LECTURE NOTES 8

POTENTIAL APPROXIMATION TECHNIQUES: THE ELECTRIC MULTIPOLE EXPANSION AND MOMENTS OF THE ELECTRIC CHARGE DISTRIBUTION

There are often situations that arise where an "observer" is far away from a localized charge distribution $\rho(\vec{r})$ and wants to know what the potential $V(\vec{r})$ and / or the electric field intensity $E(\vec{r})$ are far from the localized charge distribution.

If the localized charge distribution has a net electric charge Q_{net}, then far away from this localized charge distribution, the potential $V(\vec{r})$ to a good approximation will behave very much like that of a point charge,

$$\boxed{V_{far}(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_o}\frac{Q_{net}}{\mathbf{r}}} \quad \text{and} \quad \boxed{\vec{E}_{far}(\vec{r}) = -\vec{\nabla}V_{far}(\vec{r}) \simeq -\frac{1}{4\pi\varepsilon_o}\frac{Q_{net}}{\mathbf{r}^2}}$$

when the field point – source charge separation distance, $\mathbf{r} \gg d$, the characteristic size of the charge distribution.

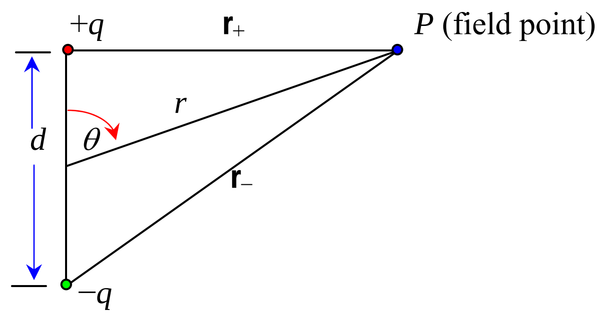
However, as the "observer" moves in closer and closer to the localized charge distribution $\rho(\vec{r}')$, he/she will discover that increasingly $V(\vec{r})$ (and hence $\vec{E}(\vec{r})$) may deviate more and more from pure point charge behavior, because $\rho(\vec{r}')$ is an ***extended*** source charge distribution.

Furthermore, $\rho(\vec{r}')$ may be such that $Q_{net} \equiv 0$, but that does NOT necessarily imply that $V(\vec{r}) = 0$ (and $\vec{E}(\vec{r})=0$)!

Example:

A pure, *physical* electric dipole is a spatially-extended, simple charge distribution where $Q_{net} = 0$ but $V(\vec{r}) \neq 0$ and $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \neq 0$, as shown in the figure below:

A pure physical electric dipole is composed of two opposite electric charges separated by a distance d:



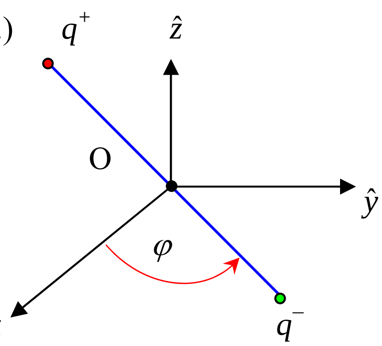
The Potential $V(\vec{r})$ and Electric Field $\vec{E}(\vec{r})$ of a Pure Physical Electric Dipole

"Pure" → $Q_{net} = 0$ "*Physical*" → Spatially extended electric eipole $d \neq 0$, $d > 0$
{n.b. ∃ "point" electric dipoles with $d = 0$, e.g. neutral atoms & molecules…}

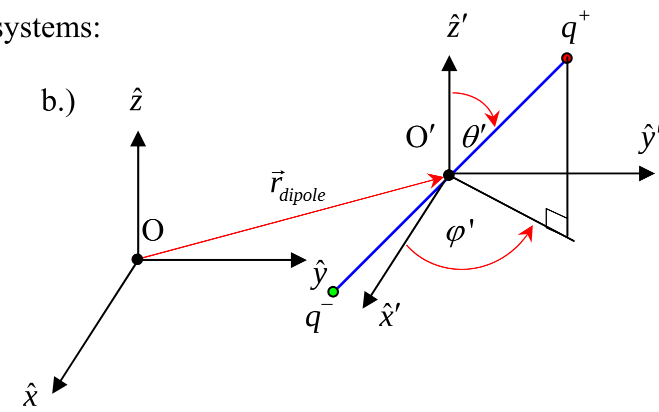
First, let us be very careful / wise as to our choice of coordinate system. A wrong choice of coordinate system will unnecessarily complicate the mathematics and obscure the physics we are attempting to learn about the nature / behavior of this system.

Examples of BAD choices of coordinate systems:

a.)



b.)



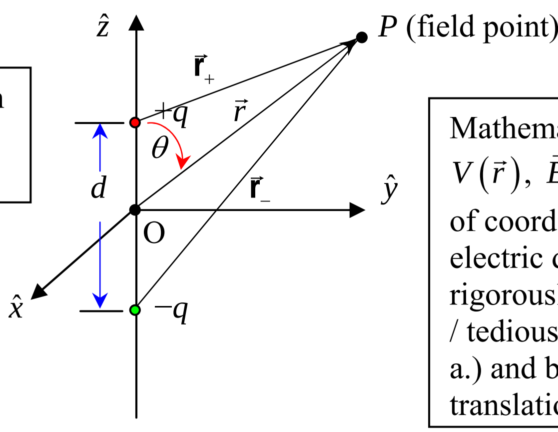
Dipole lying in $x-y$ plane has φ-dependence, but (at least it) is centered at the origin.

Even more mathematically complicated!! Origin is not conveniently chosen (arbitrary?) Angle the dipole axis makes with respect to $\hat{z}$ & $\hat{x}$ axes must be described by two angles - θ and φ.

Smart / wise choice of coordinate system: Exploit intrinsic symmetry of problem.

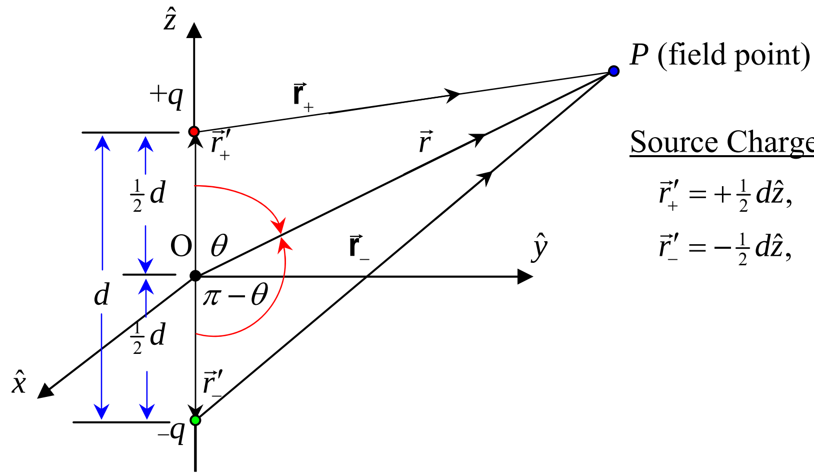
Physical electric dipole has axial symmetry – choose $\hat{z}$ axis to be along line separating q^+ and q^-. Choose *x*-*y* plane to lie mid-way between q^+ and q^-:

n.b. This problem now has no φ-dependence



Mathematical expressions obtained for $V(\vec{r})$, $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ for this choice of coordinate system for the physical electric dipole can be explicitly and rigorously related to more complicated / tedious mathematical expressions for a.) and b.) above – via coordinate translations & rotations!

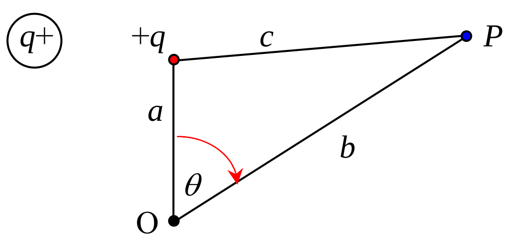
Pure, Physical Electric Dipole:

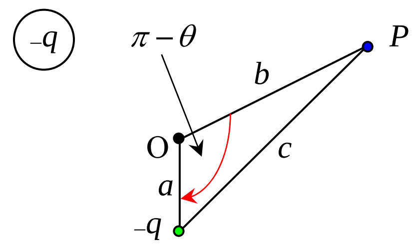


Source Charge Locations

$$\vec{r}'_+ = +\tfrac{1}{2}d\hat{z}, \quad |\vec{r}'_+| = \tfrac{1}{2}d$$
$$\vec{r}'_- = -\tfrac{1}{2}d\hat{z}, \quad |\vec{r}'_-| = \tfrac{1}{2}d$$

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$





$$\mathbf{r}_+^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2\left(\frac{d}{2}\right) r\cos\theta$$
$$= \left(\frac{d}{2}\right)^2 + r^2 - dr\cos\theta$$
$$= r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta$$

$$\mathbf{r}_-^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2\left(\frac{d}{2}\right) r\cos(\pi-\theta)$$
$$= \left(\frac{d}{2}\right)^2 + r^2 + dr\cos\theta$$
$$= r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta$$

Use Principle of Linear Superposition for Total Potential:

$$V_{TOT}(\vec{r}) = V_{+q}(\vec{r}) + V_{-q}(\vec{r}) \equiv V_{dipole}(\vec{r})$$

$$V_{+q}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{+q}{\mathbf{r}_+} = \frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}} = \frac{+q}{4\pi\varepsilon_o}\frac{1}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}}$$
$$V_{-q}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{-q}{\mathbf{r}_-} = \frac{1}{4\pi\varepsilon_o}\frac{-q}{\sqrt{r^2 + (d/2)^2 + rd\cos\theta}} = \frac{-q}{4\pi\varepsilon_o}\frac{1}{\sqrt{r^2 + (d/2)^2 + rd\cos\theta}}$$

$$\therefore \quad V_{dipole}(\vec{r}) = V_{+q}(\vec{r}) + V_{-q}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{+q}{\mathbf{r}_+} - \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathbf{r}_-}$$
$$= \frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}} - \frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{r^2 + (d/2)^2 + rd\cos\theta}}$$
$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}} - \frac{1}{\sqrt{r^2 + (d/2)^2 + rd\cos\theta}}\right]$$

This is an <u>exact</u> analytic mathematical expression for the potential associated with a pure $(Q_{net} = 0)$ physical electric dipole with charges $+q$ and $-q$ separated from each other by a distance d. Note further that, because of the judicious choice of coordinate system and the intrinsic (azimuthal) symmetry, $V_{dipole}(\vec{r})$ has no φ-dependence.

The exact analytic expression for potential associated with pure physical electric dipole:

$$V_{dipole}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}} - \frac{1}{\sqrt{r^2 + (d/2)^2 + rd\cos\theta}}\right\}$$

As mentioned earlier, often we are / will be interested <u>only</u> in knowing (<u>approximately</u>) $V_{dipole}(\vec{r})$ when $|\vec{r}| \gg d$. For example, many kinds of neutral molecules have permanent electric dipole <u>moments</u> $\vec{p} \equiv q\vec{d}$ (Coulomb-meters) and (obviously) for such molecules, the dipole's separation distance d is (typically) on the order of ~ few Ångstroms, i.e. $d \sim O(5Å)$ $\{1 Å \equiv 10^{-10}\, m = 10\, nm\ (1\, nm = 10^{-9}\, m)\}$. So even if the field point P is e.g. $|\vec{r}| = 1\mu m = 10^{-6} m$ away from such a molecular dipole, $|\vec{r}| = 1\mu m \gg d \sim 5nm$, since $d/|\vec{r}| \simeq 0.005$!

In such situations, when $|\vec{r}| \gg d$ an <u>*approximate*</u> solution for $V_{dipole}(\vec{r})$ which has the benefit of <u>reduced</u> <u>mathematical</u> <u>complexity</u>, will <u>suffice</u> to give a <u>*good*</u> / <u>*reasonable*</u> physical description of the intrinsic physics, accurate e.g. <u>to 1% (or better)</u> when compared directly to the <u>exact analytical expression</u> over the <u>range</u> of distance scales $|\vec{r}| \gg d$ that are of interest to us.

Thus for $|\vec{r}| > d$, the exact expressions for the $\mathbf{r}_+$ and $\mathbf{r}_-$ separation distances are:

$$\mathbf{r}_+ = \sqrt{r^2 + (d/2)^2 - rd\cos\theta} = r\sqrt{1 + \left(\frac{d}{2}\Big/r\right)^2 - \left(\frac{d}{r}\right)\cos\theta} = r\sqrt{1 + \frac{1}{4}\left(\frac{d}{r}\right)^2 - \left(\frac{d}{r}\right)\cos\theta}$$

$$\mathbf{r}_- = \sqrt{r^2 + (d/2)^2 + rd\cos\theta} = r\sqrt{1 + \left(\frac{d}{2}\Big/r\right)^2 + \left(\frac{d}{r}\right)\cos\theta} = r\sqrt{1 + \frac{1}{4}\left(\frac{d}{r}\right)^2 + \left(\frac{d}{r}\right)\cos\theta}$$

Now if $(d/r) \ll 1$, then let us define:

$$\varepsilon_+ \equiv \frac{1}{4}\left(\frac{d}{r}\right)^2 - \left(\frac{d}{r}\right)\cos\theta \qquad \text{and:} \qquad \varepsilon_- \equiv \frac{1}{4}\left(\frac{d}{r}\right)^2 + \left(\frac{d}{r}\right)\cos\theta$$

Then: $\dfrac{1}{\mathbf{r}_+} = \dfrac{1}{r\sqrt{1+\varepsilon_+}}$ and: $\dfrac{1}{\mathbf{r}_-} = \dfrac{1}{r\sqrt{1+\varepsilon_-}}$

with: $\varepsilon_+ \ll 1$ and: $\varepsilon_- \ll 1$

Now if $\varepsilon_+ \ll 1$ and $\varepsilon_- \ll 1$, we can use the Binomial Expansion (a specific version of the more generalized Taylor Series Expansion) of the expression:

$$\frac{1}{\sqrt{1+\varepsilon_\pm}} = (1+\varepsilon_\pm)^{-1/2} = 1 - \frac{1}{2}\varepsilon_\pm + \frac{1\bullet 3}{2\bullet 4}\varepsilon_\pm^{\ 2} - \frac{1\bullet 3\bullet 5}{2\bullet 4\bullet 6}\varepsilon_\pm^{\ 3} + \ldots - \ldots \text{ (Valid on the interval: } -1 \le \varepsilon_\pm \le +1\text{)}$$

Since $\varepsilon_\pm$ is already <<1, then the higher-order terms $(\varepsilon_\pm)^2, (\varepsilon_\pm)^3, (\varepsilon_\pm)^4, \ldots$ etc. are incredibly small (<<<<<1), so negligible error is incurred by *neglecting* these higher-order terms, i.e. keeping *only* terms *linear* in $\varepsilon_\pm$ in the binomial expansion of $\dfrac{1}{\sqrt{1+\varepsilon_\pm}}$, we have:

$$\frac{1}{\mathbf{r}_+} = \frac{1}{r\sqrt{1+\varepsilon_+}} \simeq \frac{1}{r}\left(1-\tfrac{1}{2}\varepsilon_+\right) \qquad \text{and:} \qquad \frac{1}{\mathbf{r}_-} = \frac{1}{r\sqrt{1+\varepsilon_-}} \simeq \frac{1}{r}\left(1-\tfrac{1}{2}\varepsilon_-\right)$$

Then:

$$V^{dipole}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\mathbf{r}_+} - \frac{1}{\mathbf{r}_-}\right\} \simeq \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{r}\left(1-\tfrac{1}{2}\varepsilon_+\right) - \frac{1}{r}\left(1-\tfrac{1}{2}\varepsilon_-\right)\right\}$$

$$= \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{x}\right)\left\{\cancel{1} - \tfrac{1}{2}\varepsilon_+ - \cancel{1} + \tfrac{1}{2}\varepsilon_-\right\} = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left\{\left(\tfrac{1}{2}\right)\left(\varepsilon_- - \varepsilon_+\right)\right\}$$

Now: $\varepsilon_+ \equiv \dfrac{1}{4}\left(\dfrac{d}{r}\right)^2 - \left(\dfrac{d}{r}\right)\cos\theta$ and: $\varepsilon_- \equiv \dfrac{1}{4}\left(\dfrac{d}{r}\right)^2 + \left(\dfrac{d}{r}\right)\cos\theta$

$$V_{dipole}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left\{\left(\frac{1}{2}\right)\left[\left(\cancel{\left(\frac{1}{4}\right)\left(\frac{d}{r}\right)^2} + \left(\frac{d}{r}\right)\cos\theta\right) - \left(\cancel{\left(\frac{1}{4}\right)\left(\frac{d}{r}\right)^2} - \left(\frac{d}{r}\right)\cos\theta\right)\right]\right\}$$

Then:

$$= \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left(\frac{1}{2}\right)\left\{\left(\frac{d}{r}\right)\cos\theta + \left(\frac{d}{r}\right)\cos\theta\right\}$$

$$= \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left(\frac{1}{\cancel{2}}\right)\left\{\cancel{2}\left(\frac{d}{r}\right)\cos\theta\right\} = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left(\frac{d}{r}\right)\cos\theta$$

Thus:

$$\boxed{V_{dipole}(\vec{r}) \simeq \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left(\frac{d}{r}\right)\cos\theta = \frac{q}{4\pi\varepsilon_o}\left(\frac{d}{r^2}\right)\cos\theta = \frac{qd}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\cos\theta}$$

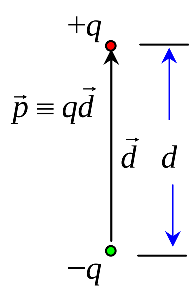
<u>The Magnitude of the Electric Dipole Moment</u>: $\boxed{p \equiv qd = |\vec{p}|}$

Thus, we may also express the potential of a pure physical dipole as:

$$\boxed{V_{dipole}(\vec{r}) = \frac{qd}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\cos\theta = \frac{p}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\cos\theta} \qquad \text{(valid for } d << r)$$

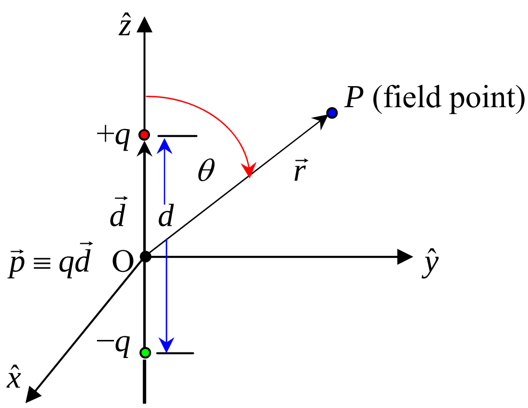
Note that: $\boxed{V_{dipole}(\vec{r}) \sim \frac{1}{r^2}}$ whereas $\boxed{V_{monopole}(\vec{r}) \sim \frac{1}{r}}$ (valid for point charge q located at origin)

We define the vector electric dipole moment as: $\boxed{\vec{p} \equiv q\vec{d}}$ where the charge-separation distance vector $\vec{d}$ points (by convention) <u>*from*</u> $-q$ <u>*to*</u> $+q$:



SI Units of p = Coulomb-meters

In our current situation here we see that $\vec{d} = d\hat{z}$:



Thus here if: $\vec{p} = q\vec{d} = qd\hat{z}$ but: $\hat{z} = \cos\theta\,\hat{r}$ then: $\vec{p} = q\vec{d} = qd\hat{z} = qd\cos\theta\,\hat{r} = p\cos\theta\,\hat{r}$

<u>Then</u>: $V_{dipole}(\vec{r}) \simeq \frac{qd}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\cos\theta = \frac{qd\cos\theta}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right) = \frac{p\cos\theta}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)$

The potential $V_{dipole}(\vec{r})$ associated with an electric dipole moment $\vec{p}$ $\left(\vec{p} = q\vec{d} = qd\hat{z}\right)$ from a <u>pure</u>, <u>physical</u> electric dipole oriented with $\vec{d} = d\hat{z}$, for $|\vec{r}| \gg |\vec{d}|$ is thus given by:

$$\boxed{V_{dipole}(\vec{r}) \simeq \frac{p\cos\theta}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right) = \frac{\vec{p}\bullet\hat{r}}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)} \quad \text{where: } \boxed{\vec{p}\bullet\hat{r} = p\cos\theta = qd\cos\theta}$$

The electric field $\vec{E}_{dipole}(\vec{r})$ associated with a pure, physical electric dipole, with electric dipole moment $\vec{p} = q\vec{d} = qd\hat{z}$ is:

$\vec{E}_{dipole}(\vec{r}) = -\vec{\nabla}V_{dipole}(\vec{r}) = E_r^{dipole}(\vec{r})\hat{r} + E_\theta^{dipole}(\vec{r})\hat{\theta} + E_\varphi^{dipole}(\vec{r})\hat{\varphi}$ in spherical-polar coordinates.

The components of $\vec{E}_{dipole}(\vec{r})$ in spherical-polar coordinates are:

$$E_r^{dipole}(\vec{r}) = -\frac{\partial V_{dipole}(\vec{r})}{\partial r} = \frac{1}{4\pi\varepsilon_o}\frac{2p}{r^3}\cos\theta$$

$$E_\theta^{dipole}(\vec{r}) = -\frac{1}{r}\frac{\partial V_{dipole}(\vec{r})}{\partial \theta} = \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}\sin\theta$$

$$E_\varphi^{dipole}(\vec{r}) = -\frac{1}{r\sin\theta}\frac{\partial V_{dipole}(\vec{r})}{\partial \varphi} = 0$$

Explicitly, the electric field intensity of a pure, physical electric dipole with electric dipole moment $\vec{p} = q\vec{d} = qd\hat{z}$ (in spherical-polar coordinates) is:

$$\vec{E}_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{2p}{r^3}\cos\theta\hat{r} + \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}\sin\theta\hat{\theta} = \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

Note that: $\left|\vec{E}_{dipole}(\vec{r})\right| \sim \frac{1}{r^3}$ (*c.f. w/* $\vec{E}_{monopole}(\vec{r}) \sim \frac{1}{r^2}$ for single point charge q at $\vec{r} = 0$).

Note also that $V_{dipole}(\vec{r})$ and $\vec{E}_{dipole}(\vec{r})$ have no explicit φ-dependence, since the charge configuration for an electric dipole is manifestly axially / azimuthally symmetric (i.e. charge configuration for electric dipole is invariant under arbitrary φ-rotations).

Now: $V_{dipole}(\vec{r}) = \frac{\vec{p}\bullet\hat{r}}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)$ with electric dipole moment $\vec{p} = qd\hat{z}$, and $\vec{p}\bullet\hat{r} = p\cos\theta = qd\cos\theta$, (since $\hat{z}\bullet\hat{r} = \cos\theta$), and $r^2 = x^2 + y^2 + z^2$ in Cartesian/rectangular coordinates.

In Cartesian/rectangular coordinates the electric field intensity of a pure, physical electric dipole with electric dipole moment $\vec{p} = q\vec{d} = qd\hat{z}$ (in spherical-polar coordinates) is:

$$\vec{E}_{dipole}(\vec{r}) = -\vec{\nabla}V_{dipole}(\vec{r}) = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)V_{dipole}(\vec{r}) = E_x^{dipole}\hat{x} + E_y^{dipole}\hat{y} + E_z^{dipole}\hat{z}$$

Transformation from Spherical-Polar → Cartesian Coordinates:

$x = r\sin\theta\cos\varphi$	$\hat{x} = \sin\theta\cos\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} - \sin\varphi\hat{\varphi}$
$y = r\sin\theta\sin\varphi$	$\hat{y} = \sin\theta\sin\varphi\hat{r} + \cos\theta\sin\varphi\hat{\theta} + \sin\varphi\hat{\varphi}$
$z = r\cos\theta$	$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$

It is a straight-forward exercise to show that the electric field components associated with a pure physical electric dipole with electric dipole moment $\vec{p} = q\vec{d} = qd\hat{z}$ (in Cartesian coordinates) are:

$$E_x^{dipole} = \frac{p}{4\pi\varepsilon_o}\left(\frac{3xz}{r^5}\right) = \frac{p}{4\pi\varepsilon_o}\left(\frac{3\sin\theta\cos\theta}{r^3}\right)$$

$$E_y^{dipole} = \frac{p}{4\pi\varepsilon_o}\left(\frac{3yz}{r^5}\right) = \frac{p}{4\pi\varepsilon_o}\left(\frac{3\sin\theta\cos\theta}{r^3}\right) = E_x^{dipole} \Leftarrow$$

$$E_z^{dipole} = \frac{p}{4\pi\varepsilon_o}\left(\frac{3z^2 - r^2}{r^5}\right) = \frac{p}{4\pi\varepsilon_o}\left(\frac{3\cos^2\theta - 1}{r^3}\right)$$

(since charge configuration of electric dipole is axially / azimuthally symmetric)

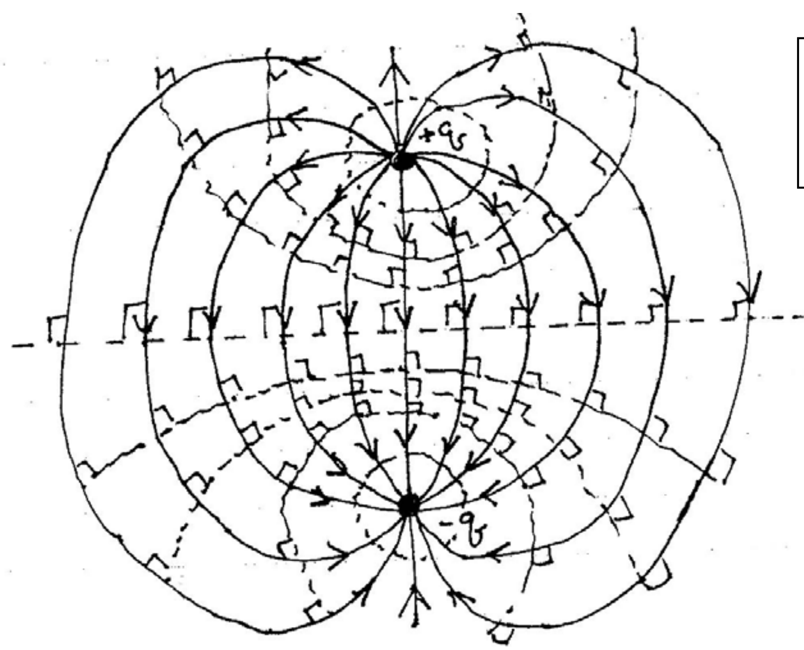
In *coordinate-free* form, it is also a straight-forward exercise (try it!!!) to show that the electric field intensity of a pure physical electric dipole with electric dipole moment $\vec{p} = q\vec{d} = qd\hat{z}$ is of the form:

$$\vec{E}_{dipole}^{physical}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right]$$

whereas the *coordinate-free* form of a *point* electric dipole is of the form:

$$\vec{E}_{dipole}^{point}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right] - \frac{1}{3\varepsilon_o}\vec{p}\,\delta^3(\vec{r})$$

$\vec{E}$ – Field Lines & Equipotentials Associated with a Pure, Physical Electric Dipole:



n.b. Equipotentials are $\perp$ to lines of $\vec{E}(\vec{r})$ everywhere!

We explicitly show here that the electric field associated with a pure physical electric dipole with electric dipole moment $\vec{p} = p\hat{z} = qd\hat{z}$ can be written in coordinate-free form as:

$$\vec{E}_{dipole}^{physical}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right]$$

We have already shown (above) that:

$$\vec{E}_{dipole}^{physical}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{p}{r^3}\right)\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

Now: $\vec{p} = p\hat{z}$ and $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$ (in spherical-polar coordinates)

Thus: $\vec{p}\bullet\hat{r} = p\hat{p}\bullet\hat{r} = p\hat{z}\bullet\hat{r}$

But: $\hat{z}\bullet\hat{r} = \left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\bullet\hat{r} = \cos\theta$

And: $\hat{r}\bullet\hat{r} = 1$, $\hat{\theta}\bullet\hat{r} = 0$

Thus: $\vec{p}\bullet\hat{r} = p\cos\theta$

And: $\vec{p} = (\vec{p}\bullet\hat{r})\hat{r} + \left(\vec{p}\bullet\hat{\theta}\right)\hat{\theta} = \overbrace{p\cos\theta\hat{r}}^{=(\vec{p}\bullet\hat{r})\hat{r}} - p\sin\theta\hat{\theta}$

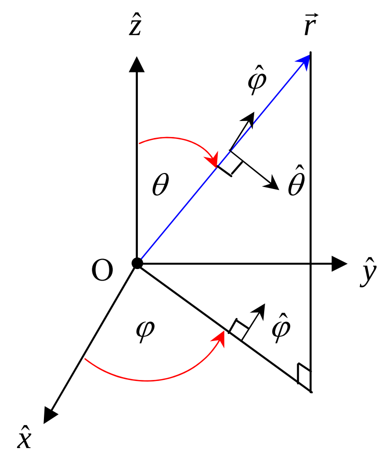
So therefore:

$$\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right] = 3p\cos\theta\hat{r} - p\cos\theta\hat{r} + p\sin\theta\hat{\theta}$$

$$= 2p\cos\theta\hat{r} + p\sin\theta\hat{\theta}$$

$$= p\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

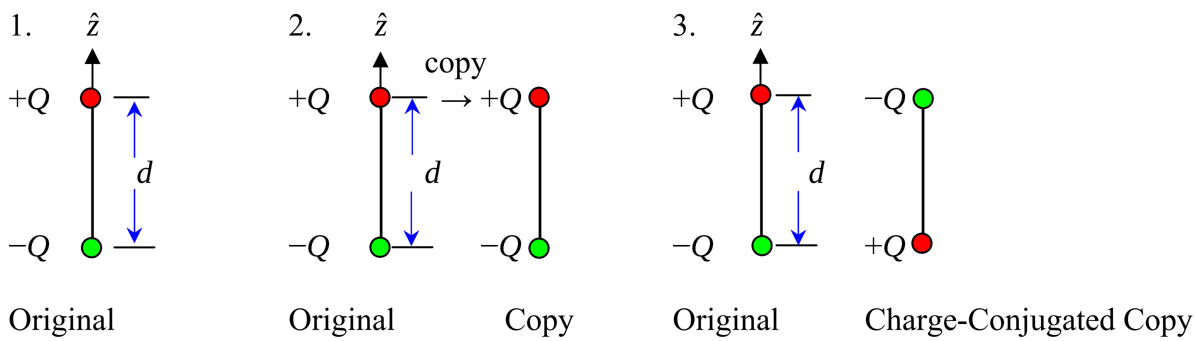
Thus: $\boxed{\vec{E}_{dipole}^{physical}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}\right]}$ *Q.E.D.*

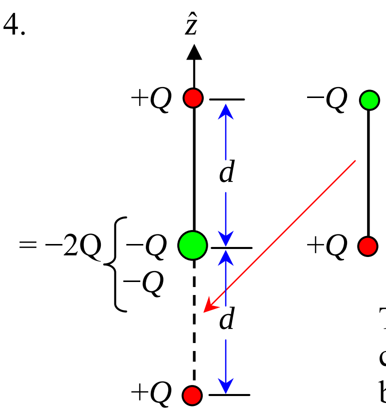


The Potential $V_{quad}(\vec{r})$ and Electric Field $\vec{E}_{quad}(\vec{r})$ Associated with a Pure, Linear Physical Electric Quadrupole

We have seen that a pure, physical electric dipole was constructed by:
1. Starting with a monopole electric moment (i.e. charge $+Q$)
2. "Copying it"
3. Charge-conjugating ($+Q \rightarrow -Q$) the "copied" charge
4. Displacing the conjugated charge $-Q$ from the original charge $+Q$ by a separation distance d

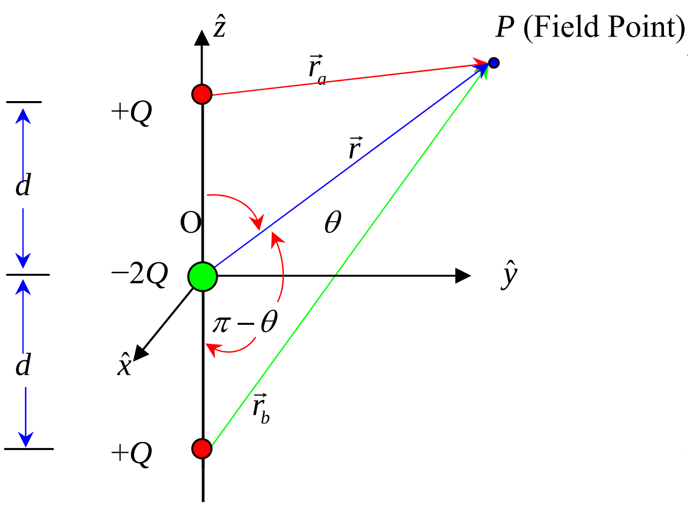
Likewise, we can construct a pure, physical, linear electric quadrupole by:
1. Starting with a pure, physical, linear electric dipole
2. "Copying it"
3. Charge-conjugating the charges associated with the "copied" electric dipole
4. Translating the charge conjugated electric dipole along the symmetry axis of the original electric dipole by amount d, as shown in the figures below:





Translation of charge-conjugated copy along axis of original dipole by amount *d*.

<u>Pure, Physical, Linear Electric Quadrupole</u>:



Note that this linear electric quadrupole has axial / aximuthal symmetry – i.e. because all charges ($+Q$, $-2Q$, $+Q$) are co-linear (all on $\hat{z}$ axis), problem is invariant under (arbitrary) φ-rotations.

$\Rightarrow$ $V_{quad}(\vec{r})$ and $\vec{E}_{quad}(\vec{r})$ will have no explicit φ-dependence for the linear electric quadrupole.

n.b. $Q_{TOT} = 0$ for *<u>pure</u>* electric quadrupole.

Again, we use the principle of (linear) superposition to obtain $V_{quad}(\vec{r})$:

$$V_{quad}(\vec{r}) = V_{TOT}(\vec{r}) = V_{+Q}(@\, z = +d) + V_{-2Q}(@\, z = 0) + V_{+Q}(@\, z = -d)$$

$$= \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r_a} - \frac{2Q}{r} + \frac{Q}{r_b}\right) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r}\right)\left[\left(\frac{r}{r_a}\right) - 2 + \left(\frac{r}{r_b}\right)\right]$$

Again, using the Law of Cosines: $r_a^2 = r^2 + d^2 - 2rd\cos\theta$ and $r_b^2 = r^2 + d^2 + rd\cos\theta$

We obtain:

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r}\right)\left\{\frac{r}{\sqrt{r^2+d^2-2rd\cos\theta}} - 2 + \frac{r}{\sqrt{r^2+d^2+2rd\cos\theta}}\right\} \Leftarrow$$

Exact analytic expression

Again, for regime where the observation point P is *far away* from pure, physical, linear electric quadrupole, i.e. $r >> d$, we expand $\left(\frac{r}{r_a}\right)$ and $\left(\frac{r}{r_b}\right)$ in a binomial (i.e. Taylor) series (as was done previously for the case of a pure, physical electric dipole).

Neglecting terms in these expansions that are higher order than linear (i.e. > $(d/r)^2$) we obtain:

$$\left(\frac{r}{r_a}\right) \simeq 1 - \left(\frac{d}{r}\right)\cos\theta + \left(\frac{d}{r}\right)^2 \frac{\left(3\cos^2\theta - 1\right)}{2}$$

$$\left(\frac{r}{r_b}\right) \simeq 1 + \left(\frac{d}{r}\right)\cos\theta + \left(\frac{d}{r}\right)^2 \frac{\left(3\cos^2\theta - 1\right)}{2}$$

Recall that the Ordinary Legendré Polynomials $P_\ell\overbrace{(x)}^{x=\cos\theta}$ are:

$$P_0(x) = 1 \quad \rightarrow P_0(\cos\theta) = 1$$

$$P_1(x) = x \quad \rightarrow P_1(\cos\theta) = \cos\theta$$

$$P_2(x) = \frac{\left(3x^2-1\right)}{2} \rightarrow P_2(\cos\theta) = \frac{\left(3\cos^2\theta - 1\right)}{2}$$

Shorthand notation: $P_\ell(\cos\theta) = P_\ell(\theta)$

$$\therefore \quad \left(\frac{r}{r_a}\right) \simeq P_0(\theta) - \left(\frac{d}{r}\right)P_1(\theta) + \left(\frac{d}{r}\right)^2 P_2(\theta) \quad \text{and} \quad \left(\frac{r}{r_b}\right) \simeq P_0(\theta) + \left(\frac{d}{r}\right)P_1(\theta) + \left(\frac{d}{r}\right)^2 P_2(\theta)$$

$$\therefore \quad V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r}\right)\left[\left(\frac{r}{r_a}\right) - 2 + \left(\frac{r}{r_b}\right)\right] = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r}\right)\left[2\left(\frac{d}{r}\right)^2 \frac{\left(3\cos^2\theta-1\right)}{2}\right]$$

$$= \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left(\frac{3\cos^2\theta - 1}{2}\right)$$

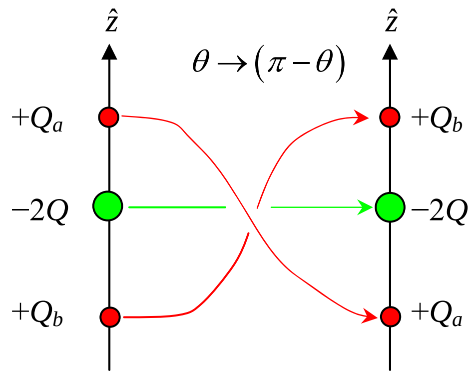
Then for $r >> d$:

$$V_{quad}(\vec{r}) \simeq \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\overbrace{\left(\frac{3\cos^2\theta-1}{2}\right)}^{P_2(\theta)} = \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)P_2(\theta)$$

Note that: $V_{quad}(\vec{r}) \sim \frac{1}{r^3}$ (c.f. with $V_{monopole}(\vec{r}) \sim 1/r$ and $V_{dipole}(\vec{r}) \sim 1/r^2$)

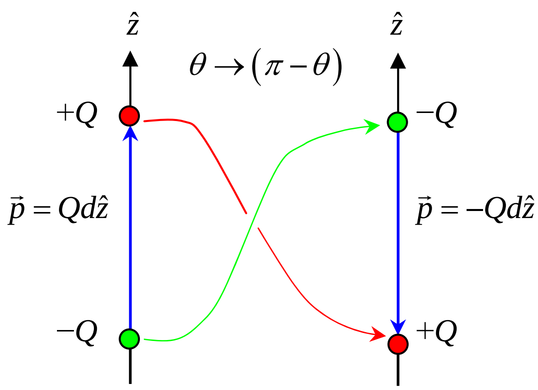
Note also that: $\boxed{V_{quad}(\vec{r}) \sim \underbrace{P_2(\theta)}_{\frac{1}{2}(3\cos^2\theta - 1)}}$ (c.f. with $\boxed{V_{monopole}(\vec{r}) \sim \underbrace{P_0(\theta)}_{=1}, \quad V_{dipole}(\vec{r}) \sim \underbrace{P_1(\theta)}_{=\cos\theta}}$)

Note *further* that: $V_{quad}(\vec{r})$ *must* be proportional to an *even* power of l, i.e. $P_{l=even}(\theta)$ because a pure, physical, linear electric quadrupole has reflection symmetry about the $\hat{z}$-axis (i.e. about $\theta = \pi/2$) (i.e. a rotation from / by a vector lying in $x-y$ plane e.g. $\hat{x}$ or $\hat{y}$ axis).



$P_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1)$
is an *even* function under
$\theta \to (\pi - \theta)$ reflection:
$P_2(\pi - \theta) = +P_2(\theta)$

We can also see that $V_{dipole}(\vec{r})$ *must* be proportional to an odd power of l, i.e. $P_{l=odd}(\theta)$ because a pure, physical, linear electric dipole has a sign change under reflection symmetry about $\theta = \pi/2$

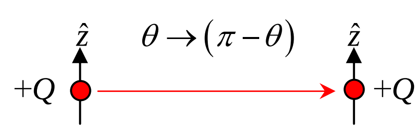


$P_1(\theta) = \cos\theta$
is an *odd* function under
$\theta \to (\pi - \theta)$ reflection:
$P_1(\pi - \theta) = -P_1(\theta)$

$$\cos(\pi - \theta) = \cos\pi\cos\theta + \sin\theta \overset{=0}{\cancel{\sin\pi}}$$
$$= -\cos\theta$$

Likewise, $V_{monopole}(\vec{r})$ *must* be proportional to an even power of l:

$P_0(\theta) = P_0(\pi - \theta) = 1$



As we have seen for the two previous cases, that of:

1. The electric monopole, with its accompanying electric monopole moment, the electric charge Q (n.b. Q is a scalar quantity) (SI units of Q: Coulombs)

2. The electric dipole with its accompanying electric dipole moment $\vec{p} \equiv Q\vec{d}$, $p = |\vec{p}| = Qd$ (n.b. $\vec{p}$ is a vector quantity) (SI units of $\vec{p}$: Coulomb-meters)

3. The electric quadrupole also has an accompanying electric quadrupole moment $\overleftrightarrow{Q} \equiv 2Q\vec{d}\vec{d}$ (n.b. $\overleftrightarrow{Q}$ is a tensor quantity) (SI units of $\overleftrightarrow{Q}$: Coulomb-meters^2)

Tensor $\overleftrightarrow{Q} \equiv 2Q\vec{d}\vec{d}$ = "double vector" $\quad |\overleftrightarrow{Q}| \equiv 2Qdd = 2Qd^2$

2-dimensional matrix

Formally speaking, $\overleftrightarrow{Q}$ is a rank-2 tensor (i.e. a 2-dimensional matrix) - the 9 elements of the $\overleftrightarrow{Q}$ tensor (in general) are:

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_{xx} & Q_{yz} & Q_{zx} \\ Q_{xy} & Q_{yy} & Q_{zy} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix}$$

n.b. $\overleftrightarrow{Q}$ has only six *independent* components, because $Q_{ij} = Q_{ji}$

i.e. $Q_{xy} = Q_{yx}$
$Q_{xz} = Q_{zx}$
$Q_{yz} = Q_{zy}$

Also, note that: $Q_{xx} + Q_{yy} + Q_{zz} = 0$ or: $Q_{zz} = -(Q_{xx}+Q_{yy})$ {i.e. $\overleftrightarrow{Q}$ is *traceless*}

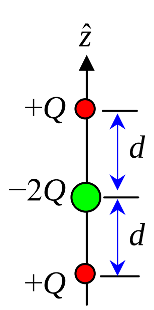
The quadrupole moment tensor can also be written in *coordinate-free* form, e.g. in Cartesian coordinates as:

n = # discrete charges q_i

$$\overleftrightarrow{Q} \equiv \tfrac{1}{2}\sum_{i=1}^{n}\left(3\vec{r}_i\vec{r}_i - \overleftrightarrow{1}r_i^2\right)q_i \quad \text{with} \quad r_i^2 = \vec{r}_i \bullet \vec{r}_i$$

Unit Dyadic: $\overleftrightarrow{1} \equiv \begin{pmatrix} \hat{x}\hat{x} & 0 & 0 \\ 0 & \hat{y}\hat{y} & 0 \\ 0 & 0 & \hat{z}\hat{z} \end{pmatrix}$

For the case of a pure, linear (i.e. axially/azimuthally symmetric) electric quadrupole with quadrupole moment $\overleftrightarrow{Q}$ (e.g. oriented along the $\hat{z}$-axis):



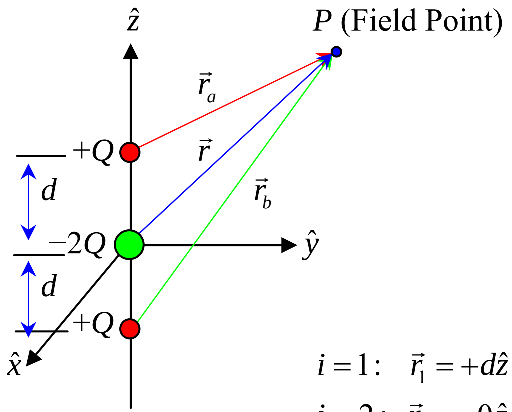
Here, $Q_{xx} = Q_{yy}$, and since: $Q_{xx} + Q_{yy} + Q_{zz} = 0$

Then: $Q_{zz} = -2Q_{xx} = -2Q_{yy} \equiv 2Qd^2$ All other Q_{ij} vanish (= 0) for $i \neq j$

i.e. $\overleftrightarrow{Q}_{quad}^{linear} = Qd^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +2 \end{pmatrix}$

n.b. conventions / definitions of $\overleftrightarrow{Q}_{quad}^{linear}$ differ in different textbooks!!!

For the case of a pure, linear (i.e. axially/azimuthally symmetric) electric quadrupole with quadrupole moment $\vec{\vec{Q}}$ (oriented along the $\hat{z}$ -axis), expressed in Cartesian coordinates:



$$\boxed{\vec{\vec{Q}} \equiv \tfrac{1}{2}\sum_{i=1}^{n}\left(3\vec{r}_i\vec{r}_i - \vec{\vec{1}}r_i^2\right)q_i} \text{ with } \boxed{r_i^2 = \vec{r}_i \bullet \vec{r}_i}$$

\# discrete charges (→ n)

Unit Dyadic (→ $\vec{\vec{1}}$):

$$\vec{\vec{1}} \equiv \begin{pmatrix} \hat{x}\hat{x} & 0 & 0 \\ 0 & \hat{y}\hat{y} & 0 \\ 0 & 0 & \hat{z}\hat{z} \end{pmatrix}$$

$$i=1: \quad \vec{r}_1 = +d\hat{z} \qquad q_1 = +Q \qquad r_i = |\vec{r}_i|$$
$$i=2: \quad \vec{r}_2 = 0\hat{z} \qquad q_2 = -2Q$$
$$i=3: \quad \vec{r}_3 = -d\hat{z} \qquad q_3 = +Q$$

Thus: $$\vec{\vec{Q}} = \underbrace{\frac{1}{2}Q\left(3d^2\hat{z}\hat{z} - d^2\vec{\vec{1}}\right)}_{\substack{\text{for charge 1:}\\ +Q\,@\,\vec{r}_1=+d\hat{z}}} - \underbrace{\frac{2}{2}Q\left(3\bullet \overset{=0}{\cancel{0}}\hat{z}\hat{z} - \overset{=0}{\cancel{0}}\bullet\vec{\vec{1}}\right)}_{\substack{\text{for charge 2:}\\ -2Q\,@\,\vec{r}_2=0\hat{z}}} + \underbrace{\frac{1}{2}Q\left(3d^2\hat{z}\hat{z} - d^2\vec{\vec{1}}\right)}_{\substack{\text{for charge 3:}\\ +Q\,@\,\vec{r}_3=-d\hat{z}}} = Qd^2\left(3\hat{z}\hat{z} - \vec{\vec{1}}\right)$$

$$\therefore \quad \vec{\vec{Q}} = Qd^2\left(3\hat{z}\hat{z} - \vec{\vec{1}}\right) = 2Qd^2\left(\frac{3\hat{z}\hat{z} - \vec{\vec{1}}}{2}\right)$$

Then: $$\boxed{V_{quad}(\vec{r}) \simeq \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\frac{\left(3\cos^2\theta - 1\right)}{2} = \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)P_2(\cos\theta)} \qquad P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

We can express $V_{quad}(\vec{r})$ in a different (but totally equivalent manner), using the fact(s) that:

$$\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$$

$$\hat{z}\bullet\hat{r} = \hat{r}\bullet\hat{z} = \cos\theta \qquad \hat{x}\bullet\hat{x} = 1,\ \hat{x}\bullet\hat{y} = 0,\ \hat{x}\bullet\hat{z} = 0$$

$$3(\hat{r}\bullet\hat{z})(\hat{z}\bullet\hat{r}) = 3\cos^2\theta \qquad \hat{y}\bullet\hat{x} = 0,\ \hat{y}\bullet\hat{y} = 1,\ \hat{y}\bullet\hat{z} = 0$$

$$\hat{r}\bullet\vec{\vec{1}}\bullet\hat{r} = 1 \qquad \hat{z}\bullet\hat{x} = 0,\ \hat{z}\bullet\hat{y} = 0,\ \hat{z}\bullet\hat{z} = 1$$

Then for observation/field point P far from quadrupole, i.e. $r >> d$:

$$V_{quad}(\vec{r}) \approx \overbrace{\frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left(\hat{r}\bullet\vec{Q}\bullet\hat{r}\right)}^{\text{coordinate-free form}} = \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[\frac{\hat{r}\bullet\left(3\hat{z}\hat{z}-\vec{1}\right)\bullet\hat{r}}{2}\right]$$

$$= \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[\frac{3\left(\hat{r}\bullet\hat{z}\right)\left(\hat{z}\bullet\hat{r}\right)-\hat{r}\bullet\vec{1}\bullet\hat{r}}{2}\right] = \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\underbrace{\left[\frac{3\cos^2\theta-1}{2}\right]}_{\equiv P_2(\cos\theta)}$$

$$= \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)P_2\left(\cos\theta\right)$$

axially-symmetric

$V_{quad}(\vec{r})$ as given above is valid for a pure, linear, physical electric quadrupole oriented along the $\hat{z}$ -axis, for r (observation / field point) >> d.

The potential $V_{quad}(\vec{r})$ and electric field intensity $\vec{E}_{quad}(\vec{r})$ associated with a pure, physical, linear electric quadrupole with quadrupole moment $\vec{Q}$ (oriented along the $\hat{z}$ -axis) are:

$$V_{quad}(\vec{r}) = \frac{2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^3}\right)\left[\frac{3\cos^2\theta-1}{2}\right]$$

$\vec{E}_{quad}(\vec{r}) = E_r\hat{r} + E_\theta\hat{\theta} + E_\varphi\hat{\varphi} = -\vec{\nabla}V_{quad}(\vec{r})$, in spherical-polar coordinates:

$$E_r(\vec{r}) = -\frac{\partial V(\vec{r})}{\partial r} = \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)\left[\frac{3\cos^2\theta-1}{2}\right] = \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)P_2\left(\cos\theta\right)$$

$$E_\theta(\vec{r}) = -\frac{1}{r}\frac{\partial V(\vec{r})}{\partial\theta} = \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)\sin\theta\cos\theta$$

$$E_\varphi(\vec{r}) = -\frac{1}{r\sin\theta}\frac{\partial V(\vec{r})}{\partial\varphi} = 0$$

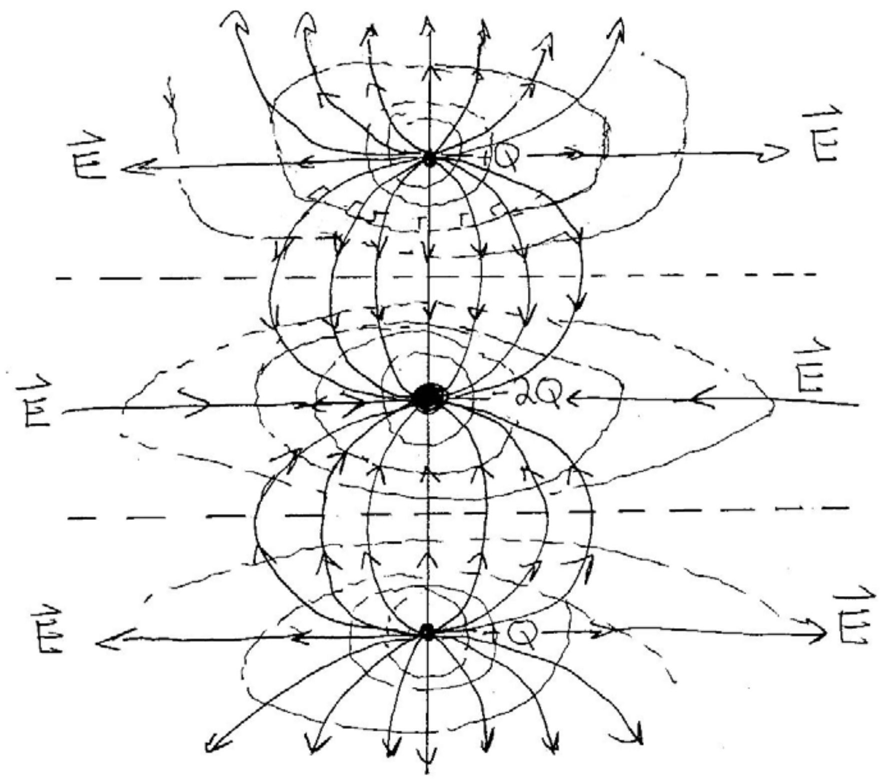
← No φ -dependence because charge configuration is manifestly axially / azimuthally symmetric (invariant under arbitrary φ -rotations)

Explicitly writing out the form of the electric field intensity $\vec{E}_{quad}(\vec{r})$ for a pure, linear, physical electric quadrupole oriented along the $\hat{z}$ -axis, for r (observation / field point) >> d:

$$\vec{E}_{quad}(\vec{r}) = \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)\left[\frac{3\cos^2\theta-1}{2}\right]\hat{r} + \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)\sin\theta\cos\theta\hat{\theta}$$

$$= \frac{3\bullet 2Qd^2}{4\pi\varepsilon_o}\left(\frac{1}{r^4}\right)\left[\left(\frac{3\cos^2\theta-1}{2}\right)\hat{r} + \sin\theta\cos\theta\hat{\theta}\right]$$

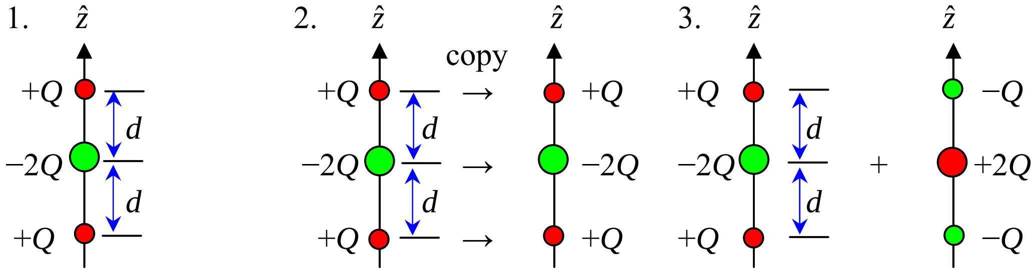
$\vec{E}$ -field lines & equipotentials associated with a pure, physical, linear electric quadrupole:
n.b. $\vec{E}$ -field lines $\perp$ to equipotentials everywhere in space



Higher-Order Pure, Linear Physical Electric Multipoles

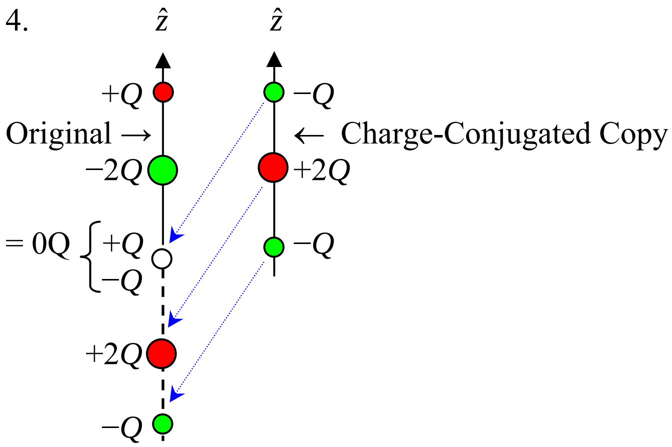
The next higher order pure, linear physical multipole is known as the pure, linear physical electric ***octupole***. We can construct / create it (as before) by:

1. Starting with a pure, linear, physical electric ***quadrupole***
2. "Copying it"
3. Charge-conjugating ($Q \rightarrow -Q$) the charges associated with the "copied" electric quadrupole
4. Translating the charge-conjugated electric quadrupole along the symmetry axis of the original electric quadrupole, this time by an amount 2d:

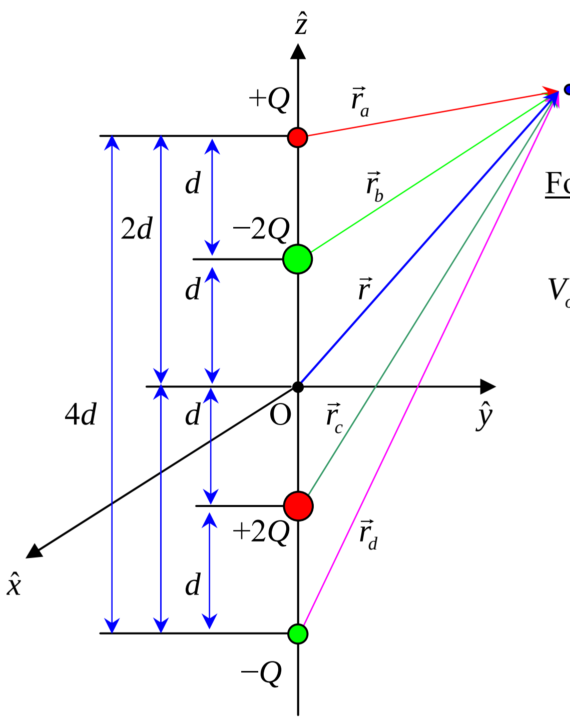


Original Original Copy Original Charge-Conjugated Copy

4.



Pure, Linear (Axially/Azimuthally-Symmetric) Physical Electric Octupole:



P (Observation / Field point)

Following the methodology as used in previous cases:

$$V_{octupole}(\vec{r}) = \sum_{i=1}^{4} V_i(\vec{r}) \sim \left(\frac{1}{r^4}\right) \underbrace{P_3(\cos\theta)}_{=\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)} * \frac{1}{4\pi\varepsilon_o} * \vec{\vec{\vec{O}}}$$

$$\vec{E}_{octupole}(\vec{r}) = -\vec{\nabla} V_{octupole}(\vec{r}) \sim \left(\frac{\vec{\vec{\vec{O}}}}{r^5}\right) * \frac{1}{4\pi\varepsilon_o}$$

$\vec{\vec{\vec{O}}}$ = Octupole Moment $\sim Q\vec{d}\vec{d}\vec{d}$ (Rank-3 tensor)

$\left|\vec{\vec{\vec{O}}}\right| \sim Qd^3$ (SI units: coulomb-meter^3)

Note: $Q_{TOT} = 0$

In general, for l^{th}-order electric multipole, $\ell = 0, 1, 2, 3, \ldots$ defining $M_\ell \equiv \ell^{th}$-order multipole moment (SI units: coulomb-(meters)$^\ell$) then the potential associated with a pure, physical, linear multipole moment is of the form:

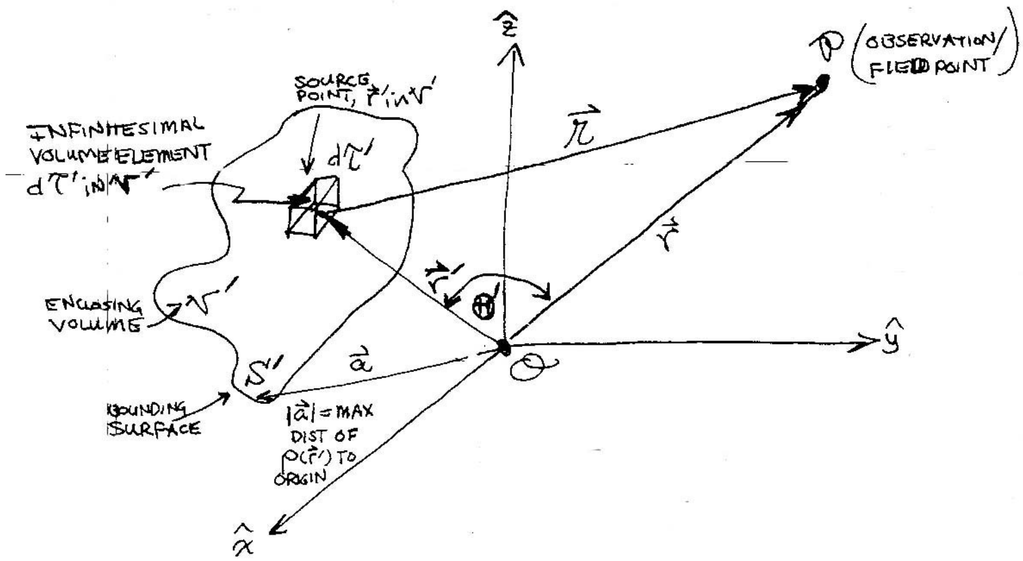
$$\boxed{V_\ell(\vec{r}) \sim \frac{M_\ell}{4\pi\varepsilon_o}\left(\frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

The electric field intensity associated with a pure, physical, linear multipole moment is of the form:

$$\boxed{\vec{E}_\ell(\vec{r}) = -\vec{\nabla} V_\ell(\vec{r}) \sim \frac{1}{4\pi\varepsilon_o}\frac{M_\ell}{r^{\ell+2}}}$$

Multipole Moments, Potential and Electric Field Associated with an Arbitrary Localized Electric Charge Distribution $\rho(\vec{r}')$ - Outside of $\rho(\vec{r}')$

Suppose we have an arbitrary, but localized electric charge distribution $\rho(\vec{r}')$ somewhere in space, contained within the volume v' and bounded by the surface S' :



$\vec{\mathbf{r}} = \vec{r} - \vec{r}'$ $\qquad\qquad$ $\mathbf{r} = |\vec{\mathbf{r}}| = |\vec{r} - \vec{r}'|$

$\cos\Theta' = \hat{r}\cdot\hat{r}' =$ cosine of opening angle between vectors $\vec{r}$ and $\vec{r}'$.
Θ' = opening angle between vectors $\vec{r}$ and $\vec{r}'$ - very important!

Law of Cosines: $\quad \mathbf{r}^2 = r^2 + r'^2 - 2rr'\cos\Theta' = r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'$

If the observation / field point P is far away from electric charge distribution $\rho(\vec{r}')$ such that: $r = |\vec{r}| \gg a = |\vec{a}|$ = maximum distance of $\rho(\vec{r}')$ to origin ϑ then for $r >> a$ (a = max value of $\vec{r}'$):

$$\mathbf{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\Theta'\right] \qquad \text{or:} \qquad \mathbf{r} = r\sqrt{1 + \underbrace{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\Theta'}_{\equiv\varepsilon\ll 1 \text{ for } r\gg a}}$$

Define: $\varepsilon \equiv \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\Theta'$ for $r >> a$ (a = max value of $\vec{r}'$)

Now: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho(\vec{r}')}{\mathbf{r}}d\tau'$ with: $\frac{1}{\mathbf{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2}$

Carry out a (full) binomial expansion of $1/\mathbf{r}$ (for $r >> a$):

$$\frac{1}{\mathbf{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\sum_{n=0}^{\infty}\binom{-1/2}{n}\varepsilon^n = \frac{1}{r}\left(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2-\frac{5}{16}\varepsilon^3+...\right)$$

<u>where</u>: $\binom{-1/2}{n} = \frac{(-1)^n}{n!}\frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)}$ is the binomial coefficient and $\Gamma(x)$ is the gamma function.

<u>and</u>: $\frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)....\left(-\frac{1}{2}+n-1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)....\left(n-\frac{3}{2}\right)$

<u>Then</u>: $$\frac{1}{\mathbf{r}} = \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\Theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r}-2\cos\Theta'\right)^2-\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r}-2\cos\Theta'\right)^3+...\right]$$

Collecting together like powers of r'/r :

$$\frac{1}{\mathbf{r}} = \frac{1}{r}\left[1+\left(\frac{r'}{r}\right)\cos\Theta'+\left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\Theta'-1}{2}\right)+\left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\Theta'-3\cos\Theta'}{2}\right)^3+...\right]$$

Thus we see that:

$$\frac{1}{\mathbf{r}} = \frac{1}{r}\left[P_0(\cos\Theta')+\left(\frac{r'}{r}\right)P_1(\cos\Theta')+\left(\frac{r'}{r}\right)^2 P_2(\cos\Theta')+\left(\frac{r'}{r}\right)^3 P_3(\cos\Theta')+...\right] \quad !!!!$$

<u>Hence</u>: $\boxed{\frac{1}{\mathbf{r}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell}P_\ell(\cos\Theta')}$ where Θ' = opening angle between $\vec{r}$ and $\vec{r}'$.

This remarkable result occurs because $\frac{1}{\sqrt{1+\varepsilon}}$ (where $\varepsilon \equiv \left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\cos\Theta'$) is known as the <u>Generating Function</u> for the Legendré Polynomials!!!

Then, since $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\rho(\vec{r}')\left(\frac{1}{\mathbf{r}}\right)d\tau'$ for $r >> a$ (a = max value of $\vec{r}'$), the potential <u>outside</u> the volume v' containing the charge distribution $\rho(\vec{r}')$ is given by:

$$\begin{aligned} V_{outside}(\vec{r}) &= \frac{1}{4\pi\varepsilon_o}\int_{v'}\left(\frac{1}{r}\right)\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell}\rho(\vec{r}')P_\ell(\cos\Theta')\,d\tau' \\ &= \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}(r')^{\ell}\rho(\vec{r}')P_\ell(\cos\Theta')\,d\tau' \end{aligned}$$

<u>Then defining</u>: $V_\ell^{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}(r')^{\ell}\rho(\vec{r}')P_\ell(\cos\Theta')\,d\tau'$

We obtain (for $r >> a$): $$V_{outside}(\vec{r}) = \sum_{\ell=0}^{\infty} V_{\ell}^{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}(r')^{\ell}\rho(\vec{r}')P_{\ell}(\cos\Theta')d\tau'$$

Linear superposition of multipole potentials!!!

Θ' = opening angle between $\vec{r}$ and $\vec{r}'$.

This expression is known as the Multipole Expansion of $V_{outside}(\vec{r})$ in powers of $1/r$.
It is valid / useful when $r >> a$ (a = max value of $\vec{r}'$). Note that this is an exact expression.

Having obtained $V_{outside}(\vec{r})$, we can then obtain $\vec{E}_{outside}(\vec{r}) = -\vec{\nabla}V_{outside}(\vec{r})$, and thus we see that:

$$\vec{E}_{outside}(\vec{r}) = \sum_{\ell=0}^{\infty} E_{\ell}^{outside}(\vec{r}) = -\sum_{\ell=0}^{\infty}\vec{\nabla}V_{\ell}^{outside}(\vec{r}) \quad \text{i.e.} \quad \vec{E}_{\ell}^{outside}(\vec{r}) = -\vec{\nabla}V_{\ell}^{outside}(\vec{r})$$

Linear superposition of multipole electric fields!!!

Thus, we see that, for observation / field point distances far away from the (arbitrary) localized electric charge distribution $\rho(\vec{r}')$ (i.e. $r >> a$ (a = max value of $\vec{r}'$)) the electrostatic potential $V_{outside}(\vec{r})$ and associated electric field $\vec{E}_{outside}(\vec{r}) = -\vec{\nabla}V_{outside}(\vec{r})$ are linear superpositions of multipole electrostatic potentials $V_{\ell}^{outside}(\vec{r})$ and multipole electric fields $\vec{E}_{\ell}^{outside}(\vec{r})$ respectively, each arising from the ℓ^{th} electric multipole moment M_{ℓ} associated with the localized electric charge distribution $\rho(\vec{r}')$!!!

Order of Electric Multipole	Electrostatic Potential $V_{\ell}^{outside}(\vec{r})$	Electric Field $\vec{E}_{\ell}^{outside}(\vec{r}) = -\vec{\nabla}V_{\ell}^{outside}(\vec{r})$	Electric Multipole Moment M_{ℓ}
$\ell = 0$ Monopole	$P_0 = 1$ $= \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r}\right)$	$= \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r^2}\right)$	$M_0 = Q$ (total/net charge, coulombs) (scalar)
$\ell = 1$ Dipole	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd}{r^2}\right)$	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd}{r^3}\right)$	$M_1 = Q\vec{d} = \vec{p}$ (coulomb-meters) (vector)
$\ell = 2$ Quadrupole	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^2}{r^3}\right)$	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^2}{r^4}\right)$	$M_2 = 2Q\vec{d}\vec{d} = \vec{Q}$ (coulomb-meters^2) (rank-2 tensor)
$\ell = 3$ Octupole	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^3}{r^4}\right)$	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^3}{r^5}\right)$	$M_3 \sim Q\vec{d}\vec{d}\vec{d} = \vec{O}$ (coulomb-meters^3) (rank-3 tensor)
$\ell = 4$ Sextupole	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^4}{r^5}\right)$	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^4}{r^6}\right)$	$M_4 \sim Q\vec{d}\vec{d}\vec{d}\vec{d} = \vec{S}$ (coulomb-meters^4) (rank-4 tensor)
....	....	....	....
ℓ^{th} Order Multipole	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^{\ell}}{r^{\ell+1}}\right)$	$\sim \frac{1}{4\pi\varepsilon_o}\left(\frac{Qd^{\ell}}{r^{\ell+2}}\right)$	$M_{\ell} \sim Q(\vec{r})^{\ell} = \vec{M}$ (coulomb-meters$^{\ell}$) (rank-ℓ tensor)

Thus we see that:

→ The higher-order multipole fields fall off $1/r$ faster than those associated with next lower order multipole.

→ Must get in closer and closer to charge distribution $\rho(\vec{r}')$ in order to sense / observe / detect the higher-order moments!

We can write the electrostatic potential yet another way:

For $r >> a$ (a = max value of $\vec{r}'$)

$$V_{outside}(\vec{r}) = \sum_{l=0}^{\infty} V_l^{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{1}{r}\int_{v'}\rho(\vec{r}')d\tau' + \frac{1}{r^2}\hat{r}\cdot\int_{v'}\vec{r}'\rho(\vec{r}')d\tau' + \frac{1}{r^3}\int_{v'}\frac{\left(3(\hat{r}\cdot\vec{r}')^2 - \vec{r}'^2\right)}{2}\rho(\vec{r}')d\tau' + \ldots\right]$$

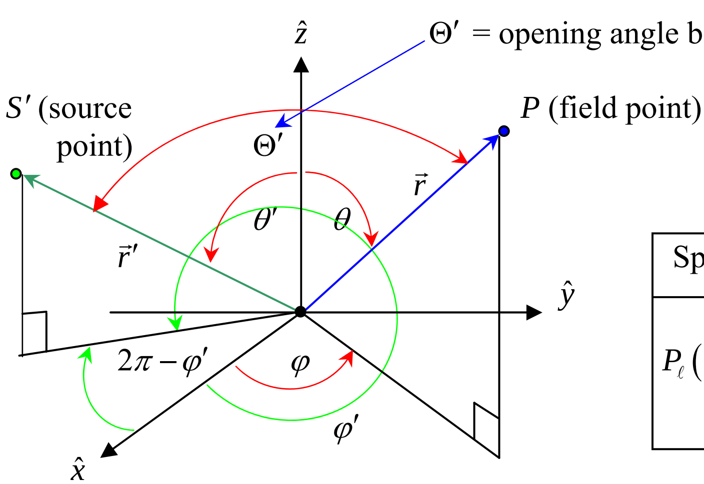
Thus, we see that:

$$\boxed{V_{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\underbrace{\frac{Q_{Net}}{r}}_{\substack{monopole\\ term}} + \underbrace{\frac{\vec{p}\cdot\hat{r}}{r^2}}_{\substack{dipole\\ term}} + \underbrace{\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3}}_{\substack{quadrupole\\ term}} + \ldots\right]}$$

with: $\boxed{Q_{Net} \equiv \int_{v'}\rho(\vec{r}')d\tau'}$, $\boxed{\vec{p} \equiv \int_{v'}\vec{r}'\rho(\vec{r}')d\tau'}$ and $\boxed{\overleftrightarrow{Q} \equiv \int_{v'}\frac{\left(3(\hat{r}\cdot\vec{r}')^2 - \vec{r}'^2\right)}{2}\rho(\vec{r}')d\tau'}$ .....

Recall / note: $\hat{r}\cdot\vec{r}' = \vec{r}'\cdot\hat{r} = r'\cos\Theta'$ where Θ' = opening angle between $\vec{r}$ and $\vec{r}'$.

The multipole expansion of $V_{outside}(\vec{r})$ which contains the opening angle Θ' between $\vec{r}$ (field point) and $\vec{r}'$ (source point) can be rewritten in terms of (θ and φ) for $\vec{r}$ and (θ' and φ') for $\vec{r}'$ using the so-called Addition Theorem for Spherical Harmonics:



Spherical Harmonics Addition Theorem:
$P_\ell(\cos\Theta') = \left(\frac{4\pi}{2\ell+1}\right)\sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\theta,\varphi)Y_{\ell m}(\theta',\varphi')$ n.b. complex conjugate

Then: $$\boxed{\frac{1}{\mathbf{r}} = \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\frac{(r')^\ell}{r^{\ell+1}}P_\ell(\cos\Theta') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{4\pi}{2\ell+1}\right)\frac{(r')^\ell}{r^{\ell+1}}Y_{\ell m}^*(\theta,\varphi)Y_{\ell m}(\theta',\varphi')}$$

Thus:

$$V_{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}(r')^{\ell}\rho(\vec{r}')P_{\ell}(\cos\Theta')\,d\tau'$$
$$= \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}\sum_{m=-\ell}^{+\ell}\left(\frac{4\pi}{2l+1}\right)(r')^{\ell}\rho(\vec{r}')Y^*_{\ell,m}(\theta,\varphi)Y_{\ell,m}(\theta',\varphi')\,d\tau'$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}V^{outside}_{\ell m}(\vec{r})$$

where:
$$V^{outside}_{\ell m}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{4\pi}{2\ell+1}\right)\int_{v'}\frac{(r')^{\ell}}{r^{\ell+1}}\rho(\vec{r}')Y^*_{\ell,m}(\theta,\varphi)Y_{\ell,m}(\theta',\varphi')\,d\tau'$$

Thus:

$$V_{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{4\pi}{2l+1}\right)\left(\frac{1}{r^{\ell+1}}\right)\int_{v'}(r')^{\ell}\rho(\vec{r}')\sum_{m=-\ell}^{+\ell}Y^*_{\ell,m}(\theta,\varphi)Y_{\ell,m}(\theta',\varphi')\,d\tau'$$
$$= \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{4\pi}{2l+1}\right)\left(\frac{1}{r^{\ell+1}}\right)\sum_{m=-\ell}^{+\ell}Y^*_{\ell,m}(\theta,\varphi)\left[\int_{v'}(r')^{\ell}\rho(\vec{r}')Y_{\ell,m}(\theta',\varphi')\,d\tau'\right]$$

The $Y_{l,m}(\theta,\varphi)$ are the Spherical Harmonics; θ and φ are the polar & azimuthal angles for $\vec{r}$, the vector from the origin to the field point, P and θ' and φ' are the polar & azimuthal angles for $\vec{r}'$, the vector from the origin to the source point, S'.

We can then define $q_{\ell m}$ - the Electric Multipole Moment of order ℓ & m:

$$q_{\ell m} \equiv \int_{v'}(r')^{\ell}\rho(\vec{r}')Y_{\ell m}(\theta',\varphi')\,d\tau'$$

Because of the properties of the $Y_{\ell,m}(\theta,\varphi)$, namely that:

$$Y_{\ell-m}(\theta,\varphi) = (-1)^{\ell}Y^*_{\ell,m}(\theta,\varphi) \qquad Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2(\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}P_{\ell}(\cos\theta)e^{im\varphi}$$

We see that: $q_{\ell-m} = (-1)^{\ell}q^*_{\ell,m}$

Thus:
$$V^{outside}_{\ell,m} = \frac{1}{4\pi\varepsilon_o}\left(\frac{4\pi}{2\ell+1}\right)\frac{1}{r^{\ell+1}}Y^*_{\ell,m}(\theta,\varphi)q_{\ell m}$$

Then:
$$V_{outside}(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}V^{outside}_{\ell,m}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{4\pi}{2\ell+1}\right)\frac{1}{r^{\ell+1}}Y^*_{\ell,m}(\theta,\varphi)q_{\ell,m}$$

Again, $\vec{E}_{outside}(\vec{r}) = -\vec{\nabla}V_{outside}(\vec{r})$ which by the principle of linear superposition becomes:

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\vec{E}^{outside}_{\ell,m}(\vec{r}) = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\vec{\nabla}V^{outside}_{\ell,m}(\vec{r})$$

i.e. $\vec{E}^{outside}_{\ell,m}(\vec{r}) = -\vec{\nabla}V^{outside}_{\ell,m}(\vec{r})$

The main advantage of using these seemingly more complex expressions for $V_{\ell,m}^{outside}(\vec{r})$ involving the $Y_{\ell m}^{*}(\theta,\varphi)$ and $Y_{\ell m}(\theta',\varphi')$ spherical harmonics is that they are directly connected to a right-handed $\hat{x}-\hat{y}-\hat{z}$ coordinate system. The earlier expression for $V_{outside}(\vec{r})$ involving the $P_\ell(\cos\Theta')$ Legendré Polynomials, it must be kept in mind *at all times* that Θ' = opening angle between field point $\vec{r}$ and source point $\vec{r}'$.

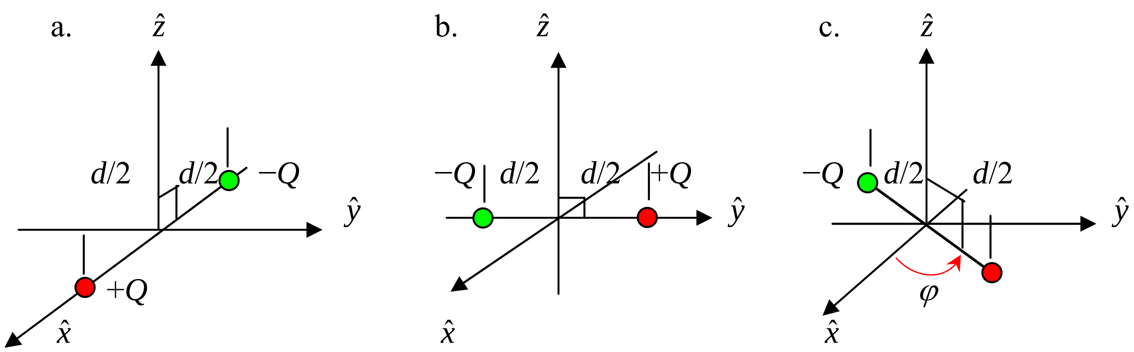
The explicit derivation of $V_{outside}(\vec{r})$ using the Addition Theorem for Spherical Harmonics:

$$V_{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{4\pi}{2\ell+1}\right)\left(\frac{1}{r^{\ell+1}}\right)\sum_{m=-\ell}^{+\ell} Y_{\ell m}^{*}(\theta,\varphi)\underbrace{\int_{v'}(r')^{\ell}\rho(\vec{r}')Y_{\ell m}(\theta',\varphi')\,d\tau'}_{\substack{\equiv q_{\ell m} \\ \text{(electric multipole moment of order } \ell\&m)}}$$

thus makes it explicitly clear that $V_{outside}(\vec{r}) = fcn(r,\theta,\varphi)$ only – all source variable (r',θ',φ') dependence has been integrated out, in carrying out the integral over the volume v' !!!

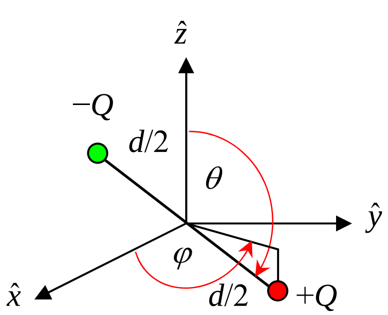
Thus $V_{outside}(\vec{r})$ is fully capable of correctly/exactly describing many other kinds of multipole moments we have not yet discussed, e.g.:

A. Pure Physical Electric Dipole(s) Lying in the *x*-*y* Plane:



(*x*-axis) (*y*-axis) (*x*-*y* plane)

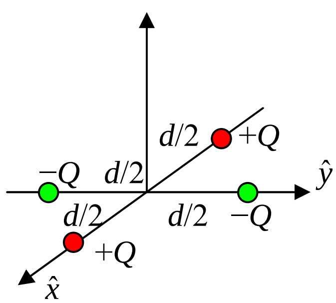
B. Pure, Physical Electric Dipole Randomly Oriented in Space:



(3-*D* dipole)

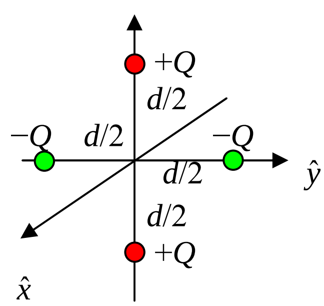
C. <u>Pure Physical, but Non-Colinear Electric Quadrupoles</u>:

a.



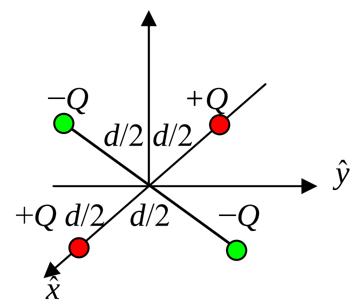
(*x*-*y* plane)

b.



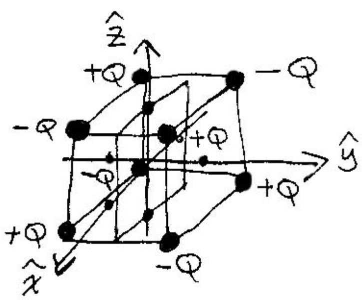
(*y*-*z* plane)

c.



(*x*-*z* plane)

D. <u>Pure Physical, but Non-Colinear Electric Octupoles</u>:



Cube Centered on $(x,y,z) = (0,0,0)$

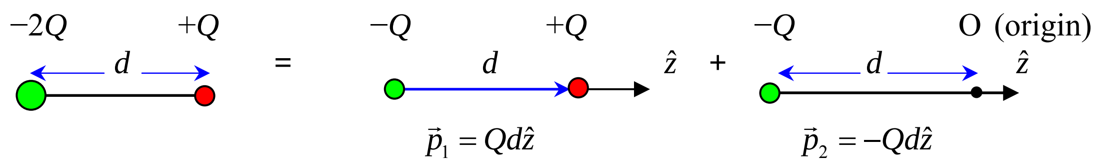
<u>The Choice of Origin of Coordinates Does Matter!!!</u>

Note that the choice of origin of coordinates in the electric multipole expansion of $V_{outside}(\vec{r})$ <u>does</u> matter – can affect e.g. determination of electric dipole moment, $\vec{p}$ if $Q_{NET} \neq 0$!!

A point charge Q located at the origin of coordinates O $(x,y,z) = (0,0,0)$ is a pure electric monopole. However, a point charge Q located some distance $|\vec{d}|$ along $\hat{d}$ from the origin is no longer a <u>*pure*</u> electric monopole! The monopole moment $Q = Q_{TOT}$ does <u>*not*</u> change, but $V_0(\vec{r})$ (where $\ell = 0$) <u>does</u> change, because $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\left(\dfrac{Q}{r}\right)$ is <u>not</u> quite correct – the exact potential is $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\left(\dfrac{Q}{\boldsymbol{r}}\right)$ and $\mathbf{r} \neq r$; however $\boldsymbol{r} \simeq r$ when $r >> r'$.

- For higher electric moments, if (and only if) $Q_{TOT} = Q_{NET} = 0$, then (pure) electric moment M_ℓ (where $\ell > 0$) <u>is</u> independent of choice of origin of coordinate system.
- If net / total charge $Q_{NET} = Q_{TOT} \neq 0$, then the higher-order electric moment(s) M_ℓ (where $\ell > 0$) can be made to vanish if one chooses the origin or coordinates to be located at the charge-weighted <u>center of charge</u>, then $\vec{r}' = 0$.

$$\boxed{q_{\ell m} = \int_{v'} (r')^{\ell} \rho(\vec{r}') Y_{\ell m}(\theta, \varphi) d\tau' = 0 \ \text{ if } r' = 0}$$



Note here that: $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0$!!!

If the origin is displaced from the center of charge for electric dipole by an amount $\vec{a}$:

e.g. $\vec{r}^{*\prime} = \vec{r}' + \vec{a}$ where $\vec{a}$ = vector displacement of origin of coordinate system,

<u>then</u>:

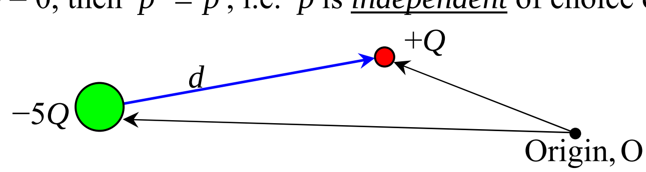
$$\vec{p}^* = \int_{v'} \vec{r}^{*\prime} \rho(\vec{r}') d\tau' \ \rightarrow \ \vec{p}'^* = \int_{v'} (\vec{r}' + \vec{a}) \rho(\vec{r}') d\tau'$$
$$= \int_{v'} \vec{r}' \rho(\vec{r}') d\tau' + \int_{v'} \vec{a} \rho(\vec{r}') d\tau'$$
$$= \vec{p} + \vec{a} \underbrace{\int_{v'} \rho(\vec{r}') d\tau'}_{=Q_{Net}(=Q_{Tot})} = \vec{p} + Q_{Net}\vec{a} = \vec{p} + \vec{p}_{origin}$$

- If $Q_{NET} \neq 0$, then $\vec{p}^* = \vec{p} + \vec{p}_{origin} \neq \vec{p}$ because the <u>origin-dependent electric dipole moment</u>, $\vec{p}_{origin} \equiv Q_{Net}\vec{a} \neq 0$!!!

If $Q_{NET} \neq 0$, then the choice of origin <u>*does*</u> matter; because the electric dipole moment $\vec{p}$ depends on the choice of origin !!!

If $Q_{NET} \neq 0$, then higher-order electric multipole moments <u>*must*</u> be accompanied by <u>*explicitly* specifying the choice of origin of coordinates</u>!!!

- **Iff** $Q_{NET} = 0$, then $\vec{p}^* = \vec{p}$, i.e. $\vec{p}$ is <u>*independent*</u> of choice or origin of coordinate system.



The Potential for a Pure Physical Electric Quadrupole (in Cartesian Coordinates) Not Necessarily With Colinear Charges

The potential for a pure, physical electric quadrupole (not necessarily with collinear charges) can be written in Cartesian coordinates as:

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\left(\frac{x_i x_j}{r^5}\right)\int_{v'}\left(3x_i x_j - r'^2\delta_{ij}\right)\rho(\vec{r}')\,d\tau'$$

or as: $$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\left(\frac{x_i x_j}{r^5}\right)Q_{ij}$$

with elements of the quadrupole moment tensor $Q_{ij} \equiv \int_{v'}\left(3x_i' x_j' - r'^2\delta_{ij}\right)\rho(\vec{r}')\,d\tau'$

with $r'^2 = x'^2 + y'^2 + z'^2 = x_1'^2 + x_2'^2 + x_3'^2$

and where the summations i = 1, 2, 3 and j = 1, 2, 3 represent sums over the $\left\{\begin{smallmatrix}1,2,3\\ x,y,z\end{smallmatrix}\right\}$ components respectively; i.e. $i, j = 1$: $x_1 \equiv x$ $i, j = 2$: $x_2 \equiv y$ and $i, j = 3$: $x_3 \equiv z$

and where δ_{ij} = Kroenecker δ − function $\left\{\begin{smallmatrix}=0 \text{ if } i\neq j\\ =1 \text{ if } i=j\end{smallmatrix}\right\}$

The 9 elements of the quadrupole moment tensor $\overleftrightarrow{Q}$ are the Q_{ij}'s: $\overleftrightarrow{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}$

<u>Where</u>: $\sum_{i=1}^{3} Q_{ii} = 0$ *i.e.* $\overbrace{Q_{11} + Q_{22} + Q_{33}}^{\text{sum of diagonal elements}=0} = 0$ (i.e. $\overleftrightarrow{Q}$ is a *<u>traceless</u>* rank-2 tensor / 3×3 matrix)

<u>and also</u>: $Q_{ij} = Q_{ji}$ for $i \neq j$, i.e. $Q_{12} = Q_{21}$, $Q_{13} = Q_{31}$ and $Q_{23} = Q_{32}$.

In general, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$ then:

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^5}\right)\left\{3xy\int_{v'} x'y'\rho(\vec{r}')\,d\tau' + 3zx\int_{v'} x'z'\rho(\vec{r}')\,d\tau' + 3yz\int_{v'} y'z'\rho(\vec{r}')\,d\tau' \right.$$
$$\left. + \frac{1}{2}\left(3x^2 - 1\right)\int_{v'} x'^2\rho(\vec{r}')\,d\tau' + \frac{1}{2}\left(3y^2 - 1\right)\int_{v'} y'^2\rho(\vec{r}')\,d\tau' + \frac{1}{2}\left(3z^2 - 1\right)\int_{v'} z'^2\rho(\vec{r}')\,d\tau'\right\}$$

The 9 elements of the quadrupole moment tensor $\overleftrightarrow{Q}$ (in Cartesian coordinates) are thus:

Mean square of $x_i x_j$ (multiplied by q).

$$Q_{xx} = \int_{v'} x'^2 \rho(\vec{r}')\,d\tau' = q\overline{x'}^2 = q\langle x'\rangle^2$$

$$Q_{yy} = \int_{v'} y'^2 \rho(\vec{r}')\,d\tau' = q\overline{y'}^2 = q\langle y'\rangle^2$$

$$Q_{zz} = \int_{v'} z'^2 \rho(\vec{r}')\,d\tau' = q\overline{z'}^2 = q\langle z'\rangle^2$$

$$Q_{xy} = \int_{v'} x'y' \rho(\vec{r}')\,d\tau' = q\overline{x'y'} = q\langle x'y'\rangle = Q_{yx}$$

$$Q_{yz} = \int_{v'} y'z' \rho(\vec{r}')\,d\tau' = q\overline{y'z'} = q\langle y'z'\rangle = Q_{zy}$$

$$Q_{zx} = \int_{v'} z'x' \rho(\vec{r}')\,d\tau' = q\overline{z'x'} = q\langle z'x'\rangle = Q_{xz}$$

⇐ n.b. The Quadrupole Moment Tensor $\overleftrightarrow{Q}$ has only 6 independent components

Then:

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{1}{r^5}\right)\left[3xyQ_{xy} + 3yzQ_{yz} + 3xzQ_{xz} + \frac{1}{2}\left(3x^2-1\right)Q_{xx} + \frac{1}{2}\left(3y^2-1\right)Q_{yy} + \frac{1}{2}\left(3z^2-1\right)Q_{zz}\right.$$

A relationship exists between multipole moments expressed using spherical-polar coordinates $q_{\ell m}$ and those expressed using Cartesian coordinates Q_{ij}. The first few of these are given below:

$$q_{00} = \frac{1}{\sqrt{4\pi}} q \qquad q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} p_z \qquad q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(Q_{13} - iQ_{23}\right) \quad \text{with} \quad q_{\ell-m} = (-1)^m q^*_{\ell m}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\left(p_x - ip_y\right) \qquad q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{11} - 2iQ_{12} - Q_{22}\right)$$

The Energy / Work Associated With a Charge Distribution $\rho(\vec{r}')$ Located at (or Near) the Origin of the Coordinate System in an External Electric Field $\vec{E}_{ext}(\vec{r})$

For $r >> a$ (a = max value of $|\vec{r}'|$), the energy / work associated with a charge distribution in an external field $\vec{E}_{ext}(\vec{r})$ is given by:

$$W = Q\underbrace{V_{ext}(\vec{r}=0)}_{=-\int_{ref.\,pt.}^{\vec{r}=0}\vec{E}_{ext}\cdot d\vec{l}} - \vec{p}\cdot\vec{E}_{ext}(\vec{r}=0) - \frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}Q_{ij}\left.\frac{\partial E_j^{ext}}{\partial x_i}\right|_{x_i=0} - \ldots$$

$$\vec{E}_{ext}(\vec{r}=0) = -\vec{\nabla}V_{ext}(\vec{r}=0)$$

Where the summations i = 1, 2, 3 and j = 1, 2, 3 represent sums over the $\left\{ {1,2,3 \atop x,y,z} \right\}$ components respectively; i.e. $i, j = 1$: $x_1 \equiv x$ $i, j = 2$: $x_2 \equiv y$ $i, j = 3$: $x_3 \equiv z$

And: $Q_{ij} \equiv \int_{v'}\left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\vec{r}')\,d\tau'$ with $r'^2 = x'^2 + y'^2 + z'^2 = x_1'^2 + x_2'^2 + x_3'^2$

And with: $Q_{ij} = Q_{ji}$, and $\sum_{i=1}^{3}Q_{ii} = Q_{11} + Q_{22} + Q_{33} = Q_{xx} + Q_{yy} + Q_{zz} = 0$

Note: The multipole expansion method for $V_{outside}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{\ell=0}^{\infty}\left(\frac{4\pi}{2l+1}\right)\sum_{m=-\ell}^{+\ell}\left(\frac{1}{r^{\ell+1}}\right)Y_{\ell m}^{*}(\theta,\varphi)\,q_{\ell m}$

with $q_{\ell m} = \int_{v'}(r')^{\ell}\,Y_{lm}(\theta',\varphi')\,\rho(\vec{r}')\,d\tau'$ is analogous to the taking of an inner product!!!

It can then be seen that the electric multipole moments $q_{\ell m}$ are the strengths (i.e. coefficients) associated with the $(\ell, m)^{th}$-order multipoles of the electric charge distribution $\rho(\vec{r}')$!!!

Electrostatic Forces and Torques Acting on Multipole Moments of the Charge Distribution

The net force and torque acting on the charge distribution as an expansion in multipole moments are given below:

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}=0) + \vec{\nabla}\left(\vec{p}\cdot\vec{E}(\vec{r})\right)\Big|_{\vec{r}=0} + \vec{\nabla}\left[\frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}Q_{ij}\frac{\partial E_j^{ext}(\vec{r}=0)}{\partial x_i}\right]\Bigg|_{x_i=0} + \ldots$$

$$\begin{aligned}\vec{\tau}(\vec{r}) = \left(\vec{p}\times\vec{E}(\vec{r})\right)\Big|_{\vec{r}=0} + \frac{1}{3}\Bigg\{&\left[\frac{\partial}{\partial x_3}\left(\sum_{j=1}^{3}Q_{2j}E_j^{ext}(\vec{r}=0)\right) - \frac{\partial}{\partial x_2}\left(\sum_{j=1}^{3}Q_{3j}E_j^{ext}(\vec{r}=0)\right)\right]\Bigg|_{\vec{r}=0} \\ &+\left[\frac{\partial}{\partial x_1}\left(\sum_{j=1}^{3}Q_{3j}E_j^{ext}(\vec{r}=0)\right) - \frac{\partial}{\partial x_3}\left(\sum_{j=1}^{3}Q_{1j}E_j^{ext}(\vec{r}=0)\right)\right]\Bigg|_{\vec{r}=0} \\ &+\left[\frac{\partial}{\partial x_2}\left(\sum_{j=1}^{3}Q_{1j}E_j^{ext}(\vec{r}=0)\right) - \frac{\partial}{\partial x_1}\left(\sum_{j=1}^{3}Q_{2j}E_j^{ext}(\vec{r}=0)\right)\right]\Bigg|_{\vec{r}=0}\Bigg\} + \ldots\end{aligned}$$

