LECTURE NOTES 11

BOUNDARY VALUE PROBLEMS WITH "CLASS-*A*"/LINEAR DIELECTRICS

In an "ideal", linear, homogeneous, isotropic (≡ "Class-*A*") dielectric, we showed (in P435 Lecture Notes 10, page 21) that the bound volume charge density ρ_{bound} is proportional to the free charge volume density ρ_{free}:

$$\rho_{Bound}(\vec{r}) = -\vec{\nabla}\cdot\vec{P}(\vec{r}) = -\vec{\nabla}\cdot\left(\frac{K_e-1}{K_e}\right)\vec{D}(\vec{r}) = -\left(\frac{K_e-1}{K_e}\right)\vec{\nabla}\cdot\vec{D}(\vec{r}) = -\left(\frac{K_e-1}{K_e}\right)\rho_{free}(\vec{r})$$

$$\boxed{\rho_{Bound}(\vec{r}) = -\left(\frac{K_e-1}{K_e}\right)\rho_{free}(\vec{r}) = -\left(1-\frac{1}{K_e}\right)\rho_{free}(\vec{r})}$$

See p. 21 of P435 Lecture Notes 10

But: $K_e \equiv \dfrac{\varepsilon}{\varepsilon_o} = (1+\chi_e)$ $(= "\varepsilon_r"$ in Griffiths book)

Thus, since: $\chi_e = (K_e - 1)$ or: $\chi_e + 1 = K_e$ then we can also write this relation as:

$$\boxed{\rho_{Bound}(\vec{r}) = -\left(\frac{\chi_e}{\chi_e+1}\right)\rho_{free}(\vec{r}) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_{free}(\vec{r})}$$

{ Again, we repeat here that we do *not* necessarily have a corresponding unique/universal relationship between $\sigma_{Bound}(\vec{r})$ and $\sigma_{free}(\vec{r})$ at the boundaries of a dielectric. }

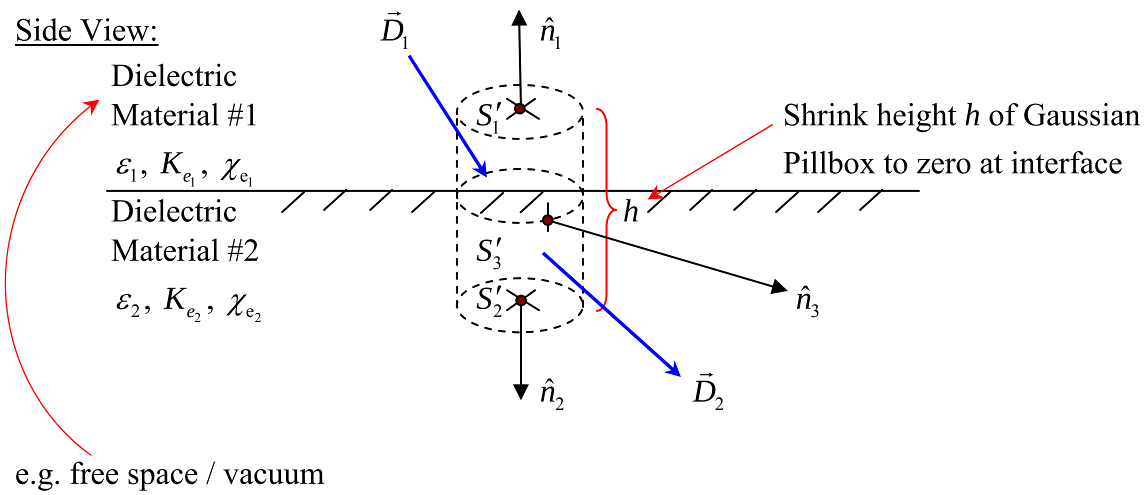
The relationship between $\rho_{Bound}(\vec{r})$ and $\rho_{free}(\vec{r})$ says that unless free charge is actually embedded within the dielectric material (somewhere), then if $\rho_{free}(\vec{r}) = 0 \rightarrow \rho_{Bound}(\vec{r}) = 0$ also, and hence any net bound charge associated with a dielectric *must* reside on the surface(s) of the dielectric. If $\rho_{free}(\vec{r}) = 0$ within the dielectric, then $\rho_{Bound}(\vec{r}) = 0$ within the dielectric and hence the potential $V(\vec{r})$ within the dielectric *must* satisfy Laplace's equation within the Class-*A* dielectric, i.e.

$$\nabla^2 V(\vec{r}) = 0$$

Thus, inside a Class-*A*/linear dielectric, *if* $\rho_{bound}(\vec{r}) = \rho_{free}(\vec{r}) = 0$ *then* Laplace's equation $\nabla^2 V(\vec{r}) = 0$ *is* valid inside the Class-*A*/linear dielectric, and *thus* we can use any/all techniques available to us to solve Laplace's equation in this region, which together with the boundary conditions, will enable us to determine everything we wish to know!

What happens at the boundary / interface between <u>*two*</u> Class-*A* dielectrics?

<u>Side View:</u>



We have the boundary condition (derived from Gauss' Law $\oint_{S'} \vec{D}(\vec{r}) \cdot d\vec{A} = Q_{free}^{enclosed}$)

$\boxed{D_{2n}(\vec{r})\big|_{interface} - D_{1n}(\vec{r})\big|_{interface} = \sigma_{free}}$ at the interface of the two dielectrics.

However, for Class-A/linear dielectrics (only):

$$\boxed{\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r})} \quad \Rightarrow \quad \boxed{\vec{D}_1(\vec{r}) = \varepsilon_1 \vec{E}_1(\vec{r}) \text{ and } \vec{D}_2(\vec{r}) = \varepsilon_2 \vec{E}_2(\vec{r})}$$

Thus we see that: $\boxed{\varepsilon_2 E_{2n}(\vec{r})\big|_{interface} - \varepsilon_1 E_{1n}(\vec{r})\big|_{interface} = \sigma_{free}}$ at the interface of the two dielectrics.

In terms of the potential, since: $\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$, then:

$\boxed{\varepsilon_2 \frac{\partial V_2(\vec{r})}{\partial n}\bigg|_{interface} - \varepsilon_1 \frac{\partial V_1(\vec{r})}{\partial n}\bigg|_{interface} = -\sigma_{free}}$ at the interface of the two dielectrics,

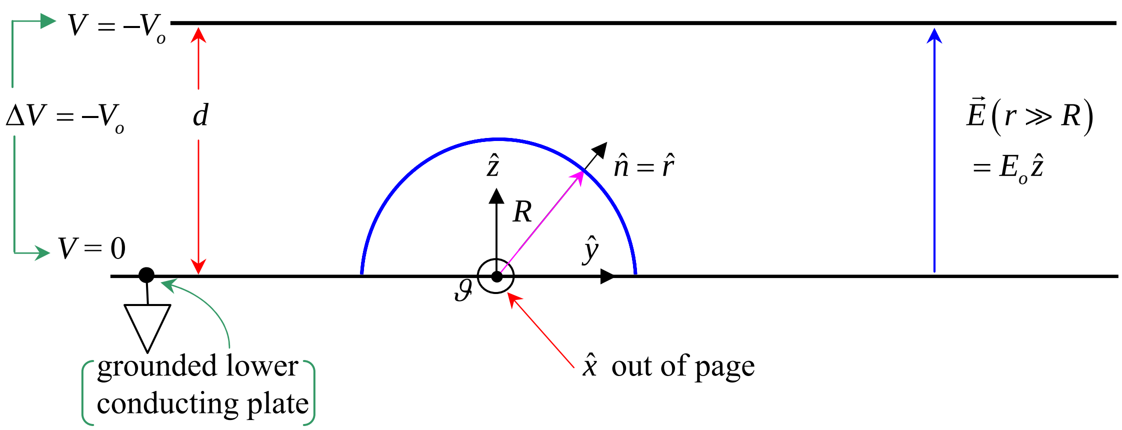
where $\frac{\partial V(\vec{r})}{\partial n}\bigg|_{interface}$ is the <u>normal</u> (i.e. $\perp$) derivative of the potential, taken <u>at/on</u> the interface.

Note (again) that the potential $V(\vec{r})$ itself is <u>continuous</u> on/at/crossing the interface:

$$\boxed{V_1(\vec{r})\big|_{interface} = V_2(\vec{r})\big|_{interface}}$$

Example:

Consider a hemispherical Class-*A*/linear dielectric of radius *R* placed in between two infinite conducting parallel plates, as shown in the figure below (n.b. figure is not drawn to scale):



The gap distance *d* between plates and radius of hemispherical dielectric *R* are such that $R \ll d$, but $d \ll \infty$. Thus, far away from the hemisphere $(r \gg R)$ we know that:

$$\vec{E}(r \gg R) = E_o\hat{z} = \left(V_o/d\right)\hat{z} \quad \text{(i.e. uniform constant } \vec{E}\text{-field far from hemisphere)}$$

We want to know/determine the following quantities:

Inside the dielectric $(r < R)$: $V(r<R)$, $\vec{E}(r<R)$, $\vec{D}(r<R)$, $\vec{P}(r<R)$

Outside the dielectric $(r > R)$: $V(r>R)$, $\vec{E}(r>R)$, $\vec{D}(r>R)$, $\vec{P}(r>R)$

Since there is no volume free charge density inside the dielectric, i.e. $\rho_{free}(r<R)=0$, we know that $\rho_{Bound}(r<R)$ *must* $=0$ inside dielectric, i.e. $\rho_{Bound}(r<R) = -\vec{\nabla}\cdot\vec{P}(r<R) = 0$.

However, at $r = R$: $\sigma_{Bound}(r=R) = \vec{P}(\vec{r})\cdot\vec{n}\big|_{r=R} = \vec{P}(\vec{r})\cdot\hat{r}\big|_{r=R}$

i.e. a bound surface charge density will exist on/at the surface of the hemispherical dielectric.

- Since $\rho_{free}(\vec{r}) = 0$ everywhere interior to the plates of the parallel plate capacitor, then $\nabla^2 V(\vec{r}) = 0$ (Laplace's Equation) holds for the volume of interest in this problem.
- Note also that this problem has azimuthal / axial symmetry (i.e. it is invariant under arbitrary rotations in φ about the $\hat{z}$-axis. Therefore, $V(\vec{r})$, $\vec{E}(\vec{r})$, $\vec{D}(\vec{r})$, $\vec{P}(\vec{r})$ and $\sigma_{Bound}(r=R)$ also cannot have any explicit φ-dependence!!

Thus, the generalized expansion / solution to Laplace's equation $\nabla^2 V(\vec{r}) = 0$ in terms of Legendré polynomials, $P_\ell(\cos\theta)$ (and not the spherical harmonics $Y_{\ell,m}(\theta,\varphi)$) suffices here:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta)$$

The boundary conditions for this problem are:

1. $V_{in}(r=R) = V_{out}(r=R)$ at/on the hemispherical surface of the dielectric
2. $V(z=0) = \quad 0$ on the lower conducting plate
3. $V(z=d) = -V_o$ on the upper conducting plate
4. $\vec{E}(r \gg R) = E_o\hat{z} \;\Rightarrow\; V_{out}(r \gg R) = -E_o \underbrace{r\cos\theta}_{z = r\cos\theta} = -E_o z$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta \qquad E_o = \dfrac{V_o}{d}$

with: $d \gg R$ and $d \ll \infty$

And since $\sigma_{free}(r=R) = 0$ on the hemispherical surface of *this* Class-*A* dielectric, then:

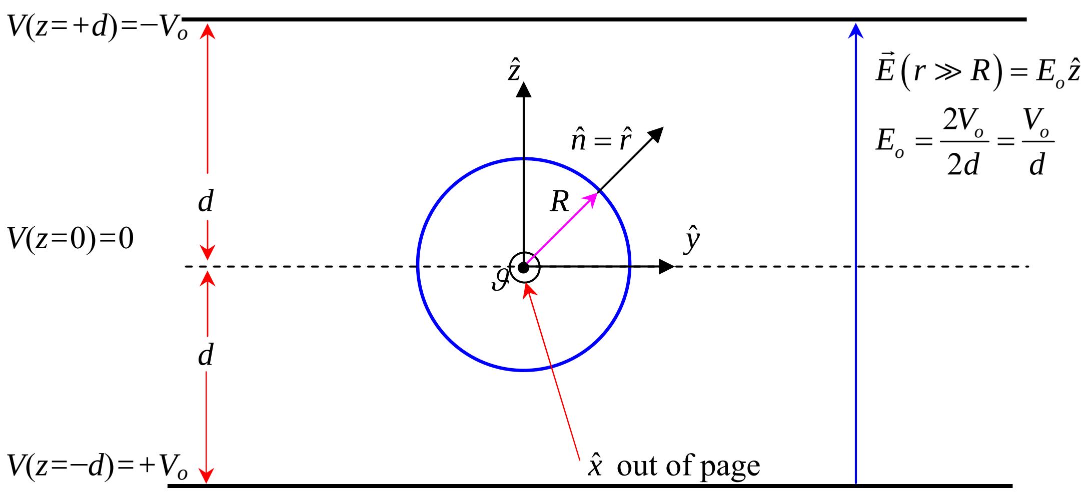
5. $D_{in}^{\perp}(r=R) = D_{out}^{\perp}(r=R) \quad\Rightarrow\quad \varepsilon E_{in}^{\perp}(r=R) = \varepsilon_o E_{out}^{\perp}(r=R)$

Or equivalently:
$$\varepsilon\frac{\partial V_{in}(r=R)}{\partial r} = \varepsilon_o\frac{\partial V_{out}(r=R)}{\partial r}$$

6. Physically, we also require $V(\vec{r})$ and $\vec{E}(\vec{r})$ to be finite everywhere.

Now, we can solve this problem directly, as is, but notice that this problem has an additional symmetry – that of reflection about the $\hat{z}$ axis! This problem is then formally equivalent in every way to that of:

Griffiths Example 4.7: Class-*A* dielectric sphere in uniform external electric field!!



This problem is very similar to one that we have already done – that of an uncharged conducting sphere in a uniform externally-applied electric field $\vec{E}_{ext} = E_o \hat{z}$.

We found for that problem, that $\vec{E} = 0$ inside the conducting sphere because the (induced) surface free charge density $\sigma_{free}(r = R)$ on the surface of the conducting sphere (n.b. an equipotential) created an electric field inside the sphere that exactly screened / canceled out the external field, thus giving a net field of zero inside this conducting sphere.

For the case of a Class-*A* dielectric sphere of radius *R* immersed in a uniform externally-applied electric field $\vec{E}(r \gg R) = E_o \hat{z}$, we will discover that $\vec{E}$ inside the Class-*A* dielectric sphere is ≠ 0, i.e. the external field is not perfectly screened out inside the Class-*A* dielectric sphere – it is only partially screened.

However, in the limit of $\chi_e \to \infty$ (i.e. the dielectric becomes essentially a conductor!), we will find that $\vec{E}$ inside the dielectric sphere does indeed → 0!!!

The boundary conditions for the Class-*A* dielectric sphere in a uniform externally-applied electric field $E(r \gg R) = \mathrm{E}_o \hat{z}$ are:

1. $V_{in}(r = R) = V_{out}(r = R)$ at the surface of the dielectric sphere

2. $V(z = 0) = 0$, but note that since $V\left(r, \theta = \frac{\pi}{2}\right) = V(z = 0)$ {since $z = r\cos\theta$ }

= Potential at/on mid-plane / midpoint (anywhere!) between conducting plates

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

This says also that: $V_{in}\left(r = R, \theta = \frac{\pi}{2}\right) = V_{out}\left(r = R, \theta = \frac{\pi}{2}\right) = 0$ but *only* at $\theta = \frac{\pi}{2}$!!!

3. $V(z = +d) = V(r\cos\theta = +d) = -V_o$ on upper plate.

3′. $V(z = -d) = V(r\cos\theta = -d) = +V_o$ on lower plate.

4. $\vec{E}_{out}(r \gg R) = E_o \hat{z} = \frac{V_o}{d}\hat{z} \quad \left(= E_o\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) = \frac{V_o}{d}\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)\right)$

$$\Rightarrow V_{out}(r \gg R) = -\left(\frac{V_o}{d}\right)z = -E_o r\cos\theta \qquad \left(E_o = \frac{V_o}{d}\right)$$

And since: $\sigma_{free}(r = R) = 0$ on the surface of this Class-*A* dielectric sphere, then:

5. $D_{in}^{\perp}(r = R) = D_{out}^{\perp}(r = R) \quad \Rightarrow \quad \varepsilon E_{in}^{\perp}(r = R) = \varepsilon_o E_{out}^{\perp}(r = R)$

Or equivalently:
$$\varepsilon \frac{\partial V_{in}(r = R)}{\partial r} = \varepsilon_o \frac{\partial V_{out}(r = R)}{\partial r}$$

6. Physically, we also require $V(\vec{r})$ and $\vec{E}(\vec{r})$ to be finite everywhere.

Note that these boundary conditions are identical to our original problem of the dielectric hemisphere, with BC 3′ added for the mirror-reflected bottom plate at $z = -d$ with potential $+V_o$

Note further, that because of the additional reflection symmetry about the $\hat{z}$ axis of this problem $z \to -z \;\; \Rightarrow \;\; V(-z) = -V(-z)$ (i.e. $V(z)$ is an odd function under $z \to -z$ reflection) because of the reflection-symmetry properties ($\theta \to -\theta$) of the Legendré Polynomials themselves, namely that $P_\ell(-\theta) = (-1)^\ell P_\ell(\theta)$ so therefore we know in advance that all ℓ = even terms *must* vanish!!! Since the electrostatic field $\vec{E}(\vec{r})$ is related to spatial derivatives of the potential $V(\vec{r})$ by $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$, we also then know in advance, from Rodrique's formula for the Legendré Polynomials that the $\vec{E}(\vec{r})$, $\vec{D}(\vec{r})$ and $\vec{\mathrm{P}}(\vec{r})$ fields will be even functions of z, i.e. $\vec{E}(-z) = \vec{E}(+z)$, $\vec{D}(-z) = \vec{D}(+z)$, $\vec{\mathrm{P}}(-z) = \vec{\mathrm{P}}(+z)$, and also that $\sigma_{free}(-d) = -\sigma_{free}(+d)$ and $\sigma_{Bound}(-\theta) = -\sigma_{Bound}(+\theta)$ will be odd functions under $z \to -z$ reflection, due to the odd reflection symmetry nature associated with the discontinuities in the fields for this problem!

Thus only the odd-ℓ $P_\ell(\theta)$ terms (for $\ell > 0$) will *survive* due to the reflection symmetry ($z \to -z$) aspects of this problem!!!

Since $\rho_{free} = 0$ here in this problem, so again, the most general solution to Laplace's Equation $\nabla^2 V(r,\theta) = 0$ is of the form:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta)$$

Inside the Class-A dielectric sphere, for $r < R$ we must have only:

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell \mathrm{P}_\ell(\cos\theta)$$

(All B_ℓ must vanish (i.e. = 0) because for $r = 0$, $V_{in}(r = 0,\theta)$ must be finite – $\exists$ no free charge inside the dielectric, and specifically, $\exists$ none at $r = 0$.)

Outside the dielectric sphere, for $r > R$, both r^ℓ and $1/r^{\ell+1}$ terms are allowed/must be included because a.) the origin is excluded and b.) the potential is explicitly finite (from BC's) at $r = \infty$:

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty}\left[A'_\ell r^\ell + \frac{B'_\ell}{r^{\ell+1}}\right]P_\ell(\cos\theta)$$

Note that when $x \to \infty$ and / or $y \to \infty$ in the gap between the parallel plates that $z = r\cos\theta = \sqrt{x^2+y^2}\cos\theta = 0$ in the gap as $r \to \infty$, when (simultaneously) $\theta \to \pi/2$.

Apply the Boundary Conditions:

First apply BC 4: $V_{out}(r \gg R) = -\left(\frac{V_o}{d}\right)z = -E_o z = -E_o \overbrace{r\cos\theta}^{=z}$

This BC tells us that we must have $A'_\ell = 0 \;\forall_\ell$, except for the $\ell = 1$ term, which gives: $\boxed{A'_1 = -E_o}$

Then: $\boxed{A'_1 r P_1(\cos\theta) = -E_o r\cos\theta}$

Now apply BC 1 at $r = R$: $V_{in}(r=R) = V_{out}(r=R)$ says that:

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_o R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B'_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

Now we can either formally take an inner product here (i.e. multiply both LHS and RHS of this relation by $\mathrm{P}_{\ell'}(\cos\theta)$, $\ell' \neq \ell$ (necessarily) and integrate from $\int_0^\pi d\cos\theta$ to use the orthonormality properties of the Legendré Polynomials, i.e.

$$\int_0^\pi P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) = \left(\frac{1}{2\ell+1}\right)\delta_{\ell'\ell}$$

or, we can simply insist that coefficients on LHS and RHS must make mathematical sense, term-by-term, for each/every value of ℓ, namely:

$\ell = 0$: $\boxed{A_0 = B'_0/R}$

$\ell = 1$: $A_1 R\cos\theta = -E_o R\cos\theta + \frac{B'_1}{R^2}\cos\theta \;\Rightarrow\; \boxed{A_1 R = -E_o R + \frac{B'_1}{R^2}}$ $\quad \boxed{P_1(\cos\theta) = \cos\theta}$

$\ell \geq 2$: All $A_\ell R_\ell = \frac{B'_\ell}{R^{\ell+1}}$ or: $A_\ell = \frac{B'_\ell}{R^{2\ell+1}}$ or: $\boxed{B'_\ell = A_\ell R^{2\ell+1}}$

Applying BC 5: $\varepsilon \frac{\partial V_{in}(r=R)}{\partial r} = \varepsilon_o \frac{\partial V_{out}(r=R)}{\partial r}$

$$= \varepsilon \frac{\partial}{\partial r} \sum_{\ell=0}^{\infty} A_\ell r^\ell \mathrm{P}_\ell(\cos\theta)\bigg|_{r=R} = \varepsilon_o \frac{\partial}{\partial r}\left[-E_o r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B'_\ell}{r^{\ell+1}} P_\ell(\cos\theta)\right]\bigg|_{r=R}$$

$$= \varepsilon \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -\varepsilon_o E_o \cos\theta - \varepsilon_o \sum_{\ell=0}^{\infty} \frac{(\ell+1)B'_\ell}{R^{\ell+2}} P_\ell(\cos\theta)$$

or, since $K_e \equiv \frac{\varepsilon}{\varepsilon_o}$ ($=\varepsilon_r$ in Griffith's book)

$$\boxed{= K_e \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -E_o \cos\theta - \sum_{\ell=0}^{\infty} \frac{(\ell+1)B'_\ell}{R^{\ell+2}} P_\ell(\cos\theta)}$$

for $\ell = 0$: $0A_0 = -\frac{B'_0}{R^2}$ but from above: $A_0 = \frac{B'_0}{R} \;\Rightarrow\; \boxed{A_0 = B'_0 = 0}$

for $\ell = 1$: $K_e A_1 = -E_o - \frac{2B'_1}{R^3}$ but from above: $A_1 = -E_o + \frac{B'_1}{R^3}$

$\therefore\; -K_e E_o + \frac{K_e B'_1}{R^3} = -E_o - \frac{2B'_1}{R^3}$ solve for B'_1:

$$\frac{K_e B'_1 + 2B'_1}{R^3} = K_e E_o - E_o = (K_e - 1)E_o \;\; \text{or:} \;\; \left(\frac{K_e+2}{R^3}\right)B'_1 = (K_e-1)E_o \;\Rightarrow\; \boxed{B'_1 = \left(\frac{K_e-1}{K_e+2}\right)E_o R^3}$$

$$\therefore\quad A_1 = -E_o + \frac{B_1'}{R^3} = -E_o + \left(\frac{K_e-1}{K_e+2}\right)E_o = \frac{-(K_e+2)+(K_e-1)}{K_e+2}E_o \quad \underline{\text{or}}: \quad \boxed{A_1 = -\left(\frac{3}{K_e+2}\right)E_o}$$

for $\ell \geq 2$: All $\ell A_\ell R^{\ell+1} = -\frac{(\ell+1)B_\ell'}{R^{\ell+2}}$ <u>or</u>: $A_\ell = -\left(\frac{\ell+1}{\ell}\right)\frac{B_\ell'}{R^{2\ell+3}}$

but from BC 1 above, for $\ell \geq 2$: $B_\ell' = A_\ell R^{2\ell+1}$ <u>or</u>: $A_\ell = \frac{B_\ell'}{R^{2\ell+1}} \;\Rightarrow\; \therefore\; \frac{B_\ell'}{R^{2\ell+1}} = -\left(\frac{\ell+1}{\ell}\right)\frac{B_\ell'}{R^{2\ell+3}}$

<u>or</u>: $B_\ell' = -\left(\frac{\ell+1}{\ell}\right)\frac{B_\ell'}{R^2}$ <u>or</u>: $\boxed{B_\ell'\left(1+\left(\frac{\ell+1}{\ell}\right)\frac{1}{R^2}\right)=0}$ for $\ell \geq 2$

This relation can <u>only</u> be satisfied <u>**iff**</u> (if and only if): $\boxed{B_\ell' = 0\ \forall\ \ell \geq 2}$ and thus $\Rightarrow \boxed{A_\ell = 0\ \forall\ \ell \geq 2}$

So finally, we have the (relatively) simple relations:

$$\boxed{\begin{aligned} V_{in}(r,\theta) &= -\frac{3}{(K_e+2)}E_o\underbrace{r\cos\theta}_{=z} = -\left(\frac{3}{K_e+2}\right)E_o z \\ V_{out}(r,\theta) &= -E_o\underbrace{r\cos\theta}_{=z} + \left(\frac{K_e-1}{K_e+2}\right)E_o R\left(\frac{R}{r}\right)^2\cos\theta \\ &= -E_o z + \left(\frac{K_e-1}{K_e+2}\right)E_o R\left(\frac{R}{r}\right)^2\cos\theta \end{aligned}}$$

Note that: $V_{in}\left(r,\theta=\frac{\pi}{2}\right) = V_{out}\left(r,\theta=\frac{\pi}{2}\right)=0$	But that: $V_{in}\left(r,\theta\neq\frac{\pi}{2}\right)\neq 0$!!!

<u>Then obtain *E*-field(s) from</u>: $\vec{E}(r,\theta) = -\vec{\nabla}V(r,\theta)$

where $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \cancel{\text{phi-stuff}}$ (no <u>explicit</u> φ-dependence here, so we don't care..)

$$V_{in}(r,\theta) = -\left(\frac{3}{K_e+2}\right)E_o r\cos\theta$$

<u>So</u>: $\vec{E}_{in}(\vec{r}) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)V_{in}(r,\theta) = +\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)\left(\frac{3}{K_e+2}\right)E_o r\cos\theta$

$$\boxed{\vec{E}_{in}(\vec{r}) = \left(\frac{3}{K_e+2}\right)E_o\underbrace{\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)}_{\equiv\hat{z}} = \left(\frac{3}{K_e+2}\right)E_o\hat{z}}$$

Likewise:

$$\vec{E}_{out}(\vec{r}) = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)V_{out}(r,\theta) = +\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)\left(-E_o\underbrace{r\cos\theta}_{=z} + \left(\frac{K_e-1}{K_e+2}\right)E_oR\left(\frac{R}{r}\right)^2\cos\theta\right)$$

$$\vec{E}_{out}(\vec{r}) = +E_o\underbrace{\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)}_{\equiv\hat{z}} + \left(\frac{K_e-1}{K_e+2}\right)E_o\left(\frac{R}{r}\right)^3\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)$$

$$= E_o\hat{z} + \left(\frac{K_e-1}{K_e+2}\right)E_o\left(\frac{R}{r}\right)^3\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)$$

$$\boxed{\vec{E}_{out}(\vec{r}) = \underbrace{E_o\hat{z}}_{=\vec{E}_{ext}} + \underbrace{\left(\frac{K_e-1}{K_e+2}\right)E_o\left(\frac{R}{r}\right)^3\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)}_{\text{Electric field due to physical electric dipole moment induced in polarized dielectric sphere of radius } R \text{ and dielectric constant } K_e!!} = \vec{E}_{ext}(\vec{r}) + \vec{E}_{dipole}(\vec{r})!!!}$$

Thus we see that E_{out} is the linear superposition of the external applied field and that associated with the external electric field due to the physical electric dipole moment induced by the external applied electric field in the polarized dielectric sphere of radius R and dielectric constant K_e!!!

The second term in the above expression is:

$$\boxed{\vec{E}_{dipole}(r,\theta) = \frac{p}{4\pi\varepsilon_o r^3}\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)} \quad \text{where: } \boxed{p = 4\pi\varepsilon_o\left(\frac{K_e-1}{K_e+2}\right)E_oR^3}$$

p = magnitude of the induced electric dipole moment of the dielectric sphere!!!

Since we have a Class-A/linear dielectric, the D-field(s) are related to the E-field(s) by:

$$\boxed{\vec{D}_{in}(\vec{r}) = \varepsilon E_{in}(\vec{r}) = \left(\frac{3\varepsilon}{K_e+2}\right)E_o\hat{z} = \left(\frac{3K_e}{K_e+2}\right)\varepsilon_oE_o\hat{z}} \quad \text{since } \varepsilon = K_e\varepsilon_o$$

$$\boxed{\vec{D}_{out}(\vec{r}) = \varepsilon_o\vec{E}_{out}(\vec{r}) = \underbrace{\varepsilon_oE_o\hat{z}}_{=\vec{D}_{ext}} + \underbrace{\left(\frac{K_e-1}{K_e+2}\right)\varepsilon_oE_o\left(\frac{R}{r}\right)^3\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)}_{\text{Displacement field due to physical electric dipole moment induced in polarized dielectric sphere of radius } R \text{ and dielectric constant } K_e!!}}$$

Since the E_{in} and D_{in}-fields are now known, we can obtain the electric polarization (P_{in}) inside the dielectric sphere using:

$$\boxed{\vec{D}_{in}(\vec{r}) = \varepsilon_o\vec{E}_{in}(\vec{r}) + \vec{\mathrm{P}}_{in}(\vec{r})} \quad \underline{\text{or}}: \quad \boxed{\vec{\mathrm{P}}_{in}(\vec{r}) = \vec{D}_{in}(\vec{r}) - \varepsilon_o\vec{E}_{in}(\vec{r}) = \varepsilon_o\chi_e\vec{E}_{in}(\vec{r})}$$

$$\boxed{\begin{aligned}\vec{\mathrm{P}}_{in}(\vec{r}) &= \vec{D}_{in}(\vec{r}) - \varepsilon_o\vec{E}_{in}(\vec{r}) = \left(\frac{3K_e}{K_e+2}\right)\varepsilon_oE_o\hat{z} - \left(\frac{3}{K_e+2}\right)\varepsilon_oE_o\hat{z} \\ &= \left(\frac{3(K_e-1)}{K_e+2}\right)\varepsilon_oE_o\hat{z} = 3\left(\frac{K_e-1}{K_e+2}\right)\varepsilon_oE_o\hat{z} \quad \text{but } \chi_e = K_e - 1\end{aligned}} \quad \text{and} \quad \boxed{\vec{E}_{in}(\vec{r}) = \left(\frac{3}{K_e+2}\right)E_o\hat{z}}$$

$$\therefore \quad \boxed{\vec{\mathrm{P}}_{in}(\vec{r}) = \left(\frac{3}{K_e+2}\right)\varepsilon_o\chi_eE_o\hat{z} = \varepsilon_o\chi_e\vec{E}_{in}(\vec{r})} \quad \underline{\text{and}}: \quad \boxed{\vec{\mathrm{P}}_{out}(\vec{r}) \equiv 0}$$

Now since $\vec{P}_{in}(\vec{r}) = \varepsilon_o \chi_e \left(\dfrac{3}{K_e+2}\right) E_o \hat{z} =$ electric dipole moment <u>per unit volume</u>

$\therefore \quad \vec{p} = \vec{P}_{in}(\vec{r})$ * *Volume of sphere of radius R* $= P_{in}(\vec{r}) \bullet \frac{4}{3}\pi R^3$

<u>or</u>: $\vec{p} = \dfrac{4\pi R^3}{\cancel{3}} \varepsilon_o \chi_e \left(\dfrac{\cancel{3}}{K_e+2}\right) E_o \hat{z} = 4\pi R^3 \varepsilon_o \left(\dfrac{\chi_e}{K_e+2}\right) E_o \hat{z}$ <u>but</u>: $\chi_e = K_e - 1$

$$\therefore \quad \boxed{\vec{p} = 4\pi\varepsilon_o \left(\frac{\chi_e}{K_e+2}\right) E_o R^3 \hat{z} = 4\pi\varepsilon_o \left(\frac{\chi_e}{\chi_e+3}\right) E_o R^3 \hat{z} = 4\pi\varepsilon_o \left(\frac{K_e-1}{K_e+2}\right) E_o R^3 \hat{z}}$$

Physical macroscopic electric dipole moment of dielectric sphere induced by external applied electric field, $\boldsymbol{E}_{ext}$.

Now let us <u>explicitly</u> check whether or not any bound volume charge density $\rho_{Bound}(\vec{r})$ exists within the dielectric sphere. We know that $\rho_{free}(\vec{r}) = 0$ inside the sphere, so therefore we know that $\rho_{Bound}(\vec{r})$ <u>should</u> be = 0 inside the sphere! Let's check!

$$\boxed{\rho_{Bound}(\vec{r}) = -\vec{\nabla} \bullet \underbrace{\vec{P}_{in}(\vec{r})}_{\substack{=\,constant\,/\\ uniform\\ polarization}} = -\varepsilon_o \left(\frac{3\chi_e}{K_e+2}\right) E_o \underbrace{\vec{\nabla} \bullet \hat{z}}_{\substack{divergence\\ of\ a\ constant\\ unit\ vector=0}} = 0}$$ <u>Yup!</u>

Now let's calculate the bound surface charge density on the dielectric sphere:
We will need to use $\hat{n} = \hat{r}$ and $\hat{z} \bullet \hat{r} = \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) \bullet \hat{r} = \cos\theta$

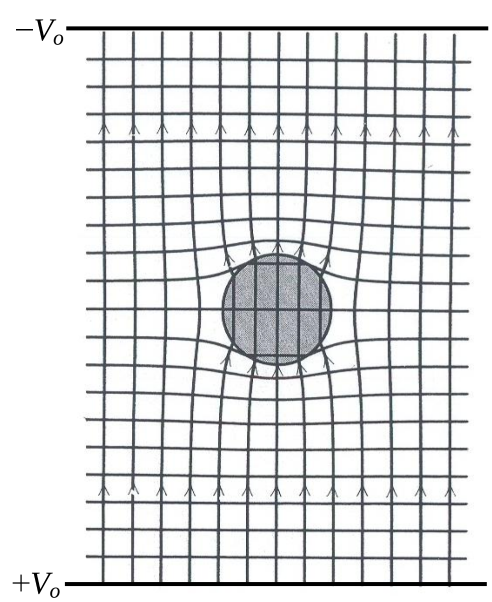
$$\sigma_{Bound}(r=R) = \vec{P}_{in}(\vec{r}) \bullet \hat{n}\Big|_{r=R} = \varepsilon_o \chi_e \left(\frac{3}{K_e+2}\right) E_o \hat{z} \bullet \hat{n}\Big|_{r=R} = \varepsilon_o \chi_e \left(\frac{3}{K_e+2}\right) E_o \hat{z} \bullet \hat{r}\Big|_{r=R}$$

$$\therefore \quad \boxed{\sigma_{Bound}(r=R) = \varepsilon_o \left(\frac{3\chi_e}{K_e+2}\right) E_o \cos\theta = 3\varepsilon_o \left(\frac{K_e-1}{K_e+2}\right) E_o \cos\theta} \quad \text{since} \quad \chi_e = K_e - 1$$

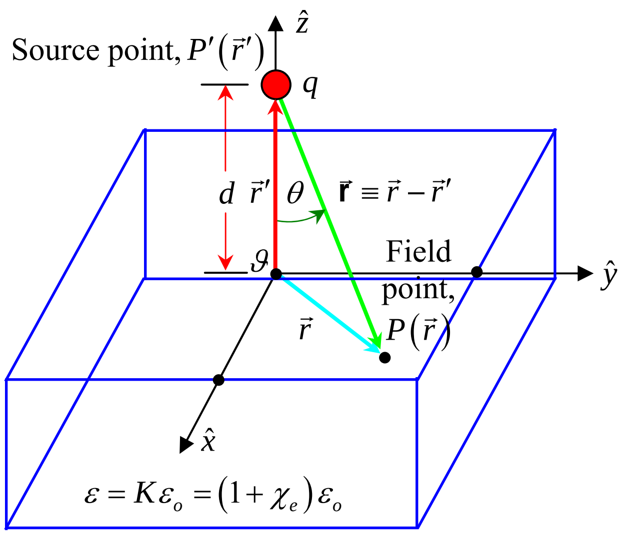
Lines of $\vec{D}$ (arrows) and equipotentials for a dielectric sphere $(K_e = 3)$ in a previously uniform external electric field. The lines of $\vec{D}$ crowd into the sphere and thus $\vec{D}$ is larger inside the sphere than outside.

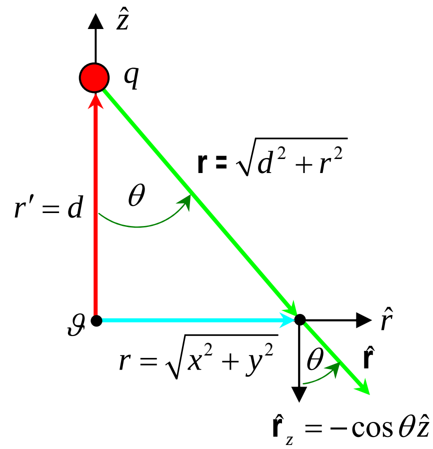
Since the equipotentials spread out inside the sphere, the magnitude of $\vec{E}$ is smaller inside the sphere than outside. A bound surface charge density exists on the sphere, hence $\vec{E}$ is also discontinuous at the surface of the sphere.

Outside the sphere $\vec{D} = \varepsilon_o \vec{E}$. Note that the field(s) outside the sphere are barely disturbed by the sphere at distances larger than ~ 1 sphere radius.



Griffiths Example 4.8 Consider an infinite Class-A dielectric located below the $x-y$ plane at $z=0$ as shown in the figure below. Calculate the force acting on a point charge q situated on the $\hat{z}$-axis, a distance d above the origin.





Solution: The bound charge on the surface of the dielectric $\sigma_{bound}(\vec{r})$ at a point r in the $x-y$ plane at $z=0$ will be opposite in charge sign to that of q, hence the force acting on the point charge q will (obviously) be attractive.

Again: $\rho_{Bound}(\vec{r})=0$ inside the dielectric (because $\rho_{free}(\vec{r})=0$ there) and:

$$\sigma_{bound}(\vec{r}) = \vec{\mathrm{P}}(\vec{r})\bullet\hat{n}\big|_{surface} = \mathrm{P}_z(\vec{r})\big|_{surface} = \varepsilon_o\chi_e E_z(\vec{r})\big|_{surface}$$

where $E_z(\vec{r})\big|_{surface}$ = z-component of $\vec{E}$-field at point r with $z=0$ (surface/interface of dielectric).

The electric field at the field point $P(\vec{r})=(x,y,z=0)$ is a linear superposition of the "direct" E-field from q itself and the $\vec{E}$-field associated with the polarized dielectric at that point.

$\vec{E}$-field from q itself: $\vec{E}^q(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\dfrac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}$ where $\boldsymbol{\mathscr{r}} \equiv \vec{r}-\vec{r}'$, $\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r}$ and $\mathscr{r} = |\boldsymbol{\mathscr{r}}| = |\vec{r}-\vec{r}'| = \sqrt{d^2+r^2}$

$\Rightarrow$ The z-component of this electric field is: $\vec{E}_z^q(\vec{r})\big|_{surface} = \dfrac{1}{4\pi\varepsilon_o}\dfrac{q}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}_z = -\dfrac{1}{4\pi\varepsilon_o}\dfrac{q}{(d^2+r^2)}\cos\theta\hat{z}$

where $\cos\theta = d\big/\sqrt{d^2+r^2}$ and $r=\sqrt{x^2+y^2}$ lies in the $x-y$ plane as shown in the figure(s) above.

$$\therefore\quad \boxed{E_z^q(r,z=0) = -\frac{1}{4\pi\varepsilon_o}\frac{qd}{(d^2+r^2)^{3/2}}}$$

The z-component of the $\vec{E}$-field due to the bound surface charge density $\sigma_{Bound}(r,z=0)$ at the point r is $\boxed{E_z^{Bound}(r,z=0) = -\dfrac{\sigma_{Bound}(r,z=0)}{2\varepsilon_o}}$ (see/read footnote on p. 89 of Griffiths book...)

Thus: $$\boxed{E_z^{TOT}(r,z=0)=E_z^q(r,z=0)+E_z^{Bound}(r,z=0)= -\frac{1}{4\pi\varepsilon_o}\frac{qd}{\left(d^2+r^2\right)^{3/2}} - \frac{\sigma_{Bound}(r,z=0)}{2\varepsilon_o}}$$

But:

$$\sigma_{Bound}(r,z=0)=\vec{\mathrm{P}}(\vec{r})\bullet\hat{n}\Big|_{surface}=\mathrm{P}_z(\vec{r})\Big|_{surface}=\varepsilon_o\chi_e E_z^{TOT}(r,z=0)$$

$$=\cancel{\varepsilon_o}\chi_e\left[\frac{1}{4\pi\cancel{\varepsilon_o}}\frac{qd}{\left(d^2+r^2\right)^{3/2}} - \frac{\sigma_{bound}(r,z=0)}{2\cancel{\varepsilon_o}}\right]$$

Thus: $$\sigma_{Bound}(r,z=0)=-\frac{\chi_e}{4\pi}\frac{qd}{\left(d^2+r^2\right)^{3/2}} - \frac{\chi_e\sigma_{Bound}(r,z=0)}{2}$$

Solve for $\sigma_{Bound}(r,z=0)$: $\left(1+\frac{\chi_e}{2}\right)\sigma_{Bound}(r,z=0)=-\frac{\chi_e}{4\pi}\frac{qd}{\left(d^2+r^2\right)^{3/2}}$ (r lies in the $x-y$ plane)

$\therefore$ $$\boxed{\sigma_{Bound}(r,z=0)=-\frac{1}{4\pi}\frac{\chi_e}{\left(1+\frac{1}{2}\chi_e\right)}\frac{qd}{\left(d^2+r^2\right)^{3/2}}=-\frac{1}{2\pi}\frac{\chi_e}{\left(2+\chi_e\right)}\frac{qd}{\left(d^2+r^2\right)^{3/2}}}$$

If we let the electric susceptibility $\chi_e\to\infty$, then the dielectric "becomes" a conductor and:

$$\boxed{\sigma_{Bound}(r,z=0)\ \to\ \sigma_{free}(r,z=0)=-\frac{1}{2\pi}\frac{qd}{\left(d^2+r^2\right)^{3/2}}=-\frac{1}{4\pi}\frac{q(2d)}{\left(d^2+r^2\right)^{3/2}}=-\frac{1}{4\pi}\frac{p}{\left(d^2+r^2\right)^{3/2}}}\quad\boxed{p\equiv q(2d)}$$

n.b. This is exactly what we obtained for charge q at a height d above ∞-conducting plane!!!
(see p. 9-12 of P435 Lecture Notes 6 and/or p.123 eqn. 3.10 of Griffiths book.)

Using the method of integrating circular strips ($dA'=dr(rd\varphi)=rdrd\varphi$) the total bound charge on the surface of the dielectric is:

$$\boxed{q_{Bound}=\int_{surface,S'}\sigma_{Bound}(r,z=0)\,dA'=\cancel{2\pi}\int_0^\infty\left[-\frac{1}{\cancel{2\pi}}\frac{\chi_e}{\left(2+\chi_e\right)}\frac{q}{\left(d^2+r^2\right)^{3/2}}\right]rdr=-\left(\frac{\chi_e}{2+\chi_e}\right)q}$$

We can then obtain $\vec{E}^{Bound}(\vec{r})=\frac{1}{4\pi\varepsilon_o}\int_{S'}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)\sigma_{Bound}(r,z=0)\,dA'$ by direct integration.

However, we can also (equivalently) accomplish this by using the <u>method of images</u> (discussed in P435 Lect. Notes 6):

To obtain the scalar potential $V(\vec{r})$ in the region $z\geq 0$, replace the entire dielectric by a single point image charge $q_{Bound}=-\left(\chi_e/(2+\chi_e)\right)q$ located at the <u>image position</u> of $(x, y, z) = (0, 0, -d)$.
i.e. q_{Bound} is located directly below the actual charge q, in the $z<0$ region at $(x, y, z) = (0, 0, -d)$.

Then, using the principle of linear superposition:

$$\boxed{\begin{aligned} V^{Tot}(\vec{r}) &= V^{q}(\vec{r}) + V^{q_{Bound}}(\vec{r}) \\ &= \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_{Bound}}{\sqrt{x^2+y^2+(z+d)^2}}\right] \end{aligned}}$$ for the region $z \geq 0$;

Similarly, to obtain the scalar potential $V(\vec{r})$ in the region $z \leq 0$, we place a point charge ($q + q_{bound}$) at (x, y, z) = (0, 0, +d), which gives:

$$\boxed{V^{Tot}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q+q_{Bound}}{\sqrt{x^2+y^2+(z-d)^2}}\right]}$$ for the region $z \leq 0$, with $\boxed{q_{Bound} = -\left(\frac{\chi_e}{2+\chi_e}\right)q}$

These two potentials do indeed satisfy Laplace's equation $\nabla^2 V^{Tot}(\vec{r}) = 0$ everywhere in their respective half-regions, and they are both finite/physically well-behaved as $r \rightarrow \infty$.

Note that the total potential $V^{Tot}(\vec{r})$ is continuous at (x, y, z = 0).

However, note also that, from using a Gaussian pillbox centered on the dielectric interface and using Gauss' law for E $\left(\oiint_{S'} \vec{E} \bullet \hat{n} dA' = Q_{tot}^{enclosed}/\varepsilon_o = Q_{boiund}^{enclosed}/\varepsilon_o\right)$ a discontinuity exists in the normal component of E (due to the bound surface charge density $\sigma_{Bound}(r, z=0)$) and thus also for the normal derivative of the potential $V^{Tot}(\vec{r})$ at the point $r = \sqrt{x^2+y^2}$ on the z = 0 boundary/ interface of the dielectric:

$$\boxed{\Delta E_z(\vec{r})\Big|_{interface} = -\varepsilon_o\left(\frac{\partial V(\vec{r})}{\partial z}\bigg|_{z=0^+} - \frac{\partial V(\vec{r})}{\partial z}\bigg|_{z=0^-}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}} = \sigma_{Bound}(r, z=0)}$$

The electric field at a field point $P(\vec{r})$ in the region above the dielectric (i.e. $z \geq 0$) due to the image charge q_{Bound} located inside the dielectric at $\vec{r}'$ = (x', y', z') = (0, 0, −d) is :

$$\boxed{\vec{E}_{q_{Bound}}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q_{Bound}}{\mathbf{r}^2}\hat{\mathbf{r}} = -\frac{1}{4\pi\varepsilon_o}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{q}{\mathbf{r}^2}\hat{\mathbf{r}}}$$

The net force acting on the "test" charge q located at $\vec{r}$ = (x, y, z) = (0, 0, +d) due to its image charge (= "source" charge, inside the dielectric) located at $\vec{r}'$ = (x', y', z') = (0, 0, −d) is attractive, acting to pull the "test" charge q downwards, towards the dielectric interface:

Coulomb's Law:

$$\boxed{\vec{F}_q(\vec{r} = (0,0,d)) = q\vec{E}_{q_{Bound}}(\vec{r} = (0,0,d)) = \frac{1}{4\pi\varepsilon_o}\frac{qq_{Bound}}{(2d)^2}\hat{z} = -\frac{1}{4\pi\varepsilon_o}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{q^2}{(2d)^2}\hat{z}}$$

Note that in the limit $\chi_e \to \infty$ (dielectric becomes a conductor): $\vec{F}_q\left(\vec{r}=(0,0,d)\right) - \frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}\hat{z}$

IMAGE CHARGE PROBLEM: POINT CHARGE NEAR INTERFACE OF 2 DIELECTRICS

Suppose a point charge, q is placed at point P a distance d away from a plane interface between two (infinite) Class-*A*/linear dielectric materials of differing electric permittivity ε_1 and ε_2 as shown in the figure below:

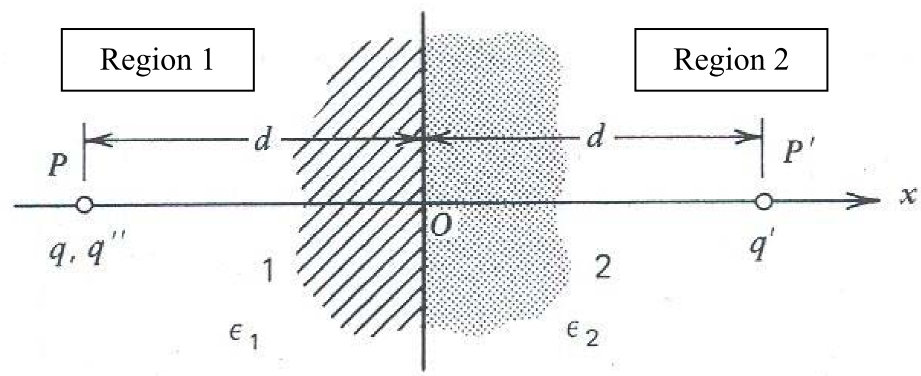


Figure 4.12 Application of the method of images to a point charge placed near a plane interface of two dielectric materials.

The potential in Region 1 ($x < 0$) is the linear superposition of the potential associated with charge q located at $(x, y, z) = (-d, 0, 0)$ and the potential associated with its image charge q' located at $(x, y, z) = (+d, 0, 0)$:

Region 1 ($x < 0$): $V_1^{Tot}(\vec{r}) = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{r} + \frac{q'}{r'}\right)$

Likewise, the potential in Region 2 ($x > 0$) will only be that associated with an image charge q'' located at $(x, y, z) = (-d, 0, 0)$ (i.e. the same location as the original charge):

Region 2 ($x > 0$): $V_2^{Tot}(\vec{r}) = \frac{1}{4\pi\varepsilon_2}\left(\frac{q''}{r''}\right) = \frac{1}{4\pi\varepsilon_2}\left(\frac{q''}{r}\right)$

Note that $r = r'' = \sqrt{(x+d)^2 + y^2 + z^2}$ and $r' = \sqrt{(x-d)^2 + y^2 + z^2}$

We impose boundary conditions to determine the strengths of the (as-yet) unknown image charges q' and q'':

Boundary Condition 1: The potential must be continuous across the interface between the two dielectrics – i.e.:

$$V_1^{Tot}(x=0, y, z) = V_2^{Tot}(x=0, y, z)$$

$$\Rightarrow \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{\sqrt{d^2+y^2+z^2}}+\frac{q'}{\sqrt{d^2+y^2+z^2}}\right)=\frac{1}{4\pi\varepsilon_2}\left(\frac{q''}{\sqrt{d^2+y^2+z^2}}\right) \quad \underline{\text{Or}}: \boxed{\frac{1}{\varepsilon_1}(q+q')=\frac{1}{\varepsilon_2}(q'')}$$

Boundary Condition 2: Since there are no *free* charges on the boundary/interface between the two planar Class-*A*/linear dielectrics, the normal component of the electric displacement, $\vec{D}(\vec{r})$ (here) is continuous across the interface:

$$D_{1n}(x=0,y,z)=D_{2n}(x=0,y,z)$$

Or:

$$\varepsilon_1 E_{1n}(x=0,y,z)=\varepsilon_2 E_{2n}(x=0,y,z)$$

Or:

$$\varepsilon_1\frac{\partial V_1(x=0,y,z)}{\partial x}=\varepsilon_2\frac{\partial V_2(x=0,y,z)}{\partial x}$$

Which is:

$$\frac{\cancel{\varepsilon_1}}{\cancel{4\pi}\,\cancel{\varepsilon_1}}\left[\frac{\cancel{d}q}{\cancel{\left[d^2+y^2+z^2\right]^{3/2}}}-\frac{\cancel{d}q'}{\cancel{\left[d^2+y^2+z^2\right]^{3/2}}}\right]=\frac{\cancel{\varepsilon_2}}{\cancel{4\pi}\,\cancel{\varepsilon_2}}\frac{\cancel{d}q''}{\cancel{\left[d^2+y^2+z^2\right]^{3/2}}}$$

Which gives the relation: $\boxed{q-q'=q''}$, which along with the first relation $\boxed{\frac{1}{\varepsilon_1}(q+q')=\frac{1}{\varepsilon_2}(q'')}$ we then simultaneously solve these to determine the strengths (and signs) of the two image charges q' and q'', relative to q. One finds:

$$\boxed{q'=-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}q} \quad \text{and} \quad \boxed{q''=\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}q}$$

Note the following two situations for q' associated with the scalar potential in Region 1 ($x < 0$):
If $\varepsilon_2 > \varepsilon_1$ then q' has the opposite sign to q, the charge q will be attracted towards the interface.
If $\varepsilon_2 < \varepsilon_1$ then q' has the same sign as q, the charge q will be repelled away from the interface.
For the scalar potential in Region 2 ($x > 0$), q'' (always) has the same sign as q.

Since the image charges are now known, the potentials $V_i(\vec{r})$ in each region (i = 1, 2) are thus also now known:

Region 1 ($x < 0$):

$$V_1^{Tot}(\vec{r})=\frac{1}{4\pi\varepsilon_1}\left(\frac{q}{r}+\frac{q'}{r'}\right)=\frac{q}{4\pi\varepsilon_1}\left[\frac{1}{r}-\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right)\frac{1}{r'}\right]$$

$$=\frac{q}{4\pi\varepsilon_1}\left[\frac{1}{\sqrt{(x+d)^2+y^2+z^2}}-\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right)\frac{1}{\sqrt{(x-d)^2+y^2+z^2}}\right]$$

Region 2 ($x > 0$):

$$V_2^{Tot}(\vec{r}) = \frac{1}{4\pi\varepsilon_2}\left(\frac{q''}{r''}\right) = \frac{1}{4\pi\varepsilon_2}\left(\frac{q''}{r}\right) = \frac{1}{4\pi\varepsilon_2}\left(\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}\right)\frac{q}{r}$$

$$= \frac{1}{4\pi\varepsilon_2}\left(\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}\right)\frac{q}{\sqrt{(x+d)^2+y^2+z^2}}$$

Hence the electric fields $\vec{E}_i(\vec{r})$ can be determined in each region (i = 1, 2) from $\vec{E}_i(\vec{r}) = -\vec{\nabla}V_i(\vec{r})$ as well as the electric displacement fields $\vec{D}_i(\vec{r}) = \varepsilon_i\vec{E}_i(\vec{r})$ and the electric polarization fields $\vec{\mathrm{P}}_i(\vec{r}) = \vec{D}_i(\vec{r}) - \varepsilon_o\vec{E}_i(\vec{r}) = \varepsilon_o\chi_{e_i}\vec{E}_i(\vec{r})$.

The bound surface charge density present on the interface can then be obtained from:

$$\boxed{\sigma_{Bound}(0,x,y) = -\left(\vec{\mathrm{P}}_2(0,x,y) - \vec{\mathrm{P}}_1(0,x,y)\right)\bullet\hat{x} = -\left(\frac{q}{2\pi}\right)\frac{(\varepsilon_2-\varepsilon_1)}{\varepsilon_1(\varepsilon_2+\varepsilon_1)}\frac{d}{\left(d^2+y^2+z^2\right)^{3/2}}}.$$

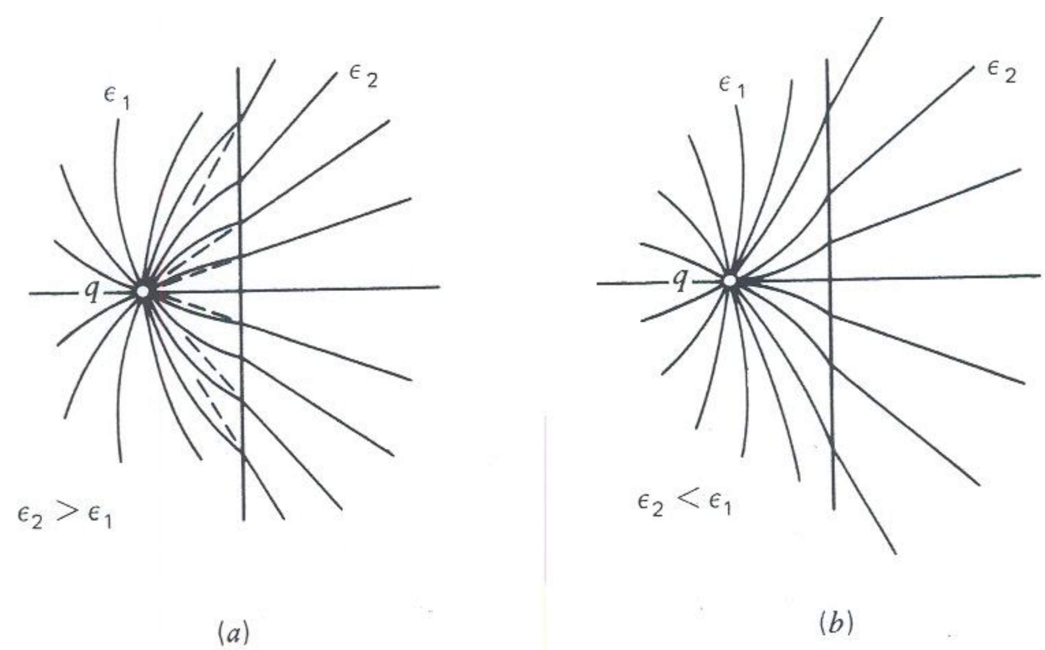


Figure 4.13 Field lines of a point charge implanted in ε_1 material near the plane interface with a material of ε_2. (*a*) $\varepsilon_2 > \varepsilon_1$. (*b*) $\varepsilon_2 < \varepsilon_1$.

ELECTROSTATIC ENERGY STORED IN DIELECTRIC MATERIALS

We know that it takes work to charge up a capacitor: $\boxed{W = \underset{\uparrow}{P.E.} = \frac{1}{2} C \Delta V^2}$

P.E. stored in *E*-field of charged capacitor

For a capacitor with a linear/Class-*A* dielectric: $\boxed{C_{DIEL} = K_e C_{VAC}}$

Thus: $$\boxed{W_{DIEL} = P.E._{DIEL} = \frac{1}{2} C_{DIEL} \Delta V^2 = K_e W_{VAC} = K_e P.E._{VAC} = \frac{1}{2} K_e C_{VAC} \Delta V^2}$$

i.e. a factor of K_e× more energy can be stored in a capacitor with a dielectric than one with same/identical geometry but with no dielectric material inside it (i.e. just free-space (vacuum)).

Recall that in free-space / vacuum, we derived:

$$\boxed{W_{VAC} = P.E._{VAC} = \frac{1}{2} \varepsilon_o \int_{v'} E^2(\vec{r})\, d\tau' = \frac{1}{2} \int_{v'} \varepsilon_o \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r})\, d\tau'}$$

In a linear/Class-*A* dielectric medium, $\varepsilon_o \to \varepsilon$ and $\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r})$, so thus:

$$\boxed{W_{DIEL} = P.E._{DIEL} = \frac{1}{2} \int_{v'} \underbrace{\varepsilon \vec{E}(\vec{r})}_{=\vec{D}(\vec{r})} \cdot \vec{E}(\vec{r})\, d\tau' = \frac{1}{2} \int_{v'} \vec{D}(\vec{r}) \cdot \vec{E}(\vec{r})\, d\tau'}$$

This can be rigorously proved as follows:

> Suppose the linear/Class-*A* dielectric is fixed in position and we bring in infinitesimal amounts of free charge, dq_{free} a little bit at a time.

As $\rho_{free}(\vec{r})$ increases by an infinitesimal amount, $\Delta\rho_{free}(\vec{r})$ the electric polarization $\vec{P}(\vec{r})$ in the dielectric will also change and with it the bound charge distribution(s) – both will increase.

However, we are interested only in the work done in association with increasing the free charge incrementally:

$$\boxed{\Delta W_{free} = \int_{v'} \Delta\rho_{free}(\vec{r}) V(\vec{r})\, d\tau'}$$

Now since $\vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho_{free}(\vec{r})$ and $\Delta\rho_{free}(\vec{r}) = \vec{\nabla} \cdot \left(\Delta\vec{D}(\vec{r})\right)$ where $\Delta\vec{D}(\vec{r})$ is the resulting infinitesimal change in the electric displacement, $\vec{D}(\vec{r})$ associated with the infinitesimal increment in volume free charge density $\Delta\rho_{free}(\vec{r})$, then $\Delta W_{free} = \int_{v'} \left(\vec{\nabla} \cdot \Delta\vec{D}(\vec{r})\right) V(\vec{r})\, d\tau'$.

Now: $$\vec{\nabla} \cdot \left[\Delta\vec{D}(\vec{r}) V(\vec{r})\right] = \left(\vec{\nabla} \cdot \Delta\vec{D}(\vec{r})\right) V(\vec{r}) + \Delta\vec{D}(\vec{r}) \cdot \vec{\nabla} V(\vec{r})$$

Thus: $$\left(\vec{\nabla} \cdot \Delta\vec{D}(\vec{r})\right) V(\vec{r}) = \vec{\nabla} \cdot \left[\Delta\vec{D}(\vec{r}) V(\vec{r})\right] - \Delta\vec{D}(\vec{r}) \cdot \vec{\nabla} V(\vec{r})$$

Then:
$$\boxed{\begin{aligned}\Delta W_{free} &= \int_{v'}\left(\vec{\nabla}\cdot\Delta\vec{D}(\vec{r})\right)V(\vec{r})\,d\tau' \\ &= \int_{v'}\vec{\nabla}\cdot\left[\Delta\vec{D}(\vec{r})V(\vec{r})\right]d\tau' - \int_{v'}\Delta\vec{D}(\vec{r})\cdot\vec{\nabla}V(\vec{r})\,d\tau' \quad \text{but} \quad \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \\ &= \int_{v'}\vec{\nabla}\cdot\left[\Delta\vec{D}(\vec{r})V(\vec{r})\right]d\tau' + \int_{v'}\Delta\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\,d\tau'\end{aligned}}$$

Using the divergence theorem and letting the surface $S' \to$ all space:

$$\Delta W_{free} = \oint_{\substack{S'\to all\\ space}} \Delta\vec{D}(\vec{r})\underbrace{\cancel{V(\vec{r})}}_{=0@r=\infty}\hat{n}dA' + \int_{\substack{v'\\ all\ space}} \Delta\vec{D}(\vec{r})\cdot E(\vec{r})\,d\tau'$$

$\therefore$
$$\boxed{\Delta W_{free} = \int_{\substack{v'\\ all\ space}} \Delta\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\,d\tau'}$$

For a Class-*A*/linear dielectric, $\vec{D}(\vec{r}) = \varepsilon\vec{E}(\vec{r})$ and thus for infinitesimal charge increments:

$$\frac{1}{2}\Delta\left[\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\right] = \frac{1}{2}\Delta\left(\varepsilon\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\right) = \frac{\varepsilon}{2}\Delta\left(E^2(\vec{r})\right) = \frac{\varepsilon}{\cancel{2}}\left(\cancel{2}\vec{E}(\vec{r})\cdot\Delta\vec{E}(\vec{r})\right)$$
$$= \varepsilon\left(\Delta\vec{E}(\vec{r})\right)\cdot\vec{E}(\vec{r}) = \Delta\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})$$

i.e.
$$\boxed{\frac{1}{2}\Delta\left[\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\right] = \Delta\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})}$$

Then: $$\Delta W_{diel} = \Delta W_{free} = \int_{\substack{v'\\ all\ space}} \Delta\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\,d\tau' = \frac{1}{2}\Delta\int_{\substack{v'\\ all\ space}}\left[\vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\right]d\tau'$$

Then: $$W_{diel}^{TOT} = \int dW_{diel} = \int \Delta W_{diel} = \frac{1}{2}\int_{\substack{v'\\ all\ space}} \vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\,d\tau' = P.E._{diel}^{TOT}$$

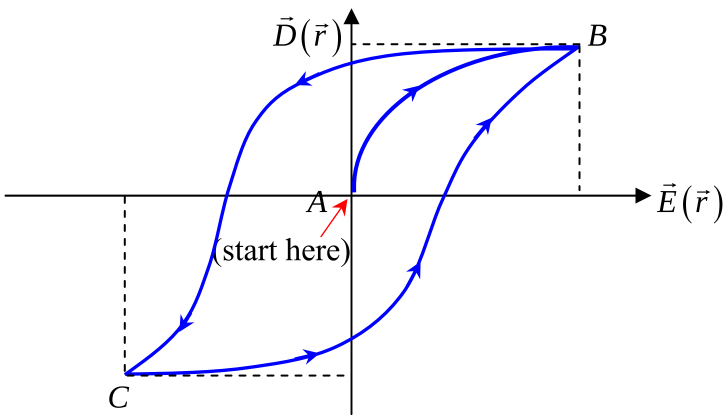
Or:
$$\boxed{W_{diel}^{TOT} = P.E._{diel}^{TOT} = \frac{1}{2}\int_{\substack{v'\\ all\ space}} \vec{D}(\vec{r})\cdot\vec{E}(\vec{r})\,d\tau'}$$

Note that for Non-Class-*A* / Non-Linear Dielectrics:

i.e.
$$\left.\left\{\begin{array}{l}\text{Non-ideal}\\ \text{Non-linear}\\ \text{Non-homogeneous}\\ \text{Non-isotropic}\\ \text{(anisotropic)}\end{array}\right.\right\}$$ dielectrics, THE ABOVE RELATION DOES ***NOT*** HOLD!!!

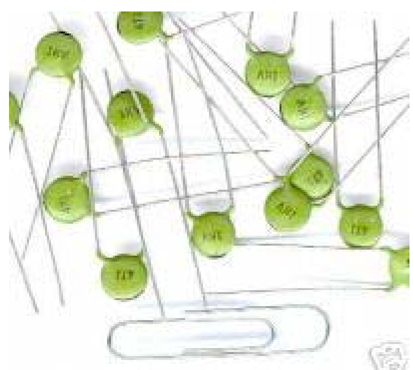
Why???

In general, the work done in polarizing (any kind of) dielectric material depends on the path taken in polarizing that dielectric (i.e. = work done in charging the capacitor with this dielectric material inside the capacitor) – because in some dielectric materials, the electric polarization $\vec{P}(\vec{r})$ is dependent on its past history and/or there may also various energy loss / dissipation / absorption mechanisms present/operative (which may also/can be frequency-dependent):



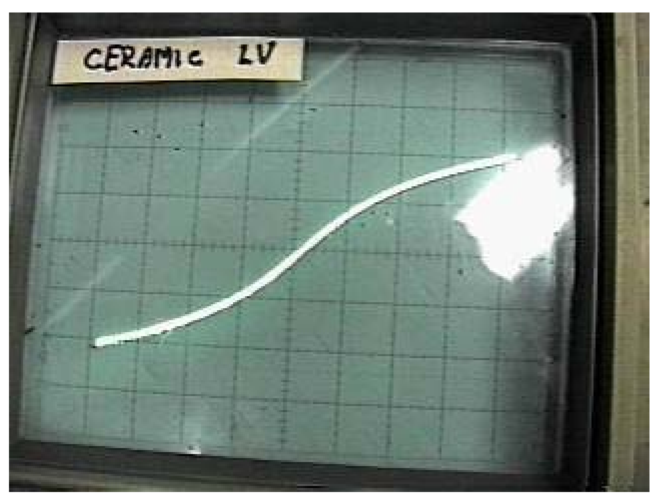
Real-world – some dielectric materials exhibit hysteresis (= "to lag behind") and have a non-linear relationship between $\vec{D}(\vec{r})$ and $\vec{E}(\vec{r})$.

Hysteresis Loop: *Area enclosed* = dissipative energy loss/energy absorbed (= Heat generated) per cycle in going around the hysteresis loop.



Physical/Practical Example: Ceramic disc capacitors (dielectric consists of un-polarized electret-type material) *do* have such a non-linear *D*-*E* relationship – one can easily observe this on an oscilloscope using e.g. a sine-wave audio signal from a sine-wave generator (for example of such a test circuit see e.g. problem 10.5 p.207 of *Electromagnetic Fields & Waves*, P. Lorrain, D. & F. Corson, 3rd Ed., W.H. Freeman, NY).

The use of ceramic disc capacitors in audio/hi-fi electronics versus *D*-*E* linear capacitors – one can (easily) *hear* the difference!!! A non-linear response generates additional harmonics (not present in original signal). In the case of ceramic disc caps, the detailed nature of additional harmonics generated by non-linear *D*-*E* response are such that they very definitely detract from the overall music/overall sound quality…



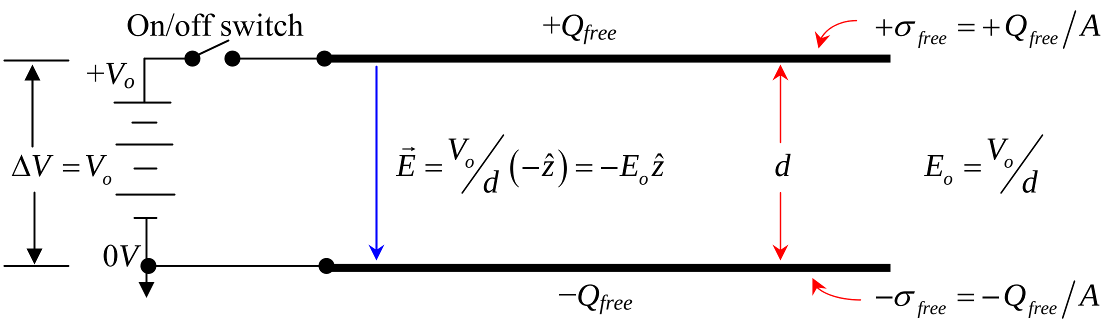
⇒ "Youz gitz watz youz payz fer!!!"

∃ Lot's of interesting info out on the internet about such things – if interested, visit/see e.g.:
http://members.aol.com/sbench102/caps.html
http://sound.westhost.com/articles/capacitors.htm

FORCES ON DIELECTRICS

Just as a conductor ($\chi_e \to \infty$) is attracted into an electric field, so too is a dielectric material ($\chi_e < \infty$). Induced bound charge on dielectrics accumulates nearest to free charges that are needed to generate the external electric field.

First, let us imagine charging up a parallel plate capacitor that initially has NO dielectric inside it. This is done e.g. by connecting up a battery to the plates of the parallel-plate capacitor, and then removing (i.e. disconnecting) the battery.



A = area of plate = $l \times w$ and $l, w >> d$

Since the electric charge on capacitor is fixed (with the battery disconnected!!!):

$$W_{vac} = \frac{1}{2}\frac{Q_{free}^2}{C_{vac}} \quad \text{where:} \quad C = C_{vac} = \frac{\varepsilon_o A}{d}$$

$$W_{vac} = \frac{1}{2}\frac{Q_{free}^2}{C_{vac}} = \frac{1}{2}\frac{Q_{free}^2}{\varepsilon_o A/d} = \frac{1}{2}\frac{Q_{free}^2 d}{\varepsilon_o A} \quad \text{with:} \quad \sigma_{free} = \frac{Q_{free}}{A} = \varepsilon_o E_o = \varepsilon_o V_o/d$$

$$\therefore \quad Q_{free} = \sigma_{free} A = \varepsilon_o E_o A = \varepsilon_o V_o\, A/d$$

If a block of Class-*A*/linear dielectric material of thickness *d* and area $A = l \times w$ is now inserted into this parallel-plate capacitor, completely filling the space between parallel plates of the capacitor (still with the battery disconnected) then Q_{free} on the plates remains the same, however there will be a drop in the potential (voltage) across the capacitor plates!

$$W_{diel} = \frac{1}{2}\frac{Q_{free}^2}{C_{diel}} \quad \text{where:} \quad C_{diel} = K_e\left(\frac{\varepsilon_o A}{d}\right) = K_e C_{vac}$$

$$\therefore W_{diel} = \frac{1}{2}\frac{Q_{free}^2}{K_e C_{vac}} = \left(\frac{1}{K_e}\right) W_{vac} \quad \text{i.e.} \quad \frac{W_{diel}}{W_{vac}} = \left(\frac{1}{K_e}\right) < 1 \quad (\text{if } K_e > 1)\ !!!$$

The stored energy in the dielectric capacitor (for constant Q_{free}) is reduced from that of the vacuum, for the same amount of free charge Q_{free} on the capacitor plates.

The "missing" energy has gone into internally polarizing the dielectric material:

i.e. $W_{\text{"miss"}} = W_{vac} - W_{diel} = (1 - 1/K_e)W_{vac}$ is the energy (temporarily) absorbed by the dielectric in polarizing itself in the $\vec{E}$-field of the capacitor. If the dielectric material is a Class-*A* / linear ideal/lossless dielectric, the capacitor will regain all of this "missing" energy when the dielectric is removed from the parallel-plate.

What is the potential difference ΔV_{diel} across the capacitor plates after the dielectric has been inserted into the capacitor gap, if the electric charge Q_{free} on capacitor is held constant?

If: $W_{vac} = \frac{1}{2}\frac{Q_{free}^2}{C_{vac}} = \frac{1}{2}C_{vac}\Delta V_{vac}^2 = \frac{1}{2}C_{vac}V_o^2$ and $W_{diel} = \frac{1}{2}\frac{Q_{free}^2}{C_{diel}} = \frac{1}{2}\frac{Q_{free}^2}{K_e C_{vac}} = \frac{1}{2}C_{diel}\Delta V_{diel}^2 = \left(\frac{1}{K_e}\right)W_{vac}$

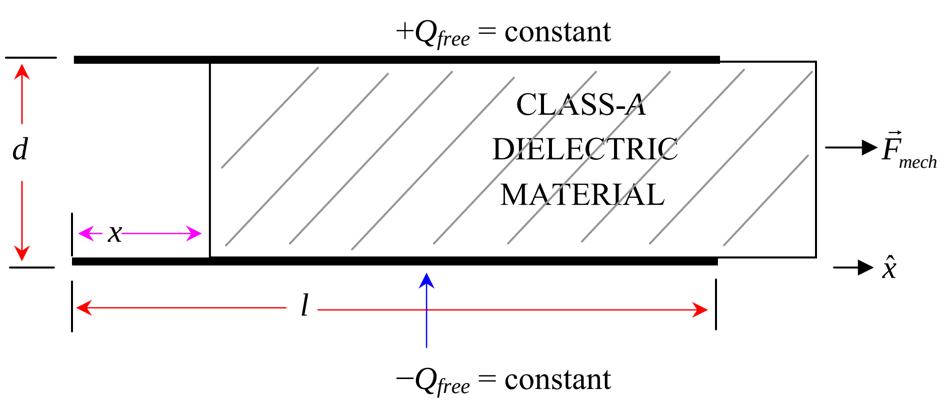
Then: $K_e W_{diel} = W_{vac} \Rightarrow \frac{1}{2}K_e C_{diel}\Delta V_{diel}^2 = \frac{1}{2}C_{vac}V_o^2$ but: $C_{diel} = K_e\left(\frac{\varepsilon_o A}{d}\right) = K_e C_{vac}$

$$\therefore \quad \frac{1}{2}K_e K_e C_{vac}\Delta V_{diel}^2 = \cancel{\frac{1}{2}}K_e^2 \cancel{C_{vac}}\Delta V_{diel}^2 = \cancel{\frac{1}{2}}\cancel{C_{vac}}V_o^2 \Rightarrow \boxed{\Delta V_{diel} = \Delta V_{vac}/K_e = V_o/K_e}$$

Now, since the overall energy of a parallel-plate capacitor is lower (by an amount $1/K_e$) with a dielectric material inside it than without it, because any isolated system wants to have minimum overall energy associated with it, there will be a force acting on the dielectric as it is inserted into the gap of the parallel-plate capacitor, acting so as to pull the dielectric into the parallel-plate capacitor, i.e. sucking it into the capacitor!!!

Thus, once the dielectric material is fully inside the parallel-plate capacitor, it will require a mechanical force to pull it out again.

SIDE VIEW:



$0 \le x \le l$
$x = 0$: Dielectric fully inside ‖-plate capacitor.
$x = l$: Empty ‖-plate capacitor (no dielectric).

The infinitesimal amount of mechanical work done in pulling out the slab of Class-*A* dielectric material of thickness *d* and area $A = l \times w$ an infinitesimal distance *dx* from the parallel-plate capacitor is:

$$dW_{mech} = \vec{F}_{mech} \bullet d\vec{x} = F_{mech} dx \quad \text{since } d\vec{x} = dx\hat{x} \text{ and } \vec{F}_{mech} \parallel \hat{x}\text{, i.e. } \vec{F}_{mech} = F_{mech}\hat{x}$$

Turning this around: $F_{mech} = \dfrac{dW_{mech}}{dx}$

Now for a partially-removed block of dielectric pulled out a distance *x* from this capacitor (where $0 \le x \le l$), the total capacitance (n.b. for capacitors wired in parallel – their capacitances add) is:

$$\begin{aligned} C_{TOT}(x) &= C_{diel}(x) + C_{vac}(x) \\ &= K_e \frac{\varepsilon_o A}{d}\left(\frac{l-x}{l}\right) + \frac{\varepsilon_o A}{d}\left(\frac{x}{l}\right) \\ &= K_e \frac{\varepsilon_o (l-x) w}{d} + \frac{\varepsilon_o x w}{d} \\ &= K_e \frac{\varepsilon_o l w}{d} + \frac{\varepsilon_o (x - K_e x) w}{d} \quad \text{but: } A = lw \\ &= K_e \frac{\varepsilon_o A}{d} - \frac{\varepsilon_o (K_e - 1) x w}{d} \qquad (K_e - 1) = \chi_e \\ &= K_e \frac{\varepsilon_o A}{d} - \frac{\varepsilon_o \chi_e x w}{d} \qquad \chi_e = (K_e - 1) \end{aligned}$$

$$\boxed{\begin{array}{l} 0 \le x \le l \\ x = 0\text{: Dielectric fully inside } \parallel\text{-plate capacitor.} \\ x = l\text{: Empty } \parallel\text{-plate capacitor (no dielectric).} \end{array}}$$

Thus: $W_{mech}(x) = \dfrac{1}{2}\dfrac{Q_{free}^2}{C_{TOT}(x)} = \dfrac{1}{2}\dfrac{Q_{free}^2}{\left[K_e \dfrac{\varepsilon_o A}{d} - \dfrac{\varepsilon_o \chi_e x w}{d}\right]}$ (Q_{free} = constant, here)

The electrostatic force must be equal in magnitude but opposite in direction to the mechanical force, i.e.:

$$\vec{F}_{EM}(x) = -\vec{F}_{mech}(x) = F_{mech}(x)(-\hat{x}) = -\frac{dW_{mech}(x)}{dx}\hat{x}$$

Then: $$F_{EM}(x) = -\frac{dW_{mech}(x)}{dx} = -\frac{d}{dx}\frac{1}{2}\left(\frac{Q_{free}^2}{C_{TOT}(x)}\right) = +\frac{1}{2}\left(\frac{Q_{free}}{C_{TOT}(x)}\right)^2 \frac{dC_{TOT}(x)}{dx}$$

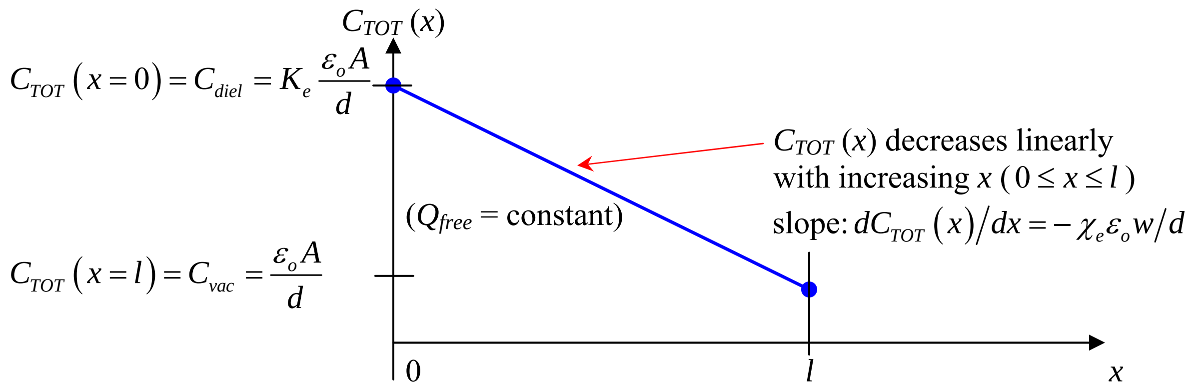
But: $C_{TOT}(x) = K_e \dfrac{\varepsilon_o A}{d} - \dfrac{\varepsilon_o \chi_e x w}{d}$ and $Q_{free} = C(x)\Delta V(x)$ or: $\Delta V(x) = \left(\dfrac{Q_{free}}{C_{TOT}(x)}\right)$

$$\therefore \frac{dC_{TOT}(x)}{dx} = -\frac{\varepsilon_o \chi_e w}{d}$$

$$\therefore F_{EM}(x) = \frac{1}{2}\left(\frac{Q_{free}}{C_{TOT}(x)}\right)^2 * \left(-\frac{\varepsilon_o \chi_e w}{d}\right)$$

or: $$\boxed{\vec{F}_{EM}(x) = -\frac{\varepsilon_o \chi_e w}{2d}\left(\frac{Q_{free}}{C_{TOT}(x)}\right)^2 \hat{x}} \quad \text{with: } \boxed{C_{TOT}(x) = K_e \frac{\varepsilon_o A}{d} - \frac{\varepsilon_o \chi_e x w}{d}}$$

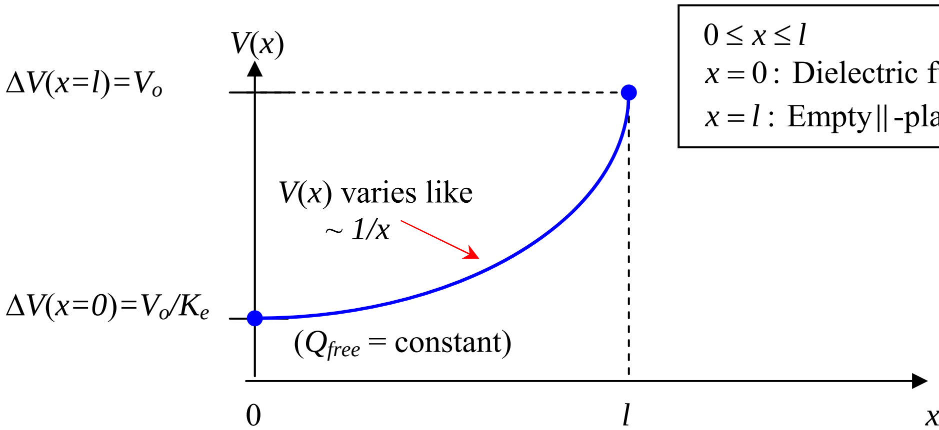
THE ELECTROSTATIC FORCE ON DIELECTRIC (Q_{free} fixed) WITH $0 \le x \le l$:



Initial potential difference of battery

$$\Delta V(x) = \left(\frac{Q_{free}}{C_{TOT}(x)}\right)$$

Fixed / constant, $Q_{free} = \varepsilon_o E_o A = \varepsilon_o V_o A/d$



$0 \le x \le l$
$x = 0$: Dielectric fully inside ||-plate capacitor.
$x = l$: Empty || -plate capacitor (no dielectric).

The *EM* force acts such as to "suck" the dielectric into the gap between the parallel plate capacitor. Just as the dielectric is being inserted into the gap of the parallel-plate capacitor (i.e. when $x = l$), the magnitude of this force is:

$$\boxed{\left|F_{EM}(x=l)\right| = \frac{\varepsilon_o \chi_e w}{2d}\left(\frac{Q_{free}}{C_{TOT}(x=l)}\right)^2 = \frac{\chi_e Q_{free}^2 d}{2\varepsilon_o l^2 w}} \quad \text{with} \quad \boxed{C_{TOT}(x=l) = \varepsilon_o \frac{A}{d} = \varepsilon_o \frac{lw}{d}}$$

However, when the dielectric is fully inserted into the capacitor gap (when $x = 0$) the *EM* force does not vanish! The magnitude of this force is:

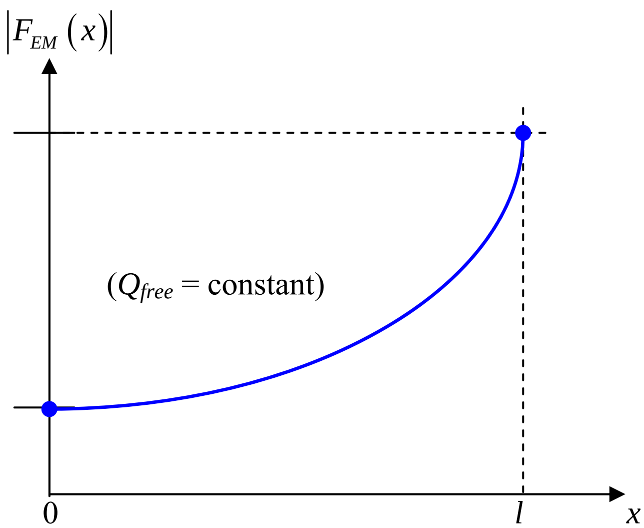
$$\boxed{\left|F_{EM}(x=0)\right| = \frac{\varepsilon_o \chi_e w}{2d}\left(\frac{Q_{free}}{C_{TOT}(x=0)}\right)^2 = \frac{\chi_e Q_{free}^2 d}{2K_e^2 \varepsilon_o l^2 w}} \quad \text{with} \quad \boxed{C_{TOT}(x=0) = K_e \varepsilon_o \frac{A}{d} = K_e \varepsilon_o \frac{lw}{d}}$$

Note that $\left|F_{EM}(x=0)\right| = \left|F_{EM}(x=l)\right|/K_e^2$, i.e. the *EM* force when $x = 0$ (dielectric fully inserted into the capacitor gap) is reduced by a factor of K_e^2 relative to the EM force when $x = l$ (no dielectric inside the gap of the parallel-plate capacitor).

The magnitude of the *EM* force acting on the dielectric as a function of the displacement distance *x* of the dielectric in the gap of the parallel-plate capacitor is shown in the figure below, for the case where the free charge on the plates of the capacitor is held constant:

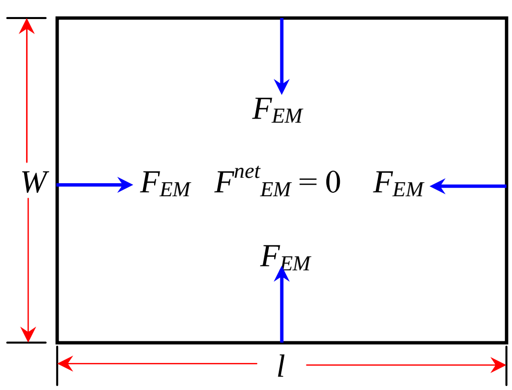
$$\left|F_{EM}\left(x=l\right)\right|=\frac{\chi_e Q_{free}^2 d}{2\varepsilon_o l^2 w}$$

$$\left|F_{EM}\left(x=0\right)\right|=\frac{\chi_e Q_{free}^2 d}{2K_e^2\varepsilon_o l^2 w}=\frac{1}{K_e^2}\left|F_{EM}\left(x=l\right)\right|$$



$0\le x\le l$
$x=0$: Dielectric fully inside $\|$-plate capacitor.
$x=l$: Empty $\|$-plate capacitor (no dielectric).

The *EM* force acting on the dielectric does <u>not</u> vanish when $x = 0$ (as one might <u>*initially*</u> think, from Newton's 2nd Law of Motion: $\vec{F}_{mech}\left(x=0\right)=-\vec{F}_{EM}\left(x=0\right)=m\vec{a}_{dielectric}\left(x=0\right)$ because the *EM* force simultaneously acts <u>on all four sides of the dielectric</u>. Thus, when $x = 0$ these four negative inward forces <u>balance</u> each other – compressing the dielectric – such that the <u>*net*</u> *EM* force on the dielectric when $x = 0$ is <u>zero</u>, as shown below in an overhead view of the parallel plate capacitor:



Thus, we can now see that the *EM* force acting on each side of the parallel-plate capacitor constitutes a negative (i.e. inward) pressure $P_{side}^{\perp}=F_{EM}/A_{side}^{\perp}$ acting on each side of the dielectric – i.e. trying to compress it inwards, towards the center of the dielectric.

The *EM* pressure acting on the dielectric when the dielectric is fully inserted into the gap of the parallel-plate capacitor (i.e. at $x = 0$) is:

$$P_{side}^{\perp}(x=0) = F_{EM}(x=0)/A_{side}^{\perp} = \frac{\chi_e Q_{free}^2 \cancel{d}}{2K_e^2 \varepsilon_o l^2 w}\left(\frac{1}{w\cancel{d}}\right) = \frac{\chi_e Q_{free}^2}{2K_e^2 \varepsilon_o l^2 w^2} = \frac{\chi_e Q_{free}^2}{2K_e^2 \varepsilon_o A^2} = \frac{\varepsilon_o \chi_e V_o^2}{2K_e^2 d^2}$$

The *EM* pressure acting on the dielectric when the dielectric is fully inserted into the gap of the parallel-plate capacitor (i.e. at $x = 0$) when there is no dielectric inside the gap of the parallel-plate capacitor (i.e. at $x = l$) is:

$$P_{side}^{\perp}(x=l) = F_{EM}(x=l)/A_{side}^{\perp} = \frac{\chi_e Q_{free}^2 \cancel{d}}{2\varepsilon_o l^2 w}\left(\frac{1}{w\cancel{d}}\right) = \frac{\chi_e Q_{free}^2}{2\varepsilon_o l^2 w^2} = \frac{\chi_e Q_{free}^2}{2\varepsilon_o A^2} = \frac{\varepsilon_o \chi_e V_o^2}{2d^2}$$

Thus we see that $P_{side}^{\perp}(x=l) = K_e^2 P_{side}^{\perp}(x=0)$ - i.e. the *EM* pressure acting on the dielectric is largest (by a factor of K_e^2) when it is held just outside the gap of the parallel plate capacitor (when $x = 0$) compared to when the dielectric is fully inside the gap of the capacitor (when $x = l$).

We now consider the situation when the battery is <u>always</u> connected to the parallel-plate capacitor during the removal / insertion of the dielectric material. Then $\Delta V = V_o =$ <u>constant</u> potential difference across the capacitor plates <u>at all times</u>, thus the free charge on the plates $Q_{free}(x)$ varies/is <u>not</u> constant as the dielectric is inserted/removed!

$$\Rightarrow \frac{Q_{free}(x)}{C_{TOT}(x)} = \Delta V(x) = V_o = \text{constant}$$

$$\Rightarrow Q_{free}(x) = V_o C_{TOT}(x) \neq \text{constant for } \Delta V = V_o = \text{constant}$$

For $\Delta V = V_o =$ constant, note that the battery <u>also</u> does work on the dielectric:

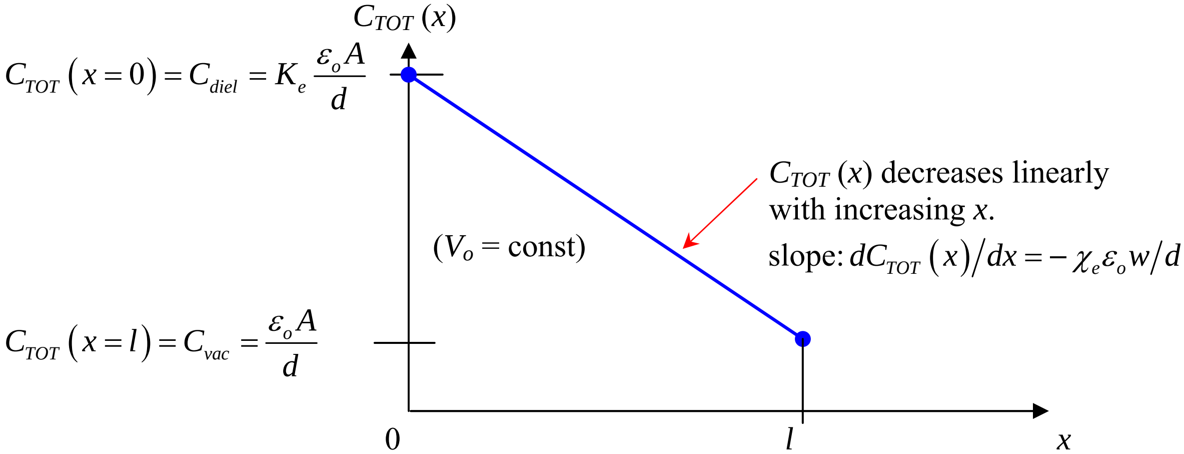
$$F_{EM}(x) = -\frac{dW_{mech}(x)}{dx} - \frac{dW_{batt}(x)}{dx} = -\frac{dW_{mech}(x)}{dx} + V_o\frac{dQ_{free}(x)}{dx}$$

<u>Here</u>: $\boxed{W_{mech}(x) = \frac{1}{2}C_{TOT}(x)V_o^2}$ for V_o held constant

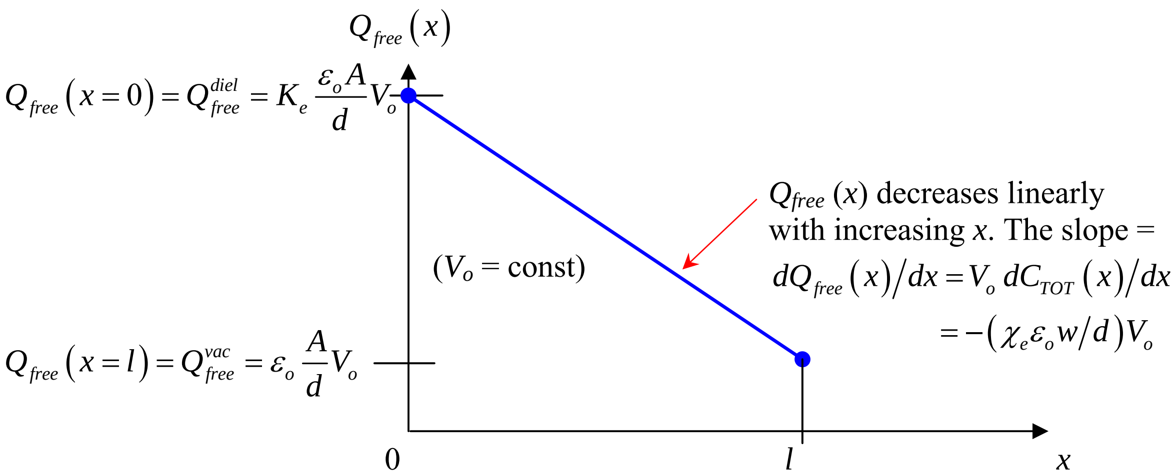
$\therefore$ $\boxed{F_{EM}(x) = -\frac{1}{2}V_o^2\frac{dC_{TOT}(x)}{dx} + V_o^2\frac{dC_{TOT}(x)}{dx} = +\frac{1}{2}V_o^2\frac{dC_{TOT}(x)}{dx}}$ <u>but</u>: $\boxed{\frac{dC_{TOT}(x)}{dx} = -\frac{\varepsilon_o \chi_e w}{d}}$

<u>Thus</u>: $\boxed{F_{EM}(x) = \frac{1}{2}V_o^2\left(-\frac{\varepsilon_o \chi_e w}{d}\right) = -\frac{\varepsilon_o \chi_e w}{2d}V_o^2}$ = constant force (!!!) for V_o held constant.

<u>Or</u>: $\boxed{\vec{F}_{EM}(x) = -\frac{\varepsilon_o \chi_e w}{2d}V_o^2\hat{x}}$ = constant force (!!!) for V_o held constant.



$0 \le x \le l$
$x=0$: Dielectric fully inside ||-plate capacitor.
$x=l$: Empty ||-plate capacitor (no dielectric).

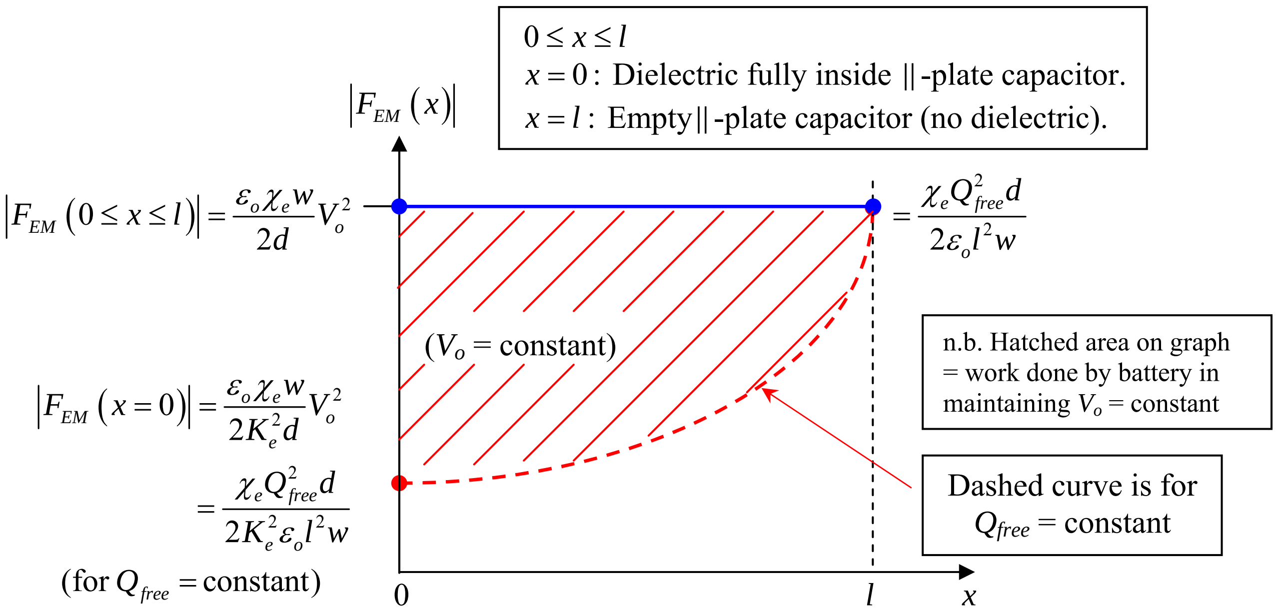


Here again, for this situation where V_o = constant, a net EM force acts on the capacitor, trying to "suck" it into the gap of the parallel plate capacitor:

$$\boxed{F_{EM}(x)=\frac{1}{2}V_o^2\left(-\frac{\varepsilon_o\chi_e w}{d}\right)=-\frac{\varepsilon_o\chi_e w}{2d}V_o^2}=\text{constant force (!!!) for } V_o \text{ held constant.}$$

Thus we see that the *EM* force $F_{EM}(x)$ acting on the dielectric is independent of the position x of the dielectric in the gap of the parallel plate capacitor, for the case of for V_o held constant across the plates of the parallel plate capacitor.

A plot of $F_{EM}(x)$ vs. x for V_o held constant across the plates of the parallel plate capacitor is shown in the figure below:



The work done by the battery in maintaining constant potential difference V_o across the plates of the parallel plate capacitor while the dielectric is inserted into the gap of the capacitor is:

$$\boxed{W_{batt} = \int_{x=l}^{x=0} \vec{F}_{batt}(x) \bullet d\vec{x}} \quad \underline{\text{but}}: \boxed{\vec{F}_{batt}(x) = +V_o \frac{dQ_{free}(x)}{dx}} \quad \underline{\text{where}}: \boxed{Q_{free}(x) = C_{TOT}(x) V_o}$$

<u>Thus</u>: $$\boxed{\frac{dQ_{free}(x)}{dx} = V_o \frac{dC_{TOT}(x)}{dx} = -\frac{\varepsilon_o \chi_e w}{d} V_o = \text{constant!}}$$

<u>Then</u>: $$\boxed{\vec{F}_{batt}(x) = +V_o \frac{dQ_{free}(x)}{dx} \hat{x} = -\frac{\varepsilon_o \chi_e w}{d} V_o^2 \hat{x}}$$

<u>Thus</u>:

$$\boxed{\begin{aligned} W_{batt} &= \int_{x=l}^{x=0} \vec{F}_{batt}(x) \bullet d\vec{x} = \int_{x=l}^{x=0} \left(-\frac{\varepsilon_o \chi_e w}{d} V_o^2 \hat{x} \right) \bullet d\vec{x} \\ &= -\frac{\varepsilon_o \chi_e w}{d} V_o^2 [0 - l] = \frac{\varepsilon_o \chi_e l w}{d} V_o^2 = \chi_e \frac{\varepsilon_o A}{d} V_o^2 = \chi_e C(x = l) V_o^2 \end{aligned}}$$

Since the *EM* force acting on the dielectric is constant (i.e. independent of *x*) for the case where the potential difference across the capacitor plates is held constant as the dielectric is inserted into the gap of the parallel plate capacitor, then we see that the *EM* pressure (= *EM* force/unit area) is also a constant, independent of *x* for this situation:

$$\boxed{P_{side}^{\perp}(x) = F_{EM}(x) / A_{side}^{\perp} = \frac{\varepsilon_o \chi_e \cancel{w}}{2d} V_o^2 \left(\frac{1}{\cancel{w} d} \right) = \frac{\varepsilon_o \chi_e V_o^2}{2d^2}} \quad (\text{for } V_o = \text{constant})$$

Note that this *EM* pressure is the <u>same</u> as that for the case when $Q_{free} = \text{constant}$ when $x = l$ (i.e. no dielectric in the gap of the parallel plate capacitor):

$$\boxed{P_{side}^{\perp}(x = l) = F_{EM}(x = l) / A_{side}^{\perp} = \frac{\chi_e Q_{free}^2 \cancel{d}}{2\varepsilon_o l^2 w} \left(\frac{1}{w \cancel{d}} \right) = \frac{\chi_e Q_{free}^2}{2\varepsilon_o l^2 w^2} = \frac{\chi_e Q_{free}^2}{2\varepsilon_o A^2} = \frac{\varepsilon_o \chi_e V_o^2}{2d^2}}$$