LECTURE NOTES 13

ELECTRIC CURRENTS

Before launching into a full-blown discussion of magnetic phenomena, we first want to discuss electric currents…

The free electric current I_{free} at a given point in space (call this point $\vec{r}$, defined relative to a local origin of coordinates) is defined as the time-rate of change of free charge Q_{free} at that point in space.

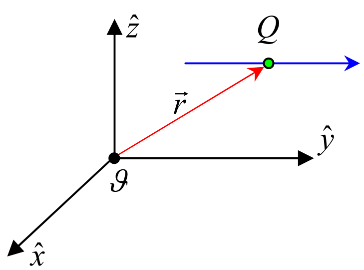
$$\boxed{I_{free} = \frac{dQ_{free}}{dt}}$$ or more explicitly:

$$\boxed{I_{free}(\vec{r},t) = \frac{dQ_{free}(\vec{r},t)}{dt}}$$ = instantaneous electric current at point $\vec{r}$ at time t

The S.I. unit of current I is the Ampere, in honor of André Marie Ampere, for his 1820 work on understanding the nature of electric currents.

1 Ampere ≡ 1 Coulomb of charge per second (crossing/passing through an imaginary surface)

An electric current implies that electric charge is in motion, relative to an observer whose rest frame is in the fixed coordinate system with ϑ at its origin:

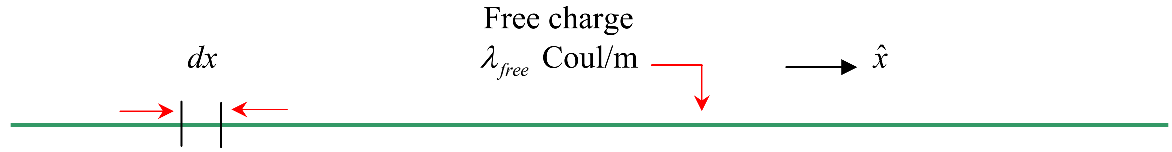


Moving charge Q relative to origin, ϑ

n.b. ∃ no absolute reference frame anywhere in the universe – we can only speak of relative motions between objects.

<u>Line Currents</u>:

Consider an infinitely long straight filamentary (i.e. infinitesimally thin) wire. Imagine that this wire is electrically charged with λ_{free} Coulombs/meter of <u>free</u> charge per unit length, as shown below in the figure:



If the charge is static, i.e. not moving, then there is no electric current I flowing in the wire. However, if a potential difference ΔV is imposed across the ends of the wire, then a <u>line</u> current will begin to flow by an amount:

$$\boxed{I_{free} = \lambda_{free} v}$$

where v = (relative) speed of charge moving down the wire, i.e. <u>relative</u> to the wire <u>itself</u>.

But speed is just the magnitude of the velocity, so actually, this relation is a <u>vector</u> relation:

$$\boxed{\vec{I}_{free}(\vec{r},t) = \lambda_{free}\vec{v}(\vec{r},t)}$$

(assuming λ_{free} = constant, i.e. $\lambda_{free} \neq \lambda_{free}(\vec{r},t)$)

Now, an infinitesimal segment $dx = vdt$ instantaneously carries charge $dq = \lambda_{free}dx = \lambda_{free}vdt$ past an observation point P in the time interval dt:

$$\boxed{I_{free} = \frac{dq_{free}}{dt} = \frac{\lambda_{free}dx}{dt} = \lambda_{free}v} \quad \text{where} \quad \boxed{v = \frac{dx}{dt}}$$

Vectorizing this (assuming λ_{free} = constant): $\vec{I} = \dfrac{\lambda d\vec{x}}{dt} = \lambda\vec{v}$ or more generally:

$$\boxed{\vec{I}_{free}(\vec{r},t) = \lambda_{free}\frac{d\vec{r}(t)}{dt} = \lambda_{free}\vec{v}(\vec{r},t)} \quad \text{for line currents}$$

n.b. in principle $\lambda_{free} = \lambda_{free}(\vec{r},t)$ also!

In a (circular) particle accelerator, e.g. a <u>cyclotron</u>, charged particles (e.g. <u>protons</u>) circulate in an evacuated ring of radius R at speeds nearly that of the speed of light i.e. $v_p \lesssim c$.

Suppose R = 1 meter, and also suppose that we have I = 1 Ampere of (proton) current circulating in the cyclotron, and assume further that the protons are <u>uniformly</u> distributed around the <u>circumference</u> of the machine. If we additionally assume (for simplicity's sake) that $v_p = c = 3\times10^8\, m/\sec$ then:

$$I_p = \lambda_{free}c \quad \text{or:} \quad \lambda_{free} = \frac{I}{c} = \frac{1\,Ampere}{3\times10^8\, m/s} \quad \underline{\text{but}}\text{: } 1 \text{ Ampere} = 1 \text{ Coulomb / sec}$$

$$\therefore\ \lambda_{free} = \frac{10^{-8}\,Coulombs}{3\,\cancel{sec}}\bullet\frac{\cancel{sec}}{meters} = 0.\overline{333}\times10^{-8}\,Coulombs/meter$$

The circumference of the proton accelerator is $C = 2\pi R =$ total length of line charge

$$\therefore\quad Q_p^{TOT} = \lambda_{free} C = 0.\overline{333}\times 10^{-8}\frac{Coulombs}{\cancel{meter}}\times 2\times\pi\times 1\ \cancel{meter} = \frac{2\pi}{3}\times 10^{-8}\,Coulombs$$

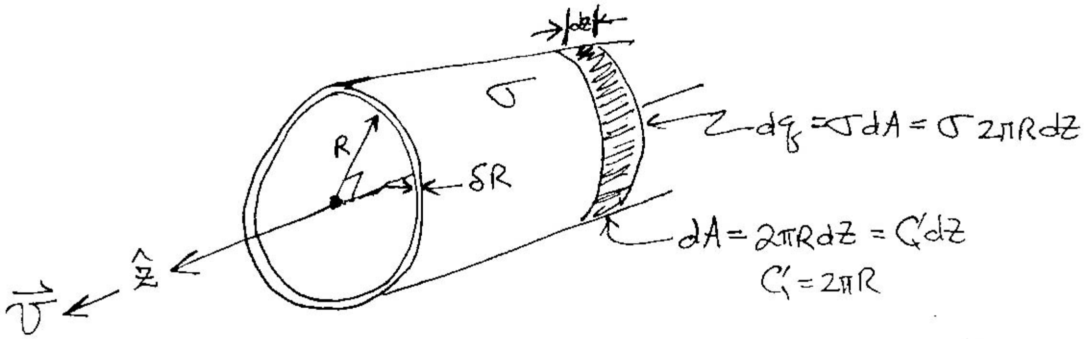
If there are N_p = total # protons circulating in the cyclotron accelerator, then:

$$N_p = \frac{Q_p^{TOT}}{q_p} = \frac{\frac{2\pi}{3}\times 10^{-8}\,Coulombs}{1.602\times 10^{-19}\,Coulombs} \simeq 1.3\times 10^{11}\ \text{protons circulating in the cyclotron}$$

Thus, I_p = 1 Ampere of proton current circulating in this cyclotron corresponds to $N_p \simeq 1.3\times 10^{11}$ protons circulating in the cyclotron (if they are all traveling at the speed of light, *c*).

<u>Surface Currents</u>:

Imagine an infinitely long, straight, hollow conducting cylindrical tube of radius *R* with infinitesimally thin walls (of thickness δR) carrying σ_{free} Coulombs/meter^2 of (initially) stationary electric charge. A potential difference ΔV is placed across the length of this hollow tube, causing an electric <u>surface</u> current to flow, of magnitude *I* Amperes.



An infinitesimal section of this hollow tube *dz* = *vdt* instantaneously carries charge $dq_{free} = \sigma_{free} 2\pi R dz = \sigma_{free} dA$ past an observation point *P* (on the tube somewhere) in the time interval *dt*. Then:

$$I \equiv \frac{dq_{free}}{dt} = \frac{\sigma_{free} 2\pi R dz}{dt} = 2\pi R\sigma\frac{dz}{dt} \quad \text{but} \quad v = \frac{dz}{dt}$$

$$\therefore\quad I = (2\pi R)\sigma_{free} v = C\sigma_{free} v$$

In principle, $\sigma_{free} = \sigma_{free}(\vec{r},t)$

Vectorizing this, <u>*assuming*</u> σ_{free} = constant (i.e. $\sigma_{free} \neq \sigma_{free}(\vec{r},t)$) then:

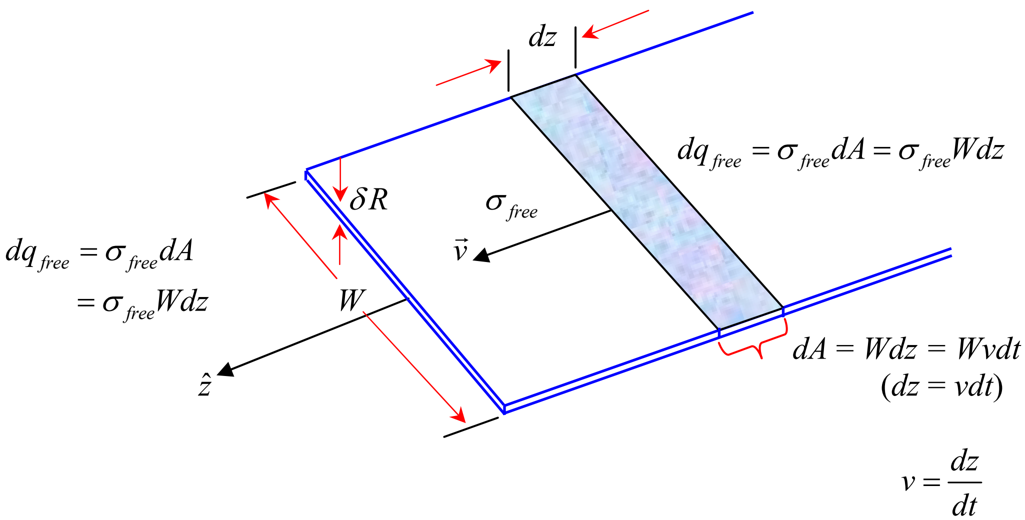
$$\boxed{\vec{I}(\vec{r},t) = C\sigma_{free}\vec{v}(\vec{r},t) = (2\pi R)\sigma_{free}\vec{v}(\vec{r},t)}$$ for a hollow conducting tube of radius *R*.

However, note that we can also define a <u>surface current density</u> $\vec{K}(\vec{r},t)$ as:

$$\boxed{\vec{K}(\vec{r},t) \equiv \frac{\vec{I}(\vec{r},t)}{C} = \frac{\vec{I}(\vec{r},t)}{2\pi R} = \sigma_{free}\vec{v}(\vec{r},t) \quad \left(\text{Amperes}/\text{meter}\right)}$$

In principle, $\sigma_{free} = \sigma_{free}(\vec{r},t)$

Instead of a surface current flowing on a long, hollow conducting tube of radius R, suppose we had a surface current flowing on a flat (i.e. planar) conductor of width W. This is simply equivalent to e.g. cutting the long hollow conducting tube (with infinitesimally thin walls of thickness δR) and unrolling it out into a flat plane. Then the width W of the flat sheet = circumference C of the original tube, i.e. $W = C = 2\pi R$ and thus:



$$\boxed{I \equiv \frac{dq_{free}}{dt} = \frac{\sigma_{free} W dz}{dt} = \sigma_{free} W \frac{dz}{dt} = \sigma_{free} W v}$$

Vectorizing this (assuming σ = constant, i.e. $\sigma_{free} \neq \sigma_{free}(\vec{r},t)$), the current flowing through the flat sheet is:

$$\boxed{\vec{I}(\vec{r},t) = \sigma_{free} W\, \vec{v}(\vec{r},t)}$$ for a planar conducting sheet of width W

n.b. in principle $\sigma_{free} \neq \sigma_{free}(\vec{r},t)$

Here again, we can also define a <u>lineal surface current density</u> $\vec{K}(\vec{r},t)$ as:

$$\boxed{\vec{K}(\vec{r},t) \equiv \frac{I(\vec{r},t)}{W} = \sigma_{free} \vec{v}(\vec{r},t)}$$ Amperes / meter $\Leftarrow$ n.b. not Amperes / m^2!!

This can also be written in <u>differential</u> form as:

$$\boxed{\vec{K}(\vec{r},t) \equiv \frac{d\vec{I}(\vec{r},t)}{dW} = \sigma_{free} \vec{v}(\vec{r},t)}$$ Amperes / meter

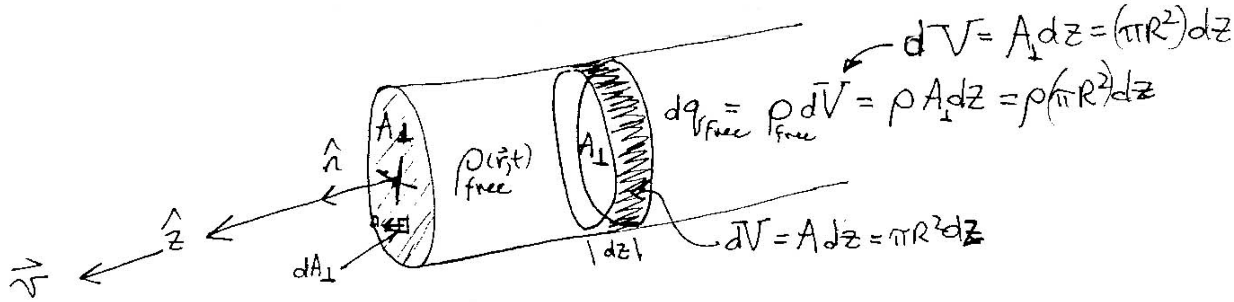
n.b. in principle $\sigma_{free} \neq \sigma_{free}(\vec{r},t)$

To <u>explicitly</u> tie in with Griffiths book: $\boxed{dW = d\ell_\perp}$ or $\boxed{W = \ell_\perp}$ (see p. 211-212).

<u>Then</u>: $\boxed{\vec{I}(\vec{r},t) = \int_0^W \vec{K}(\vec{r},t)\, dW = \int_0^{\ell_\perp} \vec{K}(\vec{r},t)\, d\ell_\perp}$ and $\boxed{\vec{K}(\vec{r},t) \equiv \frac{d\vec{I}(\vec{r},t)}{dW} = \frac{d\vec{I}(\vec{r},t)}{d\ell_\perp}}$

Volume Currents:

Consider an infinitely long conducting circular rod of radius R of cross-sectional area $A_\perp = \pi R^2$. For simplicity, let us assume that the volume free charge density $\rho_{free}(\vec{r},t)$ = constant , i.e. $\rho_{free}(\vec{r},t) = \rho_o^{free}\left(Coulombs/m^3\right)$ = uniform volume free charge density initially charging this conductivity rod. (n.b. this is impossible for static electric charge, but is not impossible when the electric charge is moving, en-mass!)



We again place a potential difference ΔV across the ends of the rod, and a volume current I flows in the rod.

$$\boxed{I \equiv \frac{dq_{free}}{dt} = \frac{\rho_{free} dV}{dt} = \frac{\rho_{free} A_\perp dz}{dt} = \rho_{free} A_\perp \frac{dz}{dt}} \quad \text{but:} \quad \vec{v} = \frac{dz}{dt}$$

$$\therefore \quad I = \rho_{free} AV = \rho_{\underset{free}{0}} AV \quad \text{(here)}$$

Vectorizing this: $\boxed{\vec{I}(\vec{r},t) = \rho_{free} A_\perp \vec{v}(\vec{r},t)}$ for a conducting rod of cross-sectional area $A_\perp$

If $\rho_{free} = \rho_{free}(\vec{r},t)$ then more generally: $\boxed{\vec{I}(\vec{r},t) = \rho_{free}(\vec{r},t) A_\perp \vec{v}(\vec{r},t)}$

Here again, we can define an areal current density $\vec{J}(\vec{r},t)$ as:

$$\boxed{\vec{J}(\vec{r},t) \equiv \frac{\vec{I}(\vec{r},t)}{A_\perp} = \rho_{free}(\vec{r},t)\vec{v}(\vec{r},t)} \text{ Amperes/m}^2$$

$A_\perp$ = cross-sectional area of conductor

We can also define this in differential form as:

$$\boxed{\vec{J}(\vec{r},t) \equiv \frac{d\vec{I}(\vec{r},t)}{dA_\perp}}$$

$dA_\perp$ is an infinitesimal cross-sectional area element of $A_\perp$

Then: $\boxed{I(\vec{r},t) = \int_S \vec{J}(\vec{r},t) \cdot d\vec{A}_\perp}$ where: $\boxed{d\vec{A}_\perp = \hat{n} dA_\perp}$

Since we're taking a dot product, we can drop the "$\perp$" subscript on $d\vec{A}_\perp = \hat{n} dA_\perp \rightarrow d\vec{A} = \hat{n} dA$

(Always remember / keep in mind that A (here) is the cross-sectional area of the conductor!!)

Thus:

$$\boxed{I(\vec{r},t) = \int_S \vec{J}(\vec{r},t) \cdot d\vec{A}}$$

From electric charge conservation (i.e. the empirical fact that electric charge can neither be created, nor destroyed), the total charge per unit time leaving a volume V is:

$$\boxed{\oint_S \vec{J}(\vec{r},t)\cdot d\vec{A} = \int_V \left(\vec{\nabla}\cdot\vec{J}(\vec{r},t)\right) d\tau} \quad \text{(by the divergence theorem)}$$

n.b. Closed surface integral!

Because electric charge is conserved, whatever electric charge flows out of / flows into the surface S *must* come from / go into the volume V respectively, i.e.:

$$\int_V \left(\vec{\nabla}\cdot\vec{J}(\vec{r},t)\right) d\tau = -\frac{d}{dt}\int_V \rho_{free}(\vec{r},t)\, d\tau = -\int_V \left(\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}\right) d\tau$$

e.g. a current flowing out through surface $S \rightarrow$ decrease in the charge density in volume V

Since this holds for any volume (arbitrary), then integrands on LHS = RHS:

Electric Charge Conservation:	$\vec{\nabla}\cdot\vec{J}(\vec{r},t) = -\dfrac{\partial \rho_{free}(\vec{r},t)}{\partial t}$	$\Leftarrow$ Continuity Equation

"Dictionary" for point, line, surface and volume currents:

$\sum_{i=1}^{n}(\sim)\, q_i\vec{v}_i$ $\quad\int_{line}(\sim)\,\vec{I}dl$ $\quad\int_{surface}(\sim)\,\vec{K}dA$ $\quad\int_{volume}(\sim)\,\vec{J}d\tau$

Correspondence to: $dq \sim \lambda dl$ $\quad dq \sim \sigma dA$ $\quad dq \sim \rho d\tau$

Griffiths Example 5.4

a.) A current I is uniformly distributed over a wire of radius R and circular cross-section $A = \pi R^2$. Find the volume current density $\vec{J}$.

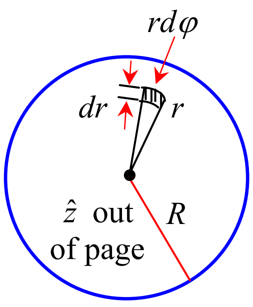
Since I is uniformly distributed over cross-sectional area A of wire $\rightarrow \vec{J}$ must also be uniform:

$$\boxed{J = \frac{I}{A} = \frac{I}{\pi R^2}}$$

b.) If $\vec{J}(\vec{r}) = kr\hat{z}$ where r = radial distance from cylindrical symmetry axis and k = constant

Thus J is not uniform/constant here. Compute I from: $I = \int_S \vec{J}(\vec{r})\cdot d\vec{A}$ where $d\vec{A} = rd\varphi dr\,\hat{z}$

End-View of Conductor:



$$I = \int_S \vec{J}(\vec{r})\cdot d\vec{A} = \int_0^{2\pi} d\varphi \int_0^R (kr\hat{z})\cdot(rdr\hat{z}) = \int_0^{2\pi} d\varphi \int_0^R kr^2 dr \quad (\hat{z}\cdot\hat{z} = 1)$$

$$= 2\pi k\int_0^R r^2 dr = \frac{2\pi}{3}kr^3\Big|_{r=0}^{r=R} = \frac{2\pi}{3}kR^3$$

MAGNETISM AND THE MAGNETIC FIELD $\vec{B}$

The Lorentz Force on a Charged Particle, $\vec{E}$ and $\vec{B}$-Fields:

When a charged particle is moving in an external magnetic field $\vec{B}$, $\exists$ (there exists) a magnetic force acting on the particle $\vec{F}_m(\vec{r})$ which (in MKSA (SI) units) is:

$\boxed{\vec{F}_m(\vec{r}) = q\,\vec{v}(\vec{r})\times\vec{B}(\vec{r})}$ where: q = electric charge of the particle

$\vec{v}(\vec{r})$ = laboratory velocity of charged particle at $\vec{r}$

$\vec{B}(\vec{r})$ = magnetic field intensity at $\vec{r}$

If the charged particle is also in an external $\vec{E}$-field, then $\exists$ an electric force acting on the particle $\vec{F}_e(\vec{r})$ which is:

$$\boxed{\vec{F}_e(\vec{r}) = q\vec{E}(\vec{r})}$$

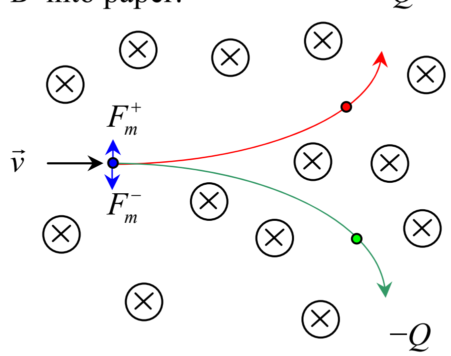
Thus the net force acting on a moving charged particle simultaneously in both an $\vec{E}$ and $\vec{B}$ field is (by the principle of linear superposition):

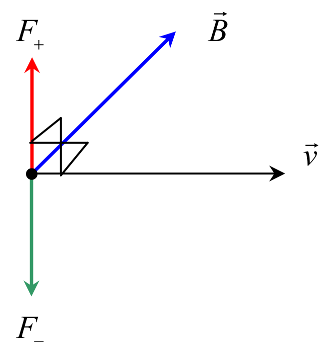
$$\boxed{\vec{F}_{TOT}(\vec{r}) = \vec{F}_e(\vec{r}) + \vec{F}_m(\vec{r}) = q\vec{E}(\vec{r}) + q\vec{v}(\vec{r})\times\vec{B}(\vec{r})} \quad \Leftarrow \quad \underline{\text{Lorentz Force}}$$

Note that $\vec{F}_e(\vec{r})$ is along $\vec{E}(\vec{r})$ (i.e. $\vec{F}_e(\vec{r}) \parallel \vec{E}(\vec{r})$) while $\vec{F}_m(\vec{r})$ is $\perp$ to $\vec{v}(\vec{r})$ and also $\vec{B}(\vec{r})$!

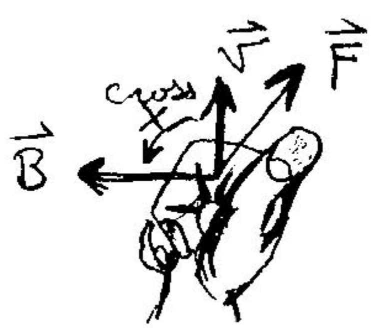
$\rightarrow$ Magnetic forces do no work, because $W_m = \int_C \vec{F}_m(\vec{r})\bullet d\vec{\ell} = 0$, since $\vec{F}_m(\vec{r})$ is $\perp$ to $d\vec{\ell}$!

$\vec{B}$ into paper:





Cross-product $\vec{v}\times\vec{B}$ of $\vec{F}_m = q\,\vec{v}\times\vec{B}$ is defined by the right-hand rule:



Curl fingers of your right hand for the cross product, right hand thumb points in the direction of $\vec{F}/q = \vec{v}\times\vec{B}$

Note that the Lorentz Force $\vec{F}_{TOT} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{v}\times\vec{B}$ is valid for <u>all</u> velocities $0 \le v \le c$ - even for <u>fully</u> relativistic particles!

Now $\vec{F}_{TOT} = q\vec{E} + q\vec{v}\times\vec{B}$ is the total force acting on a charged particle in the Lab frame, which contains <u>both</u> an $\vec{E}$-field and a $\vec{B}$-field. Note that if $\vec{v} \rightarrow 0$, then $\vec{F}_m \rightarrow 0$ also.

What does this look like to a charged particle in its <u>own</u> <u>*rest*</u> frame?

If we make a (Galilean) transformation into the rest frame of a moving charged particle, then there is <u>only</u> an electric field $\vec{E}'$ seen by the particle!

$$\boxed{\vec{E}' = \vec{E} + \vec{v}\times\vec{B}}$$

The force on the charged particle in the charged particle's own rest frame is:

$$\boxed{\vec{F}'_{TOT} = q\vec{E}' = q\vec{E} + q\vec{v}\times\vec{B} = \vec{F}_{TOT}}$$

$\Rightarrow$ Net force acting on the particle in the charged particle's own rest frame
$\equiv$ Net force acting on the particle in the lab frame!

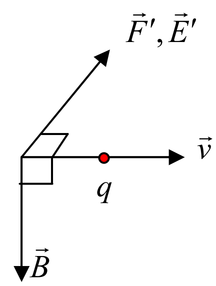
Note that this expression for $\vec{E}'$ is true <u>only</u> for non-relativistic velocities, i.e. $v/c \equiv \beta \ll 1$

i.e. $v << c = 3 \times 10^8$ m/sec

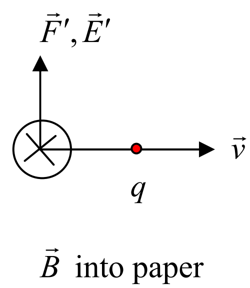
Thus, suppose a charged particle is moving in a uniform $\vec{B}$-field (No $\vec{E}$-field is present in lab)

<u>Then</u>: $\boxed{\vec{F}_{TOT} = q\vec{v}\times\vec{B}}$ and $\boxed{\vec{F}'_{TOT} = q\vec{E}' = q\vec{v}\times\vec{B}}$ thus $\boxed{\vec{E}' = \vec{v}\times\vec{B}}$ here!

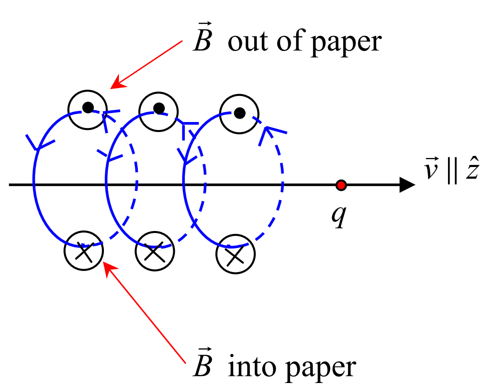
$\Rightarrow$ A charged particle moving in <u>magnetic</u> field $\vec{B}$ (only) in the lab frame "sees" an <u>electric</u> field $\vec{E}' = \vec{v}\times\vec{B}$ in its own rest frame!!



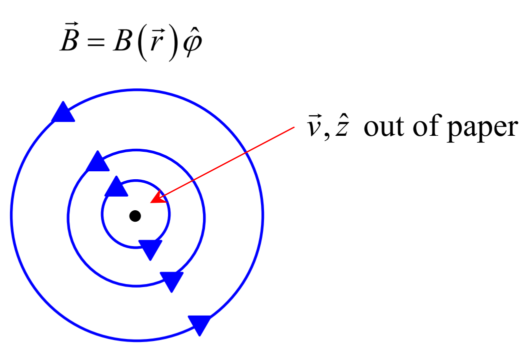
or:



<u>Conversely</u>, an electrically charged particle <u>moving</u> in the lab with velocity $\vec{v}$ <u>generates</u> a (solenoidal) magnetic field $\vec{B}$ in the lab frame! (n.b. lines of $\vec{B}$ defined by right-hand rule!):



<u>or</u>:



The macroscopic magnetic field $\vec{B}$ generated by a moving charged particle is a solenoidal magnetic field, i.e. $\vec{B} = B(\vec{r})\hat{\varphi}$!

For $v << c$ (i.e. $\beta = v/c << 1$) the $\vec{B}$-field at a point $\vec{r}$ away from a moving charged particle is:

$$\boxed{\vec{B}(\vec{r}) = \frac{1}{c^2}\left(\vec{v}\times\vec{E}(\vec{r})\right)}$$

where $\vec{E}$ (here) is the electrostatic field of charged particle as observed in its own rest frame, i.e.

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}}$$

<u>Then</u>: $\boxed{\vec{B}(\vec{r}) = \frac{q}{4\pi\varepsilon_o c^2}\vec{v}\times\left(\frac{\hat{r}}{r^2}\right)}$ <u>but</u>: $\boxed{\frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{\vec{r}}{|\vec{r}|^3} \quad \text{since } \vec{r} = |\vec{r}|\hat{r} = r\hat{r}}$

<u>Now</u>: $\boxed{c = \frac{1}{\sqrt{\varepsilon_o\mu_o}}}$ $\therefore$ $\boxed{c^2 = \frac{1}{\varepsilon_o\mu_o}}$

where:

ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m,

μ_o = magnetic permeability of free space $\equiv 4\pi\times10^{-7}$ N/Ampere^2 (= Henrys/m)

$$\therefore \quad \vec{B}(\vec{r}) = \frac{q\cancel{\varepsilon_o}\mu_o}{4\pi\cancel{\varepsilon_o}}\left(\vec{v}\times\frac{\vec{r}}{r^3}\right) \quad \text{or:} \quad \boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}\times\frac{\vec{r}}{r^3}\right) = \left(\frac{\mu_o}{4\pi}\right)\left(q\vec{v}\times\frac{\hat{r}}{r^2}\right)}$$

This is the (macroscopic) magnetic field observed in the lab frame due to a charged particle moving with velocity $\vec{v}$ with $v << c$.

Note one similarity between the macroscopic $\vec{E}$ and $\vec{B}$-fields of a point electrically charged particle:

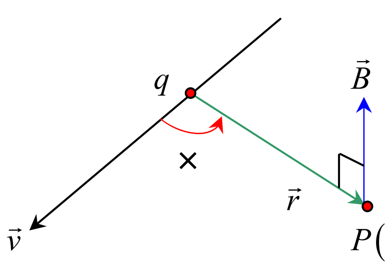
$$\vec{E}(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{q\hat{r}}{r^2}\right)$$

<u>Both</u> fields decrease as $1/r^2$ from point charge!

$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{q\vec{v}\times\hat{r}}{r^2}\right)$$

For a point charged particle moving with velocity $\vec{v}$ ($v << c$) in the lab:

$$\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left(\frac{q\vec{v}\times\hat{r}}{r^2}\right)$$



$P(\vec{r})$ = observation point

We emphasize that the macroscopic $\vec{B}$-field seen in the lab frame is generated by the motion of the electrically charged particle through space-time. Note that the $\vec{B}$-field appears only in the lab frame. In the rest frame of the charged particle, the only field it sees is its own electric field!

The phenomenon of magnetism and magnetic fields is not *solely* a property of moving electrically charged particles/is not solely a property of electromagnetism!

Moving strong, weak and/or gravitational charges will (respectively) also produce strong, weak and/or gravitational magnetic fields too!!!

Furthermore, *c*, the speed of "light" $c = 1/\sqrt{\varepsilon_o \mu_o}$ is not "just" the maximum allowable/maximum possible speed for the electromagnetic interaction, but it is also the maximum possible speed for any/all of the four known fundamental forces of nature, i.e. the *E* & *M*, strong, weak and gravitational forces. Thus "*c*" = speed of "light" is actually a misnomer, because it applies to any/all of the fundamental forces of nature! *c* is the maximum speed that anything may travel in this universe – thus the true physics origin(s) of *c* have nothing to do with electromagnetism *per se*, but everything to do with the very structure of space-time, i.e. the vacuum ("empty" space-time) itself !!! (n.b. However, the microscopic vacuum is not "empty"!!)

Since *c*, the speed of "light" is related to the macroscopic parameters ε_o and μ_o associated with the *E&M* aspects of the vacuum $c = 1/\sqrt{\varepsilon_o \mu_o}$ and *c* is the same for any/all of the fundamental forces of nature, i.e. $c_e = c_s = c_w = c_g = c = 3\times 10^8 \, m/\sec$, there must *also* be analogous macroscopic quantities to that of ε_o and μ_o for the macroscopic "electric" and "magnetic" properties of the vacuum associated with each of the other fundamental forces of nature, i.e.

$$c \equiv \left\{ c_e = \underbrace{\frac{1}{\sqrt{\varepsilon_o \mu_o}}}_{E\&M} \right\} \equiv \left\{ c_s = \underbrace{\frac{1}{\sqrt{\varepsilon_s \mu_s}}}_{\substack{strong\\ force}} \right\} \equiv \left\{ c_w = \underbrace{\frac{1}{\sqrt{\varepsilon_w \mu_w}}}_{\substack{weak\\ force}} \right\} \equiv \left\{ c_g = \underbrace{\frac{1}{\sqrt{\varepsilon_g \mu_g}}}_{gravity} \right\}$$

Note that: $\varepsilon_o \neq \varepsilon_s \neq \varepsilon_w \neq \varepsilon_g \leftarrow$ "electric" permittivities not necessarily equal/identical

$\mu_o \neq \mu_s \neq \mu_w \neq \mu_g \leftarrow$ "magnetic" permeabilities not necessarily equal/identical

Thus, we from this perspective, we can see that e.g. for the *E&M* force, the macroscopic $\vec{B}$-field associated with an electrically charged particle moving through space-time is associated with the response of the vacuum (i.e. space-time and its structure - at the microscopic level) to the passage of the electrically charged particle through space-time! This says something very deep about the fundamental nature of our universe!

Just as electromagnetism (*E&M*) has electric and magnetic fields, so do the strong, weak and gravitational interactions also have "electric" and "magnetic" fields too!

For the weak interactions (responsible for radioactivity and β-decay), there exist "weak" charges, and there is "weak" electricity and "weak" magnetism – i.e. static "weak" electric field(s) associated with the "weak" charge(s) and "weak" magnetic field(s) associated with moving 'weak" charge(s)!

For QCD (Quantum Chromo-Dynamics) – (i.e. the strong interactions / nuclear forces) there exist strong charges with associated so-called "chromo-electric" fields and "chromo-magnetic" fields!

For gravity, gravitational charge = mass! There exist gravito-electric and gravito-magnetic fields. The "every-day" gravitational force we experience living on the surface of our own planet is due to the gravito-electric field of the Earth! The tides on the earth are due (primarily) to the gravito-electric field of the moon!

Thus it can be seen that "electric charge", "electric" and "magnetic" fields are not *solely* the "property" of electromagnetism; indeed these are fundamental aspects/properties of any/all/each of the four known forces of nature!!!

Macroscopic "Electric" fields are associated with the "charges" of each fundamental force. Macroscopic "Magnetic" fields are produced when a "charge" (of any kind) moves through space-time!

This undergraduate physics course is devoted to studying the myriad phenomena associated with just one of the four fundamental forces known to exist (today) in nature (n.b. are there more??) What we teach you in this class is relevant (in many ways) to all forces of nature!!!

The (static) electric field associated with a point electric charge q, in its own rest frame, at the microscopic level, is comprised of (large numbers of) virtual photons (= quanta of the electromagnetic field).

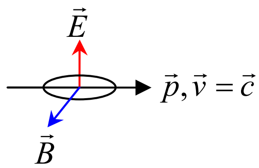
Virtual photons carry momentum $p = h/\lambda$ where h = Plank's constant = 6.626×10^{-34} Joule-sec and λ = (DeBroglie) wavelength of the virtual photon.

While virtual photons also carry ***kinetic*** energy (like $p^2/2mc$ {non-relativistically}), they have zero ***total*** energy, since the (total energy)^2 is $E^2 = p^2c^2 + m^2c^4 = 0$; thus a ***complex*** relation exists between the momentum p and mass m of virtual photons: $pc = imc^2$ (where $i = \sqrt{-1}$), and $E = 0$ also implies that virtual photons have no vibrational/oscillatory frequency f associated with them, since $E = hf = 0$ for virtual photons – i.e. for virtual photons, the relation $c = f\lambda$ does not exist (whereas this relation does exist/holds for real photons). Note that $f = 0$ for virtual photons does in face make physical sense, since zero-frequency virtual photons are associated with the ***static*** macroscopic electric field $\vec{E}(\vec{r})$ of a point electric charge, q.

Both real and virtual photons have associated with them electric and magnetic field vectors, which are orthogonal (i.e. perpendicular) to each other:

Real Photons:

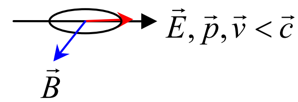
$\vec{E} \perp \vec{B} \perp (\vec{p}, \vec{v} = \vec{c})$



Virtual Photons:

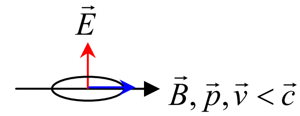
Static $\vec{E}$-field:

$\vec{B} \perp (\vec{E} \parallel \vec{p}, \vec{v} = \vec{c})$

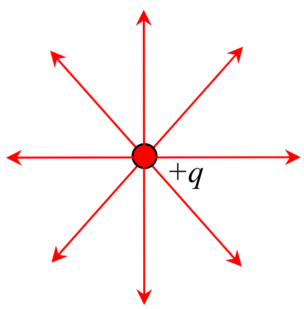


Static $\vec{B}$-field:

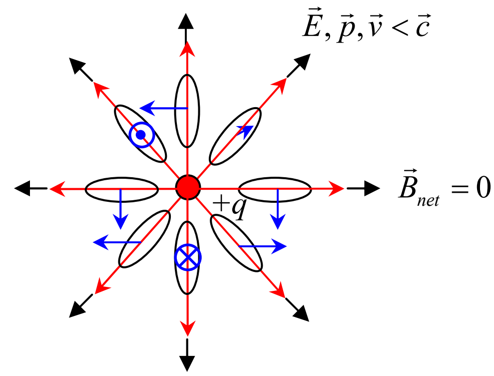
$\vec{E} \perp (\vec{B} \parallel \vec{p}, \vec{v} = \vec{c})$



Static Radial Macroscopic $\vec{E}$-Field of a "classical" point electric charge, q:



"Static" "Radial" Microscopic $\vec{E}$-Field of a "classical" point electric charge, q:



Microscopically, averaging over statistically significant numbers of virtual photons, (with manifest quantum mechanical/wave-like behavior – {n.b. individual photons do not follow/obey classical particle trajectories!}) a "classical" point electric charge q in its own rest frame has:

A radial electric field: $\vec{E}(\vec{r}) = E(\vec{r})\hat{r}$

A radial (inward/outward) momentum field: $\vec{p}(\vec{r}) = p(\vec{r})\hat{r}$ (n.b. $\vec{p}_{net} = 0$)

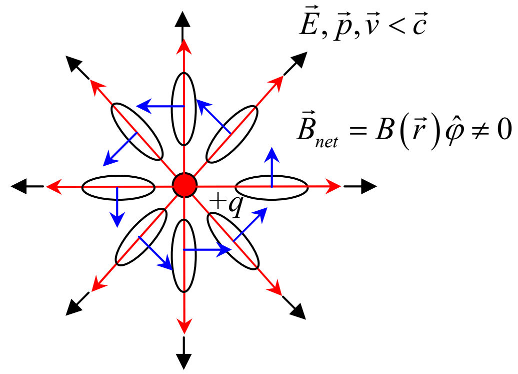
Depends on sign of electric charge, q

A null magnetic field ($\vec{B}_{net} = 0$, virtual photon $\vec{B}$-field vectors statistically cancel each other out). {n.b. A *real* electron has a (point) intrinsic magnetic dipole moment $\mu = e\hbar / m_e c$ (relativistic effect!) and corresponding non-zero macroscopic (and thus microscopic) magnetic dipole field!}

Note that, as with any statistical average, that on short time scales there are moment-to-moment fluctuations on *all* of these quantities!

For a moving "classical" point electric charge (i.e. neglect/ignore the intrinsic point magnetic dipole moment of real electron), a net macroscopic magnetic field $\vec{B}$ (= statistical average over microscopic virtual photons) is observed in the lab frame (but not in the rest frame of the point electric charge). The relative motion of the electric charge and observer breaks the rotational invariance associated with the electric charge at rest!

End view – point charge coming toward reader:



The macroscopic $\vec{B}$-field associated with a moving point electric charge q arises from Einstein's relativity & the fundamental nature/fundamental aspects of space-time itself!

The Macroscopic Magnetic "Induction" / Magnetic Intensity / Magnetic Field, $\vec{B}$

Units of $\vec{B}$ in (*SI*) MKSA system: $\boxed{\dfrac{N-\text{sec}}{Coulomb-meter} = \dfrac{N}{Ampere\ meter} \equiv Teslas = Weber/m^2}$

$\boxed{\text{1 Tesla} = 10^4 \text{ Gauss ("old" cgs units of } B)}$

The macroscopic magnetic induction $\vec{B}$ is defined in terms of the force acting on a test charge Q_T moving with velocity $\vec{v}$ from $\boxed{\vec{F}_m(\vec{r}) = Q_T \vec{v}(\vec{r}) \times \vec{B}(\vec{r})}$.

A Weber is the (*SI*) MKSA unit of magnetic flux, Φ_m:

$\boxed{\Phi_m \equiv \int_S \vec{B} \cdot d\vec{A}}$ - counts $\vec{B}$-field lines passing through a surface *S*.

(SI units: Weber = Volt-sec = Tesla-m^2)

Magnetic flux is defined analogous to that for electric flux:

$\boxed{\Phi_e \equiv \int_S \vec{E} \cdot d\vec{A}}$ - counts $\vec{E}$-field lines passing through a surface *S*.

(SI units: (Volts/m)*m^2 = Volt-meters)

If the surface *S* is a planar area *A*, then the magnetic flux $\Phi_m = \vec{B} \cdot \vec{A} \leftarrow A$ = area through which lines of $\vec{B}$ pass through.

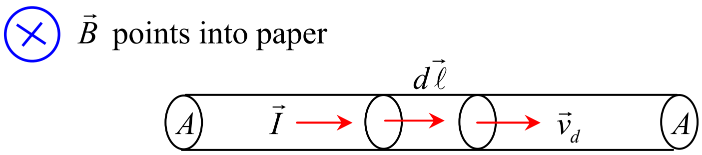
Note that:

1Weber = 1 Volt-sec.

∴ 1 Tesla = 1 Newton/Ampere-meter = 1 Weber/m^2 = 1 Volt-sec / m^2

Macroscopic Magnetic Force Acting on a Wire Carrying a Steady Current *I*:

Consider a filamentary wire carrying steady current I immersed in a magnetic field, $\vec{B}$:



$d\vec{\ell}$ is (infinitesimal) line segment of wire defined to be ∥ to direction of (conventional) current flow, $\vec{I}$.

Section of conducting wire

<u>By convention (blame goes to none other than Benjamin Franklin!)</u>:
The drift velocity $\vec{v}_d$ of <u>+ve</u> electric charge carriers in a conductor gives the <u>direction</u> of flow of current in the wire:

$$\boxed{\vec{v}_d \parallel d\vec{\ell} \parallel \vec{I}}$$

{In a <u>real</u> wire, electrons ($q = e^-$) comprise current. The e^- flow in the direction <u>opposite</u> to that of conventional current. The electron was discovered in 1897 by J.J. Thompson, long after Benjamin Franklin's time!)

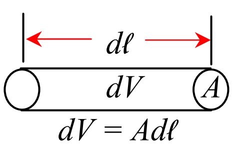
The elemental magnetic force $d\vec{F}_m$ acting on a infinitesimal line segment $d\vec{\ell}$ of current-carrying wire in the presence of an external, macroscopic $\vec{B}$ -field is:

$$\boxed{d\vec{F}_m = dQ_{TOT}\vec{v}_d \times \vec{B}}$$

Where dQ_{TOT} = total free charge contained within infinitesimal volume $dV = Ad\ell$ associated with infinitesimal line segment $d\vec{\ell}$ of wire of cross sectional area A containing electric charge carriers moving with a drift velocity $\vec{v}_d$ in/along the wire.

What is dQ_{TOT}?? $\boxed{dQ_{TOT} = qndV = qn(Ad\ell)}$

<u>Line segment</u>:



q = charge of individual carriers
n = # charge carriers / unit volume (= # density, #/m^3)
dV = elemental volume containing Q_{TOT} charge
A = cross-sectional area of wire
dl = elemental length of infinitesimal line segment of wire

$$\boxed{d\vec{F}_m = dQ_{TOT}\left(\vec{v}_d \times \vec{B}\right) = qndV\left(\vec{v}_d \times \vec{B}\right) = qn\left(Ad\ell\right)\left(\vec{v}_d \times \vec{B}\right) = nq\left(Ad\ell\right)\left(\vec{v}_d \times \vec{B}\right)}$$

Now $d\vec{\ell} \parallel \vec{v}_d$, so $d\vec{\ell} \times \vec{B}$ points in the <u>same</u> direction as $\vec{v}_d \times \vec{B}$, i.e.:

$$\left|\vec{v}_d\right|\left(d\vec{\ell} \times \vec{B}\right) = \left|d\vec{\ell}\right|\left(\vec{v}_d \times \vec{B}\right)$$

$$= v_d\left(d\vec{\ell} \times \vec{B}\right) = d\ell\left(\vec{v}_d \times \vec{B}\right) \quad \text{because} \quad v_d = \left|\vec{v}_d\right| \quad \text{and} \quad d\ell = \left|d\vec{\ell}\right|$$

$$\therefore \quad \boxed{d\vec{F}_m = nqAv_d\left(d\vec{\ell} \times \vec{B}\right)}$$

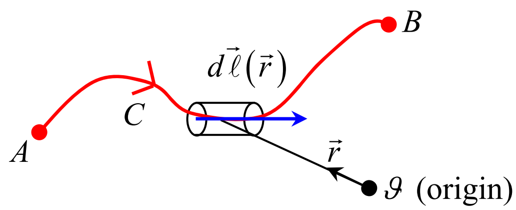
What is $nqAv_d$?? $\boxed{I_{TOT} = \frac{dQ_{TOT}}{dt} = \frac{\left(nqA\left|d\vec{\ell}\right|\right)}{dt} = nqA\frac{\left|d\vec{\ell}\right|}{dt} = nqA\left|\vec{v}_d\right| = nqAv_d}$!!!

i.e. $\boxed{\frac{\left|d\vec{\ell}\right|}{dt} = \left|\vec{v}_d\right| = v_d}$ $\therefore$ $\boxed{I = nqA\left|\vec{v}_d\right| = nqAv_d}$

$\therefore$ $\boxed{d\vec{F}_m = Id\vec{\ell}\times\vec{B}}$ = Elemental magnetic force on infinitesimal line segment $d\vec{\ell}$ of wire carrying a (steady) electric current I in an external magnetic field $\vec{B}$.

The net magnetic force acting on a wire carrying a <u>steady</u> (i.e. constant) electric current I in an external magnetic field, $\vec{B}$ is obtained by summing up all the infinitesimal force contributions $d\vec{F}_m$ along the length of the wire, i.e. integrating:

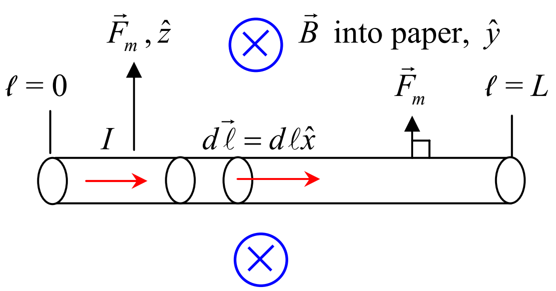
$$\boxed{\vec{F}_m = \int d\vec{F}_m = \int_C Id\vec{\ell}(\vec{r})\times\vec{B}(\vec{r}) = I\int_C d\vec{\ell}(\vec{r})\times\vec{B}(\vec{r})}$$

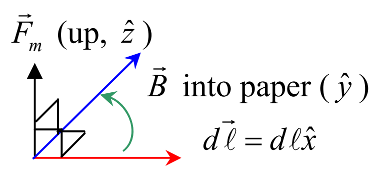


The net force acting on a straight wire carrying steady current I in external field $\vec{B}$:

$$\vec{F}_m = \int d\vec{F}_m = I\int_C d\vec{\ell}(\vec{r})\times\vec{B}(\vec{r}) = I\int_{\ell=0}^{\ell=L} d\vec{\ell}(\vec{r})\times\vec{B}(\vec{r})$$

Let: $d\vec{\ell}(\vec{r}) = d\vec{\ell} = d\ell\hat{x}$ and $\vec{B}(\vec{r}) = B_o\hat{y}$ (uniform)



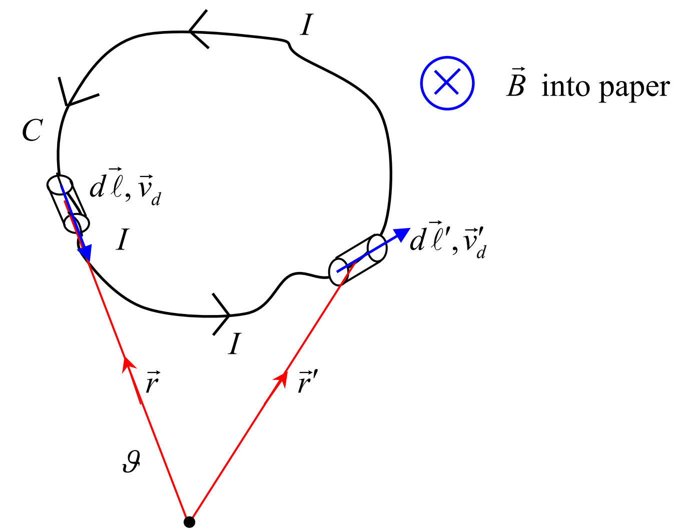


$$d\vec{\ell}\times\vec{B} = (d\ell\hat{x})\times(B_o\hat{y}) = d\ell B_o\underbrace{\hat{x}\times\hat{y}}_{=\hat{z}} = d\ell B_o\hat{z} \text{ (i.e. up)}$$

Thus the net magnetic force acting on a straight wire of length L carrying steady current I in the $+\hat{x}$ direction, immersed in a uniform field $\vec{B} = B_o\hat{y}$ is:

$$\boxed{\vec{F}_m = IB_o\hat{z}\int_0^L dx = IB_oL\hat{z}}$$

Suppose the current-carrying wire is not straight, but instead e.g. is a closed current loop:



Then (here): $\vec{F}_m = \oint_C I d\vec{\ell}(\vec{r}) \times \vec{B}(\vec{r})$

In general, $\vec{B}(\vec{r})$ may not be uniform, but is a function of position, $\vec{r}$

i.e. in general: $\vec{B} = \vec{B}(\vec{r})$

Note that in general, $d\vec{\ell} = d\vec{\ell}(\vec{r})$ too!

$\vec{F}_m = \oint_C I d\vec{\ell}(\vec{r}) \times \vec{B}(\vec{r})$ for a closed loop

In general, both $d\vec{\ell}(\vec{r})$ and $\vec{B}(\vec{r})$ can/will be functions of position

However, suppose $\vec{B}$ is uniform, e.g. $\vec{B} = B_o \hat{z}$

Then: $\vec{F}_m = \underbrace{I}_{\text{constant}} \left\{ \oint_C d\vec{\ell}(\vec{r}) \right\} \times \underbrace{B_o \hat{z}}_{\text{constant}}$

But: $\oint_C d\vec{\ell}(\vec{r}) \equiv 0$!!!

$\therefore$ $\boxed{\vec{F}_m = \oint_C I d\vec{\ell}(\vec{r}) \times \vec{B}(\vec{r}) \equiv 0}$ for uniform/constant $\vec{B}$ (Not true for non-uniform $\vec{B} = \vec{B}(\vec{r})$!)

The Macroscopic Magnetic Torque Acting on a Circuit

The infinitesimal (or elemental) magnetic torque $d\vec{\tau}_m$ acting on an infinitesimal line segment $d\vec{\ell}$ carrying a steady current I:

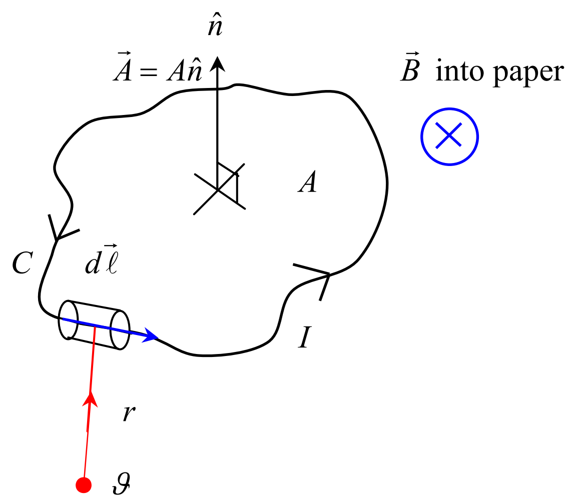
$$\boxed{d\vec{\tau}_m = \vec{r} \times d\vec{F}_m = \vec{r} \times \left(I d\vec{\ell} \times \vec{B} \right)}$$

Total torque: $\boxed{\vec{\tau}_m = \int_C d\vec{\tau}_m}$

Total torque for a current-carrying wire: $\boxed{\vec{\tau}_m = I \int_C \vec{r} \times \left(d\vec{\ell} \times \vec{B} \right)}$

For a closed steady current loop/closed steady current-carrying circuit: $\boxed{\vec{\tau}_m = I \oint_C \vec{r} \times \left(d\vec{\ell} \times \vec{B} \right)}$

Define $\vec{A} = A\hat{n}$ = vectorial area enclosed by loop (defined by contour C), and:
Define $\hat{n}$ = unit normal vector, n.b. direction is defined by right-hand rule of taking contour C.



Thus: $\vec{\tau}_m = I \oint_C \underbrace{\vec{r} \times \left(d\vec{\ell} \right.}_{\text{n.b. Units of area}} \left. \times \vec{B} \right) = I \left(\vec{A} \times \vec{B} \right) \Leftarrow \{ \text{if } \vec{B} \neq \vec{B}(\vec{r}) \}$

Components of torque, $\vec{\tau}_m$:

$$\vec{\tau}_{m_x} = I \oint_C \left[\vec{r} \times \left(d\vec{\ell} \times \vec{B} \right) \right]_x = I \overbrace{\left(A_y B_z - A_z B_y \right)}^{\text{If } \vec{B} \text{ is uniform}}$$

$$\vec{\tau}_{m_y} = I \oint_C \left[\vec{r} \times \left(d\vec{\ell} \times \vec{B} \right) \right]_y = I \left(A_z B_x - A_x B_z \right)$$

$$\vec{\tau}_{m_z} = I \oint_C \left[\vec{r} \times \left(d\vec{\ell} \times \vec{B} \right) \right]_z = I \left(A_x B_y - A_y B_x \right)$$

Since:

$$\left[\vec{r}\times\left(d\vec{\ell}\times\vec{B}\right)\right]_x = y\left(dxB_y - dyB_x\right) - z\left(dzB_x - dxB_z\right) = ydxB_y - ydyB_x - zdzB_x + zdxB_z$$

$$\left[\vec{r}\times\left(d\vec{\ell}\times\vec{B}\right)\right]_y = z\left(dyB_z - dzB_y\right) - x\left(dxB_y - dyB_x\right) = zdyB_z - zdzB_y - xdxB_y + xdyB_x$$

$$\left[\vec{r}\times\left(d\vec{\ell}\times\vec{B}\right)\right]_z = x\left(dzB_x - dxB_z\right) - y\left(dyB_z - dzB_y\right) = xdzB_x - xdxB_z - ydyB_z + ydzB_y$$

$A_x = \vec{A}\cdot\hat{x},\;\; A_y = \vec{A}\cdot\hat{y},\;\; A_z = \vec{A}\cdot\hat{z}$

$\vec{A}$ is an areal vector whose x, y, z-components are the areas A_x, A_y, A_z enclosed by C and projected onto the y-z, z-x, and x-y planes, respectively.

Thus for a uniform $\vec{B}$ field: $\boxed{\vec{\tau}_m = I\left(\vec{A}\times\vec{B}\right)}$

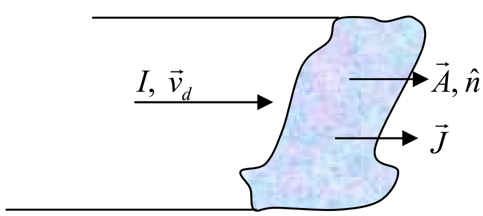
Define: $\boxed{\vec{m} \equiv I\vec{A} = IA\hat{n}}$ = magnetic dipole moment associated with a current loop of area A. (SI Units: Ampere-meters^2)

It can be shown geometrically that: $\boxed{2\vec{A} = \oint_C \vec{r}\times d\vec{\ell}}$ or: $\boxed{\vec{A} = \frac{1}{2}\oint_C \vec{r}\times d\vec{\ell}}$

Thus: $\boxed{\vec{m} \equiv I\vec{A} = \frac{1}{2}I\oint_C \vec{r}\times d\vec{\ell}}$

If an electric current I is not confined to a zero-diameter/filamentary wire, but instead is associated with an extended medium, then define current density $\vec{J}$ (Amps/m^2) such that:

$$I = \vec{J}\cdot\vec{A}$$
$$Id\vec{\ell} = \left(\vec{J}\cdot\vec{A}\right)d\vec{\ell} = \vec{J}\left(Ad\ell\right) = \vec{J}dV$$



$I,\ \vec{J},\ \vec{v}_d,\ d\vec{\ell},\ \vec{A}$ all parallel

Hence, we see that:

$$\boxed{\vec{m} \equiv I\vec{A} = \frac{1}{2}I\oint_C \vec{r}\times d\vec{\ell} = \frac{1}{2}\oint_C \vec{r}\times\left(Id\vec{\ell}\right) = \frac{1}{2}\int_V \vec{r}\times\left(\vec{J}dV\right)}$$

Thus, an infinitesimal volume element dV of current-carrying conductor has associated with it an infinitesimal magnetic dipole moment of:

$\boxed{d\vec{m} = \frac{1}{2}\vec{r}\times\vec{J}dV}$ and thus: $\boxed{\vec{m} = \int d\vec{m} = \frac{1}{2}\int_V \vec{r}\times\vec{J}dV}$

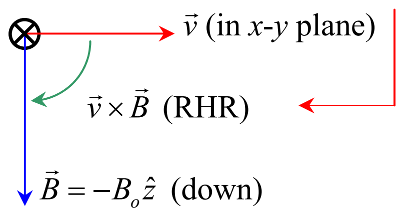
{This result will be very useful for discussing magnetic phenomena in the future…}

Griffiths Example 5.1: Cyclotron Motion

Consider an electrically charged particle with charge Q moving in a uniform magnetic field $\vec{B} = -B_o\hat{z}$ (down) with velocity $\vec{v} \neq v_z\hat{z}$ (i.e. has velocity components only in *x*-*y* plane).

Magnetic (i.e. Lorentz) force acting on the moving charged particle is $\vec{F}_m = Q\vec{v}\times\vec{B}$

Side View: $\vec{F}_m$ (into page) (by the Right-Hand Rule)



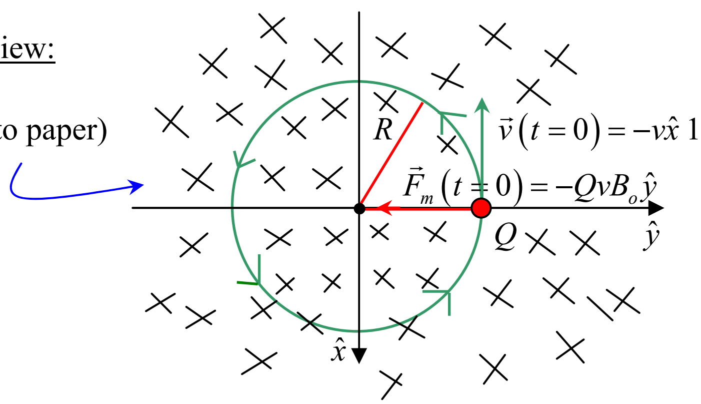
$\vec{F}_m = \left|\vec{F}_m\right|(-\hat{r})$ {i.e. $\vec{F}_m$ is radially inward} $\quad \vec{v}\,(\text{in } x-y \text{ plane}) \perp \vec{B} = -B_o\hat{z} \;\Rightarrow\; \vec{v}\times\vec{B} = vB(-\hat{r})$

$$\left|\vec{F}_m\right| = QvB = ma = m\frac{v^2}{R}$$

Magnetic force provides centripetal (i.e. radial inward) acceleration – bends particle around in a circle of radius R:

Top View:

($\vec{B}$ into paper)



Initially, suppose $\vec{v}(t=0) = -v\hat{x}$.

Then: $\vec{F}_m(t=0) = Q\vec{v}(t=0)\times\vec{B} = Q(-v\hat{x})\times(-B_o\hat{z}) = +QvB_o\left(\underbrace{\hat{x}\times\hat{z}}_{=-\hat{y}}\right) = -QvB_o\hat{y}$

As time increases from $t = 0$, can explicitly show that the orbit of charged particle in this *B*-field lies on circle of radius R; motion is CCW around circle.

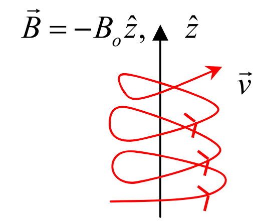
For Non-Relativisitic Motion:

Momentum $\vec{p} = m\vec{v}$ and $p = |\vec{p}| = mv$, $v = |\vec{v}|$. Since $\left|\vec{F}_m\right| = QvB = mv^2/R$ then: $QB = mv/R$

or: $\boxed{p = mv = QBR}$

If measure the radius of curvature of a charged particle (and +/− sign of curvature!) then know:
a.) momentum p of the charged particle and b.) charge-sign of the charged particle
$\Rightarrow$ Very important "tool" for use in particle / high energy / nuclear physics experiments!!

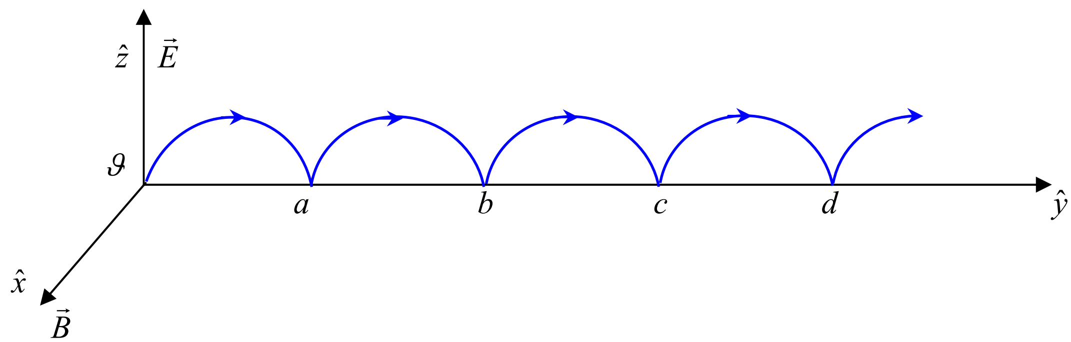
If the charged particle's velocity vector $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ has a component parallel to $\vec{B}\left(= B_o\hat{z}\right)$ (i.e. $v_z \neq 0$) then this z-component of the motion (here) is <u>unaffected</u> by $\vec{B}$ because $\vec{F}_m = Q\vec{v}\times\vec{B} = 0$ for $\vec{v} \parallel \vec{B}$ - then charged particle moves in a helix / spiral in $\hat{z}$ as shown in the figure on the right:



<u>Griffiths Example 5.2: Cycloid Motion</u>

Suppose we now include an electric field $\vec{E}$ so that $\exists$ a force $\vec{F}_e = Q\vec{E}$ on the charged particle (charge Q) in addition to the magnetic/Lorentz force, $\vec{F}_m = Q\vec{v}\times\vec{B}$

We orient uniform $\vec{E}$ in the $\hat{z}$ direction, i.e. $E = E_o\hat{z}$
We orient uniform $\vec{B}$ in the $\hat{x}$ direction, i.e. $\vec{B} = B_o\hat{x}$ } $\vec{E} \perp \vec{B}$ as shown in the figure below:



The charged particle is released <u>from rest</u> (i.e. $v\left(t=0\right)=0$) at the origin ϑ. Initially the $\vec{E}$-field accelerates it due to $\vec{F}_e = Q\vec{E} = m\vec{a}$. However as soon as it acquires a finite velocity the $\vec{B}$-field bends it, due to $\vec{F}_m = Q\vec{v}\times\vec{B}$. Initially, for time t just after $t = 0$, the electric field gives the charged particle a $\vec{v}_{init} = v_z$ and then $\left(\vec{v}_z\times\vec{B}\right)\Rightarrow\hat{y}$ direction. As the charged particle curves over to the $\hat{y}$ direction, the charged particle begins to lose speed (kinetic energy) as it curves over more. Its speed / kinetic energy actually goes to <u>zero</u> when it touches the $\hat{y}$ axis at point $y = a$. Then the process starts all over again . . . the process repeats over and over….

Note that there is <u>no</u> force acting on the charged particle in the $\hat{x}$ direction – only in the $\hat{z}$ direction (due to $\vec{E}$ & $\vec{B}$) and the $\hat{y}$ direction (due to $\vec{B}$ {only}).

$$\therefore \quad \vec{v}(t) = v_y(t)\hat{y} + v_z(t)\hat{z} = \left(0, v_y(t), v_z(t)\right) = \left(0, \dot{y}(t), \dot{z}(t)\right)$$

$$\{\text{n.b. } \dot{y}(t) = \frac{\partial y}{\partial t}(t) \text{ and } \dot{z}(t) = \frac{\partial z}{\partial t}(t)\}$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B_o & 0 & 0 \end{vmatrix} = B_o\dot{z}\hat{y} - B_o\dot{y}\hat{z}$$

Newton's Second Law: $\vec{F} = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$

$F = Q\vec{E} + Qv \times \vec{B} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = m\vec{a}$ with $E = E_o\hat{z}$ and $\vec{B} = B_o\hat{x}$

Then:

$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = Q\left(E_o\hat{z} + B_o\dot{z}\hat{y} - B_o\dot{y}\hat{z}\right) = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$

Treating the $\hat{y}$ and $\hat{z}$ components separately (since they are independent):

$QB_o\dot{z} = m\ddot{y}$ and $QE_o - QB_o\dot{y} = m\ddot{z}$

We define the cyclotron *angular* frequency as: $\boxed{\omega_c \equiv \frac{QB_o}{m}\left(radians/\sec\right)}$

Define the cyclotron frequency as: $\boxed{f_c = \omega_c/2\pi\left(Hz\right)}$ $\boxed{\omega_c = 2\pi f_c}$

Then: $\boxed{\ddot{y} = \omega_c\dot{z}}$ and $\boxed{\ddot{z} = \omega_c\left(\frac{E_o}{B_o} - \dot{y}\right)}$ $\Leftarrow$ coupled differential equations!

The general solutions to these coupled diff eq's are:

$$\boxed{y(t) = C_1\cos(\omega_c t) + C_2\sin(\omega_c t) + \left(\frac{E_o}{B_o}\right)t + C_3}$$

$$\boxed{z(t) = C_2\cos(\omega_c t) - C_1\sin(\omega_c t) + C_4}$$

The charged particle started from rest $\left(\dot{y}(0) = \dot{z}(0) = 0\right)$ at $t = 0$ at the origin ϑ ($y(0) = z(0) = 0$)

These are the four boundary conditions which determine / define constants C_1, C_2, C_3, and C_4:

$$\boxed{y(t) = \frac{E_o}{\omega_c B_o}\left(\omega_c t - \sin\omega_c t\right)} \quad \text{and} \quad \boxed{z(t) = \frac{E_o}{\omega_c B_o}\left(1 - \cos\omega_c t\right)}$$

Define: $\boxed{R \equiv \frac{E_o}{\omega_c B_o}}$

Then: $\boxed{y(t) = R\left(\omega_c t - \sin\omega_c t\right)}$ and $\boxed{z(t) = R\left(1 - \cos\omega_c t\right)}$

Then using the trigonometric identity $\sin^2\omega_c t + \cos^2\omega_c t = 1$, we obtain:

$\boxed{\left(y - R\omega_c t\right)^2 + \left(z - R\right)^2 = R^2}$ $\Leftarrow$ equation of a circle whose center is $(x, y, z) = \left(0, R\omega_c t, R\right)$

The circle travels in the $\hat{y}$-direction at constant speed $\boxed{v_y = \omega_c R = \frac{E_o}{B_o}}$

The motion of the charged particle is such that it is analogous to a point on the rim of a bicycle wheel rolling down the $\hat{y}$-axis at constant speed $\boxed{v_y = \omega_c R = \frac{E_o}{B_o}}$

This curve is called a cycloid of motion – the overall motion is not in the direction of $\vec{E}$ but actually perpendicular to it (because $\vec{B} \perp \vec{E}$ here)!!!