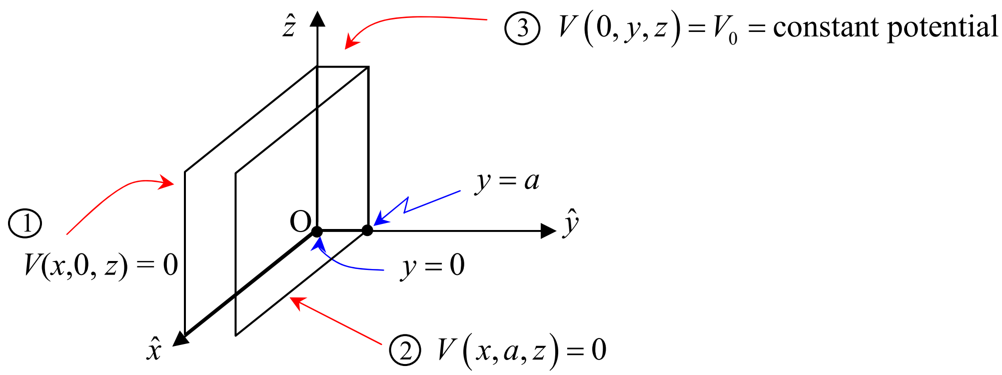
LECTURE NOTES 7.5

Laplace's Equation in Rectangular/Cartesian Coordinates

Griffiths Example 3.3:

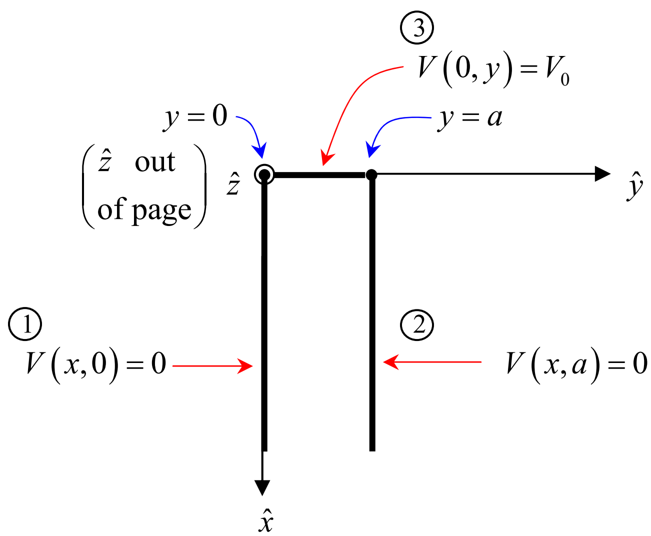
Two infinite, grounded metal plates lie parallel to the *x*-*z* plane, one located at $y = 0$ and the other located at $y = a$. The back side (at $x = 0$) is closed off with an infinite metal strip insulated from the two parallel planes, and maintained at a potential $V(0,y,z) = V_0$.



Find the potential $V(x,y,z)$ inside the "slot" region.

$\Rightarrow$ First, recognize that because $V(x,y,z)$ has *no* explicit *z*-dependence, this problem is actually only a 2-dimensional problem!

Top View of Problem:



Find $V(x,y)$ in "slot" $\Leftarrow$ independent of *z* for *any* value of *z*.

Because of this fact: $\nabla^2 V(x,y,z) \Rightarrow \nabla^2 V(x,y) = \dfrac{\partial^2 V(x,y)}{\partial^2 x} + \dfrac{\partial^2 V(x,y)}{\partial^2 y} = 0$

Boundary Conditions:

1.) $V(x, y=0)=0$

2.) $V(x, y=a)=0$

3.) $V(x=0, y)=V_0$ (constant potential)

4.) $V(x \to \infty, y) \to 0$ i.e. $V(x=\infty, y)=0$

Try product solution of form: $V(x,y)=X(x)Y(y)$

$$\frac{\partial^2 V(x,y)}{\partial x^2}+\frac{\partial^2 V(x,y)}{\partial y^2}=0$$

$$=\frac{\partial^2 \left(X(x)Y(x)\right)}{\partial x^2}+\frac{\partial^2 \left(X(x)Y(y)\right)}{\partial y^2}=0$$

$$=Y(y)\frac{\partial^2 X(x)}{\partial x^2}+X(x)\frac{\partial^2 Y(y)}{\partial y^2}=0 \quad \Leftarrow \quad \text{now divide both sides by } X(x)Y(y)$$

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{\text{function of x only}}+\underbrace{\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}}_{\text{function of y only}}=0$$

∴ must equal a constant, e.g. $= C_1$ ∴ must equal a constant, e.g. $= C_2$

Thus: $C_1+C_2=0$ or $C_1=-C_2$.

Then: $\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}=C_1$ and $\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}=C_2$ (n.b. have *total* derivatives now!)

Or: $\frac{d^2 X(x)}{dx^2}=C_1 X(x)$ $\frac{d^2 Y(y)}{dy^2}=C_2 Y(y)$

Or: $\frac{d^2 X(x)}{dx^2}-C_1 X(x)=0$ $\frac{d^2 Y(y)}{dy^2}-C_2 Y(y)=0$

but $C_1=-C_2$

$$\therefore \quad \frac{d^2 X(x)}{dx^2}-C_1 X(x)=0 \qquad \frac{d^2 Y(y)}{dy^2}+C_1 Y(y)=0$$

Let $C_1=k^2>0$: $\begin{pmatrix}\text{want exponential solutions for } X(x) \\ \text{want sine/cosine solutions for } Y(y)\end{pmatrix}$

Thus, we have:

$$\frac{d^2 X(x)}{dx^2}-k^2 X(x)=0 \quad \text{and} \quad \frac{d^2 Y(y)}{dy^2}+k^2 Y(y)=0$$

General Solutions for $X(x)$ and $Y(y)$:

Because we (deliberately) chose $C_1 = k^2 > 0$, then general solutions for $X(x)$ and $Y(y)$ are of the form: $X(x) = Ae^{kx} + Be^{-kx}$ and $Y(y) = C\sin(ky) + D\cos(ky)$ $\Leftarrow$ EXPLICITLY put these back in above diff eqn's and verify that these are correct!!

n.b. we deliberately chose $C_1 = k^2 > 0$ because of the boundary conditions:

1) $V(x, y=0) = 0$
2) $V(x, y=a) = 0$

$V(x,y)$ *must* be periodic in y. $\Rightarrow$ sine and/or cosine type solutions (not exponentials or sinh/cosh solutions).

3) $V(x=0, y) = V_0$
4) $V(x=\infty, y) = 0$ $\Leftarrow$ Cannot be sine or cosine solutions for x!

Then: $V(x,y) = X(x)Y(y) = \left[Ae^{kx} + Be^{-kx}\right]\left[C\sin(ky) + D\cos(ky)\right]$

Now impose boundary conditions:

BC ④ : $V(x=\infty, y) = 0 \quad \Rightarrow \quad \underline{\underline{A=0}} \qquad \therefore \quad V(x,y) = Be^{-kx}\left[C\sin(ky) + D\cos(y)\right]$

Absorb constant B into C and D:

$V(x,y) = e^{-kx}\left[C\sin(ky) + D\cos(y)\right]$

Impose BC ① and ② : $V(x, y=0) = 0$ and $V(x, y=a) = 0 \quad \Rightarrow \quad \underline{\underline{D=0}}$

$\therefore \quad V(x,y) = Ce^{-kx}\sin(ky)$ automatically satisfies $V(x,0) = 0$

$V(x,y) = e^{-kx}\sin(ka) = 0$ requires $\boxed{ka = n\pi.}$ Then $\sin(n\pi) = 0$ where $n = 1, 2, 3, \ldots$

$\therefore$ Define: $\boxed{k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3\ldots}$

General *specific* solution to Laplace's equation (for *this* problem):

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x}\sin(k_n y) \quad \text{where} \quad k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots\ldots$$

Note that:

$V(x,y) = \sum_{n=1}^{\infty} V_n(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x}\sin(k_n y)$ satisfies $\nabla^2 V(x,y) = 0$ (and all 4 BC's) by superposition principle, since $V_n(x,y) = C_n e^{-k_n x}\sin(k_n y)$ individually satisfy $\nabla^2 V(x,y) = 0$ and all 4 BC's for each/every value of n = 1,2,3,….

Impose final (remaining) BC: ③ $V(0,y) = V_0 = \text{constant}$

$$V(0,y) = \sum_{n=1}^{\infty} C_n e^{0}\sin(k_n y) = V_0 = \text{constant}$$

$$V(0,y)=\underbrace{\sum_{n=1}^{\infty} C_n \sin(k_n y)}_{\text{Fourier Series!!}} = V_0 = \text{constant}$$

To determine the $C_n'^s$, multiply both sides of above equation by e.g. $\sin(k_m y)$ and integrate over y from $y = 0$ to $y = a$. i.e. take *inner product*!

$$V(0,y)\sin(k_m y)=\sum_{n=1}^{\infty} C_n \sin(k_n y)\sin(k_m y) = V_0 \sin(k_m y)$$

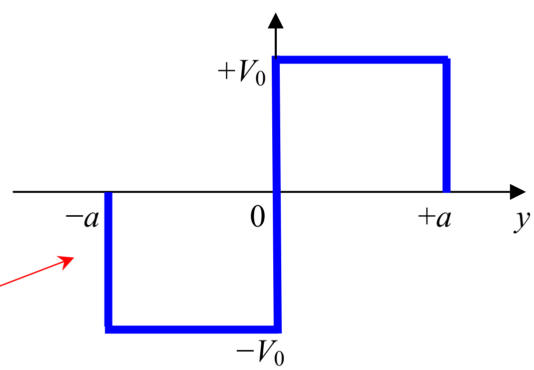
$$\int_0^a V(0,y)\sin(k_m y)\,dy=\int_0^a \sum_{n=1}^{\infty} C_n \sin(k_n y)\sin(k_m y)\,dy = \int_0^a V_0 \sin(k_m y)\,dy$$

$$=\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin(k_n y)\sin(k_m y)\,dy}_{=\frac{1}{2}\left(\frac{n\pi}{k_n}\right)\delta_{nm},\; k_n=\frac{n\pi}{a};\; =\frac{a}{2}\delta_{nm} \left(\begin{array}{l}\text{i.e. } =0 \text{ for } n\neq m \\ =\frac{a}{2} \text{ for } n=m\end{array}\right)} = V_0 \underbrace{\int_0^a \sin(k_m y)\,dy}_{=\frac{-1}{k_m}\cos(k_m y)\big|_0^a = \frac{1}{k_m}(1-\cos(k_m a)) = \frac{a}{m\pi}(1-\cos(m\pi)),\; k_m=\frac{m\pi}{a}}$$

$$\therefore\ \frac{a}{2}C_n\delta_{nm}=\frac{a}{m\pi}V_0\left(1-\cos(n\pi)\right) \qquad \underline{\text{or}}: \quad \boxed{C_n=\frac{2}{n\pi}V_0\left(1-\cos(n\pi)\right)}$$

$$\begin{pmatrix}\delta_{nm}=0 & \text{for } n\neq m \\ \quad =\frac{a}{2} & \text{for } m=n\end{pmatrix}$$

piece-wise continuous function:



Now $1-\cos(n\pi)=0$ if n = even (2, 4, 6,. . . .)

$\qquad\qquad\qquad = 2$ if n = odd (1, 3, 5,. . . .)

$$\therefore\ \boxed{C_n=\frac{4}{n\pi}V_0 \quad n=\text{odd integers only}}$$

All <u>even</u> $C_n = 0$

= Fourier Coefficients C_n for <u>Bipolar Square Wave</u>!!!

Odd function on interval $-a \le y \le a$

$\Rightarrow$ only odd n terms!!

$$C_n=\frac{4}{n\pi}V_0 \qquad n = \text{odd \#'s} = 1, 3, 5, \ldots .$$

$\therefore$ Final, fully-specified for $V(x,y)$ is:

$$\boxed{V(x,y)=\left(\frac{4V_0}{\pi}\right)\sum_{n=odd\,\#s}^{\infty}\frac{1}{n}e^{-k_n x}\sin(k_n y)} \quad \text{where:} \quad \boxed{k_n=\frac{n\pi}{a} \qquad n=1,3,5,\ldots(\text{odd integers})}$$

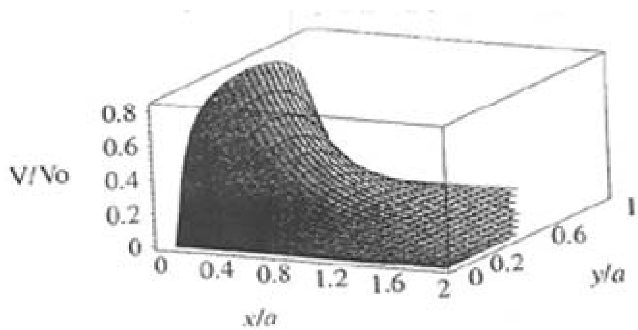
$$\boxed{V(x,y)=\left(\frac{4V_0}{\pi}\right)\sum_{n=odd\,\#s}^{\infty}\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a)}$$

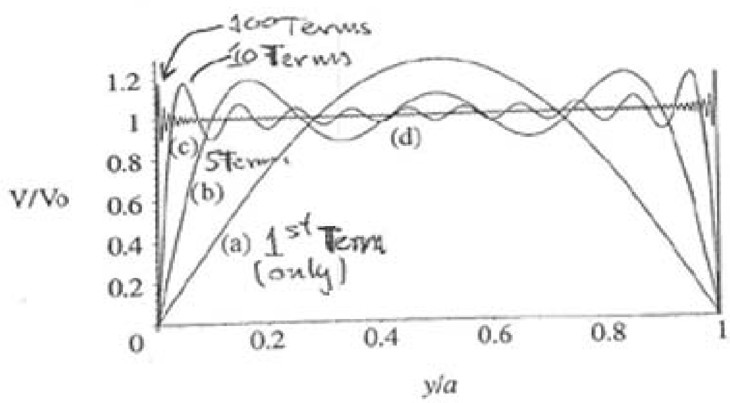
Let $\boxed{n = 2\ell + 1}$ then:

$$\boxed{V(x,y) = \left(\frac{4V_0}{\pi}\right)\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)}e^{-(2\ell+1)\pi x/a}\sin\left((2\ell+1)\pi y\Big/a\right)}$$

n.b. This ∞-series actually has an ***analytic*** representation!!!

$$\boxed{V(x,y) = \left(\frac{2V_0}{\pi}\right)\tan^{-1}\left[\frac{\sin(\pi y/a)}{\sin \pi xa}\right]}$$

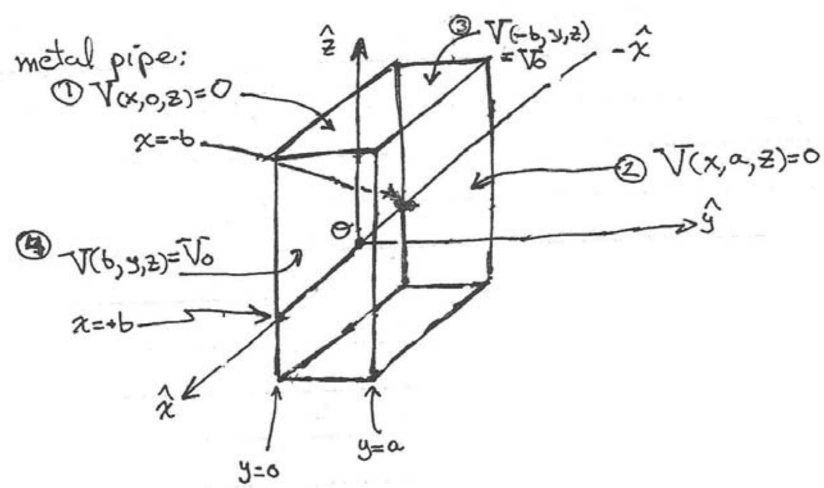




Laplace's Equation $\nabla^2 V(x,y,z)=0$ in Rectangular/Cartesian Coordinates

Griffiths Example 3.4

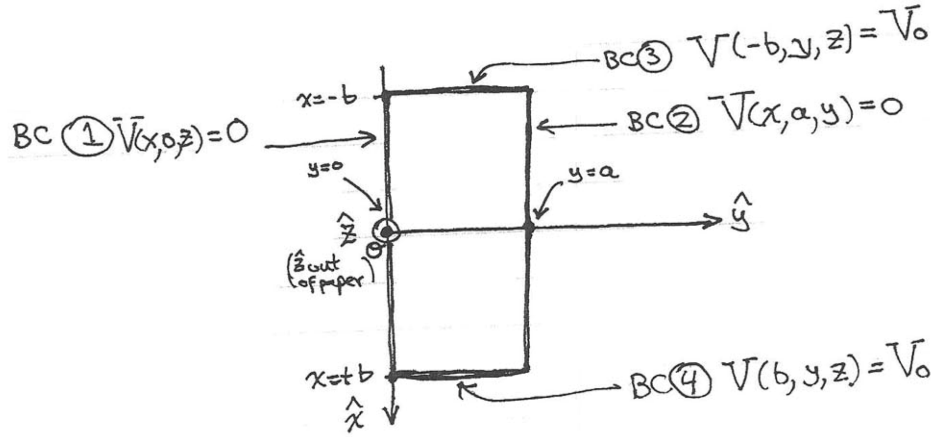
Two infinitely long, grounded metal plates lie parallel to the $x-z$ plane, one at $y=0$ and the other at $y=a$. Two additional infinitely long metal strips lie parallel to the $y-z$ plane, one at $x=-b$ and the other at $x=+b$, which are insulated from the first two grounded metal plates and maintained at a constant potential of $V(x=\pm b,y,z)=V_0$. Find the potential $V(x,y,z)$ inside the resulting rectangular metal pipe:



First, recognize (again) that $V(x,y,z)$ inside the $\overbrace{\text{rectangular pipe}}^{\text{infinitely long in } \hat{z} \text{ direction}}$ has no explicit z-dependence.

$\Rightarrow$ (again) this problem is actually a 2 dimensional problem!!!

Top View:



Note also that (again) this problem has translational invariance in z, i.e. $V(x,y,z)$ does NOT depend on z.

Find $V(x,y)$ inside the rectangular pipe, satisfying Laplace's Equation

$$\nabla^2 V(x,y) = 0 \Rightarrow \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$ in Cartesian/rectangular coordinates

subject to the boundary conditions:

1) $V(x, y=0) = 0$ 2) $V(x, y=a) = 0$

3) $V(x=-b, y) = V_0$ 4) $V(x=b, y) = V_0$

Again, try a product solution of form: $V(x,y) = X(x)Y(y)$

<u>Get</u>:

$$\frac{d^2X(x)}{dx^2} - C_1 X(x) = 0 \qquad \frac{d^2Y(y)}{dy^2} - C_2 Y(y) = 0$$

with: $C_1 + C_2 = 0$ i.e. $C_2 = -C_1$

<u>Let</u>: $\underbrace{C_1 = k^2 > 0}$ $\Rightarrow$ $C_2 = -C_1 = -k^2$

n.b. we deliberately chose this because we will need sine/cosine type solutions in y-portion (i.e. for $Y(y)$) to satisfy the <u>periodic</u> boundary conditions: 1) $V(x,y=0) = 0$ and 2) $V(x,y=a) = 0$

i.e. $Y(y=0) = 0$ i.e. $Y(y=a) = 0$

<u>only</u> sin (ky) can do *<u>this</u>*.

<u>Then</u>:

$$\frac{d^2X(x)}{dx^2} - k^2 X(x) = 0 \qquad \frac{d^2Y(y)}{dy^2} + k^2 Y(y) = 0$$

General solutions for $X(x)$ + $Y(y)$ are (again) of the form:

$X(x) = Ae^{kx} + Be^{-kx}$ and $Y(y) = C \sin(ky) + D \cos(ky)$

<u>Then</u>:

$V(x,y) = X(x)Y(y) = [Ae^{kx} + Be^{-kx}]\,[C \sin(ky) + D \cos(ky)]$

However, note that specify linear orthogonal combinations of $e^{kx} \pm e^{-kx}$

give: $$\left\{ \begin{array}{l} \sinh(kx) = \frac{1}{2}\left(e^{kx} - e^{-kx}\right) \\ \cosh(kx) = \frac{1}{2}\left(e^{kx} + e^{-kx}\right) \end{array} \right\}$$ or: $$\left\{ \begin{array}{l} e^{kx} = \frac{1}{2}\left(\cosh(kx) + \sinh(kx)\right) \\ e^{-kx} = \frac{1}{2}\left(\cosh(kx) - \sinh(kx)\right) \end{array} \right\}$$

- We could have instead, alternatively/equivalently chosen $X(x) = \left[A'\cosh(kx) + B'\sinh(kx)\right]$
- Now <u>because</u> $x = \infty$ is *<u>excluded</u>* in this problem we cannot apriori reject the Ae^{kx} term in the $X(x) = [Ae^{kx} + Be^{-kx}]$ form of the X(x) solution
- Both of the terms $Ae^{kx} + Be^{-kx}$ *<u>must</u>* be allowed here because of this.
- If we look at the $X(x) = [\left[A'\cosh(kx) + B'\sinh(kx)\right]$ version of the $X(x)$ solution, we see that BC③: $V(x = -b,y) = V_0$ and BC④: $V(x=+b,y) = V_0$ means that $V(x,y) = V(-x,y)$ is an *<u>even</u>* function of x with respect to $x \rightarrow -x$.

<u>Now</u>: $\cosh(-x) = +\cosh(x) \longleftarrow$ even fcn(x)

$\sinh(-x) = -\sinh(x) \longleftarrow$ odd fcn(x)

$\therefore$ $\underline{\underline{B' = 0}}$, or equivalently, $\underline{\underline{A = +B}}$

$$\begin{aligned} \text{i.e., } & Ae^{kx} + Be^{-kx} \\ &= Ae^{kx} + Ae^{-kx} \\ &= A(e^{kx} + e^{-kx}) \\ &= 2A(\cosh(kx)) \end{aligned}$$

Thus $\underline{\underline{A' = 2A}}$.

<u>Then</u>:

$V(x,y) = X(x)Y(y) = A\cosh(kx)\left[C\sin(ky) + D\cos(ky)\right] \longleftarrow$ absorb constant A into C & D:

$$V(x,y) = X(x)Y(y) = \cosh(kx)[C\sin(ky) + D\cos(ky)]$$

Now impose boundary conditions ① and ②:

① $V(x,y=0) = 0$, i.e. $\underbrace{Y(y=0) = 0}$ $\Rightarrow$ D <u>must</u> $= 0$

② $V(x, y=a) = 0$, i.e. $\underbrace{Y(y=a) = 0}$ $\Rightarrow$ $\sin(ka) = \sin(n\pi) = 0$

i.e. $ka = n\pi, \quad n = 1,2,3,\ldots$

i.e. $k_n = \left(\dfrac{n\pi}{a}\right), \quad n = 1,2,3,\ldots$

<u>Then:</u>

$$V_n(x,y) = C_n\cosh(k_n x)\sin(k_n y) \quad \text{with} \quad k_n = \left(\frac{n\pi}{a}\right), \quad n = 1,2,3,\ldots$$

<u>Thus</u>: $\boxed{V(x,y) = \sum_{n=1}^{\infty} V_n(x,y) = \sum_{n=1}^{\infty} C_n\cosh(k_n x)\sin(k_n y)}$ $\Leftarrow$ General Specific Solution to Laplace's equation for *<u>this</u>* problem.

This $V(x,y)$ satisfies $\nabla^2 V(x,y)=0$ with $k_n = \left(\frac{n\pi}{a}\right)$, $n=1,2,3,\ldots$ and all four BC's.

Now, from BC's ③ and ④, namely that: $V(x=\pm b, y)=V_0$

Noting that: $\cosh(k_n b)=\cosh(-k_n b)$

Then: $V(x=\pm b,y)=\sum_{n=1}^{\infty} C_n \cosh(k_n b)\sin(k_n y)=V_0$ with $k_n=\left(\frac{n\pi}{a}\right)$, $n=1,2,3,\ldots$

Now multiply both sides of this relation by sin ($k_m y$) and integrate along y from $y = 0$ to $y = a$ to project out (i.e. take inner product) in order to determine the coefficients C_n:

$$V_0 \underbrace{\int_0^a \sin(k_m y)\,dy}_{=-\frac{1}{k_m}\cos(k_m y)\big|_o^a \; = \frac{a}{m\pi}(1-\cos(m\pi))} = \sum_{n=1}^{\infty} C_n \cosh(k_n b)\underbrace{\int_0^a \sin(k_n y)\sin(k_m y)\,dy}_{=\frac{1}{2}\left(\frac{n\pi}{k_n}\right)\delta_{nm} \; = \frac{a}{2}\delta_{nm}}$$

$$\therefore \quad \frac{a}{m\pi}V_0(1-\cos(m\pi)) = \frac{a}{2}C_n\cosh(k_n b)\,\delta_{nm} \qquad \begin{pmatrix}\delta_{nm}=0 \text{ if } m\neq n \\ =1 \text{ if } m=n\end{pmatrix}$$

$$\boxed{\therefore \quad C_n = \left(\frac{2}{n\pi}\right)V_0\left[\frac{1-\cos(n\pi)}{\cosh(k_n b)}\right]} \qquad \boxed{k_n=\left(\frac{n\pi}{a}\right),\; n=1,2,3,\ldots}$$

n.b. *odd* n integers only!! (all *even* n integers have $C_n = 0$) because $[1-\cos(n\pi)] = \begin{pmatrix} 0 \text{ for even } n \\ 2 \text{ for odd } n\end{pmatrix}$

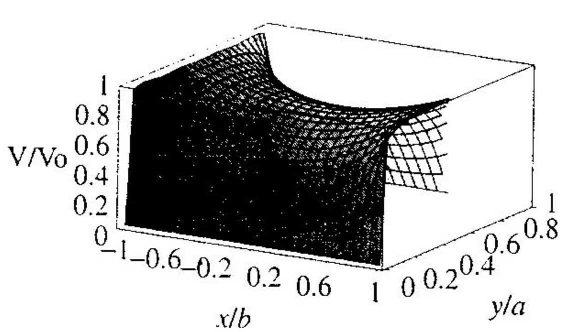
$\therefore$ Final, fully-specified solution for $V(x,y)$ (for *this* problem) is:

$$\boxed{V(x,y)=\left(\frac{4V_0}{\pi}\right)\sum_{n=odd\,\#'s}^{\infty}\left(\frac{1}{n\cosh(k_n b)}\right)\cosh(k_n x)\sin(k_n y), \quad k_n=\left(\frac{n\pi}{a}\right), \quad n=1,3,5,7,\ldots}$$

(odd #'s only!)

Let: $\boxed{n=2\ell+1}$:

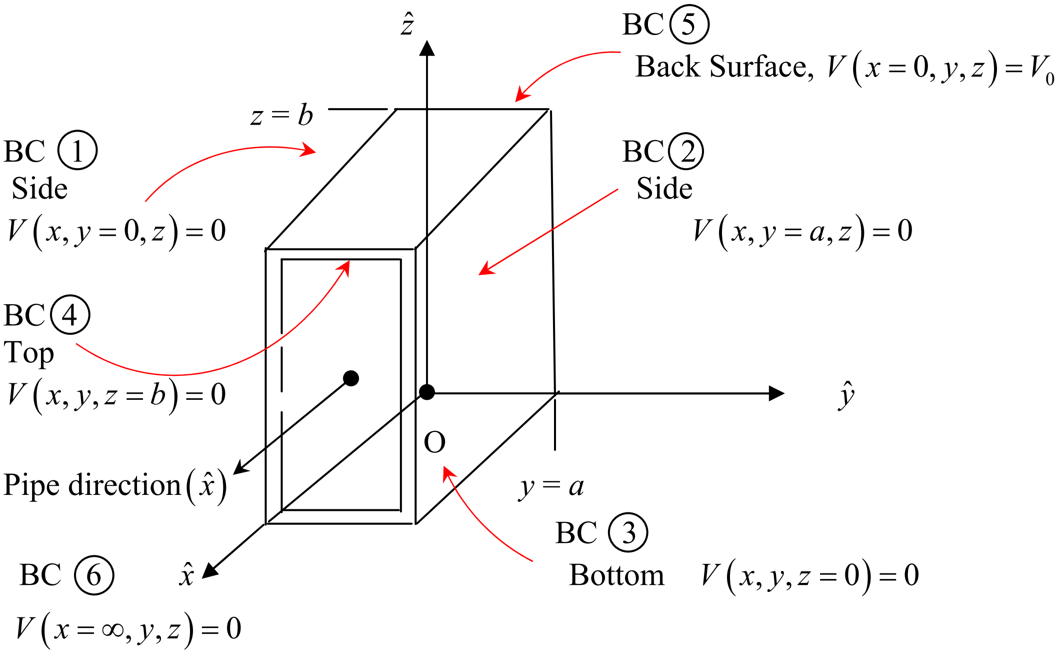
Then: $$\boxed{V(x,y)=\left(\frac{4V_0}{\pi}\right)\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)\cosh\left((2\ell+1)\pi b/a\right)}\cosh\left((2\ell+1)\pi x/a\right)\sin\left((2\ell+1)\pi y/a\right)}$$



Laplace's Equation $\nabla^2 V(x,y,z)=0$ in Rectangular/Cartesian Coordinates

Griffiths Example 3.5

An infinitely long, rectangular metal pipe (in $\hat{x}$ – direction) (of sides a and b dimensions) is grounded. At one end of the rectangular pipe, located at $x = 0$ (i.e. in the $y - z$ plane) the end is covered with a metal plate (insulated from the other surfaces) and maintained/held at a constant potential $V(0,y,z)=V_0$



Find the potential $V(x,y,z)$ inside pipe, for $x \geq 0$ satisfying Laplace's equation, $\nabla^2 V(x,y,z)=0$ and all six BC's.

NOTE: here, $V(x,y,z)$ has explicit x, y, and z-dependence $\Rightarrow$ this IS a fully three-dimensional problem. Then: $\nabla^2 V(x,y,z)=\dfrac{\partial^2 V(x,y,z)}{\partial x^2}+\dfrac{\partial^2 V(x,y,z)}{\partial y^2}+\dfrac{\partial^2 V(x,y,z)}{\partial z^2}=0$

Again, try a product solution of form: $V(x,y,z)=X(x)Y(y)Z(z)$.

Put $V(x,y,z)=X(x)Y(y)Z(z)$ into $\nabla^2 V(x,y,z)=\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0$, divide by $X(x)Y(y)Z(z)$.

Get: $$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\substack{\text{depends only on } x\\ =C_1}}+\underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{\substack{\text{depends only on y}\\ =C_2}}+\underbrace{\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}}_{\substack{\text{depends only on z}\\ =C_3}}=0 \Leftarrow \underset{\forall(x,y,z)}{\text{must}}=0$$

Then: $C_1+C_2+C_3=0$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = C_1 \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = C_2 \qquad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = C_3$$

From previous experience e.g. with Griffiths Examples 3.3 and 3.4:

Choose $C_1 > 0$ and $C_2 < 0$, $C_3 < 0$ i.e. $\underline{\underline{C_1 = -C_2 - C_3 = k^2 + \ell^2}}$, with $\left\{\begin{matrix} C_2 = -k^2 \\ C_3 = -\ell^2 \end{matrix}\right\}$

<u>Then</u>:

$$\frac{d^2X(x)}{dx^2} - \left(k^2+\ell^2\right)X(x) = 0 \qquad \frac{d^2Y(y)}{dy^2} + k^2Y(y) = 0 \qquad \frac{d^2Z(z)}{dz^2} + \ell^2Z(z) = 0$$

<u>General solutions for $X(x)$, $Y(y)$, $Z(z)$ are</u>:

$$\left.\begin{aligned} X(x) &= Ae^{\sqrt{k^2+\ell^2}x} + Be^{-\sqrt{k^2+\ell^2}x} \\ Y(y) &= C\sin(ky) + D\cos(ky) \\ Z(z) &= E\sin(\ell z) + F\cos(\ell z) \end{aligned}\right\} \; V(x,y,z) = X(x)Y(y)Z(z)$$

<u>with boundary conditions</u>:

1) Left Side: $V(x, y=0, z) = 0 \;\; \left(\text{ i.e. } Y(y=0) = 0\right)$
2) Right Side: $V(x, y=a, z) = 0 \;\; \left(\text{ i.e. } Y(y=a) = 0\right)$
3) Bottom: $V(x, y, z=0) = 0 \;\; \left(\text{ i.e. } Z(z=0) = 0\right)$
4) Top: $V(x, y, z=b) = 0 \;\; \left(\text{ i.e. } Z(z=b) = 0\right)$
5) Back: $V(x=0, y, z) = V_0 \;\; \left(\text{ i.e. } X(x=0) = V_0\right)$
6) $x \to \infty$: $V(x=\infty, y, z) = 0 \;\; \left(\text{ i.e. } X(x=\infty) = 0\right)$

<u>BC's 1) and 2) require</u>:

$D = 0$ <u>and</u> $\sin(ka) = \sin(m\pi)$ i.e. $k_n = \left(\frac{n\pi}{a}\right)$, $n = 1, 2, 3, \ldots$

<u>BC's 3) and 4) require</u>:

$F = 0$ <u>and</u> $\sin(\ell b) = \sin(m\pi)$ i.e. $\ell_m = \left(\frac{m\pi}{b}\right)$, $m = 1, 2, 3, \ldots$

<u>BC 6) requires</u>:

$A = 0$

Absorbing constants B & E into C, we have:

$$\boxed{V_{n,m}(x,y,z) = C_{n,m}e^{-\sqrt{k_n^2+\ell_m^2}x}\sin(k_n y)\sin(\ell_m y)}$$

with $\boxed{k_n = \left(\frac{n\pi}{a}\right), \quad n = 1, 2, 3, \ldots}$ and $\boxed{\ell_m = \left(\frac{m\pi}{b}\right), \quad m = 1, 2, 3, \ldots}$

Then:

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}V_{n,m}(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}e^{-\sqrt{k_n^2+\ell_m^2}\,x}\sin(k_n y)\sin(\ell_m z)$$

$$\text{with: } k_n=\left(\frac{n\pi}{a}\right),\quad n=1,2,3,\ldots$$

$$\text{and: } \ell_m=\left(\frac{m\pi}{b}\right),\quad m=1,2,3,\ldots$$

General specific solution to Laplace's equation

Now BC 5) is $V(0,y,z)=V_0,\ \ e^{-0}=1$

$$\therefore\quad V(0,y,z)=V_0=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\sin(k_n y)\sin(\ell_m z)\quad\text{with}\quad \begin{pmatrix} k_n=\left(\frac{n\pi}{a}\right), & n=1,2,3,\ldots \\ \ell_m=\left(\frac{m\pi}{b}\right), & m=1,2,3,\ldots \end{pmatrix}$$

Multiply both sides of this equation by $\sin(k_{n'}y)\sin(\ell_{m'}z)$, n.b. $\begin{pmatrix} n' \\ m' \end{pmatrix}$ are not necessarily $=\begin{pmatrix} n \\ m \end{pmatrix}$!!!

and then integrate both sides over y and z, from ($y = 0$ to $y = a$) and ($z = 0$ to $z = b$) projecting out the $C_{m,n}$ coefficients by taking this inner product:

$$V_0\int_0^a dy\int_0^b dz\sin(k_{n'}y)\sin(\ell_{m'}z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\int_0^a dy\int_0^b dz\sin(k_n y)\sin(k_{n'}y)\times\sin(\ell_m z)\sin(\ell_{m'}z)$$

$$=V_o\left\{\left(-\frac{1}{k_n{}'}\cos\left(k_n{}'y\right)\Big|_0^a\right)\left(-\frac{1}{\ell_m{}'}\cos\left(\ell_m{}'z\right)\Big|_0^b\right)\right\}=\sum_{n=1}^{\infty}C_{nm}\left(\frac{a}{2}\right)\delta_{nn'}\left(\frac{b}{2}\right)\delta_{mm'}$$

$$=V_o\left(\frac{a}{n'\pi}\right)\left(1-\cos(n'\pi)\right)\left(\frac{b}{m'\pi}\right)\left(1-\cos(m'\pi)\right)=\left(\frac{ab}{4}\right)C_{nm}\delta_{nn'}\delta_{mm'}$$

$$\left(\text{using } k_{n'}=\left(\frac{n'\pi}{a}\right)\quad\text{and}\quad \ell_{m'}=\left(\frac{m'\pi}{b}\right)\right)$$

Now: $\left(1-\cos(n\pi)\right)=\left\{\begin{matrix}0\text{ for } n = \text{even \#}\\ 2\text{ for } n = \text{odd \#}\end{matrix}\right\}$ and $\left(1-\cos(m\pi)\right)=\left\{\begin{matrix}0\text{ for } m = \text{even \#}\\ 2\text{ for } m = \text{odd \#}\end{matrix}\right\}$

$$\therefore\quad \frac{4\cancel{ab}}{(n\pi)(m\pi)V_o}=\left(\frac{\cancel{ab}}{4}\right)C_{nm}\ \text{ for } n,m=\text{odd integers: } 1,3,5,\ldots$$

Or: $$C_{n,m}=\frac{16}{\pi^2 nm}V_0\quad\text{for}\quad n,m=\text{odd integers: } 1,3,5\ldots.$$

Thus, for *this* problem the final fully-specified solution for $V(x,y,z)$ satisfying Laplace's Equation $\nabla^2 V(x,y,z)=0$ and all six boundary conditions is:

$$V(x,y,z)=\left(\frac{16V_0}{\pi^2}\right)\sum_{n=\text{odd}\,\#}^{\infty}\sum_{m=\text{even}\,\#}^{\infty}\frac{1}{nm}e^{-\sqrt{k_n^2+\ell_m^2}\,x}\sin(k_n y)\sin(\ell_m z)\qquad k_n=\left(\frac{n\pi}{a}\right),\quad \ell_m=\frac{m\pi}{b}$$

Let $n,m\to(2i+1),(2j+1)$ respectively, then:

$$V(x,y,z)=\left(\frac{16V_0}{\pi^2}\right)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\frac{1}{(2i+1)(2j+1)}e^{-\pi\sqrt{(2i+1)^2/a^2+(2j+1)^2/b^2}\,x}\sin\left(\frac{(2i+1)\pi y}{a}\right)\sin\left(\frac{(2j+1)\pi z}{b}\right)$$

with: $k_i=\left(\frac{(2i+1)\pi}{a}\right),\quad \ell_j=\left(\frac{(2j+1)\pi}{b}\right)\qquad i=1,2,3,\ldots\quad j=1,2,3,\ldots$

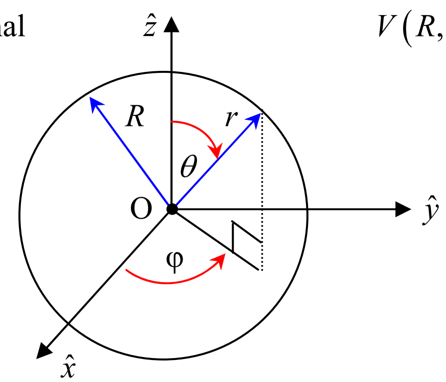
Laplace's Equation in Spherical Coordinates

Griffiths Example 3.6

= some constant (SI units = Volts)

A θ-dependent potential, $V_0(\theta)=k\sin^2\left(\frac{\theta}{2}\right)$ is specified on the surface of a hollow sphere of radius, R. Find the potential inside the sphere, i.e. for $r\le R$.

Note that because of azimuthal symmetry (φ – rotational invariance) $V(\vec r)$ has *no* explicit φ-dependence, i.e. $V(\vec r)\ne \text{fcn}(\varphi)$



$V(R,\theta,\varphi)=V(R,\theta)=k\sin^2(\theta/2)$

$\Rightarrow$ 2-dimensional problem, $V(\vec r)=\text{fcn}(r,\theta)$ *only*, i.e:

$\nabla^2V(r,\theta,\varphi)\;\Rightarrow\;\nabla^2V(r,\theta)=0.$

Try a product solution of the form: $V(r,\theta)=R(r)P(\theta)$, then Laplace's equation becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\theta)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r,\theta)}{\partial\theta}\right)=0$$

See P435 Lecture Notes #7 pages 18-25 for details of separation of variables in spherical coordinates, etc.

General Solution to Laplace's Equation, $\nabla^2V(r,\theta)=0$ with no explicit φ-dependence ($m = 0$) is:

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)\underbrace{P_\ell(\cos\theta)}_{\text{Ordinary Legendre' Polynomial of order},\,\ell}$$

Now apply BC's:

BC ⓪: $V(r,\theta)$ must be finite @ $r = 0$ (no charges @ origin!)

$\Rightarrow B_\ell = 0 \;\; \forall \; \ell \;\; (\forall = \text{for all } \ell)$

BC ①:

$$V(r=R,\theta) = k\sin^2\left(\frac{\theta}{2}\right) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

In order to determine the A_ℓ coefficients, take inner product – i.e. multiply both sides of this equation by $P_{\ell'}(\cos\theta)$ $(\ell' \neq \ell)$ and integrate over θ (with $d\cos\theta = \sin\theta d\theta$) from $(\theta = 0$ to $\theta = \pi)$ - i.e. project out the $A_\ell{}'^s$: (n.b. ℓ' not necessarily = ℓ)

$$k\int_0^\pi \sin^2\left(\frac{\theta}{2}\right) * P_{\ell'}(\cos\theta)\sin\theta d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_0^\pi P_\ell(\cos\theta)P_\ell'(\cos\theta)\sin\theta d\theta}_{=\left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}} \quad \text{(Kroenecker } \delta\text{-fcn)}$$

$$\delta_{\ell\ell'} \begin{Bmatrix} = 0 \text{ if } \ell' \neq \ell \\ = 1 \text{ if } \ell' = \ell \end{Bmatrix} \quad \text{Kroenecker } \delta\text{-fcn}$$

$$\therefore \quad A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) = k\int_0^\pi P_\ell(\cos\theta)\sin^2\left(\frac{\theta}{2}\right)\sin\theta d\theta$$

Or: $$A_\ell = \left(\frac{2\ell+1}{2R^\ell}\right)k\int_0^\pi P_\ell(\cos\theta)\sin^2\left(\frac{\theta}{2}\right)\sin\theta d\theta$$

Now:

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta) \quad \text{using the half-angle formula}$$

$$= \frac{1}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right) \qquad \begin{pmatrix} P_0(\cos\theta) = 1 \\ P_1(\cos\theta) = \cos\theta \end{pmatrix}$$

$$\therefore \quad A_\ell = \left(\frac{2\ell+1}{2R^\ell}\right)k\int_0^\pi \left(\frac{1}{2}\right)P_\ell(\cos\theta)\left(P_0(\cos\theta) - P_1(\cos\theta)\right)\sin\theta d\theta$$

$$= \left(\frac{2\ell+1}{4R^\ell}\right)k\left[\underbrace{\int_0^\pi P_o(\cos\theta)\left(P_\ell(\cos\theta)\right)\sin\theta d\theta}_{=\left(\frac{2}{1}\right)\delta_{0\ell}} - \underbrace{\int_0^\pi P_1(\cos\theta)\left(P_\ell(\cos\theta)\right)\sin\theta d\theta}_{=\left(\frac{2}{3}\right)\delta_{1\ell}}\right]$$

$A_0 = \left(\frac{k}{2R^0}\right) = \left(\frac{k}{2}\right)$ and $A_1 = -\left(\frac{k}{2R^1}\right) = -\left(\frac{k}{2R}\right)$, all other $A_\ell = 0$ for $\ell > 1$ (i.e. $\ell \geq 2$).

The fully-specified solution $V(r,\theta)$ satisfying Laplace's Eqn $\nabla^2 V(r,\theta) = 0$ and the BC's is given by:

$$\boxed{V(r,\theta) = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta), \quad A_0 = k/2, \quad A_1 = -k/2R}$$

Or: $\boxed{V(r,\theta) = \left(\frac{k}{2}\right)\left[1 - \left(\frac{r}{R}\right)\cos\theta\right]}$ for $r \leq R$ with the BC: $V(R,\theta) = k\sin^2\left(\frac{\theta}{2}\right)$.

Griffiths Example 3.7

A θ-dependent potential $V_0(\theta) = k\sin^2\left(\frac{\theta}{2}\right)$ is specified on the surface of a hollow sphere of radius, R.

Find the potential outside the sphere, i.e. for $r \ge R$. $\Rightarrow$ This is the same problem as Griffiths Example 3.6, just asking for $V(r,\theta)$ outside the sphere.

BC⓪: Here, $V(r,\theta)$ must be finite for $r \to \infty$

$$\Rightarrow \text{ All } A_\ell = 0$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

BC①: $V(r=R,\theta) = k\sin^2\left(\frac{\theta}{2}\right) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$

In order to determine the $B_\ell{}^{'s}$, take inner product – i.e. multiply both sides of this equation by $P_{\ell'}(\cos\theta)$ and integrate over θ from $(\theta = 0 \text{ to } \theta = \pi)$ - project out the $B_\ell{}^{'s}$:

$$k\int_0^\pi \sin^2\left(\frac{\theta}{2}\right) P_{\ell'}(\cos\theta)\sin\theta d\theta = \sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{R^{\ell+1}}\right)\int_0^\pi P_{\ell'}(\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta$$

$$= \sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{R^{\ell+1}}\right)\underbrace{\int_0^\pi P_\ell'(\cos\theta)\sin\theta d\theta}_{=\left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}}$$

Kroenecker δ-fcn

$$\delta_{\ell\ell'} = \begin{cases} =0 \text{ if } \ell' \ne \ell \\ =1 \text{ if } \ell' = \ell \end{cases}$$

$$\therefore \quad \left(\frac{B_\ell}{R^{\ell+1}}\right)\left(\frac{2}{2\ell+1}\right) = k\int_0^\pi \sin^2\left(\frac{\theta}{2}\right) P_\ell(\cos\theta)\sin\theta d\theta$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta) = \frac{1}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$$

$$B_\ell = \left[\frac{(2\ell+1)R^{\ell+1}}{4}\right]k\left[\int_0^\pi P_0(\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta - \int_0^\pi P_1(\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta\right]$$

$$B_\ell = -\left[\frac{(2\ell+1)R^{\ell+1}}{4}\right]k\left[\left(\frac{2}{1}\right)\delta_{0\ell} - \left(\frac{2}{3}\right)\delta_{1\ell}\right]$$

$$\therefore \quad B_0 = \left(\frac{kR^2}{2}\right), \quad B_1 = -\left[\frac{kR^2}{2}\right], \text{ all other } B_\ell = 0 \text{ for } \ell > 1 \text{ i.e. } (\ell \ge 2).$$

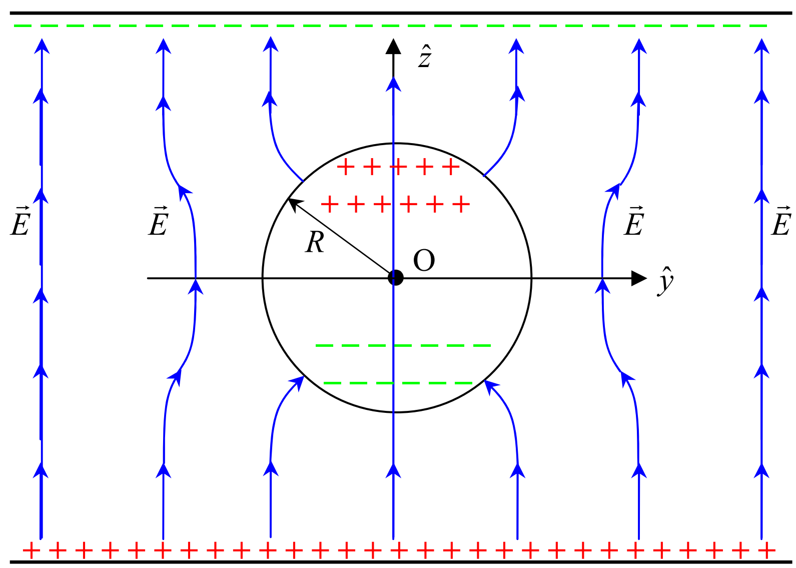
Fully-specified solution $V(r,\theta)$ satisfying Laplace's equation $\nabla^2 V(r,\theta) = 0$ and BC's is given by:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta) = \frac{B_0}{r}\underbrace{P_0(\cos\theta)}_{=1} + \frac{B_1}{r^2}\underbrace{P_1(\cos\theta)}_{=\cos\theta}$$

Or: $V(r,\theta) = \left(\frac{kR}{2r}\right)\left[1 - \left(\frac{R}{r}\right)\cos\theta\right]$ for $r \ge R$ with BC $V(r=R,\theta) = k\sin^2\left(\frac{\theta}{2}\right)$.

Griffiths Example 3.8

An (initially) uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\vec{E} = E_o\hat{z}$. A surface charge distribution $\sigma(r,\theta)$ will result, with +ve excess free surface charge in the "northern" hemisphere and −ve excess free surface charge in the "southern" hemisphere. The induced surface charge distribution, in turn, distorts/changes the electric field in the neighborhood of the sphere. Find the potential outside the sphere, i.e. for $r > R$.



Note that this problem (again) has a azimuthal symmetry – i.e. it is invariant under arbitrary rotations in φ, $\Rightarrow \ \therefore \ V(\vec{r})$ has no $\varphi-$dependence. Thus the potential $V(r,\theta,\varphi) \rightarrow V(r,\theta)$ (i.e. this is a 2-D problem).

Note also that the conducting sphere is an equipotential surface at $r = R$ $(\forall \ \theta + \varphi)$. The actual value of the potential on the surface of the conducting sphere is arbitrary (as long as it is finite) so we can chose $V(r = R, \theta) = 0$ without loss of generality.

Since $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

BC⓪: Far from the sphere, we know $\vec{E} = E_0\hat{z}$
$\therefore$ Far from the sphere, we know $V(r \rightarrow \infty) = -E_0 z + C$ $\quad$ (with $z = r\cos\theta$)
where C is a constant (to be determined).

Consider now the reflection symmetry of this problem (i.e. $z \rightarrow -z$). Note that in/on the equatorial plane (i.e. the x-y plane @ $z = 0$), the surface of the conducting sphere has potential: $V(r = R, \theta = \pi/2) = 0$ (i.e. at $z = 0$). Due to the $z \rightarrow -z$ reflection symmetry of this problem, the entire equatorial plane is *also* at this same potential, i.e. the entire equatorial plane is also at this *equipotential*. Thus, we must *also* have $V\left(r \rightarrow \infty, \theta = \dfrac{\pi}{2}\right) = 0$ (i.e. at $z = 0$).

This tells us that the constant $C = 0$ and thus: $\boxed{V(r \rightarrow \infty) = -E_0 z.}$

The general solution for the potential, $V(r,\theta)=0$ in spherical coordinates (for $m = 0$, i.e. *no* φ-dependence) is: $\boxed{V(r,\theta)=\sum_{\ell=0}^{\infty}V_\ell(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$

Note that for *this* problem, the origin $(r=0)$ is *excluded* (we want $V(r,\theta)$ for $r > R$, so the $B_\ell \neq 0$ in general for such situations).

Furthermore, far from the sphere $V(r,\theta)=-E_0 z=-E_0 r\cos\theta \quad (z=r\cos\theta)$, thus we can see that $A_\ell \neq 0$ in general.

Apply BC① on the surface of sphere:

$$V(r=R,\theta)=0=\sum_{\ell=0}^{\infty}\left(A_\ell R^\ell+\frac{B_\ell}{R^{\ell+1}}\right)P_\ell(\cos\theta)$$

The *only* way this can be satisfied for all θ (arbitrary) is if: $\left(A_\ell R^\ell+\frac{B\ell}{R^{\ell+1}}\right)=0\ \forall\ \ell$

Or: $\boxed{B_\ell=-R^{2\ell+1}A_\ell}$ *

Then: $V(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell\left(r^\ell-\left(\frac{R^{2\ell+1}}{r^{\ell+1}}\right)\right)P_\ell(\cos\theta)$

NOTE: for $r \gg R$ the second term in parentheses is negligible. Thus far from sphere we must have $V(r\to\infty,0)=-E_0 z=-E_0 r\cos\theta=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)$, and since $P_1(\cos\theta)=\cos\theta$ we don't have to bother with inner product "stuff" − we just have to realize that only the $\ell=1$ term survives in the sum on the RHS!!!

$\therefore\ V(r\to\infty)=-E_0 r\cos\theta=A_1 r\cos\theta \ \Rightarrow\ A_1=-E_0$

$\therefore\ \boxed{B_1=-R^3A_1=+E_0R^3}$ (from * above).

Thus:

$$V(r,\theta)=-E_0\left(r-\frac{R^3}{r^2}\right)\cos\theta=-E_0 r\left(1-\left(\frac{R}{r}\right)^3\right)\cos\theta=\underbrace{-E_0 r\cos\theta}+E_0R\left(\frac{R}{r}\right)^2\cos\theta$$

$$=\underbrace{-E_0 z}_{\substack{\text{potential for}\\ \text{external field}}}+\underbrace{E_0R\left(\frac{R}{r}\right)^2\cos\theta}_{\substack{\text{potential associated with}\\ \text{electric dipole }(1/r^2\text{ pot'l})\\ \text{from }\sigma(r,\theta)\text{ on cond. sphere}}}=V_{ext}(r,\theta)+V_{sphere}(r,\theta)$$

We can now calculate/determine the surface charge density $\sigma(\theta)$ on surface of conducting sphere:

$$\sigma(\theta)=-\varepsilon_0\left.\frac{\partial V(r,\theta)}{\partial r}\right|_{r=R}=\varepsilon_0 E\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\theta\Bigg|_{r=R}=3\varepsilon_0 E_0\cos\theta$$

Note that $\sigma(\theta)>0$ in the "northern" hemisphere and that $\sigma(0)<0$ in the "southern" hemisphere.

Griffiths Example 3.9

A specified surface charge density $\sigma_0(\theta) = k\cos\theta$ is "glued" over the surface of a spherical shell of radius *R*. Find the resulting potential inside and outside the sphere.

[n.b. We *could* do this problem by direct integration of $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_0(\theta)}{\mathrm{r}} dA$, but we'll use the Laplace equation series solution method instead…]

Here again, note that we have azimuthal symmetry (i.e. no explicit φ-dependence), Thus this is a 2-*D* problem, i.e. $V(r,\theta,\varphi) \Rightarrow V(r,\theta)$.

First, for $r < R$ (inside sphere):

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

(Here, all $B_\ell = 0$ because $1/r^{\ell+1} \to \infty @ r = 0$, i.e. $V(0,\theta)$ must be finite)

Second, for $r > R$ (outside sphere):

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

(Here, all $A_\ell = 0$ because $r^\ell \to \infty @ r \to \infty$, i.e. $V(r \to \infty,\theta) \to 0$.

The potential $V(r = R,\theta)$ must be continuous @ $r = R$:

i.e. $V_{in}(r = R,\theta) = V_{out}(r = R,\theta)$

$$\therefore \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

This can only be true $\forall\theta$ **iff** (if and only if):

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}, \text{ i.e. } \boxed{B_\ell = R^{2\ell+1} A_\ell}$$

Now since the surface charge density $\sigma_0(\theta) = k\cos\theta$ is glued onto the surface of sphere @ $r = R$, the radial derivative of $V(r,\theta)\big|_{r=R}$ suffers a discontinuity at this surface

$$\text{i.e. } \left(\frac{\partial V_{out}(r,\theta)}{\partial r} - \frac{\partial V_{in}(r,\theta)}{\partial r}\right)\Bigg|_{r=R} = -\frac{\sigma_0(\theta)}{\varepsilon_0} = -\frac{k\cos\theta}{\varepsilon_0}$$

Then: $$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell \left(\frac{R^{2\ell+1}}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

and: $$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad \text{with} \quad B_\ell = R^{2\ell+1} A_\ell$$

Thus: $$\frac{\partial V_{out}(r,\theta)}{\partial r}\Bigg|_{r=R} = -\sum_{\ell=0}^{\infty} A_\ell \left(\frac{R^{2\ell+1}}{r^{\ell+2}}\right)(\ell+1) P_\ell(\cos\theta)\Bigg|_{r=R} = -\sum_{\ell=0}^{\infty} (\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

$$\therefore \quad \left[\frac{\partial V_{out}(r,\theta)}{\partial r} - \frac{\partial V_{in}(r,\theta)}{\partial r}\right]\Bigg|_{r=R} = -\sum_{\ell=0}^{\infty}\left\{(\ell+1)A_\ell R^{\ell-1} + \ell A_\ell R^{\ell-1}\right\}P_\ell(\cos\theta)$$

$$= -\sum_{\ell=-}^{\infty}(2\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta) = -\frac{k\cos\theta}{\varepsilon_0}$$

$\Rightarrow$ only the $\ell = 1$ $\left(i.e.\ P_1(\cos\theta) = \cos\theta\right)$ term is non-zero!! All other A_ℓ & $B_\ell = 0$ for $\ell \neq 1$.

$$\therefore \quad -3A_1R^0P_1(\cos\theta) = -3A_1\cos\theta = -\frac{k}{\varepsilon_0}\cos\theta$$

$$\therefore \quad \boxed{A_1 = \frac{k}{3\varepsilon_0}} \quad \text{and} \quad \boxed{B_1 = R^3A_1 = \frac{1}{3}R^3\left(\frac{k}{\varepsilon_0}\right)}$$

Then:

For $r < R$: $\boxed{V_{in}(r,\theta) = A_1rP_1(\cos\theta) = \frac{k}{3\varepsilon_0}r\cos\theta = \frac{k}{3\varepsilon_0}z \qquad (z = r\cos\theta)}$

For $r > R$: $\boxed{V_{out}(r,\theta) = \frac{B_1}{r^2}P_1(\cos\theta) = \left(\frac{k}{3\varepsilon_0}\right)R\left(\frac{R}{r}\right)^2\cos\theta}$

Note that if $k = 3\varepsilon_0E_0$, then $\sigma_0(\theta) = k\cos\theta = 3\varepsilon_0E_0\cos\theta$

and: $V_{in}(r,\theta) = E_0z = E_0r\cos\theta \ \Rightarrow\ \vec{E}_{in}(\vec{r}) = -\vec{\nabla}V_{in}(\vec{r}) = -E_0\hat{z} \Leftarrow$ constant/uniform inside sphere!

Proof:

$$\vec{E}_{in}(\vec{r}) = -\vec{\nabla}V_{in}(\vec{r}) = -\vec{\nabla}V_{in}(r,\theta) = -\vec{\nabla}(E_0z) = -\vec{\nabla}(E_0r\cos\theta)$$

$$= -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right)(E_0r\cos\theta)$$

$$= -E_0\cos\theta\hat{r} + E_0\sin\theta\hat{\theta} = -E_0\underbrace{\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)}_{=\hat{z}} = -E_0\hat{z}$$

Similarly: $V_{out}(r,\theta) = E_0R\left(\frac{R}{r}\right)^2\cos\theta \ \Rightarrow\ \vec{E}_{out}(\vec{r}) =$ electric dipole field, $\sim 1/r^3$