LECTURE NOTES 6

THE METHOD OF IMAGES

- A useful technique for solving (i.e. finding) $\vec{E}(\vec{r})$ and/or $V(\vec{r})$ for a certain class/special classes of electrostatic (and magnetostatic) problems that have some (or high) degree of mirror-reflection symmetry. ← Exploit “awesome” power of symmetry intrinsic to the problem, if present.

- Idea is to convert a (seemingly) difficult electrostatic problem involving spatially-extended charged objects (e.g. charged conductors) and then replace them with a finite number of carefully, intelligently chosen/well-placed discrete point charges!!

- Solving the simpler point-charge problem is the solution for the original, more complicated problem!!!

- Can replace e.g. a charged surface of a conductor (which is at constant potential – an equipotential) by an equivalent/identical equipotential surface (at same potential) due to one (or more) such/so-called “image charges”.

→ By doing this replacement, the original boundary conditions associated with original problem are retained/conserved.

.: $\vec{E}$-fields and potentials $V$ of the original and “surrogate” problems must be the same/identical!!

- The method of images is best learned by example…
Example 1: Point Charge $q$ Located Near An Infinite, Grounded Conducting Plane:

n.b. A grounded conductor is a special type of equipotential: infinite amounts of electrical charge ($\pm Q$) can flow from / to ground to / from the conducting surface so as to maintain electrostatic potential $V = 0$ (Volts) at all times. (In reality / real life, $\exists$ no such thing. e.g. especially / particularly for AC / time-varying electromagnetic fields for frequencies $f \gg 1$ MHz, due to inductance effects (magnetic analog of capacitance)).

What are: $E(r), V(r)$ and $\sigma(x,y,z = 0)$ ($\sigma(x,y,z = 0)$ = induced surface charge density on $\infty$-plane)

$\infty$-conducting plane @ $V(x,y,z = 0) = 0$

wire

Earth ground Symbol

\[ \bar{E}(\bar{r}) @ Field \text{ Point, } P \]

\[ \bar{r} \equiv \bar{r} - \bar{r}' \]

Source Point, $S$
Side View of Problem:

Lines of $\vec{E}$:
* point radially outward @ $+q$
* are $\perp$ to surface of $\infty$-conducting plane @ $z = 0 \ \forall \ (x, y)$

$\infty$-conducting plane $V(x, y, 0) = 0$

$\hat{x}$ points outward from paper)

Now *mirror-reflect* above problem:

i.e. Let $z \to -z$ and simultaneously let $+q(z) \to -q(-z)$

(more generally let $\vec{r} \to -\vec{r}$ for objects in problem)

Now (mentally) remove the grounded, $\infty$-conducting plane:
Side View:

Original Charge @ (0,0,+d)  

Image Charge @ (0,0,−d)

We have replaced ∞-conducting grounded plane (equipotential, \( V(x, y, z = 0) = 0 \)) with an image point charge \( -q \) located at \( (x, y, z) = (0, 0, −d) \).

⇒ The method of images is highly analogous to mirror-type optics problem!!

− here we have point “object” \( +q \) at \( (x, y, z) = (0, 0, +d) \) and plane mirror at \( (x, y, z) = (x, y, 0) \).

An image of point object is formed as a point image a distance \( d \) behind the mirror; point image \( -q \) is located at \( (x, y, z) = (0, 0, −d) \).

Optical mirrors are essentially equipotentials – optics works for electrostatic problems!!!

Mathematical constraint (boundary condition) on \( V \): \( V(x, y, z = 0) = 0 \ \forall \ (x, y, z = 0) \).

(i.e. everywhere on ∞-conducting grounded plane.)
3-D View of Image Charge Problem for Grounded Infinite Conducting Plane:

We already know how to solve / do this problem!!

Use Principle of Superposition to solve / determine total potential at observation / field point \( \vec{r} \):

\[
V_{\text{TOT}}(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r})
\]

\[
= \frac{1}{\varepsilon_0} \frac{+q}{r_1} + \frac{1}{\varepsilon_0} \frac{-q}{r_2}
\]

\[
= \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

What are \( r_1 \) & \( r_2 \) ??

Use law of cosines \( c^2 = a^2 + b^2 - 2ab \cos \theta \) to obtain expressions for \( r_1 + r_2 \).
Here it is easier simply to use basic definition of separation distance, i.e.

Let: \( \theta = \theta_1 \)

and: \( \theta_2 = \pi - \theta_1 = \pi - \theta \)

\[
\begin{align*}
\theta_1 &= \theta_1 \\
\theta_2 &= \pi - \theta_1 = \pi - \theta \\
\end{align*}
\]

\[
\begin{align*}
\Delta r &= \Delta x + \Delta y + \Delta z \\
\end{align*}
\]

\[
\begin{align*}
\cos(\pi - \theta) &= \cos \pi \cos \theta + \sin \pi \sin \theta \\
&= -\cos \theta \\
\therefore \quad \Delta r^2 &= d^2 + r^2 + 2dr \cos \theta
\end{align*}
\]

Then: \( V_{\text{tot}}(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{\Delta r_1} - \frac{1}{\Delta r_2} \right) \)

Drop “p” on field point subscript:

\[
V_{\text{tot}}(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} \right)
\]

Note that:

1. \( V_{\text{tot}}(\vec{r} @ z = 0) = 0 \) i.e. \( V_{\text{tot}}\left( \theta_1 = \theta_2 = \theta = \frac{\pi}{2} = 90^\circ \right) = 0 \)

2. \( V_{\text{tot}}(\vec{r} \rightarrow \infty) = 0 \)

\[
V_{\text{tot}}(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} \right) = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{\Delta r_1} - \frac{1}{\Delta r_2} \right) = V_1(\vec{r}) + V_2(\vec{r})
\]

We can now determine \( \vec{E}_{\text{tot}}(\vec{r}) \) from either:

(a.) \( \vec{E}_{\text{tot}}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \left( \frac{\vec{f}_1}{\Delta r_1} - \frac{\vec{f}_2}{\Delta r_2} \right) \) (using the Principle of Superposition)

or:

(b.) \( \vec{E}_{\text{tot}}(\vec{r}) = -\vec{V}_{\text{tot}}(\vec{r}) = -\left[ \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right] V_{\text{tot}}(\vec{r}) \) (e.g. in Cartesian coordinates)
Because we will see this same problem again (in the near future) from a different perspective, let us rewrite the problem in spherical polar coordinates, using the law of cosine results:

\[ r_1 = \sqrt{r^2 + d^2 - 2rd \cos \theta} \quad \text{and} \quad r_2 = \sqrt{r^2 + d^2 + 2rd \cos \theta} \]

\[
V_{\text{tot}}(\vec{r}) = \frac{q}{4\pi \epsilon_0} \left\{ \frac{1}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd \cos \theta}} \right\}
\]

\[
\vec{E}_{\text{tot}}(\vec{r}) = -\nabla V_{\text{tot}}(\vec{r}) = -\left\{ \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi} \right\} V_{\text{tot}}(\vec{r})
\]

\[
= E_r^\text{tot} \hat{r} + E_\theta^\text{tot} \hat{\theta} + E_\phi^\text{tot} \hat{\phi}
\]

1. \[ E_r^\text{tot} = -\frac{\partial}{\partial r} V_{\text{tot}}(\vec{r}) = -\frac{q}{4\pi \epsilon_0} \frac{\partial}{\partial r} \left\{ \frac{1}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd \cos \theta}} \right\} \]

\[
= -\frac{q}{4\pi \epsilon_0} \left\{ -\frac{1}{2} \left[ \frac{(2r - 2d \cos \theta)}{\left( r^2 + d^2 - 2rd \cos \theta \right)^{3/2}} + \frac{1}{2} \left[ \frac{(2r + 2d \cos \theta)}{\left( r^2 + d^2 + 2rd \cos \theta \right)^{3/2}} \right] \right\}
\]

\[
= -\frac{q}{4\pi \epsilon_0} \left\{ \frac{(r-d \cos \theta)}{\left[ r^2 + d^2 - 2rd \cos \theta \right]^{3/2}} - \frac{(r+d \cos \theta)}{\left[ r^2 + d^2 + 2rd \cos \theta \right]^{3/2}} \right\} \quad \text{n.b.} \quad \left\{ \frac{\partial}{\partial \theta} \cos \theta = -\sin \theta \right\}
\]

2. \[ E_\theta^\text{tot} = -\frac{1}{r} \frac{\partial}{\partial \theta} V_{\text{tot}}(\vec{r}) = -\frac{q}{4\pi \epsilon_0 r} \frac{\partial}{\partial \theta} \left\{ \frac{1}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd \cos \theta}} \right\} \]

\[
= -\frac{q}{4\pi \epsilon_0} \left( \frac{1}{r} \right) \left\{ -\frac{1}{2} \left[ \frac{(2rd \sin \theta)}{\left( r^2 + d^2 - 2rd \cos \theta \right)^{3/2}} + \frac{1}{2} \left[ \frac{(-2rd \sin \theta)}{\left( r^2 + d^2 + 2rd \cos \theta \right)^{3/2}} \right] \right\}
\]

\[
= +\frac{q}{4\pi \epsilon_0} \left( \frac{1}{r} \right) \left\{ \left( +rd \sin \theta \right) \left[ \frac{1}{\left( r^2 + d^2 - 2rd \cos \theta \right)^{3/2}} + \frac{1}{\left( r^2 + d^2 + 2rd \cos \theta \right)^{3/2}} \right] \right\}
\]

\[
= +\frac{qd \sin \theta}{4\pi \epsilon_0} \left\{ \frac{1}{\left( r^2 + d^2 - 2rd \cos \theta \right)^{3/2}} + \frac{1}{\left( r^2 + d^2 + 2rd \cos \theta \right)^{3/2}} \right\}
\]

3. \[ E_\phi^\text{tot} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} V_{\text{tot}}(\vec{r}) = 0 \quad \text{because} \quad V_{\text{tot}}(\vec{r}) \quad \text{has no} \quad \phi \quad \text{-dependence.}
\]

i.e. \[ V_{\text{tot}}(\vec{r}) \quad \text{and} \quad E_{\text{tot}}(\vec{r}) \quad \text{are azimuthally symmetric (invariant under} \quad \phi \rightarrow \phi^{'} \quad \text{rotations)}
\]

because original charge distribution is azimuthally symmetric – no \( \phi \)-dependence.
Thus, \( \vec{E}_{TOT}(\vec{r}) = E_r^{TOT} \hat{r} + E_\theta^{TOT} \hat{\theta} + E_\phi^{TOT} \hat{\phi} \), or:

\[
E_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{(r - d \cos \theta)}{\left(r^2 + d^2 - 2rd \cos \theta\right)^{3/2}} - \frac{(r + d \cos \theta)}{\left(r^2 + d^2 + 2rd \cos \theta\right)^{3/2}} \right] \hat{r} + \frac{(d \sin \theta)}{\left(r^2 + d^2 - 2rd \cos \theta\right)^{3/2}} \hat{\theta} + \frac{(d \sin \theta)}{\left(r^2 + d^2 + 2rd \cos \theta\right)^{3/2}} \hat{\phi}
\]

with:

\[
V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd \cos \theta}} \right\}
\]

The above expressions are potential and electric field associated with a spatially-extended electric dipole, with electric dipole moment \( \vec{p} \equiv +q\vec{r}_1 - q\vec{r}_2 = q\Delta\vec{r}_{21} \) (SI Units: Coulomb-meters) with separation distance \( \Delta\vec{r}_{21} \equiv \vec{r}_1 - \vec{r}_2 = d\hat{z} - d(-\hat{z}) = d\hat{z} + d\hat{z} = 2d\hat{z} \). Here (i.e. in this problem), the separation distance \( |\Delta r_{21}| = 2d \).

n.b. The vector \( \Delta\vec{r}_{21} \equiv \vec{r}_1 - \vec{r}_2 \) points from \( -q \) to \( +q \) ← important convention!!!
Electric Field $\vec{E}(\vec{r})$ and Equipotentials of an Electric Dipole with Electric Dipole Moment, $\vec{p} = p\hat{z}$:

We can now determine the surface free charge density $\sigma_{\text{free}}(x, y, z = 0)$ on the infinite, grounded conducting plane e.g. via two methods:

**METHOD 1:**

$$\sigma_{\text{free}}(x, y, z = 0) = -\varepsilon \frac{\partial V_{\text{TOT}}(\vec{r})}{\partial n}$$

where $\frac{\partial}{\partial n} = \left[\begin{array}{l}
\text{gradient normal to surface of } \infty\text{-conducting plane} \\
\frac{\partial}{\partial z} (\text{here}).
\end{array}\right.$

i.e. here,

$$\sigma_{\text{free}}(x, y, z = 0) = -\varepsilon \frac{\partial V_{\text{TOT}}(\vec{r})}{\partial z}$$

In Cartesian coordinates:

$$V_{\text{TOT}}(x, y, z) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right\}_{z=0}$$

Thus:

$$\sigma_{\text{free}}(x, y, z = 0) = -\frac{q}{4\pi} \frac{\partial}{\partial z} \left\{ \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right\}_{z=0}$$
\[
\begin{align*}
&= -\frac{q}{4\pi} \left\{ \frac{1}{2} \left[ \frac{2(z-d)}{x^2 + y^2 + (z-d)^2} - \frac{2(z+d)}{x^2 + y^2 + (z+d)^2} \right] \right\}_{z=0} \\
&= +\frac{q}{4\pi} \left\{ \frac{-d}{x^2 + y^2 + d^2} - \frac{d}{x^2 + y^2 + d^2} \right\} \\
&= -\frac{2qd}{4\pi \left[ x^2 + y^2 + d^2 \right]^{3/2}} \frac{1}{\left[ x^2 + y^2 + d^2 \right]^{3/2}} = -\frac{p}{4\pi \left[ x^2 + y^2 + d^2 \right]^{3/2}}
\end{align*}
\]

Electric dipole moment \( p = 2qd = q(2d) \) where \( 2d \) = charge separation distance.

\(\rightarrow\) Note that the sign of the induced surface free charge on \( \infty \)-conducting plane is opposite to that of original charge \( q \).

\(\rightarrow\) Note also that \( \sigma_{\text{free}}(x,y,z = 0) \) is greatest at \( (x = 0,y = 0,z = 0) \) - directly underneath original charge, \( q \). i.e. \( \sigma_{\text{max}} = -\frac{2qd}{4\pi \left( \frac{1}{d^3} \right)} = \frac{-2q}{4\pi \left( \frac{1}{d^3} \right)} \)

\( \Rightarrow \) See plot of \( \sigma_{\text{free}}(x,y,z = 0) \) below (on p. 12).

**METHOD 2:** Use Gauss’ Law: \( \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \)

Use “shrunken” Gaussian Pillbox of height \( \delta h \) centered on / around \( \infty \)-conducting plane:

On the conducting surface (\( @ z = 0 \), \( \theta = \frac{\pi}{2} \)) \( \Rightarrow \sin \theta = \sin \left( \frac{\pi}{2} \right) = 1 \) and \( \cos \theta = \cos \left( \frac{\pi}{2} \right) = 0 \).
\[
\vec{E}_{\text{surf}} (x, y, z = 0) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{r}{\left[ r^2 + d^2 \right]^{3/2}} - \frac{r}{\left[ r^2 + d^2 \right]^{3/2}} \right] \hat{\mathbf{r}} + \frac{d}{\left[ r^2 + d^2 \right]^{3/2}} \hat{\theta}
\]
\[
= \frac{q}{4\pi\varepsilon_0} 0\hat{\mathbf{r}} + \frac{2d}{\left[ r^2 + d^2 \right]^{3/2}} \hat{\theta}
\]

When \( \theta = \frac{\pi}{2} = 90^\circ \)

Now \( \hat{z} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta} \)

Please remember / derive this!!

Consider \( \hat{\mathbf{r}}, \hat{\theta} \)

to lie in \( \hat{y} - \hat{z} \)

plane:

\[
\hat{z} = \hat{\mathbf{r}} \cos \theta
\]

\[
\alpha = \left( \pi - \frac{\pi}{2} - \theta \right) = \frac{\pi}{2} - \theta
\]

\[
\cos \alpha = \cos \left( \frac{\pi}{2} - \theta \right) = \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta = \sin \theta
\]

Thus, when \( \theta = 90^\circ = \frac{\pi}{2} \), \( \hat{z} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta} = \cos \left( \frac{\pi}{2} \right) \hat{\mathbf{r}} - \sin \left( \frac{\pi}{2} \right) \hat{\theta} = -\hat{\theta} \)

On conducting plane \( z = 0 \)

\[
\vec{E}_{\text{surf}} (x, y, z = 0) = -\frac{q(2d)}{4\pi\varepsilon_0} \frac{1}{\left[ r^2 + d^2 \right]^{3/2}} \hat{z}
\]

\[
\vec{E}_{\text{surf}} (x, y, z = 0) = -\frac{q(2d)}{4\pi\varepsilon_0} \frac{1}{\left[ x^2 + y^2 + d^2 \right]^{3/2}} \hat{z}
\]

\( r^2 = x^2 + y^2 + z^2 \) on conducting plane
Gaussian Pillbox Surface:

\[
\oint S \vec{E} \cdot d\vec{A} = \int_{S_1} \vec{E}_0 \cdot d\vec{A}_1 + \int_{S_2} \vec{E}_2 \cdot d\vec{A}_2 + \int_{S_3} \vec{E}_3 \cdot d\vec{A}_3
\]

\[
= \int_{S_1} \vec{E}_1 \cdot d\vec{A}_1
\]

\[
\sigma_{\text{free}} \sim \frac{1}{r^3}
\]

\[\vec{E} \text{ just } \varepsilon \text{ above surface}\]

\[\hat{x}, \hat{y}\]

\[\text{area of surface } S_1\]

Gauss Law:

\[
\oint \vec{E} \cdot d\vec{A} = \int_{S_1} \vec{E}_0 \cdot d\vec{A}_1 = -\frac{q(2d)}{4\pi \varepsilon_o} \frac{1}{\left[\chi^2 + y^2 + d^2\right]^{1/2}} \hat{z} \cdot A_1 \hat{z}
\]

\[
= Q_{\text{encl}}^{\text{free}} = \frac{\sigma_{\text{free}}(x, y, z = 0) A_1}{\varepsilon_o}
\]

\[
\therefore \sigma_{\text{free}}(x, y, z = 0) = -\frac{q(2d)}{4\pi} \frac{1}{\left[\chi^2 + y^2 + d^2\right]^{1/2}} = -\frac{p}{4\pi} \frac{1}{\left[\chi^2 + y^2 + d^2\right]^{1/2}}
\]

Electric dipole moment \( p = q(2d) \) (Coulomb-meters) where \( 2d \) = charge separation distance

\[
\Rightarrow \text{Same answer as obtained in Method 1.}
\]

NOTE: \( Q_{\text{plane}}^{\text{plane}} = \int_{\text{plane}} \sigma_{\text{free}}(x, y, 0) dA = -\frac{p}{4\pi} \int_0^\infty \int_0^{2\pi} \frac{1}{r^2} r dr d\phi = \frac{qd}{\sqrt{r^2 + d^2}}\bigg|_0^\infty = -q
\]

The net force of attraction of charge +q to \( \infty \)-conducting plane is just that of force of charge +q attracted to its image charge, −q a separation distance \( |\Delta \vec{r}_2| = |\vec{r}_1 - \vec{r}_2| = r = 2d \) away!!!

\[
\vec{F}_{+q}^{\text{NET}}(r) = +q \vec{E}_{-q}(r = 2d) = \frac{1}{4\pi \varepsilon_o} \frac{+q(-q)}{(2d)^2} \hat{z} = -\frac{1}{4\pi \varepsilon_o} \frac{q^2}{(2d)^2} \hat{z}
\]
We can also obtain the net force of attraction of the charge \( +q \) and grounded, infinite conducting plane by adding up all of the individual contributions \( q d\vec{E}(\vec{r} = (0,0,d)) \) due to \( \sigma_{\text{free}}(x,y,z=0) \):

\[
\sigma_{\text{free}}(x,y,z=0) = -\frac{q(2d)}{4\pi} \frac{1}{\left[x^2 + y^2 + d^2\right]^{\frac{3}{2}}} = -\frac{2qd}{4\pi} \frac{1}{\left[x^2 + y^2 + d^2\right]^{\frac{3}{2}}}
\]

\[
\vec{F}_{\text{NET}}^q(\vec{r}) = q \int d\vec{E}_{\text{TOT}} = q \int \frac{1}{4\pi\varepsilon_o} \sigma_{\text{free}}(x,y,0) \left(\frac{1}{r^2}\right) \hat{r} dA \quad \text{where} \quad dA = 2\pi rd\theta \quad \text{and} \quad \hat{r} = \cos \theta \hat{z} = \left(\frac{d}{r}\right) \hat{z}
\]

\[
\vec{F}_{\text{NET}}^q(\vec{r}) = -\frac{q^2(2d)}{4\pi} \left(\frac{1}{4\pi\varepsilon_o}\right) \int_0^\infty \frac{1}{\left[r^2 + d^2\right]^{\frac{3}{2}}} \left(\frac{1}{r^2 + d^2}\right)^* \frac{d}{\sqrt{r^2 + d^2}} \ast 2\pi rd\theta \hat{z}
\]

\[
= -\frac{q^2d^2}{4\pi\varepsilon_o} \int_0^\infty \frac{rdr}{\left[r^2 + d^2\right]^{\frac{3}{2}}} \hat{z} = -\frac{1}{4\pi\varepsilon_o} \frac{q^2}{(2d)^3} \hat{z}
\]

Integration over the conducting plane:

The work done to assemble the Image Charge Problem (i.e. put \(+q\) first at \((x,y,z = d)\) and then bring in \(-q\) at \((x,y,z = -d)\) from \(\infty\)) is:

\[
W_{\text{ICP}} = \int_{x^2 + y^2 + d^2}^{2d} \vec{F}_{\text{mech}} \cdot d\vec{l} = F_{\text{mech}} * (2d) = -\frac{1}{4\pi\varepsilon_o} \frac{q^2}{(2d)^3} * 2d = -\frac{1}{4\pi\varepsilon_o} \frac{q^2}{(2d)} \quad (\text{Joules})
\]

Also:

\[
W_{\text{ICP}} = \frac{\varepsilon_o}{2} \int_{\text{all space}} E^2 d\tau = \frac{\varepsilon_o}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} d\tau = \frac{\varepsilon_o}{2} \int_{\text{all space}} (E_r \cdot E_r + E_\theta \cdot E_\theta) d\tau
\]

Note that this integral includes both the \( z > 0 \) and \( z < 0 \) regions for the image charge problem.
However, for the actual problem, i.e. the charge $+q$ above grounded $\infty$-conducting plane, there is NO ELECTRIC FIELD in the $z < 0$ region!

Thus \[ W_{\text{actual}} = \frac{1}{2} W_{\text{ICP}} \] i.e. \[ W_{\text{actual}} = -\frac{1}{2} \left( \frac{1}{4\pi\varepsilon_0} \right) \left( \frac{q^2}{2d} \right) \]

For the actual problem we can obtain $W_{\text{actual}}$ by calculating the work required to bring $+q$ in from infinity to a distance $d$ above the grounded $\infty$-conducting plane. The mechanical force required to oppose the electrical force of attraction is:

\[
\vec{F}_{\text{mech}} = -\vec{F}_E = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2z)^2} \hat{\hat{z}} \quad \text{(along } \hat{\hat{z}} \text{ axis)}
\]

\[
W_{\text{actual}} = \int_{\infty}^{d} \vec{F}_{\text{mech}} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \int_{\infty}^{d} \frac{q^2}{(2z)^2} dz = \frac{q^2}{16\pi\varepsilon_0} \int_{\infty}^{d} \frac{dz}{z^2}
\]

\[
= \frac{q^2}{16\pi\varepsilon_0} \left[ -\frac{1}{z} \right]_{\infty}^{d} = -\frac{q^2}{16\pi\varepsilon_0 d}
\]

\[ W_{\text{actual}} = -\frac{1}{2} \left( \frac{1}{4\pi\varepsilon_0} \right) \left( \frac{q^2}{2d} \right) \] Same answer as that obtained above!

**IMPORTANT NOTES / COMMENTS ON IMAGE CHARGE PROBLEMS**

1.) Image charges are always located outside of regions(s) where $V(\vec{r})$ and $\vec{E}(\vec{r})$ are to be calculated!!

→ Image charges **cannot / must not** be located inside region where $V(\vec{r})$ and $\vec{E}(\vec{r})$ are to be calculated (no longer the same problem!!)

2.) $W_{\text{ICP}}$ (all space) = $2 \times W_{\text{actual}}$ (half space).

→ In general, this is not true $\forall$ image charge problems. Be careful here! Depends on detailed geometry of conducting surfaces.

3.) Depending on nature of problem, image charge(s) **may or may not** be opposite charge sign!!

4.) Depending on nature of problem, image charge(s) **may or may not** be same strength as original charge Q.
Image Charge Problem

Example 2: (Griffiths Example 3.2 p. 124-126)

Point charge \( +q \) situated a distance \( a \) away from the center of a grounded conducting sphere of radius \( R < a \). Find the potential outside the sphere.

\[
\hat{z} \quad V(r = R) = 0 \quad \text{on sphere (equipotential)}
\]

\[
\hat{y} \quad \text{Take origin of coordinates to be @ center of sphere}
\]

From spherical \( \text{and} \ \hat{y} \) axial symmetry (rotational invariance) of problem, if solution for image charge \( q' \) is to exist, it must be:

1.) \underline{inside} spherical conductor \( (r < R) \)
2.) \( q' \) image charge \underline{must} lie along \( \hat{y} \) axis (i.e. along line from charge \( +q \) to center of sphere).
3.) because \( V(r = R) = 0 \) on sphere, \( q' \ \underline{must} \) be opposite charge sign of \( +q \).
4.) want to replace grounded conducting sphere with equipotential \( V(r = R) = 0 \) by use of image charge \( q' \) at distance \( b \) away from center of sphere:
NOTE: Two points on the surface of sphere where the potential $V_{TOT}(r = R) = 0$ is easy to calculate - is on the $\hat{y}$ axis at the field points $P_1$ and $P_2$: 

$$r_1 = \vec{r} - \vec{r}_1 \quad \text{and} \quad r_2 = \vec{r} - \vec{r}_2$$

At point $P_1$: 

$$\vec{r}_1 = a\hat{y}, \quad \vec{r} = R(-\hat{y}) = -R\hat{y}, \quad \vec{r}_1 = \vec{r} - \vec{r}_1 = -R\hat{y} - a\hat{y} = -(R + a)\hat{y}$$

$$V_{r_1}(r = R) = 0$$

$$r_1 = |r_1| = (R + a)$$

$$\vec{r}_2 = b\hat{y}, \quad \vec{r} = R(-\hat{y}) = -R\hat{y}, \quad \vec{r}_2 = \vec{r} - \vec{r}_2 = -R\hat{y} - b\hat{y} = -(R + b)\hat{y}$$

$$r_2 = |r_2| = (R + b)$$

For general: 

$$V_{TOT} = V_1 + V_2 = \frac{1}{4\pi\varepsilon_0} \left( \frac{q + q'}{r_1} + \frac{q'}{r_2} \right)$$

At point $P_1$: 

$$V_{r_1}(r = R) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{R + a} + \frac{q'}{R + b} \right) = 0 \quad \Rightarrow \quad \frac{q}{R + a} = -\frac{q'}{R + b}$$

Relation #1

At point $P_2$: 

$$\vec{r}_1 = a\hat{y}, \quad \vec{r} = R(\hat{y}) = +R\hat{y}, \quad \vec{r}_1 = \vec{r} - \vec{r}_1 = R\hat{y} - a\hat{y} = (R - a)\hat{y}$$

$$V_{r_2}(r = R) = 0$$

$$r_1 = |r_1| = (a - R) \quad (a > R) !!$$

$$\vec{r}_2 = b\hat{y}, \quad \vec{r} = R(\hat{y}) = +R\hat{y}, \quad \vec{r}_2 = \vec{r} - \vec{r}_2 = R\hat{y} - b\hat{y} = (R - b)\hat{y}$$

$$r_2 = |r_2| = (R - b)$$

$$V_{r_2}(r = R) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{R - a} + \frac{q'}{R - b} \right) = 0 \quad \Rightarrow \quad \frac{q}{a - R} = -\frac{q'}{R - b}$$

Relation #2
We now have two equations (Relations # 1 & 2), and we have two unknowns: $q'$ and $b$. Solve equations simultaneously!

- First, we eliminate $q'$:

  From Relation #1 we have:  
  $$q' = -\frac{R+b}{R+a}q$$

  From Relation #2 we have:  
  $$q' = -\frac{R-b}{a-R}q$$

  \[ \therefore \frac{R+b}{R+a} = \frac{R-b}{a-R} \]

  OR:  
  $$(R+b)(a-R) = (R+a)(R-b)$$

  $$-R^2 + ab - bR = R^2 - bR - ab$$

  $$-2R^2 + 2ab = 0$$

  OR:  
  $$ab = R^2$$

  OR:  
  $$b = \frac{R^2}{a}$$

Then:  
$$q' = -\frac{R+b}{R+a}q = -\frac{R+\frac{R^2}{a}}{R+a}q = -\frac{(\frac{R}{a})\frac{1+\frac{R}{a}}{\frac{R}{a}}}{\frac{R}{a}}q = -\left(\frac{R}{a}\right)q$$

Thus:  
$$q' = -\left(\frac{R}{a}\right)q$$
CHECK:

Does $q' = -\left(\frac{R}{a}\right)q$, located at $\vec{r}_2 = b\hat{y} = \left(\frac{R^2}{a}\right)\hat{y}$ satisfy the B.C. that $V(r = R) = 0$ for any $r = R$?

$$V_{\text{tor}}(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q}{r_1} + \frac{q'}{r_2} \right\}$$

At an arbitrary field point $P_3$ anywhere on the surface of sphere, $r = R$:

$\begin{align*}
r_1 &= \sqrt{a^2 + R^2 - 2aR\cos\theta} \\
r_2 &= \sqrt{b^2 + R^2 - 2bR\cos\theta}
\end{align*}$

Note that we changed to $\hat{z}$ axis here in order to define (& use) the polar angle, $\theta$ !!!

Then: $V_{\text{tor}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q}{r_1} + \frac{q'}{r_2} \right\}$ with:

$$\begin{align*}
q' &= -\left(\frac{R}{a}\right)q \\
b &= \left(\frac{R^2}{a}\right)
\end{align*}$$

$$V_{\text{tor}}(r = R) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{\left(\frac{R}{a}\right)}{\sqrt{b^2 + R^2 - 2bR\cos\theta}} \right\}$$

$$= \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{\left(\frac{R}{a}\right)}{\sqrt{\left(\frac{R^2}{a}\right)^2 + R^2 - 2\left(\frac{R^2}{a}\right)R\cos\theta}} \right\}$$

$$= \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{1}{\left(\frac{a}{R}\right)\sqrt{\left(\frac{R^2}{a}\right)^2 + R^2 - 2\left(\frac{R^2}{a}\right)\cos\theta}} \right\}$$
$V_{TOT}(r = R) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + R^2 - 2aR \cos \theta}} - \frac{1}{\sqrt{(a/R)^2 \left( R^4/a^4 \right) + a^2 - 2aR \cos \theta}} \right\}$

\[ = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + R^2 - 2aR \cos \theta}} - \frac{1}{\sqrt{a^2 + R^2 - 2aR \cos \theta}} \right\} \]

\[ = 0 \quad \forall \quad \theta, \varphi \quad (@ \, r = R) \quad \text{YES!!!} \]

The scalar potential for an arbitrary point outside the grounded, conducting sphere \((r > R)\) is:

$V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{1}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} - \frac{(R/a)}{\sqrt{(R/a)^2 + r^2 - 2(R/a)r \cos \theta}} \right\}$

Then: $\vec{E}_{TOT}(\vec{r}) = -\nabla V_{TOT}(\vec{r})$ and thus: $E_r(\vec{r}) = -\frac{\partial V_{TOT}(\vec{r})}{\partial r}$

And thus: $\sigma_{\text{free}}(r = R) = -\varepsilon_o \frac{\partial V_{TOT}(\vec{r})}{\partial n} \bigg|_{r=R} = -\varepsilon_o \frac{\partial V_{TOT}(\vec{r})}{\partial r} \bigg|_{r=R} = +\varepsilon_o E_r(\vec{r}) \bigg|_{r=R}$

Total charge on surface of the conducting sphere: $Q_{\text{total}}^\text{sphere} = \int_{\text{sphere}}^{\text{total}} \sigma_{\text{free}} \, dA = \left( \frac{R}{a} \right) q$
Image Charge Problem

Example #3: Point charge $+q$ near a charged conducting sphere of radius $R$.

(variation on image charge Example #2)

→ Use the superposition principle for image charges!!

**Step 1:** Replace the conducting sphere by an image charge $q' = -\left(\frac{R}{a}\right)q$ located at $\vec{r}_q = b\hat{y} = \left(\frac{R^2}{a}\right)\hat{y}$ (same as in Example #2)

→ This makes surface of sphere an equipotential surface $V(r = R) = 0$.

**Step 2:** Add a second image charge $q''$ at center of sphere to raise potential on surface of sphere to achieve required potential $V(r = R) = V$ (positive or negative constant potential on sphere)

Note: $q''$ is also on same axis ($\hat{y}$) as $q$ and $q'$.

Then: $\sigma_{\text{free}}^{\text{TOT}} = \sigma_{\text{free}}^{\text{TOT}}(q') + \sigma_{\text{free}}^{\text{TOT}}(q'')$

Surface free charge density due to $q'$

Surface free charge density due to $q''$

Then: $V_{\text{TOT}}(\vec{r}) = V_q(\vec{r}) + V_{q'}(\vec{r}) + V_{q''}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q}{r_q} + \frac{q'}{r_q} + \frac{q''}{r_q} \right\}$

But: $q' = -\left(\frac{R}{a}\right)q$ and $Q_{\text{sphere}} = q' + q''$

Then: $\vec{E}_{\text{TOT}}(\vec{r}) = -\nabla V_{\text{TOT}}(\vec{r})$ and $\sigma_{\text{free}}^{\text{TOT}} = -\varepsilon_o \frac{\partial V_{\text{TOT}}(\vec{r})}{\partial n} \bigg|_{r=R} = -\varepsilon_o \frac{\partial V_{\text{TOT}}(\vec{r})}{\partial r} \bigg|_{r=R} = +\varepsilon_o E_r(\vec{r}) \bigg|_{r=R}$

Since: $E_r(\vec{r}) = -\frac{\partial V_{\text{TOT}}(\vec{r})}{\partial r}$