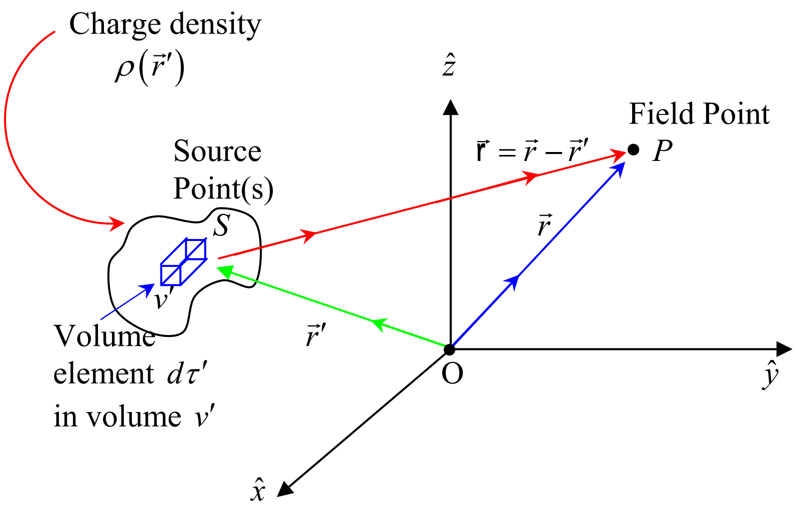
LECTURE NOTES 7

LAPLACE'S EQUATION

As we have seen in previous lectures, very often the primary task in an electrostatics problem is *e.g.* to determine the electric field $\vec{E}(\vec{r})$ of a given stationary/static charge distribution – *e.g.* via Coulomb's Law:



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'} \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\rho(\vec{r}')\,d\tau' \qquad \boldsymbol{\mathfrak{r}} = \vec{r} - \vec{r}' \qquad \hat{\boldsymbol{\mathfrak{r}}} = \frac{\boldsymbol{\mathfrak{r}}}{|\boldsymbol{\mathfrak{r}}|} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$|\boldsymbol{\mathfrak{r}}| = |\vec{r} - \vec{r}'| = \sqrt{(x_p - x_s)^2 + (y_p - y_s)^2 + (z_p - z_s)^2}$$

Oftentimes $\rho(\vec{r}')$ is complicated, and *analytic* calculation of $\vec{E}(\vec{r})$ is painful / tedious (or just plain hard). (Numerical integration on a computer is likely faster/easier. . .)

Oftentimes it is easier to *first* calculate the potential $V(\vec{r})$, and *then* use $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

Here: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{v'} \frac{1}{\boldsymbol{\mathfrak{r}}}\rho(\vec{r}')\,d\tau'$

But even doing this integral analytically often can be very challenging. . .

Furthermore, often in problems involving conductors, $\rho(\vec{r}')$ may not *apriori* (*i.e.* beforehand) be known! Charge is free to move around, and often only the total *free* charge Q_{free} is controlled / known in the problem.

In such cases, it is usually better to recast the problem in DIFFERENTIAL form, using Poisson's equation:

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = -\vec{\nabla}\cdot\vec{\nabla}V(\vec{r}) = -\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o}$$

Or: $\boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o}}$ $\Leftarrow$ Poisson's Equation

Poisson's equation, together with the *boundary conditions* associated with the value(s) allowed for $V(\vec{r})$ *e.g.* on various conducting surfaces, or at $r = \infty$, *etc*. enables one to uniquely determine $V(\vec{r})$ (we'll see how / why shortly. . .).

The Poisson equation is an inhomogeneous second-order differential equation – its solution consists of a particular solution for the inhomogeneous term (RHS of Poisson's Equation) plus the general solution for the homogeneous second-order differential equation:

$$\boxed{\nabla^2 V(\vec{r}) = 0} \quad \Leftarrow \quad \text{Laplace's Equation}$$

commensurate with the boundary conditions for the specific problem at hand.

Very often, in fact, we are interested in finding the potential $V(\vec{r})$ in a *charge-free* region, containing no electric charge, *i.e.* where $\rho(\vec{r}\,') = 0$.

If $\rho(\vec{r}\,') = 0$, then $\nabla^2 V(\vec{r}) = 0$ and the TRIVIAL solution is $V(\vec{r}) = 0 \;\forall\; \vec{r}$, which is boring / useless!

We seek *physically meaningful* / *non-trivial* solutions $V(\vec{r}) \neq 0$ that satisfy $\nabla^2 V(\vec{r}) = 0$ and the boundary conditions on $V(\vec{r})$ for a given physical problem.

Now, before we go any further on this discussion, let's back up a bit and take a (very) broad generalized MATHEMATICAL view (or approach) to find $V(\vec{r})$.

First, let's simplify the discussion, by talking about *one-dimensional* problems:

If $\rho(x) = 0$, Laplace's Equation in one-dimension becomes (in rectangular/Cartesian coordinates):

$$\nabla^2 V(\vec{r}) = 0 \quad \Rightarrow \quad \boxed{\frac{d^2 V(x)}{dx^2} = 0} \quad \Leftarrow \text{ Note the \textit{total} (not partial) derivative with regards to } x.$$

Integrating this equation (both sides) once, we have:

$$\int \frac{d^2 V(x)}{d^2 x} dx = \int \frac{d}{dx}\left(\frac{dV(x)}{dx}\right) dx = \int d\left(\frac{dV}{dx}\right) = \frac{dV(x)}{dx} = \int dx = m = 1^{\text{st}} \text{ constant of integration}$$

Then: $\displaystyle \int \frac{dV(x)}{\cancel{dx}} \cancel{dx} = \int m dx = m \int dx$

Or: $\displaystyle \int dV(x) = V(x) = mx + b \leftarrow 2^{\text{nd}}$ constant of integration

So: $V(x) = b + mx$ (equation for a straight line) is the general solution for $\dfrac{d^2 V(x)}{dx^2} = 0$.

y-intercept slope

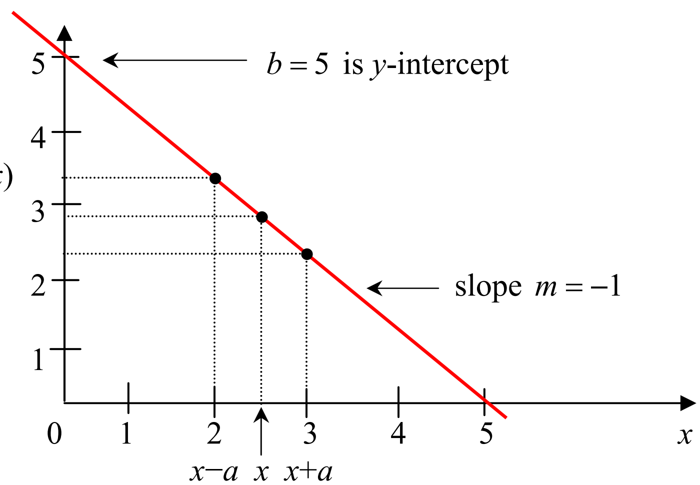
Depending on the boundary conditions for the problem, *e.g.* suppose $V(x=5)=0$ Volts and $V(x=1)=4$ Volts, then together, these two boundary conditions *uniquely* specify what b and m must be – we have two equations, and two unknowns (m & b) – solve simultaneously:

$$V(x) = b + mx \quad \leftarrow \text{equation for a straight line}$$

y-intercept slope

$$V(x=5) = 0 = b + 5m \quad \rightarrow \quad b = -5m$$
$$V(x=1) = 4 = b + 1m \quad \rightarrow \quad 4 = -5m + 1m = -4m$$
$$\therefore \quad V(x) = 5 - 1x \quad \text{or:} \quad m = -1 \text{ and } b = 5.$$

$V(x) = 5 - 1x$ is the equation of a straight line for *this* problem.



General features of 1-*D* Laplace's Equation $\nabla^2 V(x) = 0$ and potential $V(x)$:

1. From above one-dimensional case $V(x) = b + mx$ (general solution = straight line eqn.) we can see that:

   $V(x)$ is the <u>*average*</u> of $V(x+a)$ and $V(x-a)$ *i.e.* $V(x) = \frac{1}{2}\{V(x+a) + V(x-a)\}$

$\Rightarrow$ Laplace's Equation is a kind of <u>*averaging instruction*</u>

The solutions of $V(x)$ are as "boring" as possible, but fit the endpoints (boundary conditions) properly.

This may be "obvious" in one-dimension, but it is <u>also</u> true / <u>also</u> holds in 2-*D* and 3-*D* cases of $\nabla^2 V(\vec{r}) = 0$.

2. $\nabla^2 V(\vec{r})$ tolerates / allows NO local maxima or minima – extrema <u>*must*</u> occur at <u>endpoints</u>

   i.e. $\nabla^2 V(\vec{r}) = 0$ requires the <u>*second*</u> <u>*spatial*</u> <u>*derivative(s)*</u> of $V(\vec{r})$ to be <u>zero</u>.

   - <u>Not</u> a proof, because *e.g.* $\exists$ $fcns(x)$ where the second derivative <u>***vanishes***</u> <u>other</u> than at endpoints - *e.g.* $f(x) = x^4$ (has a minimum at $x = 0$).

<u>Laplace's Equation in Two Dimensions (in Rectangular/Cartesian Coordinates)</u>

If $V = V(x,y)$ then $\nabla^2 V = 0 \Rightarrow \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

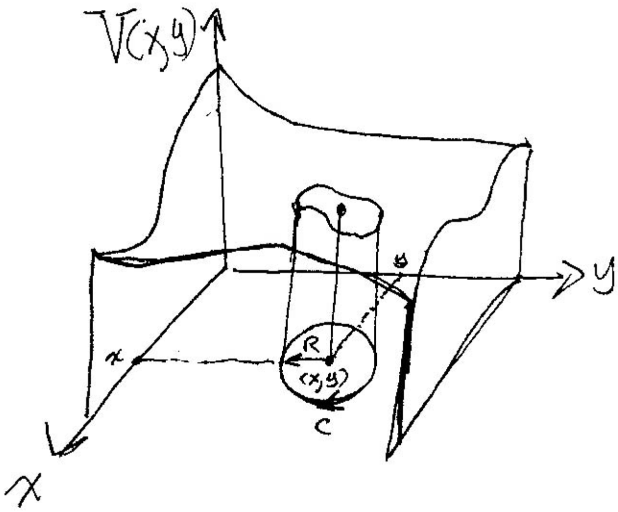
n.b. now have <u>*partial*</u> derivatives of $V(\vec{r})$.

Because $\nabla^2 V = 0 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) V(x,y)$ now contains <u>*partial*</u> derivatives, the general solution does <u>*not*</u> contain just two arbitrary constants or any finite number - $\exists$ an <u>*infinite*</u> number of possible solutions (in general)

– the most general solution is a linear combination of harmonic functions (sine and cosine functions of *x* and *y* in rectangular coordinates and other functions (Bessel Functions) in cylindrical coordinates).

Nevertheless, $V(x,y)$ will still wind up being the <u>*average value of V*</u> around a point (x,y) within a <u>*circle*</u> of radius *R* <u>*centered*</u> on the point (x,y).

The Method of Relaxation - Iterative Computer Algorithm for Finding $V(x,y)$:



$$V(x,y)=\frac{1}{2\pi R}\oint_{\substack{\text{circle C of}\\ \text{radius R}\\ \text{centered}\\ \text{on (x,y)}}} V(\vec{r})\,dl$$

- Start with $V(x,y)$ as <u>specified on boundary</u> (fixed)
- Choose reasonable "interpolated" values of $V(x,y)$ (from boundary) on <u>*interior*</u> (x,y) points <u>*away*</u> from the boundaries.
- 1st pass reassigns $V(x,y)$ = <u>*average value at interior point*</u> (x,y) of its <u>*nearest neighbors*</u>.
- 2nd pass repeats this process . .
- 3rd pass repeats this process . . .
- *etc.* ….

After few iterations, $V(x,y)$ of n^{th} iteration settles down, *e.g.* when:

$$\Delta V(x,y)=\left|\overbrace{V_n(x,y)}^{\textit{iteration } n}-\overbrace{V_{n-1}(x,y)}^{\textit{iteration } n-1}\right| \le \text{ tolerance}$$

then QUIT iterating, $V(x,y)$ is determined after n^{th} iteration is "good enough".

$V(x,y)$ again will have no local maxima or minima – all <u>extrema</u> will occur on boundaries. $\nabla^2 V(x,y)=0$ has solution $V(x,y)$ which is the most featureless function – as smooth as possible.

Laplace's Equation in Three Dimensions

Can't draw this on 2-*D* sheet of paper (because now this is a 4 dimensional problem!), but:

$V(x,y,z) = V(\vec{r}) =$ average value of V over a spherical surface of radius R centered on $\vec{r}$.

i.e. $$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\substack{\text{sphere at } \vec{r} \\ \text{of radius R}}} V da$$

Again $V(\vec{r})$ will have no local maxima or minima

- all extrema ***must*** occur at boundaries of problem (see work-through proof in Griffiths, *p*. 114)
- The average potential produced by a collection of charges, averaged over a sphere of radius R is equal to the value of the potential at the center of that sphere!

Boundary Conditions on the Potential $V(\vec{r})$

Dirichlet Boundary Conditions on $V(\vec{r})$:

$V(\vec{r})$ itself is specified (somewhere) on the boundary - *i.e.* the value of $V(\vec{r})$ is specified (somewhere) on the boundary.

Neumann Boundary Conditions on $V(\vec{r})$:

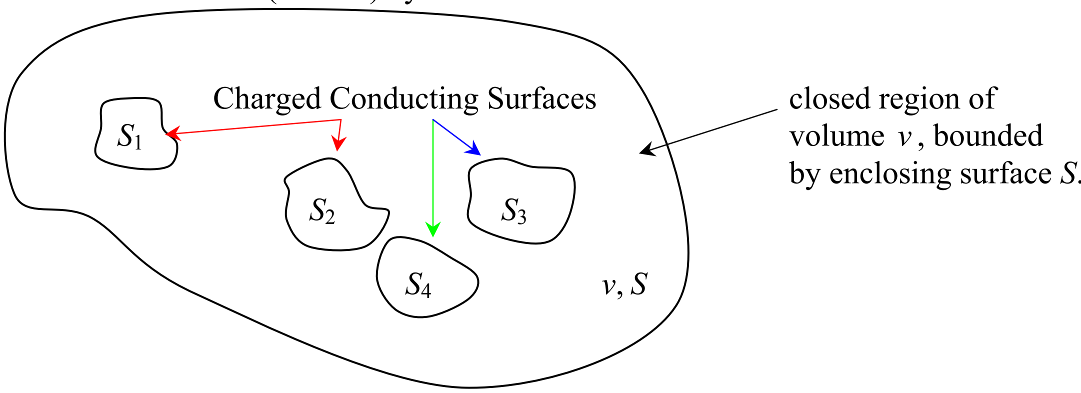
The ***normal*** derivative of $V(\vec{r})$ is specified somewhere on the boundary - *i.e.*

$\vec{\nabla} V(\vec{r}) \bullet \hat{n} = -E^{\perp}(\vec{r})$ is specified somewhere on the boundary.

Uniqueness Theorem(s):

Suppose we have ***two*** solutions of Laplace's equation, $V_1(\vec{r})$ and $V_2(\vec{r})$, each satisfying the *same* boundary condition(s), *i.e*. the potentials $V_1(\vec{r})$ and $V_2(\vec{r})$ are specified on the boundaries. We assert that the two solutions can at most differ by a constant. (*n.b*. Only ***differences*** in the scalar potential $V(\vec{r})$ are important / physically meaningful!)

Proof: Consider a closed region of space with volume v which is exterior to n charged conducting surfaces S_1, S_2, S_3. . . S_n that are responsible for generating the potential V. The volume v is bounded (outside) by the surface S.



Suppose we have ***two*** solutions $V_1(r)$ and $V_2(r)$ both satisfying $\nabla^2 V(\vec{r})=0$ *i.e.* $\nabla^2 V_1(\vec{r})=0$ and $\nabla^2 V_2(\vec{r})=0$ in the *charge-free* region(s) of the volume v.

$V_1(r)$ and $V_2(r)$ satisfy either Dirichlet boundary conditions or satisfy Neumann boundary conditions $\vec{\nabla}V(\vec{r})\bullet\hat{n}$ on the surfaces S_1, S_2, S_3. . . S_n. We also demand that $V(r)$ be *finite* at $r=\infty$.

Let us define: $V_\Delta(\vec{r})\equiv V_1(\vec{r})-V_2(\vec{r})=$ difference in the two potential solutions at the point $\vec{r}$.

Since both $\nabla^2 V_1(\vec{r})=0$ and $\nabla^2 V_2(\vec{r})=0$ then:

$$\nabla^2 V_\Delta(\vec{r})=\nabla^2\left(V_1(\vec{r})-V_2(\vec{r})\right)=\underbrace{\nabla^2 V_1(\vec{r})}_{\substack{separately\\=0}}-\underbrace{\nabla^2 V_2(\vec{r})}_{\substack{separately\\=0}}=0 \quad \underline{\text{Note that:}}\ \nabla^2 V(\vec{r})=\vec{\nabla}\bullet\left(\vec{\nabla}V(\vec{r})\right)$$

The potentials $V_{i=1,2}$ are uniquely specified on charged (equipotential) surfaces S_1, S_2, S_3, . . . S_n in the volume v.

Now apply the divergence theorem to the quantity $\left(V_\Delta\vec{\nabla}V_\Delta\right)$; we also define: $\vec{E}_\Delta(\vec{r})\equiv-\vec{\nabla}V_\Delta(\vec{r})$

$$\int_v \vec{\nabla}\bullet\left(V_\Delta\vec{\nabla}V_\Delta\right)d\tau=\underbrace{\int_{\substack{S+\\S_1+S_2+S_3+\dots S_n}}}\left(V_\Delta\vec{\nabla}V_\Delta\right)\bullet d\vec{A}=\int_{\substack{S+\\S_1+S_2+S_3+\dots S_n}}V_\Delta\left(-\vec{E}_\Delta\right)\bullet d\vec{A}$$

Volume integral over enclosing volume v

Surface integral over ALL surfaces in v

Then:

$$-\int_{\substack{S+\\S_1+S_2+S_3+\ldots S_n}} V_\Delta\left(\vec{E}_\Delta \bullet d\vec{A}\right) = -\int_S V_\Delta\left(\vec{E}_\Delta \bullet d\vec{A}_S\right) - \int_{S_1} V_\Delta\left(\vec{E}_\Delta \bullet d\vec{A}_{S_1}\right) - \int_{S_2} V_\Delta\left(\vec{E}_\Delta \bullet d\vec{A}_{S_2}\right) - \ldots - \int_{S_n} V_\Delta\left(\vec{E}_\Delta \bullet d\vec{A}_{S_n}\right)$$

<u>Recognizing that</u>:

1. The conducting surfaces S_1, S_2, S_3, . . . S_n, are <u>*equipotentials*</u>.
   Thus: $V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$ (= a constant on surfaces S_1, S_2, S_3, . . . S_n) <u>*must*</u> = 0 at/on those surfaces!!!
2. The volume v is arbitrary, so let's choose volume $v \to \infty$, and thus surface area $S \to \infty$ as well.
3. $\int_{S_i} \vec{E}_\Delta \bullet d\vec{A}_i = \Phi_{E_i}$ = electric flux through i^{th} surface.
4. $V_\Delta(r \to \infty) = V_1(r \to \infty) - V_2(r \to \infty)$ (= constant on surface $S \to \infty$) <u>*must*</u> = 0
   because $V_1(r \to \infty) = V_2(r \to \infty)$.

$$\therefore \int_{\substack{v\\ all\ space}} \vec{\nabla} \bullet \left(V_\Delta \vec{\nabla} V_\Delta\right) d\tau = -\underbrace{V_\Delta^S}_{=0} \underbrace{\int_{\substack{S\\ all\ space}} \vec{E}_\Delta \bullet d\vec{A}_S}_{=\Phi_E^S} - \underbrace{V_\Delta^{S_1}}_{=0} \underbrace{\int_{S_1} \vec{E}_\Delta \bullet d\vec{A}_{S_1}}_{=\Phi_E^{S_1}} - \underbrace{V_\Delta^{S_2}}_{=0} \underbrace{\int_{S_2} \vec{E}_\Delta \bullet d\vec{A}_{S_2}}_{=\Phi_E^{S_2}} - \ldots - \underbrace{V_\Delta^{S_n}}_{=0} \underbrace{\int_{S_n} \vec{E}_\Delta \bullet d\vec{A}_{S_n}}_{=\Phi_E^{S_n}}$$

Thus: $\int_{\substack{v\\ all\ space}} \vec{\nabla} \bullet \left(V_\Delta \vec{\nabla} V_\Delta\right) d\tau = 0$

However, using the identity $\vec{\nabla} \bullet \left(V_\Delta \vec{\nabla} V_\Delta\right) = V_\Delta\left(\nabla^2 V_\Delta\right) + \overbrace{\left(\vec{\nabla} V_\Delta\right)^2}^{=\vec{\nabla} V_\Delta \bullet \vec{\nabla} V_\Delta}$

<u>Then</u>:
$$\int_{\substack{v\\ all\ space}} \vec{\nabla} \bullet \left(V_\Delta \vec{\nabla} V_\Delta\right) d\tau = \int_{\substack{v\\ all\ space}} V_\Delta \underbrace{\cancel{\left(\nabla^2 V_\Delta\right)}}_{=0} d\tau + \int_{\substack{v\\ all\ space}} \left(\vec{\nabla} V_\Delta\right)^2 d\tau = 0$$

$$= \int_{\substack{v\\ all\ space}} \underbrace{\left(\vec{\nabla} V_\Delta\right)^2}_{\substack{mathematically\\ \geq 0}} d\tau = \int_{\substack{v\\ all\ space}} \left(\vec{\nabla} V_\Delta \bullet \vec{\nabla} V_\Delta\right) d\tau = 0$$

The <u>*only*</u> way $\int_V \underbrace{\left(\vec{\nabla} V_\Delta\right)^2}_{\substack{mathematically\\ \geq 0}} d\tau \overset{\text{can be}}{=} 0$ is **iff** (*i.e.* if and only if) the integrand $\left(\vec{\nabla} V_\Delta(\vec{r})\right)^2 = \left(\vec{\nabla} V_\Delta(\vec{r}) \bullet \vec{\nabla} V_\Delta(\vec{r})\right) = 0$.

If $\left(\vec{\nabla} V_\Delta(\vec{r})\right)^2 = \left(\vec{\nabla} V_\Delta(\vec{r}) \bullet \vec{\nabla} V_\Delta(\vec{r})\right) = 0$, then: $\vec{\nabla} V_\Delta(\vec{r})$ itself <u>***must***</u> be = 0
$\left(i.e.\ \vec{A}(\vec{r}) \bullet \vec{A}(\vec{r}) = 0 \Rightarrow \vec{A}(\vec{r}) \equiv 0\right)$ for <u>***all***</u> points $(\vec{r})$ in volume v.

If $\vec{\nabla} V_\Delta(\vec{r}) = 0$ for all points $\vec{r}$ in volume v, then $V_\Delta(\vec{r})$ = (same) constant at all points in volume v. $\therefore$ $V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) =$ constant at all points in volume v.

Dirichlet Boundary Conditions (V specified on surfaces $S_1, S_2, S_3, \ldots, S_n$)

If $V_1(r)$ and $V_2(r)$ are specified on the surfaces $S_1, S_2, S_3, \ldots, S_n$ in the volume v enclosed by surface S (Dirichlet boundary conditions), then: $V_\Delta(\vec{r}) = \underbrace{V_1(\vec{r}) - V_2(\vec{r})}_{\underline{must} = 0!} = 0$
(*i.e.* the problem is <u>over</u>-<u>determined</u>).

$\therefore\ V_\Delta(\vec{r}) = 0$ throughout the volume v and $V_1(\vec{r}) = V_2(\vec{r})$ throughout the volume v.

i.e. the two solutions $V_1(r)$ and $V_2(r)$ for $\nabla^2 V(\vec{r}) = 0$ are ***<u>identical</u>*** – there is only ***<u>one</u>*** <u>unique</u> solution.

Neumann Boundary Conditions ($E^\perp$ specified on surfaces $S_1, S_2, S_3, \ldots, S_n$)

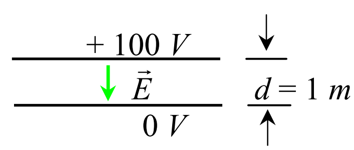
If $\vec{\nabla} V_1 \bullet \hat{n} = -\vec{E}_1^\perp$ and $\vec{\nabla} V_2 \bullet \hat{n} = -\vec{E}_2^\perp$ are specified on the surfaces $S_1, S_2, S_3, \ldots, S_n$ in the volume v enclosed by surface S (Neumann boundary conditions), then $\vec{\nabla} V_\Delta(\vec{r}) = \vec{\nabla} V_1(\vec{r}) - \vec{\nabla} V_2(\vec{r}) = 0$ at all points in volume v and $\vec{\nabla} V_\Delta \bullet \hat{n} = 0$.

Then $V_\Delta(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) =$ constant, but is *<u>not</u>* necessarily = 0 !!!

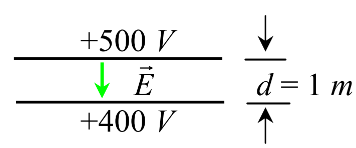
Here, solutions $V_1(r)$ and $V_2(r)$ *<u>can</u>* differ, but *<u>only</u>* by a constant V_o.
e.g. $V_1(\vec{r}) = V_2(\vec{r}) + V_o \quad \Rightarrow$ problem is NOT over-determined for $V(\vec{r})$.
($\vec{E}(\vec{r})$ *<u>is</u>* over-determined / unique, but *<u>not</u>* $V(\vec{r})$).

<u>Physical Example</u>:

<u>The Parallel Plate Capacitor</u>: $E = \Delta V/d = 100\ V/m$

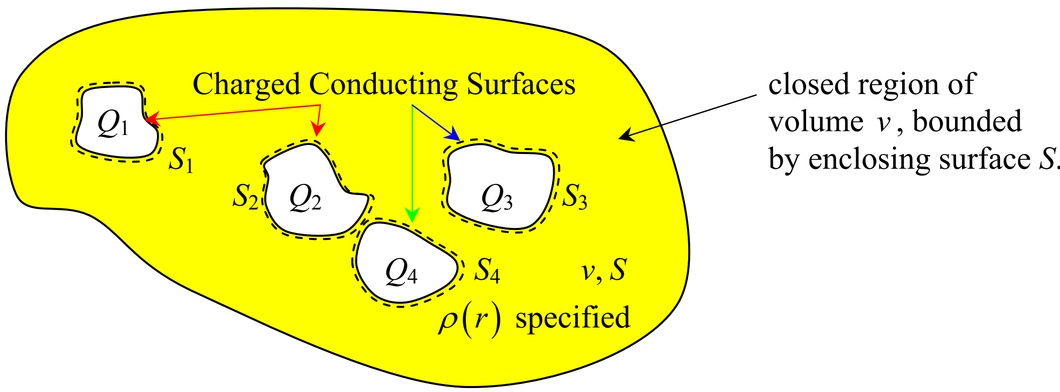


<u>Or</u>: $E = \Delta V/d = 100\ V/m$



$\Delta V = 100\,V$ in both cases – thus E-field is same/identical in both cases!

If we instead specify the charge densit(ies) $\rho(r)$ within the volume *v* (see figure below), then we also have a uniqueness theorem for the electric field associated with Poisson's equation $\left(\vec{\nabla}\bullet\vec{E}(\vec{r}) = -\nabla^2 V(\vec{r}) = \rho(r)/\varepsilon_o\right)$.



Suppose there are ***two*** electric fields $\vec{E}_1(\vec{r})$ and $\vec{E}_2(\vec{r})$, both satisfying all of the boundary conditions of this problem. Both obey Gauss' law in differential and integral form everywhere within the volume *v*:

$$\vec{\nabla}\bullet\vec{E}_1(\vec{r}) = \rho(r)/\varepsilon_o \text{ and: } \vec{\nabla}\bullet\vec{E}_2(\vec{r}) = \rho(r)/\varepsilon_o$$

$$\oint_{\substack{i^{th}\text{conducting}\\ \text{surface}, S_i}} \vec{E}_1\bullet da = \frac{1}{\varepsilon_o}Q_i^{encl} \text{ and: } \oint_{\substack{i^{th}\text{conducting}\\ \text{surface}, S_i}} \vec{E}_2\bullet da = \frac{1}{\varepsilon_o}Q_i^{encl}$$

At the outer boundary (enclosing surface *S*) we also have:

$$\int_S \vec{E}_1\bullet da = \frac{1}{\varepsilon_o}Q_{tot}^{encl} \text{ and: } \int_S \vec{E}_2\bullet da = \frac{1}{\varepsilon_o}Q_{tot}^{encl}$$

We define the difference in electric fields: $\vec{E}_\Delta(\vec{r}) \equiv \vec{E}_1(\vec{r}) - \vec{E}_2(\vec{r})$ which, in the region between the conductors, obeys $\vec{\nabla}\bullet\vec{E}_\Delta(\vec{r}) = \vec{\nabla}\bullet\vec{E}_1(\vec{r}) - \vec{\nabla}\bullet\vec{E}_2(\vec{r}) = \rho(r)/\varepsilon_o - \rho(r)/\varepsilon_o = 0$, and obeys $\int_{S_i} \vec{E}_\Delta\bullet da = \int_{S_i} \vec{E}_1\bullet da - \int_{S_i} \vec{E}_2\bullet da = \frac{1}{\varepsilon_o}Q_i^{encl} - \frac{1}{\varepsilon_o}Q_i^{encl} = 0$ over each boundary surface S_i.

Even though we do not know how the charge Q_i on the i^{th} conducing surface S_i is distributed, we *do* know that each surface S_i is an *equipotential*, hence the scalar potential $V_\Delta \equiv V_1 - V_2$ on each surface is at least a constant on each surface S_i (*n.b.* V_Δ may not necessarily be = 0, since in general V_2 may not in general be equal to V_1 on each/every surface S_i).

Using Griffith's product rule # 5: $\vec{\nabla}\bullet\left(f\vec{A}\right)=f\left(\vec{\nabla}\bullet\vec{A}\right)+\vec{A}\bullet\left(\vec{\nabla}f\right)$, then:

$$\vec{\nabla}\bullet\left(V_\Delta\vec{E}_\Delta\right)=V_\Delta\cancel{\left(\vec{\nabla}\bullet\vec{E}_\Delta\right)}+\vec{E}_\Delta\bullet\left(\vec{\nabla}V_\Delta\right)$$

However, in the region between conductors, we have shown (above) that $\vec{\nabla}\bullet\vec{E}_\Delta\left(\vec{r}\right)=0$,
and $\vec{E}_\Delta\equiv-\vec{\nabla}V_\Delta$, hence: $\vec{\nabla}\bullet\left(V_\Delta\vec{E}_\Delta\right)=\vec{E}_\Delta\bullet\left(\vec{\nabla}V_\Delta\right)=-\vec{E}_\Delta\bullet\vec{E}_\Delta=-E_\Delta^2$.

If we integrate this relation over the entire volume *v* (with associated enclosing surface *S*):

$$\int_v\vec{\nabla}\bullet\left(V_\Delta\vec{E}_\Delta\right)d\tau=\oint_{all\ S}V_\Delta\vec{E}_\Delta\bullet d\vec{a}=-\int_v E_\Delta^2 d\tau$$

Note that the surface integral covers all boundaries of the region in question – the enclosing outer surface *S* and ***all*** of the S_i inner surfaces associated with the *i* conductors. Since V_Δ is a constant on each surface, it can be pulled outside of the surface integral (*n.b.* if the outer surface *S* is at infinity, then for localized sources of charge, $V_\Delta\left(r=\infty\right)=0$). Thus:

$$V_\Delta\oint_{all\ S}\vec{E}_\Delta\bullet d\vec{a}=-\int_v E_\Delta^2 d\tau$$

But since we have shown above that $\int_{S_i}\vec{E}_\Delta\bullet da=0$ for each surface S_i, then $\oint_{all\ S}\vec{E}_\Delta\bullet d\vec{a}=0$.
Therefore: $\int_v E_\Delta^2 d\tau=0$. Note that the integrand $E_\Delta^2\left(\vec{r}\right)=\vec{E}_\Delta\left(\vec{r}\right)\bullet\vec{E}_\Delta\left(\vec{r}\right)$ is *always* non-negative.
Hence, in general, the *only* way that this integral can vanish is if $\vec{E}_\Delta\left(\vec{r}\right)\equiv\vec{E}_1\left(\vec{r}\right)-\vec{E}_2\left(\vec{r}\right)=0$ everywhere, thus, we *must* have $\vec{E}_1\left(\vec{r}\right)=\vec{E}_2\left(\vec{r}\right)$.

Solving Laplace's Equation $\left(\nabla^2 V(\vec{r})=0\right)$ in 3-*D*, 2-*D* and 1-*D* Situations

In general, when solving the potential $V(\vec{r})$ problems in 3 (or less) dimensions, first note the symmetries associated with the problem. Then, if you have:

$$\left\{\begin{matrix}\text{Rectangular}\\\text{Cylindrical}\\\text{Spherical}\end{matrix}\right\}\ \text{Symmetry} \Rightarrow \begin{matrix}\text{Solve}\\\text{Problem}\\\text{Using}\end{matrix}\ \left\{\begin{matrix}\text{Rectangular}\\\text{Cylindrical}\\\text{Spherical}\end{matrix}\right\}\ \text{Coordinates}$$

In 2-*D* and 3-*D* problems, the general solutions to $\nabla^2 V(\vec{r})=0$ are the harmonic functions (an ∞-series solution, in principle) *e.g.* of sines and cosines, Bessel functions, or Legendre Polynomials and/or Spherical Harmonics.

The boundary conditions / symmetries will select a subset of the ∞-solutions.

We will now work through derivations of finding solutions to Laplace's Equations in 3-dimensions in rectangular (*i.e.* Cartesian) coordinates, cylindrical coordinates, and spherical coordinates. We will also use / show the ***method of separation of variables***.

Laplace's Equation $\nabla^2 V(x,y,z)=0$ and Potential Problems with Rectangular Symmetry

(Rectangular / Cartesian coordinates)

In Three Dimensions: Solve Laplace's equation in rectangular / Cartesian coordinates:

$$\nabla^2 V(x,y,z)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(x,y,z)=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

The solutions of $\nabla^2 V=0$ in rectangular coordinates are known as harmonic functions (*i.e.* sines and cosines) (→ Fourier Series Solutions).

It is usually (but not always) possible to find a solution to the Laplace Equation, $\nabla^2 V=0$ which also satisfies the boundary conditions, via separation of variables technique, *i.e.* try a product solution of the form:

$$V(x,y,z)=X(x)Y(y)Z(z)$$

where: $\left\{\begin{matrix}X(x)\\Y(y)\\Z(z)\end{matrix}\right\}$ are functions only of $\left\{\begin{matrix}x\\y\\z\end{matrix}\right\}$ respectively.

Then: $\nabla^2 V(x,y,z)=0 \Rightarrow \dfrac{\partial^2 V(x,y,z)}{\partial x^2}+\dfrac{\partial^2 V(x,y,z)}{\partial y^2}+\dfrac{\partial^2 V(x,y,z)}{\partial z^2}=0$

But: $V(x,y,z)=X(x)Y(y)Z(z)$

Thus: $$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2}+\frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2}+\frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2}=0$$

$$=Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2}+X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2}+X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2}=0$$

Now divide both sides of the above equation by $X(x)Y(y)Z(z)$:

Then: $$=\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}+\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}=0$$

But: $$=\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{\substack{\text{fcn(x) only}\\ \text{independent of y, z}\\ =C_1}}+\underbrace{\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}}_{\substack{\text{fcn(y) only}\\ \text{independent of x, z}\\ =C_2}}+\underbrace{\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}}_{\substack{\text{fcn(z) only}\\ \text{independent of x, y}\\ =C_3}}=0 \quad i.e. \quad C_1+C_2+C_3=0$$

True for *all* points (x, y, z) in volume v of problem.

The only way the above equation can be true for all points (x, y, z) in volume v is if:

$$\left\{\begin{array}{l} \dfrac{1}{X(x)}\dfrac{\partial^2 X(x)}{\partial x^2} = \text{ constant } C_1 \Rightarrow \\ \dfrac{1}{Y(y)}\dfrac{\partial^2 Y(y)}{\partial y^2} = \text{ constant } C_2 \Rightarrow \\ \dfrac{1}{Z(z)}\dfrac{\partial^2 Z(z)}{\partial z^2} = \text{ constant } C_3 \Rightarrow \end{array}\right.$$

$\dfrac{d^2X(x)}{dx^2}-C_1X(x)=0$	#1
$\dfrac{d^2Y(y)}{dy^2}-C_2Y(y)=0$	#2
$\dfrac{d^2Z(z)}{dz^2}-C_3Z(z)=0$	#3

Note *total* derivatives now!!!

Subject to the constraint: $C_1 + C_2 + C_3 = 0$

Can now solve 3 ORDINARY 1-*D* differential equations, #1–3, which are subject to $C_1 + C_2 + C_3 = 0$, PLUS the specific Dirichlet / Neumann *boundary conditions* for the problem on either $V(x, y, z)$ or $\vec{\nabla}V(x,y,z)\bullet\hat{n}$ at surfaces for this 3-*D* problem.

Essentially, we have replaced the 3-*D* problem with *three* 1-*D* problems, ***and*** the constraint: $C_1 + C_2 + C_3 = 0$.

* If one has a 2-*D* rectangular coordinate problem $\left(\nabla^2 V(x,y)=0\right)$, then: $V(x,y) = X(x)Y(y)$ (only).

$$\begin{cases} \dfrac{1}{X(x)}\dfrac{d^2X(x)}{dx^2} = C_1 \Rightarrow & \dfrac{d^2X(x)}{dx^2} - C_1 X(x) = 0 \\[2ex] \dfrac{1}{Y(y)}\dfrac{d^2Y(y)}{dy^2} = C_2 \Rightarrow & \dfrac{d^2Y(y)}{dy^2} - C_2 Y(y) = 0 \end{cases}$$

Subject to the constraint: $C_1 + C_2 = 0$, *i.e.* $C_1 = -C_2$.
Plus BC's: either on $V(x,y)$ or $\vec{\nabla}V(x,y)\bullet\hat{n}$ for the 2-*D* problem.

* If one has a 1-*D* rectangular coordinate problem $\left(\nabla^2 V(x)=0\right)$, then: $V(x) = X(x)$ (only).

$$\frac{d^2V(x)}{dx^2} = 0 \;\Rightarrow\; \frac{d^2X(x)}{dx^2} = 0 \;\Rightarrow\; \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = 0 = C_1$$

$$\frac{d^2X(x)}{dx^2} = 0 \;\Rightarrow\; X(x) = V(x) = ax + b \text{ is } \underline{\textit{the}} \text{ 1-}D \text{ general solution.}$$

For 1-*D* problem $\left(\nabla^2 V(x)=0\right)$, only need to solve one *__ordinary__* differential equation subject to the constraint $C_1 = 0$ and BC's on either $V(x)$ or $\dfrac{dV(x)}{dx}$.

The General Solution $V(x,y,z) = X(x)Y(y)Z(z)$ ***for*** $\nabla^2 V(x,y,z) = 0$ **in Rectangular Coordinates**

Since we have the constraint $C_1 + C_2 + C_3 = 0$, at least one of the C_i's (i = 1, 2 or 3) *must* be less than zero.

Let us "choose" $C_1 = -\alpha^2,\ C_2 = -\beta^2,\ C_3 = \gamma^2$

Then: $\quad C_1 + C_2 + C_3 = 0$

$$-\alpha^2 - \beta^2 + \gamma^2 = 0 \qquad \text{or:} \qquad \boxed{\alpha^2 + \beta^2 = \gamma^2}$$

The boundary conditions on the surfaces will define α and β, and hence define γ.

IMPORTANT NOTE:

The *geometry* ($x - y - z$) of the problem *and* the *boundary conditions* dictate whether:

$C_1 > 0$ or $C_1 < 0$
$C_2 > 0$ or $C_2 < 0$
$C_3 > 0$ or $C_3 < 0$

i.e. have sine / cosine type solutions *vs*. sinh / cosh (or $e^x,\ e^{-x}$) type solutions for *x, y, z*.

Then *the* General Solution is (for above choice of $C_1 = -\alpha^2,\ C_2 = -\beta^2,\ C_3 = \gamma^2$):

$$V(x,y,z) = \sum_{\text{m,n=0}}^{\infty} A_{mn} \underbrace{\sin}_{\substack{\text{could be}\\ \cos}} (\alpha_n x) \underbrace{\sin}_{\substack{\text{could be}\\ \cos}} (\beta_m y) \underbrace{\sinh}_{\substack{\text{could be}\\ \cosh}} (\underbrace{\gamma_{mn}}_{=\sqrt{\alpha_n^2+\beta_m^2}} z)$$ so we *also* have the additional series solutions:

$$+\sum_{\text{m,n=0}}^{\infty} B_{mn} \cos(\alpha_n x)\cos(\beta_m y)\sinh(\underbrace{\gamma_{mn}}_{=\sqrt{\alpha_n^2+\beta_m^2}} z)$$

$$+\sum_{\text{m,n=0}}^{\infty} C_{mn} \sin(\alpha_n x)\sin(\beta_m y)\cosh(\underbrace{\gamma_{mn}}_{=\sqrt{\alpha_n^2+\beta_m^2}} z)$$

$$+\sum_{\text{m,n=0}}^{\infty} D_{mn} \cos(\alpha_n x)\cos(\beta_m y)\cosh(\underbrace{\gamma_{mn}}_{=\sqrt{\alpha_n^2+\beta_m^2}} z)$$

n.b. $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$

n.b. $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad i \equiv \sqrt{-1}$

n.b. $e^{ix} = \cos(x) + i\sin(x) \qquad e^{-ix} = \cos(x) - i\sin(x)$

n.b. $e^{x} = \cosh(x) + \sinh(x) \qquad e^{-x} = \cosh(x) - \sinh(x)$

The BC's and symmetries will determine which of the coefficients $A_{mn}, B_{mn}, C_{mn}, D_{mn} = 0$.

We solve for the non-zero coefficients A_{pq}, B_{pq}, C_{pq} *and* D_{pq} by taking *inner products*.
i.e. we multiply $V(x,y,z)=\sum(\mathit{stuff})$ by e.g. $\sin(\alpha_p x)\sin(\beta_q y)$ to project out the p-q^{th} component (*i.e.* we use the *orthogonality* properties of the individual terms in sin( ) and cos( ) Fourier Series.) and then integrate over the relevant intervals in x and y:

e.g.

$$\int_0^{x_o}\int_0^{y_o} V(x,y)\sin(\alpha_p x)\sin(\beta_q y)\,dx\,dy$$

$$\begin{aligned}
=\int_0^{x_o}\int_0^{y_o}\Bigg\{&\left(\sum_{m,n=0}^{\infty} A_{mn}\sin(\alpha_n x)\sin(\beta_m y)\underbrace{\sinh(\gamma_{mn}z)}_{\text{=constant here}} * \sin(\alpha_p x)\sin(\beta_q y)\right)\\
&+\left(\sum_{m,n=0}^{\infty} B_{mn}\cos(\alpha_n x)\cos(\beta_m y)\underbrace{\sinh(\gamma_{mn}z)}_{\text{=constant here}} * \sin(\alpha_p x)\sin(\beta_q y)\right)\\
&+\left(\sum_{m,n=0}^{\infty} C_{mn}\sin(\alpha_n x)\sin(\beta_m y)\underbrace{\cosh(\gamma_{mn}z)}_{\text{=constant here}} * \sin(\alpha_p x)\sin(\beta_q y)\right)\\
&+\left(\sum_{m,n=0}^{\infty} D_{mn}\cos(\alpha_n x)\cos(\beta_m y)\underbrace{\cosh(\gamma_{mn}z)}_{\text{=constant here}} * \sin(\alpha_p x)\sin(\beta_q y)\right)\Bigg\}\,dx\,dy
\end{aligned}$$

Fourier Functions: orthonormality properties of sin ( ) and cos ( ):

$$\int_0^{x_o}\sin(\alpha_n x)\sin(\alpha_p x)\,dx = \underbrace{\sim\sim}_{\substack{\textit{some}\\ \textit{constant}}}\ \delta_{np}\begin{pmatrix}=1 \text{ for } n=p\\ =0 \text{ for } n\neq p\end{pmatrix}$$

$$\int_0^{x_o}\cos(\alpha_n x)\sin(\alpha_p x)\,dx = 0$$

$$\boxed{\text{Kroenecker } \delta\text{-function: } \delta_{np}\begin{pmatrix}=1 \text{ for } n=p\\ =0 \text{ for } n\neq p\end{pmatrix}}$$

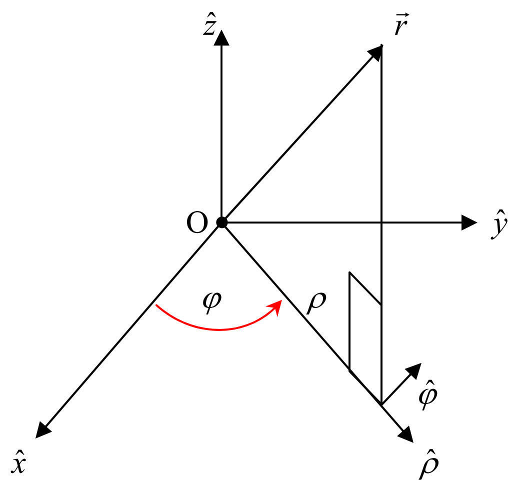
So all terms in above Σ's vanish, *except* for a single term (in each sum) – that for the A_{pq} /B_{pq} /C_{pq} /D_{pq} coefficient!!! The BC's will *e.g.* kill off 3 out of remaining 4 non-zero terms, thus only *one* term survives…

Suppose only the A_{pq} coefficient survives. Its analytic form is now known for all integers p and q.

Then the analytic form of 3-*D* potential $V(x,y,z)$ is now known – it is an infinite series solution of the form:

$$\boxed{V(x,y,z)=\sum_{m,n=0}^{\infty} A_{mn}\sin(\alpha_n x)\sin(\beta_m y)\sinh\big(\underbrace{\gamma_{mn}}_{=\sqrt{\alpha_n^2+\beta_m^2}} z\big)}$$

Laplace's Equation $\nabla^2 V(\rho,\varphi,z)=0$
And Potential Problems with Cylindrical Symmetry (Cylindrical Coordinates)



$$\vec{r}=\vec{\rho}+\vec{z}=\rho\hat{\rho}+z\hat{z} \qquad r=\sqrt{\rho^2+z^2}$$

$$\nabla^2 V(\rho,\varphi,z)=0$$
$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2}+\frac{\partial^2 V}{\partial z^2}=0$$
$$=\frac{\partial^2 V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}+\frac{\partial^2 V}{\partial\varphi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Again, we use the separation of variables technique:

$V(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z) \;\Rightarrow\; \nabla^2 V=0 \;\Rightarrow$ yields 3 ordinary differential equations:

$$\left\{\begin{array}{l}
\dfrac{d^2 Z(z)}{dz^2}-k^2 Z(z)=0 \;\Rightarrow\; Z(z)=e^{\pm kz}\\[2ex]
\dfrac{d^2 Q(\varphi)}{d\varphi^2}+\nu^2 Q(\varphi)=0 \;\Rightarrow\; Q(\varphi)=e^{\pm i\nu\varphi}\\[2ex]
\dfrac{d^2 R(\rho)}{d\rho^2}+\dfrac{1}{\rho}\dfrac{dR(\rho)}{d\rho}+\left(k^2-\dfrac{\nu^2}{\rho^2}\right)R(\rho)=0
\end{array}\right.$$

Note(s):

1.) k is arbitrary without imposing boundary conditions.
2.) k appears in both $Z(z)$ and $R(\rho)$ equations.
3.) In order for $Q(\varphi)$ to be *single-valued* (i.e. $Q(\varphi)=Q(\varphi+2\pi)$), ν *must* be an integer!

Let $x\equiv kp$ Then: $\dfrac{d^2R(x)}{dx^2}+\dfrac{1}{x}\dfrac{dR(x)}{dx}+\left(1-\dfrac{\nu^2}{x^2}\right)R(x)=0 \Leftarrow$ Bessel's Equation

$R(x)=x^\alpha\sum_{j=0}^{\infty}a_j x^j \Leftarrow$ Power Series Solution $\qquad \alpha=\pm\nu$

$a_{2j}\equiv-\dfrac{1}{4j(j+\alpha)}a_{2j-2} \qquad$ for $j=0, 1, 2, 3, \ldots$

All *odd* powers of x_j have vanishing coefficients, *i.e.* $a_1=a_3=a_5=a_{2j+1}=0$

Coefficients a_{2j} expressed in terms of a_0:

$$a_{2j} = \left[\frac{(-1)^j \Gamma(\alpha+1)}{2^{2j} j!\Gamma(j+\alpha+1)}\right] a_0 = \frac{(-1)^j}{2^{2(j+\alpha)} j!\Gamma(j+\alpha+1)}$$

where $a_0 = \dfrac{1}{2^\alpha \Gamma(\alpha+1)}$ $\Gamma(x) = \text{Gamma Function}$

There exist TWO solutions of the Radial Equation (*i.e.* Bessel's Equation):

<u>They are</u>:

Bessel Functions of 1st kind, of <u>order</u> $\pm\nu$:

$$\left.\begin{aligned} J_{+\nu}(x) &= \left(\frac{x}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)}\left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)}\left(\frac{x}{2}\right)^{2j} \end{aligned}\right\}$$

These series converge for all values of x.

n.b. If ν is <u>not</u> an integer (which is *<u>not</u>* the case here), then the $J_{\pm\nu}(x)$ form a pair of linearly independent solutions to the 2nd order Bessel's Equation:

$$R(x) = A_\nu J_\nu(x) + A_{-\nu} J_{-\nu}(x) \quad \text{for } \nu \neq \text{integer}$$

However, note that if $\nu = \text{integer}$ (which <u>is</u> the case for us here) then the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ are <u>NOT</u> linearly independent!!

If $\nu = m = \text{integer}$ (0, 1, 2, 3, …), then $J_{-m}(x) = (-1)^m J_m(x)$

∴ ⇒ We must find another <u>linearly independent</u> solution for $R(x)$ when $\nu = m = \text{integer}$

It is "customary" to replace $J \pm \nu(x)$ by just $J_\nu(x)$ and another function $N_\nu(x)$ (called Neumann Functions)

<u>Where</u>: $N_\nu(x) = \text{Bessel Function of 2}^{\text{nd}} \text{ kind} \equiv \dfrac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$

<u>NOTE</u>: $N_\nu(x)$ is <u>divergent</u> (i.e singular) at $x \to 0$

Complex Bessel Functions = Bessel Functions of 3rd kind = Hankel Functions

Hankel Functions are *<u>complex</u>* linear combinations of $J_\nu(x)$ and $N_\nu(x)$ (Bessel Functions of 1st and 2nd kind respectively). They are defined as follows:

$$\left.\begin{aligned} H_\nu^{(1)}(x) &\equiv J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) &\equiv J_\nu(x) - iN_\nu(x) \end{aligned}\right\}$$

The Hankel Functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ also form a fundamental set/basis of solutions to the Bessel equation.

The General Solution for $\nabla^2 V(\rho,\varphi,z)=0$ **in Cylindrical Coordinates**:

$$V(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z)$$

cosh(k_{mn}z) is also allowed

$$V(\rho,\varphi,z)=\sum_{m,n=0}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left[A_{mn}\sin(m\varphi)+B_{mn}\cos(m\varphi)\right]$$

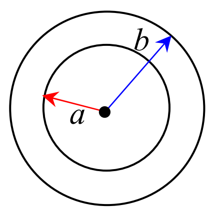
cosh(k_{mn}z) is also allowed

$$+\sum_{m,n=0}^{\infty} N_m(k_{mn}\rho)\sinh(k_{mn}z)\left[C_{mn}\sin(m\varphi)+D_{mn}\cos(m\varphi)\right]$$

Apply ALL boundary conditions on surfaces
(<u>and</u> also impose for $r=\infty$, that $V(r=\infty)$ = finite! {If $r=\infty$ is part of the problem!})

Note that sometimes we want $V(\vec{r})$ only inside some finite region of space, e.g. coaxial capacitor – if so, then <u>don't</u> have to worry about $r=\infty$ solutions being finite – an example – the Coaxial Capacitor:

<u>*End View of a Coaxial Capacitor*</u>

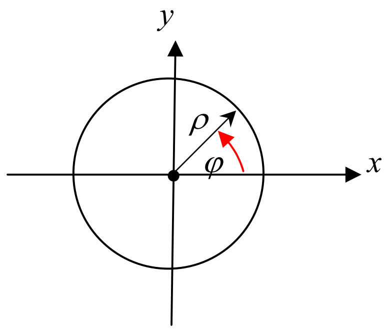


If the $\vec{r}=0$ region is an <u>*excluded region*</u> in the problem, then <u>*must*</u> include (*i.e.* allow) the $N_\nu(x)$ solutions (singular at $x=k\rho=0$)!!!

If $\vec{r}=0$ region <u>*is*</u> included in problem, then <u>ALL</u> coefficients $C_{mn}=D_{mn}\equiv 0$ (for all m, n), if $V(\rho,\varphi,z)$ is finite @ $\vec{r}=0$.

Using/imposing BC's on surfaces, orthogonality conditions on sines, cosines, $J_\nu(x)$, $N_\nu(x)$, *etc*. can find / determine values for all A_{mn}, B_{mn}, C_{mn}, D_{mn} coefficients!!

2-Dimensional Circular Symmetry
Laplace's Equation in (Circular) Cylindrical Coordinates



$\nabla^2 V(\rho,\varphi) = 0$

$V(\rho,\varphi) = R(\rho)Q(\varphi)$ Again try product solution

Potential, $V(\rho,\varphi)$ is independent of z
(*e.g.* infinitely long coaxial cable)

$$\nabla^2 V(\rho,\varphi) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

Get:

$$\frac{\rho}{R(\rho)}\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) = C_1 = -\frac{1}{Q(\varphi)}\frac{d^2 Q(\varphi)}{d\varphi^2} \qquad \text{Let } C_1 = k^2$$

Then: $\rho\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) - k^2 R(\rho) = 0$ and $\frac{d^2 Q(\varphi)}{d\varphi^2} + k^2 Q(\varphi) = 0$

Require all solutions $Q(\varphi)$ to be single-valued, i.e. $Q(\varphi) = Q(\varphi + 2\pi)$
because *must* have $V(\varphi) = V(\varphi + 2\pi)$.

Solutions for $Q(\varphi)$ are of the form:

$$Q(\varphi) = A\cos k\varphi + B\sin k\varphi$$

$Q(\varphi) = Q(\varphi + 2k\pi)$ requires k = integer = 0, $\pm$ 1, $\pm$ 2, ± 3, … $\pm$ n …

$$\frac{d^2 Q(\varphi)}{d\varphi^2} + n^2 Q(\varphi) = 0 \;\Rightarrow\; Q_n(\varphi) = A_n\cos(n\varphi) + B_n\sin(n\varphi)$$

singular @ $\rho \to \infty$ singular @ $\rho = 0$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) - n^2 R(\rho) = 0 \Rightarrow R_n(\rho) = C_n\rho^n + D_n\rho^{-n} \quad \textit{for } n \geq 1 \; (i.e.\ n = 1, 2, 3, \ldots)$$

$$R_o(\rho) = C_o + D_o\, ln(\rho) \quad \text{for } n = 0 \text{ only}$$

General Solution for $\nabla^2 V(\rho,\varphi)=0$ in Two Dimensions: Cylindrical (*a.k.a.* Zonal) Harmonics

$$V(\rho,\varphi)=V_0+V_1\, ln(\rho)+\sum_{n=1}^{\infty}\left[a_n\rho^n\cos(n\varphi)+b_n\rho^{-n}\cos(n\varphi)+c_n\rho^n\sin(n\varphi)+d_n\rho^{-n}\sin(n\varphi)\right]$$

Again, apply BC's on all relevant surfaces, impose $V(r\to\infty)=\text{finite}$, *etc*. – these will dictate / determine all coefficients, V_0, V_1, a_n, b_n, c_n and d_n.

i.e. Solve for V_0, V_1, a_n, b_n, c_n and d_n by applying all boundary conditions, $V(r\to\infty)=\text{finite}$, and using orthogonality conditions / properties:

$$a_n = \text{"}A\text{"}\int_0^{2\pi}d\varphi\int_0^{\rho_o}d\rho\;\;\rho\;\;V(\rho,\varphi)\rho^n\cos(n\varphi) \qquad\qquad dA=\rho\, d\rho\, d\varphi$$

$$b_n = \text{"}B\text{"}\int_0^{2\pi}d\varphi\int_0^{\rho_o}d\rho\;\;\rho\;\;V(\rho,\varphi)\rho^{-n}\cos(n\varphi)$$

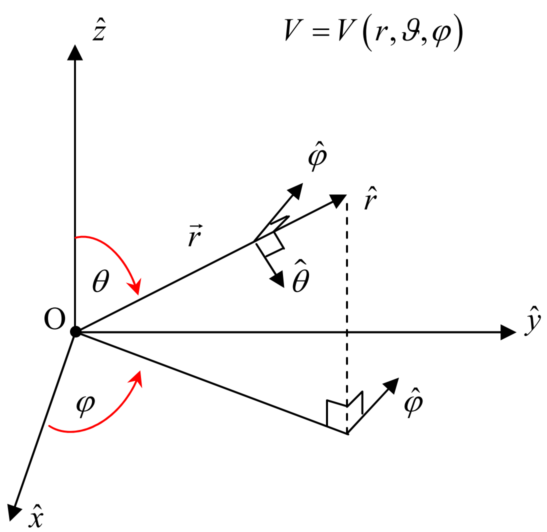
$$c_n = \text{"}C\text{"}\int_0^{2\pi}d\varphi\int_0^{\rho_o}d\rho\;\;\rho\;\;V(\rho,\varphi)\rho^n\sin(n\varphi)$$

$$d_n = \text{"}D\text{"}\int_0^{2\pi}d\varphi\int_0^{\rho_o}d\rho\;\;\rho\;\;V(\rho,\varphi)\rho^{-n}\sin(n\varphi)$$

"A", "B", "C", "D" are appropriate normalization factors (we will discuss later).

Laplace's Equation $\nabla^2 V(r,\vartheta,\varphi)=0$ In Spherical Coordinates

$$V=V(r,\vartheta,\varphi)$$



$$\nabla^2 V(r,\vartheta,\varphi)=0$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0$$

Again, try separation of variables / try product solution:

$V(r,\vartheta,\varphi)=\dfrac{U(r)}{r}P(\theta)Q(\varphi) \Leftarrow$ of this form!!

$$P(\theta)Q(\varphi)\frac{d^2U(r)}{dr^2}+\frac{U(r)Q(\varphi)}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\frac{U(r)P(\theta)}{r^2\sin^2\theta}\frac{d^2Q(\varphi)}{d\varphi^2}=0$$

Multiply by $r^2\sin^2\theta/U(r)P(\theta)Q(\varphi)$:

$$\underbrace{r^2\sin^2\theta\left[\frac{1}{U(r)}\frac{d^2U(r)}{dr^2}+\frac{1}{r^2\sin\theta}\frac{1}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)\right]}_{\text{function of } r+\theta \text{ only}}+\underbrace{\frac{1}{Q(\varphi)}\frac{d^2Q(\varphi)}{d\varphi^2}}_{\text{function of } \varphi \text{ only}}=0$$

<u>Now</u>: $\dfrac{1}{Q(\varphi)}\dfrac{d^2Q(\varphi)}{d\varphi^2}=-m^2 \Rightarrow \dfrac{d^2Q(\varphi)}{d\varphi^2}+m^2Q(\varphi)=0$

Solutions are of the form: $Q(\varphi)=e^{\pm im\varphi}$ where m = integer = 0, 1, 2, 3, …

Since $V(r,\vartheta,\varphi)=V(r,\vartheta,\varphi\pm 2\pi)$ i.e. $Q(\varphi)=Q(\varphi\pm 2\pi)$

Then: $Q(\varphi)$ <u>must</u> be single-valued!

<u>Thus</u>:

$$r^2\sin^2\theta\left[\frac{1}{U(r)}\frac{d^2U(r)}{dr^2}+\frac{1}{r^2\sin\theta}\frac{1}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)\right]=+m^2$$

$$\frac{1}{U(r)}\frac{d^2U(r)}{dr^2}=-\frac{1}{r^2\sin\theta}\frac{1}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\frac{m^2}{r^2\sin^2\theta}$$

multiply above equation by r^2:

$$\underbrace{\frac{r^2}{U(r)}\frac{d^2U(r)}{dr^2}}_{\text{function only of } r}=\underbrace{-\frac{1}{\sin\theta}\frac{1}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\frac{m^2}{\sin^2\theta}}_{\text{function only of } \theta}=-\alpha \qquad (\alpha\ge 0)$$

<u>must</u> hold for <u>any/all</u> r and θ!!

$$\therefore \quad \frac{d^2U(r)}{dr^2}-\frac{\alpha}{r^2}U(r)=0 \quad \text{and} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\left[\alpha-\frac{m^2}{\sin^2\theta}\right]P(\theta)=0$$

let / define $\alpha\equiv\ell(\ell+1)$ where ℓ = integer = 0, 1, 2, 3, …

(Trust me, ☺ I know the answer . . .)

$$\therefore \quad \frac{d^2U(r)}{dr^2}-\frac{\ell(\ell+1)}{r^2}U(r)=0 \quad \text{and} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P(\theta)=0$$

Now let $x = \cos\theta$ $\qquad x^2 = \cos^2\theta = 1 - \sin^2\theta$

$dx = d\cos\theta = \sin\theta d\theta$ $\qquad \therefore \ \sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$\sin\theta = \sqrt{1 - x^2}$

Then: $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\dfrac{\sin^2\theta}{\sin\theta}\dfrac{dP(\theta)}{d\theta}\right) + \left[\ell(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right]P(\theta) = 0$

Becomes: $\dfrac{d}{dx}\left(\left(1-x^2\right)\dfrac{dP(x)}{dx}\right) + \left[\ell(\ell+1) - \dfrac{m^2}{\left(1-x^2\right)}\right]P(x) = 0 \ \Leftarrow$ Generalized Legendre' Equation

General Solutions of the radial equation, $\dfrac{d^2U(r)}{dr^2} - \dfrac{\ell(\ell+1)}{r^2}U(r) = 0$ are of the form:

$U(r) = Ar^{\ell} + Br^{-(\ell+1)}$ $\qquad$ ($l + A + B$) are determined by boundary conditions…

For m = 0 (azimuthally-symmetric problems – no φ-dependence) the general solution for azimuthally-symmetric potential $V(r,\theta)$ is of the form:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}\right] \underbrace{P_\ell(\cos\theta)}_{\substack{\text{"ordinary" Legendre'} \\ \text{Polynomial of order } \ell}}$$

The coefficients A_ℓ and B_ℓ are determined by the boundary conditions

n.b. If $\exists$ no charges at $r = 0$, then $B_\ell = 0 \ \forall \ \ell$!!

Rodrigues' Formula is useful for "ordinary" Legendre' Polynomials:

$$P_\ell(x) \equiv \left(\frac{1}{2^\ell \ell!}\right)\frac{d^\ell}{dx^\ell}\left(x^2 - 1\right)^\ell$$

The coefficients A_ℓ and B_ℓ can be found / determined by evaluating $V(r,\theta)$ on the conducting surfaces in the problem, *e.g.* suppose we want to determine $V(\vec{r})$ inside a conducting sphere of radius $r = a$. Then on the surface of the conducting sphere at radius $r = a$ (an equipotential!):

$V(r = a,\theta) = \sum A_\ell a^\ell P_\ell(\cos\theta) = \text{constant} \ \Leftarrow$ Legendre' Series

n.b. inside conducting sphere, *e.g.* there are no charges at $r = 0$ $\qquad \therefore \ B_\ell = 0 \ \forall \ \ell$

In order to determine coefficients, take ***inner product***:

$$A_\ell = \underbrace{\left(\frac{(2\ell+1)}{2a^\ell}\right)}_{\substack{\text{normalization} \\ \text{factor}}}\int_0^\pi V(r = a,\theta) P_\ell(\cos\theta)\sin\theta d\theta$$

Orthogonality condition on $P_\ell(x)$'s:

$$\int_{-1}^{-1} P_{\ell'}(x) P_\ell(x)\,dx = \frac{2}{(2\ell+1)} \underbrace{\delta_{\ell'\ell}}_{\substack{\text{Kroenecker} \\ \delta\text{-function}}} \quad \begin{Bmatrix} \delta_{l'l} = 0 \text{ for } l' \neq l \\ \delta_{l'l} = 1 \text{ for } l' = l \end{Bmatrix}$$

n.b. The $P_\ell(\cos\theta)$ functions form a <u>complete orthonormal basis set</u> on the <u>unit circle</u> ($r = 1$) for $-1 \le \cos\theta \le 1$ or: $0 \le \theta \le \pi$

<u>"Ordinary" Legendre' Polynomials $P_\ell(x)$ $(x = \cos\theta)$ defined on the interval $-1 \le x \le 1$</u>:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 + 70x^3 + 15x)$$
$$\ldots$$

<u>Note</u>: All $P_{\ell=even}(x)$ functions are <u>*even*</u> functions of *x*: $P_{\ell=even}(-x) = +P_{\ell=even}(x)$
All $P_{\ell=odd}(x)$ functions are <u>*odd*</u> functions of *x*: $P_{\ell=odd}(-x) = -P_{\ell=odd}(x)$
under $x \to -x$ reflection. Generally speaking, $P_\ell(-x) = (-1)^\ell P_\ell(x)$.

If 3-*D* spherical coordinate problem <u>DOES</u> have azimuthal / φ-dependence, then $m^2 \neq 0$ in Associated Legendre' Equation (A.L.E.):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP(\theta)}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P(\theta) = 0 \qquad x = \cos\theta$$

Solutions to A.L.E. are Associated Legendre' Polynomials (A.L.P.'s)

Associated Legendre' Polynomials: $P_\ell^m(x) \equiv (-1)^m (1-x^2)^{\frac{m}{2}} \dfrac{d^m}{dx^m} \underbrace{P_\ell(x)}_{\substack{\text{"ordinary"} \\ \text{Legendre'} \\ \text{Polynomial}}}$

$m = \pm$ integer $\neq 0$

i.e. $m = \pm 1, \pm 2, \pm 3, \ldots$ <u>but</u> have a constraint on *m* !!! $\boxed{-\ell \le m \le +\ell}$

i.e. $m = -\ell, -\ell+1, -\ell+2, \ldots -2, -1, 0, +1, +2, \ell-2, \ell-1, \ell$

Also: $P_\ell^{-m}(x) = (-1)^m \dfrac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$

Orthogonality condition for $P_\ell^m(x)$ for fixed m:

$$\int_{-1}^{1} P_{\ell'}^m(x) P_\ell^m(x)\, dx = \frac{2}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!} \underbrace{\delta_{\ell'\ell}}_{\substack{\text{Kroenecker} \\ \delta\text{-function}}}$$

We now define *normalized* $P(\theta)Q(\varphi)$ functions known as Spherical Harmonics:

$$Y_{\ell m}(\theta,\varphi) \equiv \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) \underbrace{e^{im\varphi}}_{=Q_m(\varphi)}$$

The Spherical Harmonics $Y_{\ell m}(\theta,\varphi)$ form a complete orthonormal set of basis "vectors" on the surface of the unit sphere ($r = 1$)

Note that $Y_{\ell -m}(\theta,\varphi) = (-1)^m Y_{\ell m}^*(\theta,\varphi)$ complex conjugate

i.e. $i \to -i$ where $i \equiv \sqrt{-1}$

$Y_{\ell m}(\theta,\varphi)$ Normalization and Orthogonality Condition:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_{\ell' m'}^*(\theta,\varphi) Y_{\ell m}(\theta,\varphi) = \delta_{\ell'\ell}\delta_{m'm}$$

$$i.e.\quad \int_{\Omega=0}^{\Omega=4\pi} d\Omega Y_{\ell' m'}^*(\theta,\varphi) Y_{\ell m}(\theta,\varphi) = \delta_{\ell'\ell}\delta_{m'm} \qquad d\Omega = \sin\theta d\theta d\varphi$$

Completeness' Relation: $$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta,\varphi) Y_{\ell m}(\theta,\varphi) = \underbrace{\delta(\cos\theta - \cos\theta')}\,\underbrace{\delta(\varphi-\varphi')}$$

DIRAC δ-functions

$Y_{\ell m}(\theta,\varphi)$ Spherical Harmonics

$\ell = 0$

$$\left\{ Y_{00} = \frac{1}{\sqrt{4\pi}} \right.$$

Use $Y_{\ell -m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi)$

in order to obtain $Y_{2-2}, Y_{2-1}, Y_{3-3}, Y_{3-2}, Y_{3-1}$ *etc.*

$\ell = 1$

$$\begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\varphi} \\ Y_{10} = -\sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

Note:

$$Y_{\ell m}(\theta,\varphi) \equiv \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

$$Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}} P_\ell(\cos\theta)$$

$\ell = 2$

$$\begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\varphi} \\ Y_{21} = -\sqrt{\dfrac{5}{8\pi}} \sin\theta\cos\theta e^{i\varphi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}} \left(\dfrac{3}{2}\cos^2\theta - \dfrac{1}{2}\right) \end{cases}$$

$\ell = 3$

$$\begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}} \sin^3\theta e^{3i\varphi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}} \sin^2\theta\cos\theta e^{2i\varphi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}} \sin\theta\left(5\cos^2\theta - 1\right) e^{i\varphi} \\ Y_{30} = \sqrt{\dfrac{7}{4\pi}} \left(\dfrac{5}{2}\cos^3\theta - \dfrac{3}{2}\cos\theta\right) \end{cases}$$

…. …. etc.

General Solution for Laplace's Equation $\nabla^2 V(r,\theta,\varphi)$ in Spherical Polar Coordinates

$$V(r,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left[A_{\ell m}r^{\ell}+B_{\ell m}r^{-(\ell+1)}\right]Y_{\ell m}(\theta,\varphi)$$

Coefficients $A_{\ell m}$ and $B_{\ell m}$ are determined by / from Boundary Conditions on spherical surface(s)

If $V=V(\theta,\varphi)$ on surface (*e.g.* at $r=a$)

(*i.e.* no charge at $r=0$ in problem $\rightarrow$ $B_{\ell m}=0\ \forall_{\ell,m}$)

Then: $V(\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}A_{\ell m}Y_{\ell m}(\theta,\varphi)$ on surface ($r=a$).

And: $A_{\ell m}=\int_0^{4\pi}d\Omega Y_{\ell m}^{*}(\theta,\varphi)V(\theta,\varphi)$ on surface ($r=a$).

Note: $V(\underbrace{\theta=0}_{\substack{\text{"north"}\\ \text{pole}}},\varphi)=\sum_{\ell=0}^{\infty}\sqrt{\frac{(2\ell+1)}{4\pi}}A_{\ell 0}$

$$A_{\ell 0}=\sqrt{\frac{(2\ell+1)}{4\pi}}\int_0^{4\pi}d\Omega P_{\ell}(\cos\theta)V(\theta,\varphi)$$

General Comments: The method of separation of variables used in Laplace's equation $\nabla^2 V=0$ in rectangular, cylindrical and spherical coordinates shows up again in Poisson's Equation $\nabla^2 V=-\frac{\rho_{free}}{\varepsilon_o}$ and also in the wave equation (valid for all classical wave phenomena) $\nabla^2\psi(\vec{r},t)+\frac{1}{c^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2}=0$ and in Schrödinger's wave equation $H\psi=E\psi$ in Quantum Mechanics problems. These equations will appear again and again, in one form or another for *E&M*, Classical Mechanics, Quantum Mechanics courses as well as for Classical / Newtonian Gravity problems…

For more detailed information *e.g.* on separation of variables and solutions to 3-*D* Wave Equation $\nabla^2\psi=-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ in rectangular, cylindrical and spherical coordinates see Prof. S. Errede lecture notes (Lecture IV – parts 1 & 2) on (sound) waves in 1-*D*, 2-*D*, 3*D* Physics 406 Acoustical Physics of Music website:

http://online.physics.uiuc.edu/courses/phys406/406_lectures.html

and also see/read his Fourier Analysis Lectures on this website, if interested.