrying loop (loop #2), with current $\vec{I}_2/\vec{K}_2/\vec{J}_2$:

i.e. $\boxed{\vec{B}_{ext}(\vec{r}') = \left(\frac{\mu_o}{4\pi}\right)\oint\frac{I_2 d\vec{\ell}_2(\vec{r}'')\times\boldsymbol{\mathscr{r}}_{21}}{|\boldsymbol{\mathscr{r}}_{21}|^3}}$ where $\boxed{\boldsymbol{\mathscr{r}}_{21}\equiv(\vec{r}''-\vec{r}')}$ and $\boxed{|\boldsymbol{\mathscr{r}}_{21}| = |\vec{r}''-\vec{r}'|}$

then plug this expression for $\vec{B}_{ext}(\vec{r}')$ into any of the above relations to obtain Biot-Savart formulae for $\vec{F}_{m_1}$, $\vec{\tau}_{m_1}$, etc.

Macroscopic Magnetic Field Intensities $\vec{B}(\vec{r})$ Produced by a $\left\{\begin{array}{c}\textbf{Moving Point Charge}\\ \textbf{Line/Surface/Volume}\\ \textbf{Current Density}\end{array}\right\}$

n.b. The primed variables in the formulae below denote integration over the relevant source current distributions, and $\boxed{\boldsymbol{\mathscr{r}} \equiv \vec{r}-\vec{r}'}$, thus: $\boxed{\mathscr{r} = |\boldsymbol{\mathscr{r}}| = |\vec{r}-\vec{r}'|}$ and $\boxed{\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/|\boldsymbol{\mathscr{r}}| = \vec{r}-\vec{r}'/|\vec{r}-\vec{r}'|}$.

cf to parallel expression for ***E***-field:

1.) $\vec{B}$-field due to a moving point electric charge q ($v \ll c$):

$$\boxed{\begin{aligned}\vec{B}_{q_{free}}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}(\vec{r}')\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}(\vec{r}')\times\frac{(\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}\right)\\ &= \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}(\vec{r}')\times\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\right) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)\end{aligned}} \Leftrightarrow \boxed{\vec{E}_q(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{q\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)}$$

2.) $\vec{B}$-field due to a line current $\vec{I}(\vec{r}')$ (Amps):

$$\boxed{\begin{aligned}\vec{B}_{\vec{I}}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\int_{C'}\frac{d\vec{\ell}'(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \left(\frac{\mu_o}{4\pi}\right)\int_{C'}\frac{Id\vec{\ell}'(\vec{r}')\times(\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}\\ &= \left(\frac{\mu_o}{4\pi}\right)\int_{C'}\frac{Id\vec{\ell}'(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \left(\frac{\mu_o}{4\pi}\right)\int_{C'}\frac{Id\vec{\ell}'(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\end{aligned}} \Leftrightarrow \boxed{\vec{E}_\lambda(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\int_{C'}\frac{\lambda(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\ell'}$$

3.) $\vec{B}$-field due to a surface/sheet current $\vec{K}(\vec{r}')$ (Amps/m):

$$\boxed{\begin{aligned}\vec{B}_{\vec{K}}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\vec{K}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dA' = \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\vec{K}(\vec{r}')\times(\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}dA'\\ &= \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\vec{K}(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}dA' = \left(\frac{\mu_o}{4\pi}\right)\int_{S'}\frac{\vec{K}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dA'\end{aligned}} \Leftrightarrow \boxed{\vec{E}_\sigma(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\int_{S'}\frac{\sigma(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dA'}$$

4.) $\vec{B}$-field due to a volume current $\vec{J}(\vec{r}')$ (Amps/m^2):

$$\boxed{\begin{aligned}\vec{B}_{\vec{J}}(\vec{r}) &= \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\vec{J}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau' = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\vec{J}(\vec{r}')\times(\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}d\tau'\\ &= \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\vec{J}(\vec{r}')\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}d\tau' = \left(\frac{\mu_o}{4\pi}\right)\int_{v'}\frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}d\tau'\end{aligned}} \Leftrightarrow \boxed{\vec{E}_\rho(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\int_{v'}\frac{\rho(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'}$$

