LECTURE NOTES 6

THE METHOD OF IMAGES

- A useful technique for solving (i.e. finding) $\vec{E}(\vec{r})$ and / or $V(\vec{r})$ for a certain class / special classes of electrostatic (and magnetostatic) problems that have some (or high) degree of mirror-reflection symmetry. ← Exploit "awesome" power of symmetry intrinsic to the problem, if present.

- Idea is to convert a (seemingly) difficult electrostatic problem involving spatially-extended charged objects (e.g. charged conductors) and then replace them with a finite number of carefully, intelligently chosen / well-placed discrete point charges!!

- Solving the simpler point-charge problem is *the* solution for the original, more complicated problem!!!

- Can replace e.g. a charged surface of a conductor (which is at constant potential – an equipotential) by an equivalent / identical equipotential surface (at same potential) due to one (or more) such / so-called "image charges".

  → By doing this replacement, the original boundary conditions associated with original problem are retained / conserved.

  ∴ $\vec{E}$-fields and potentials V of the original and "surrogate" problems *must* be the same / identical!!

- The method of images is best learned by example…

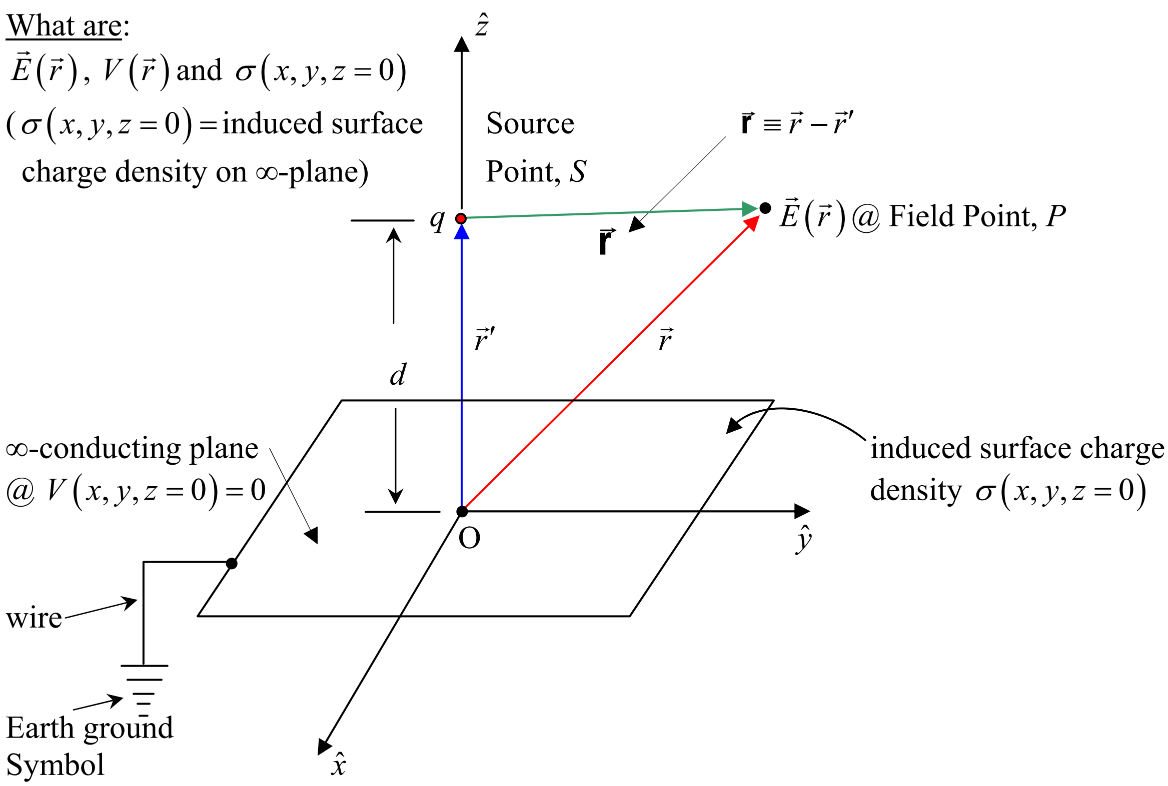
Example 1: Point Charge q Located Near An Infinite, Grounded Conducting Plane:

n.b. A grounded conductor is a special type of equipotential: infinite amounts of electrical charge ($\pm Q$) can flow from / to ground to / from the conducting surface so as to maintain electrostatic potential $V = 0$ (Volts) at all times. (In reality / real life, $\exists$ *no* such thing. e.g. especially / particularly for AC / time-varying electromagnetic fields for frequencies $f \gg 1$ MHz, due to inductance effects (magnetic analog of capacitance)).

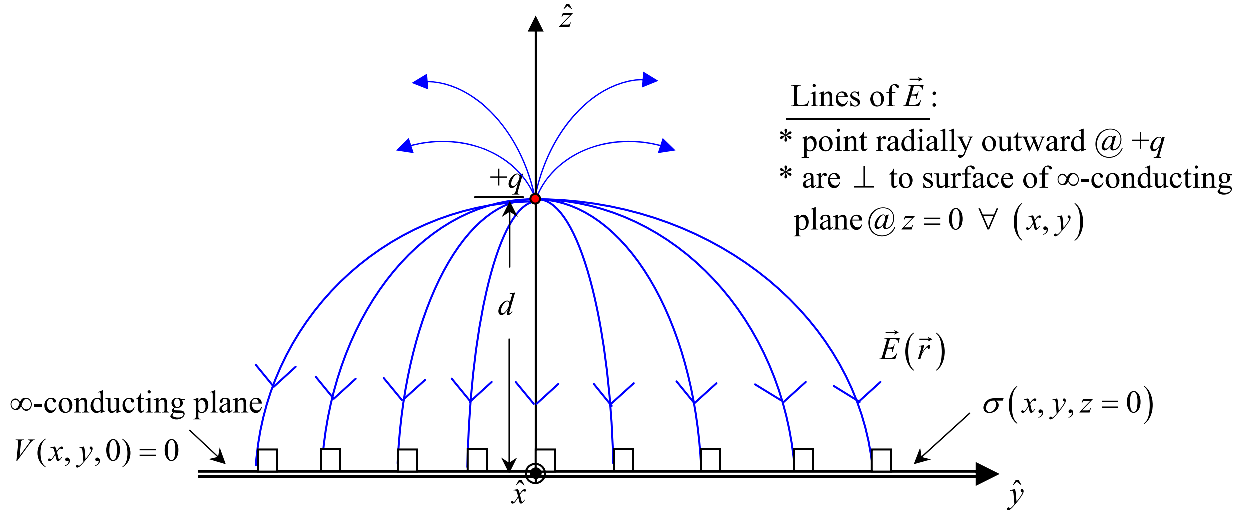
What are:

$\vec{E}(\vec{r})$, $V(\vec{r})$ and $\sigma(x, y, z=0)$

($\sigma(x, y, z=0) =$ induced surface charge density on ∞-plane)



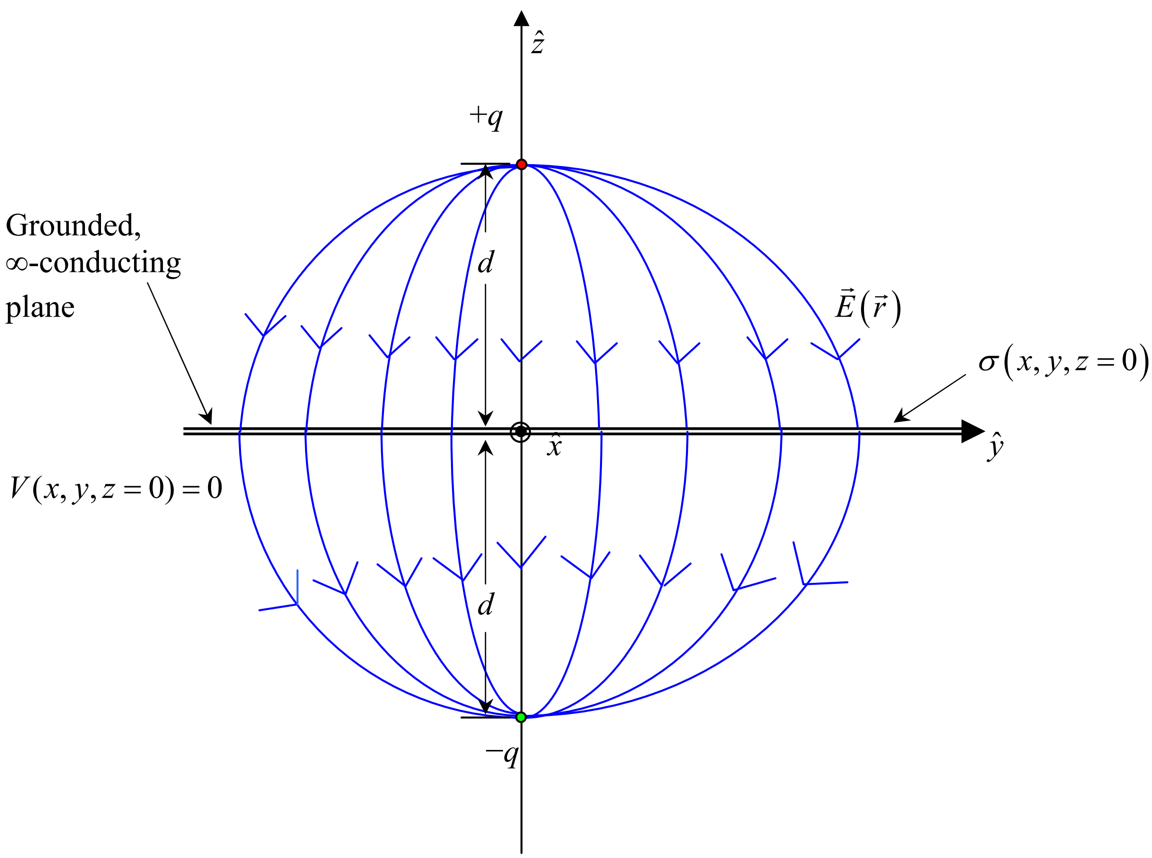
Side View of Problem:



Now *mirror-reflect* above problem:

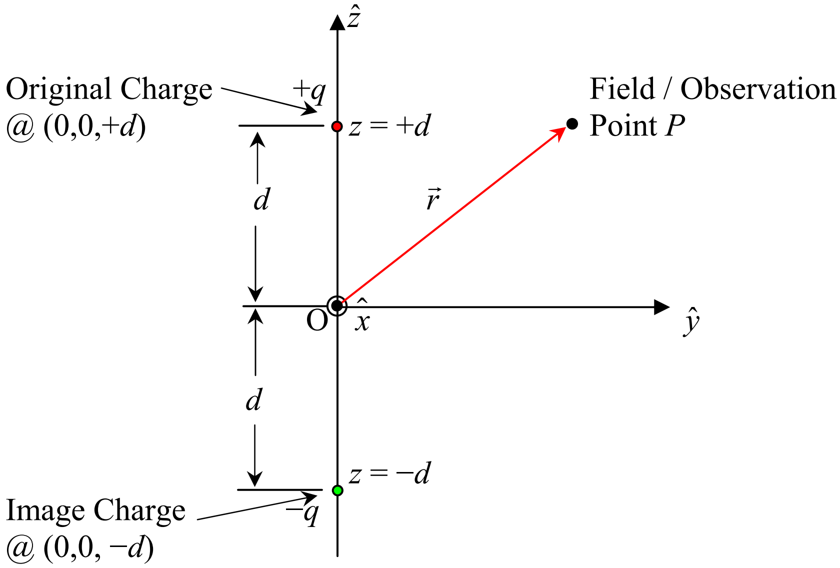
i.e. Let $z \to -z$ and simultaneously let $+q(z) \to -q(-z)$

(more generally let $\vec{r} \to -\vec{r}$ for objects in problem)



Now (mentally) remove the grounded, ∞-conducting plane:

Side View:



We have replaced ∞-conducting grounded plane (equipotential, $V(x,y,z=0)=0$) with an image point charge $-q$ located at $(x,y,z)=(0,0,-d)$.

⇒ The method of images is highly analogous to mirror-type optics problem!!

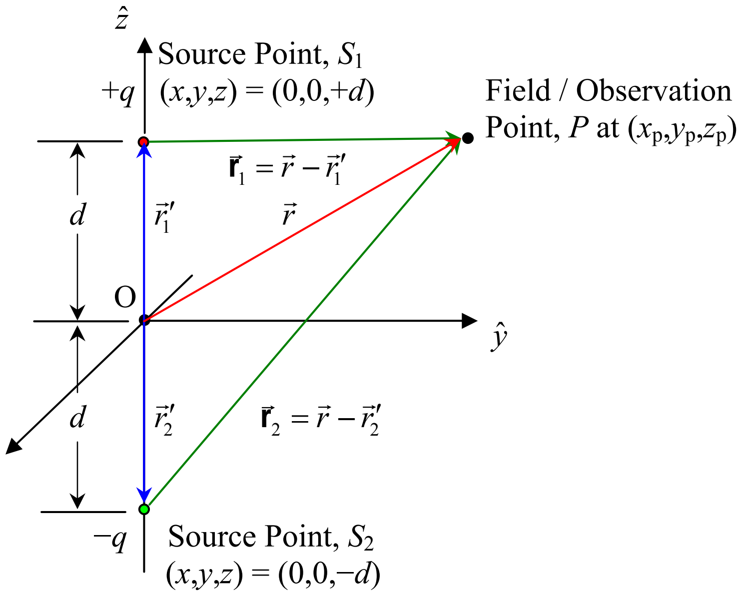
− here we have point "object" $+q$ at $(x,y,z)=(0,0,+d)$ and plane mirror at $(x,y,z)=(x,y,0)$.
An image of point object is formed as a point image a distance d behind the mirror; point image $-q$ is located at $(x,y,z)=(0,0,-d)$.

Optical mirrors are essentially equipotentials – optics works for electrostatic problems!!!

Mathematical constraint (*boundary condition*) on V: $V(x,y,z=0)=0 \;\forall\; (x,y,z=0)$.
(i.e. everywhere on ∞-conducting grounded plane.)

<u>3-D View of Image Charge Problem for Grounded Infinite Conducting Plane</u>:

We already know how to solve / do this problem!!



Use Principle of Superposition to solve / determine total potential at observation / field point $\vec{r}$:
$V_{TOT}(\vec{r}) = ?$

Potential @ point $(\vec{r})$ due to point charge $+q$ @ point $(\vec{r}_1')$

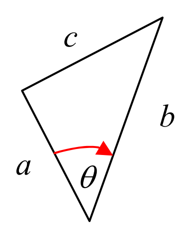
Potential @ point $(\vec{r})$ due to point charge $-q$ @ point $(\vec{r}_2')$

$$V_{TOT}(\vec{r}) = \overbrace{V_1(\vec{r})} + \overbrace{V_2(\vec{r})}$$

$$= \frac{1}{4\pi\varepsilon_o}\left(\frac{+q}{\boldsymbol{\mathscr{r}}_1}\right) + \frac{1}{4\pi\varepsilon_o}\left(\frac{-q}{\boldsymbol{\mathscr{r}}_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{\boldsymbol{\mathscr{r}}_1} - \frac{1}{\boldsymbol{\mathscr{r}}_2}\right)$$

What are $\overbrace{\boldsymbol{\mathscr{r}}_1 \ \& \ \boldsymbol{\mathscr{r}}_2}^{\text{separation distances}}$??

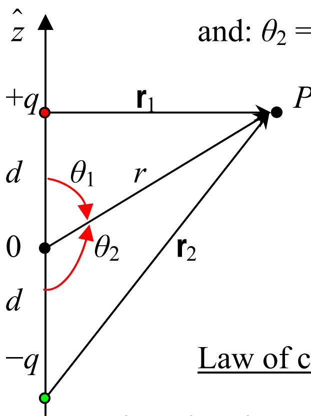


Use law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$
to obtain expressions for $\boldsymbol{\mathscr{r}}_1$ + $\boldsymbol{\mathscr{r}}_2$.

Here it is easier simply to use basic definition of separation distance, i.e.

Let: $\theta = \theta_1$

and: $\theta_2 = \pi - \theta_1 = \pi - \theta$



Law of cosines:

$$\mathbf{r}_1^2 = d^2 + r^2 - 2dr\cos\theta$$

$$\mathbf{r}_2^2 = d^2 + r^2 - 2dr\cos(\pi - \theta)$$

$$\cos(\pi - \theta) = \cos\pi\cos\theta + \overbrace{\cancel{\sin\pi}}^{=0}\sin\theta$$

$$= -\cos\theta$$

$$\therefore\ \mathbf{r}_2^2 = d^2 + r^2 + 2dr\cos\theta$$

$$\boxed{\mathbf{r} \equiv \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}$$

$$\mathbf{r}_1 \equiv \sqrt{(x_p - x_1)^2 + (y_p - y_1)^2 + (z_p - z_1)^2}$$

$$\mathbf{r}_1 = \sqrt{(x_p - 0)^2 + (y_p - 0)^2 + (z_p - d)^2}$$

$$\mathbf{r}_1 = \sqrt{x_p^2 + y_p^2 + (z_p - d)^2}$$

$$\mathbf{r}_2 \equiv \sqrt{(x_p - x_2)^2 + (y_p - y_2)^2 + (z_p - z_2)^2}$$

$$\mathbf{r}_2 = \sqrt{(x_p - 0)^2 + (y_p - 0)^2 + (z_p + d)^2}$$

$$\mathbf{r}_2 = \sqrt{x_p^2 + y_p^2 + (z_p + d)^2}$$

Then : $V_{TOT}(\vec{r}) = \dfrac{q}{4\pi\varepsilon_o}\left(\dfrac{1}{\mathbf{r}_1} - \dfrac{1}{\mathbf{r}_2}\right)$ Drop "p" on field point subscript:

$$\boxed{V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right\}}$$

Note that:

1. $V_{TOT}(\vec{r} @ z = 0) = 0$ i.e. $V_{TOT}\left(\theta_1 = \theta_2 = \theta = \frac{\pi}{2} = 90^o\right) = 0$

2. $V_{TOT}(\vec{r} \to \infty) = 0$

$$V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right\} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\mathbf{r}_1} - \frac{1}{\mathbf{r}_2}\right\} = V_1(\vec{r}) + V_2(\vec{r})$$

We can now determine $\vec{E}_{TOT}(\vec{r})$ from either:

(a.) $\vec{E}_{TOT}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) = \dfrac{q}{4\pi\varepsilon_o}\left\{\dfrac{\hat{\mathbf{r}}_1}{\mathbf{r}_1^2} - \dfrac{\hat{\mathbf{r}}_2}{\mathbf{r}_2^2}\right\}$ (using the Principle of Superposition)

What are $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$?

or:

(b.) $\vec{E}_{TOT}(\vec{r}) = -\vec{\nabla}V_{TOT}(\vec{r}) == -\left\{\dfrac{\partial}{\partial x}\hat{x} + \dfrac{\partial}{\partial y}\hat{y} + \dfrac{\partial}{\partial z}\hat{z}\right\}V_{TOT}(\vec{r})$ (e.g. in Cartesian coordinates)

Because we will see this same problem again (in the near future) from a different perspective, let us rewrite the problem in spherical polar coordinates, using the law of cosine results:

$$\left\{\mathbf{r}_1 = \sqrt{r^2+d^2-2rd\cos\theta} \quad \text{and} \quad \mathbf{r}_2 = \sqrt{r^2+d^2+2rd\cos\theta}\right\}$$

$$V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}}\right\}$$

$$\vec{E}_{TOT}(\vec{r}) = -\vec{\nabla}V_{TOT}(\vec{r}) = -\left\{\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right\}V_{TOT}(\vec{r})$$

$$= \underset{①}{E_r^{TOT}\hat{r}} + \underset{②}{E_\theta^{TOT}\hat{\theta}} + \underset{③}{E_\varphi^{TOT}\hat{\varphi}}$$

① $$E_r^{TOT} = -\frac{\partial}{\partial r}V_{TOT}(\vec{r}) = -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial r}\left\{\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}}\right\}$$

$$= -\frac{q}{4\pi\varepsilon_o}\left\{-\left(\frac{1}{2}\right)\frac{(2r-2d\cos\theta)}{\left[r^2+d^2-2rd\cos\theta\right]^{3/2}} + \left(\frac{1}{2}\right)\frac{(2r+2d\cos\theta)}{\left[r^2+d^2+2rd\cos\theta\right]^{3/2}}\right\}$$

$$= -\frac{q}{4\pi\varepsilon_o}\left\{\frac{(r-d\cos\theta)}{\left[r^2+d^2-2rd\cos\theta\right]^{3/2}} - \frac{(r+d\cos\theta)}{\left[r^2+d^2+2rd\cos\theta\right]^{3/2}}\right\}$$

n.b. $\left\{\frac{\partial}{\partial\theta}\cos\theta = -\sin\theta\right\}$

② $$E_\theta^{TOT} = -\frac{1}{r}\frac{\partial}{\partial\theta}V_{TOT}(\vec{r}) = -\frac{q}{4\pi\varepsilon_o r}\frac{\partial}{\partial\theta}\left\{\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}}\right\}$$

$$= -\frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left\{-\left(\frac{1}{2}\right)\frac{(+2rd\sin\theta)}{\left[r^2+d^2-2rd\cos\theta\right]^{3/2}} + \left(\frac{1}{2}\right)\frac{(-2rd\sin\theta)}{\left[r^2+d^2+2rd\cos\theta\right]^{3/2}}\right\}$$

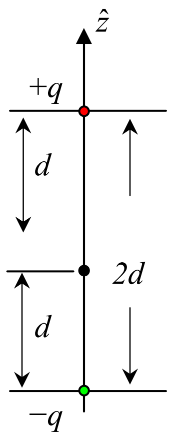
$$= +\frac{q}{4\pi\varepsilon_o}\left(\frac{1}{\cancel{r}}\right)\left\{(\cancel{r}d\sin\theta)\left[\frac{1}{\left[r^2+d^2-2rd\cos\theta\right]^{3/2}} + \frac{1}{\left[r^2+d^2+2rd\cos\theta\right]^{3/2}}\right]\right\}$$

$$= +\frac{qd\sin\theta}{4\pi\varepsilon_o}\left\{\frac{1}{\left[r^2+d^2-2rd\cos\theta\right]^{3/2}} + \frac{1}{\left[r^2+d^2+2rd\cos\theta\right]^{3/2}}\right\}$$

③ $$E_\varphi^{TOT} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}V_{TOT}(\vec{r}) = 0$$ because $V_{TOT}(\vec{r})$ has no φ-dependence.

i.e. $V_{TOT}(\vec{r})$ and $E_{TOT}(\vec{r})$ are azimuthally symmetric (invariant under $\varphi \to \varphi'$ rotations) because original charge distribution is azimuthally symmetric – no φ-dependence.

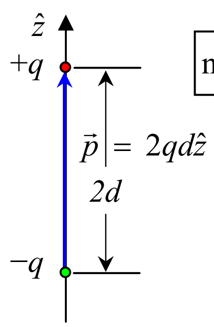
Thus, $\vec{E}_{TOT}(\vec{r}) = E_r^{TOT}\hat{r} + E_\theta^{TOT}\hat{\theta} + \underbrace{\cancel{E_\varphi^{TOT}}}_{=0}\hat{\varphi}$, or:



$$\boxed{E_{TOT}(\vec{r}) = +\frac{q}{4\pi\varepsilon_o}\left[\left\{\frac{(r - d\cos\theta)}{\left[r^2 + d^2 - 2rd\cos\theta\right]^{3/2}} - \frac{(r + d\cos\theta)}{\left[r^2 + d^2 + 2rd\cos\theta\right]^{3/2}}\right\}\hat{r} + \left\{\frac{(d\sin\theta)}{\left[r^2 + d^2 - 2rd\cos\theta\right]^{3/2}} + \frac{(d\sin\theta)}{\left[r^2 + d^2 + 2rd\cos\theta\right]^{3/2}}\right\}\hat{\theta}\right]}$$

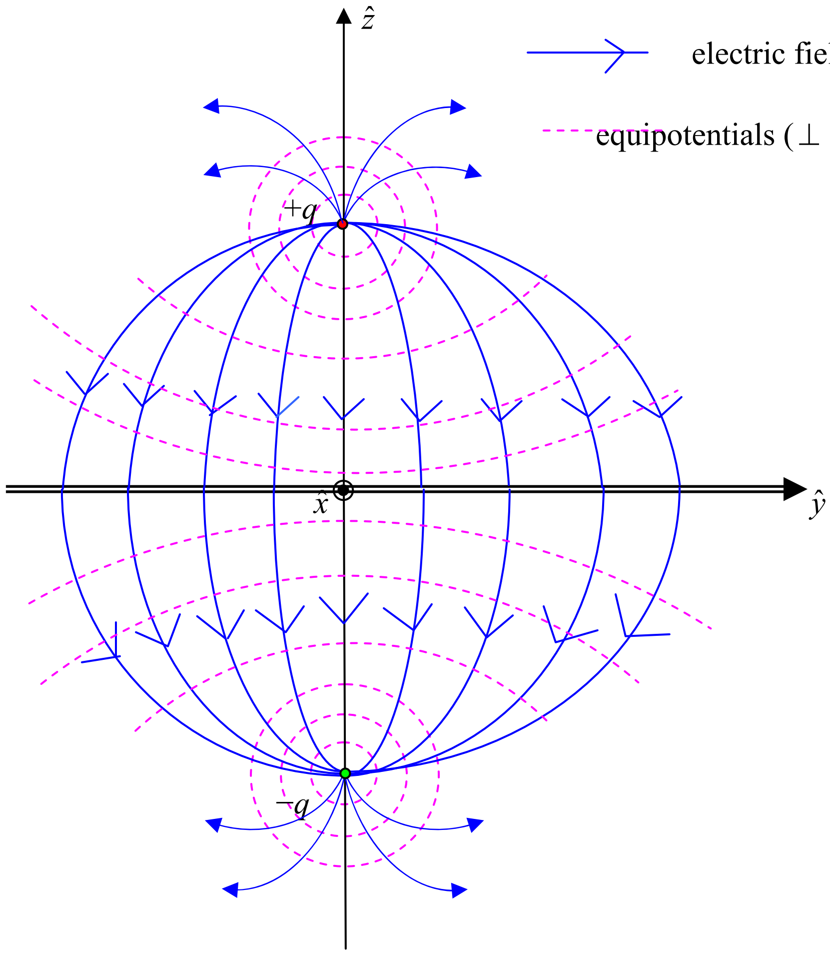
with: $\boxed{V_{TOT}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd\cos\theta}}\right\}}$

The above expressions are potential and electric field associated with a spatially-extended electric dipole, with electric dipole moment $\boxed{\vec{p} \equiv +q\vec{r}_1 - q\vec{r}_2 = q\Delta\vec{r}_{21}}$ (SI Units: Coulomb-meters) with separation distance $\Delta\vec{r}_{21} \equiv \vec{r}_1 - \vec{r}_2 = d\hat{z} - d(-\hat{z}) = d\hat{z} + d\hat{z} = 2d\hat{z}$. Here (i.e. in this problem), the separation distance $|\Delta r_{21}| = 2d$.



$\boxed{\text{n.b. The vector } \Delta\vec{r}_{21} \equiv \vec{r}_1 - \vec{r}_2 \text{ points } \textit{from } -q \textit{ to } +q \leftarrow \text{important convention!!!}}$

Electric Field $\vec{E}(\vec{r})$ and Equipotentials of an Electric Dipole with Electric Dipole Moment, $\vec{p} = p\hat{z}$:



We can now determine the surface free charge density $\sigma_{free}(x, y, z=0)$ on the infinite, grounded conducting plane e.g. via *two* methods:

METHOD 1: $\sigma_{free}(x, y, z=0) = -\varepsilon_o \left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial n}\right|_{surface}$ where $\dfrac{\partial}{\partial n} = \begin{bmatrix}\text{gradient normal to surface} \\ \text{of } \infty\text{-conducting plane}\end{bmatrix} = \dfrac{\partial}{\partial z}(\text{here})$.

i.e. here, $\sigma_{free}(x, y, z=0) = -\varepsilon_o \left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial z}\right|_{z=0}$

In Cartesian coordinates: $V_{TOT}(x, y, z) = \dfrac{q}{4\pi\varepsilon_o}\left.\left\{\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \dfrac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\right|_{z=0}$

Thus: $\sigma_{free}(x, y, z=0) = -\dfrac{q}{4\pi}\dfrac{\partial}{\partial z}\left.\left\{\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \dfrac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\right|_{z=0}$

$$= -\frac{q}{4\pi}\left\{\left(-\frac{1}{2}\right)\frac{2(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{2(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}\Bigg|_{z=0}$$

$$= +\frac{q}{4\pi}\left\{\frac{-d}{\left[x^2+y^2+d^2\right]^{3/2}} - \frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}\right\}$$

$$= -\frac{2qd}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}} = -\frac{p}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}$$

Electric dipole moment $p = 2qd = q(2d)$ where $2d =$ charge separation distance.

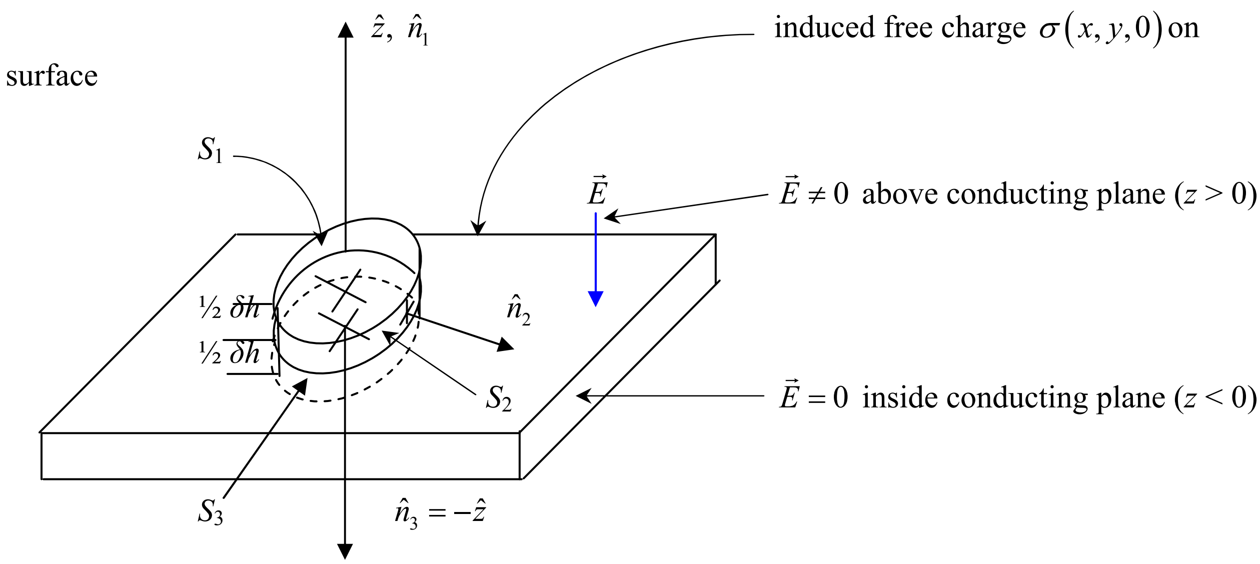
→ Note that the sign of the induced surface free charge on ∞-conducting plane is ***opposite*** to that of original charge q.

→ Note also that $\sigma_{free}(x, y, z=0)$ is greatest at $(x=0, y=0, z=0)$ - directly underneath original charge, q. i.e. $\sigma_{\max}^{free} = -\frac{2qd}{4\pi}\left(\frac{1}{d^3}\right) = -\frac{2q}{4\pi}\left(\frac{1}{d^2}\right)$

⇒ See plot of $\sigma_{free}(x, y, z=0)$ below (on p. 12).

METHOD 2: Use Gauss' Law: $\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{encl}^{free}}{\varepsilon_o}$

Use "shrunken" Gaussian Pillbox of height δh centered on / around ∞-conducting plane:



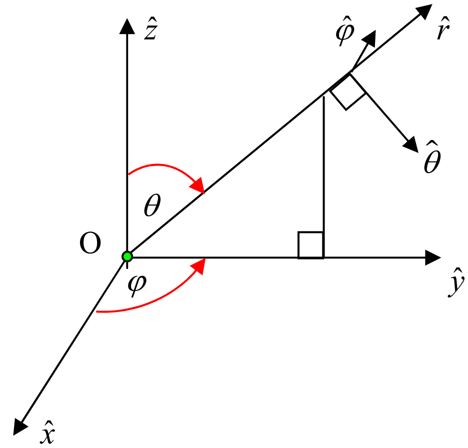
On the conducting surface (@ $z = 0$), $\theta = \frac{\pi}{2} \Rightarrow \sin\theta = \sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\theta = \cos\left(\frac{\pi}{2}\right) = 0$.

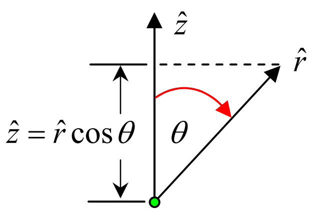
$$\vec{E}_{TOT}^{surf}(x,y,z=0)=\frac{q}{4\pi\varepsilon_o}\left[\left\{\frac{r}{\left[r^2+d^2\right]^{3/2}}-\frac{r}{\left[r^2+d^2\right]^{3/2}}\right\}\hat{r}+\left\{\frac{d}{\left[r^2+d^2\right]^{3/2}}+\frac{d}{\left[r^2+d^2\right]^{3/2}}\right\}\hat{\theta}\right]$$

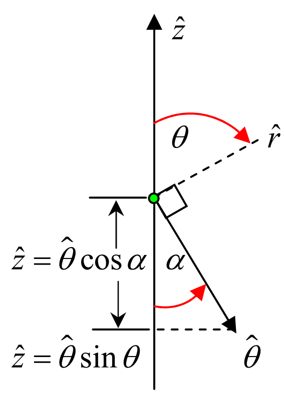
$$=\frac{q}{4\pi\varepsilon_o}\left[0\hat{r}+\frac{2d}{\left[r^2+d^2\right]^{3/2}}\hat{\theta}\right] \longleftarrow \text{When } \theta=\frac{\pi}{2}=90^o$$

Now $\boxed{\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}}$ ⟵ Please remember / derive this!!

Consider $\hat{r}$, $\vec{r}$ to lie in $\hat{y}-\hat{z}$ plane:







$$\alpha=\left(\pi-\frac{\pi}{2}-\theta\right)=\frac{\pi}{2}-\theta$$

$$\cos\alpha=\cos\left(\frac{\pi}{2}-\theta\right)=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta=\sin\theta$$

Thus, when $\theta=90^o=\frac{\pi}{2}$, $\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}=\cos\left(\frac{\pi}{2}\right)\hat{r}-\sin\left(\frac{\pi}{2}\right)\hat{\theta}=-\hat{\theta}$

On conducting plane $z=0$

$$\therefore\quad \vec{E}_{TOT}^{surf}(x,y,z=0)=-\frac{q(2d)}{4\pi\varepsilon_o}\frac{1}{\left[r^2+d^2\right]^{3/2}}\hat{z}$$

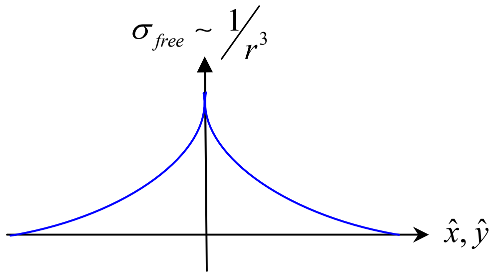
$r^2=x^2+y^2+z^2$

$$\vec{E}_{TOT}^{surf}(x,y,z=0)=-\frac{q(2d)}{4\pi\varepsilon_o}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}\hat{z}$$

$r^2=x^2+y^2$ on conducting plane

Gaussian Pillbox Surface: $$\oint_S \vec{E}\bullet d\vec{A} = \int_{S_1}\vec{E}_1\bullet d\vec{A}_1 + \underbrace{\cancelto{=0}{\int_{S_2}\vec{E}_2\bullet d\vec{A}_2}}_{\substack{\vec{E}_2\perp d\vec{A}_2 \\ S_2 \text{ shrinks} \to 0 \\ \text{anyway}}} + \underbrace{\cancelto{=0}{\int_{S_3}\vec{E}_3\bullet d\vec{A}_3}}_{\substack{\vec{E}_3=0 \\ \text{inside} \\ \text{conductor}}}$$

$$= \int_{S_1}\vec{E}_1\bullet d\vec{A}_1 \longleftarrow \vec{E} \text{ just } \varepsilon \text{ above surface}$$



area of surface S_1

Gauss Law: $$\oint \vec{E}\bullet d\vec{A} = \int_{S_1}\vec{E}_1\bullet d\vec{A}_1 = -\frac{q(2d)}{4\pi\varepsilon_o}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}\hat{z}\bullet A_1\hat{z}$$

$$= \frac{Q_{encl}^{free}}{\varepsilon_o} = \frac{\sigma_{free}(x,y,z=0)A_1}{\varepsilon_o}$$

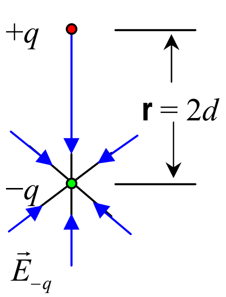
$$\therefore \ \sigma_{free}(x,y,z=0) = -\frac{q(2d)}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}} = -\frac{p}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}$$

Electric dipole moment $p = q(2d)$ $(\textit{Coulomb-meters})$ where $2d$ = charge separation distance

$\Rightarrow$ *Same answer as obtained in Method 1.*

NOTE: $$Q_{TOT}^{plane} = \int_{plane}\sigma_{free}(x,y,0)\,dA = -\frac{p}{4\pi}\int_0^{\infty}\int_0^{2\pi}\frac{1}{\left[r^2+d^2\right]}r\,dr\,d\varphi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q$$

The net force of attraction of charge $+q$ to ∞-conducting plane is just that of force of charge $+q$ attracted to its image charge, $-q$ a separation distance $|\Delta\vec{r}_{21}| \equiv |\vec{r}_1-\vec{r}_2| = \mathbf{r} = 2d$ away!!!



$$\vec{F}_{+q}^{NET}(\mathbf{r}) = +q\vec{E}_{-q}(\mathbf{r}=2d) = \frac{1}{4\pi\varepsilon_o}\frac{+q(-q)}{(2d)^2}\hat{z} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}\hat{z}$$

We can also obtain the net force of attraction of the charge $+q$ and grounded, infinite conducting plane by adding up all of the individual contributions $qd\vec{E}\left(\vec{r}=(0,0,d)\right)$ due to $\sigma_{free}(x,y,z=0)$:

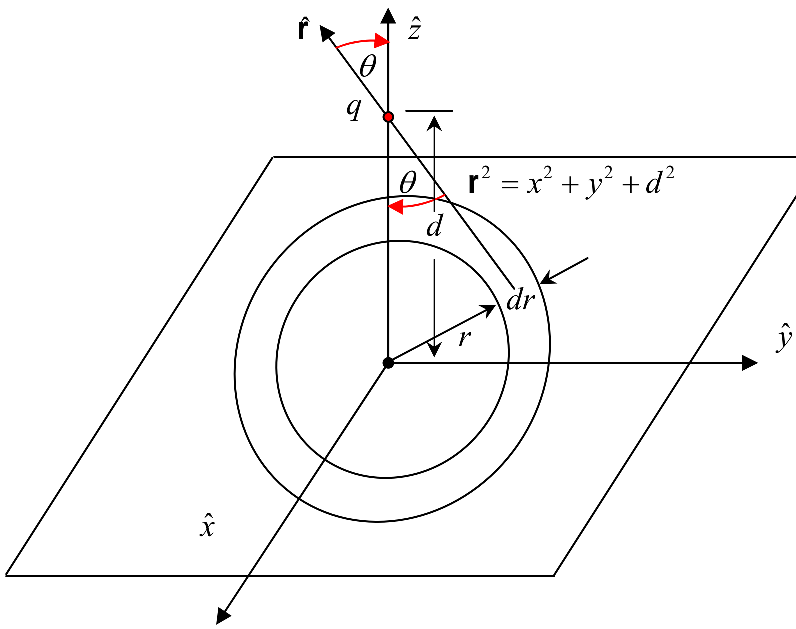
$$\sigma_{free}(x,y,z=0) = -\frac{q(2d)}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}} = -\frac{2qd}{4\pi}\frac{1}{\left[x^2+y^2+d^2\right]^{3/2}}$$

$$\vec{F}_{+q}^{NET}(\mathbf{r}) = q\int d\vec{E}_{TOT} = q\int_{\substack{S\\plane}} \frac{1}{4\pi\varepsilon_o}\sigma_{free}(x,y,0)\left(\frac{1}{\mathbf{r}^2}\right)\hat{\mathbf{r}}\,dA \text{ where } dA = 2\pi r dr \text{ and } \hat{\mathbf{r}} = \cos\theta\hat{z} = \left(\frac{d}{\mathbf{r}}\right)\hat{z}$$

$$\vec{F}_{+q}^{NET}(\mathbf{r}) = -\frac{q^2(2d)}{4\pi}\left(\frac{1}{4\pi\varepsilon_o}\right)\int_0^\infty \frac{1}{\left[r^2+d^2\right]^{3/2}} * \left(\frac{1}{r^2+d^2}\right) * \frac{d}{\sqrt{r^2+d^2}} * 2\pi r dr\,\hat{z}$$

$$= -\frac{q^2d^2}{4\pi\varepsilon_o}\int_0^\infty \frac{rdr}{\left[r^2+d^2\right]^3}\hat{z} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}\hat{z}$$

Integration over the conducting plane:



The work done to assemble the Image Charge Problem (i.e. put $+q$ first at $(x,y,z=d)$ and then bring in $-q$ at $(x,y,z=-d)$ from ∞) is:

$$W_{ICP} = \int_\infty^{2d}\vec{F}_{mech}\bullet\vec{dl} = F_{mech}*(2d) = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}*2d = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)} \text{ (Joules)}$$

Also: $$W_{ICP} = \frac{\varepsilon_o}{2}\int_{\substack{all\\space}} E^2 d\tau = \frac{\varepsilon_o}{2}\int_{\substack{all\\space}} \vec{E}\bullet\vec{E}\,d\tau = \frac{\varepsilon_o}{2}\int_{\substack{all\\space}} \left(E_r\bullet E_r + E_\theta\bullet E_\theta\right)d\tau$$

Note that this integral includes *both* the $z > 0$ and $z < 0$ regions for the image charge problem.

However, for the *actual* problem, i.e. the charge $+q$ above grounded ∞-conducting plane, there is NO ELECTRIC FIELD in the $z < 0$ region!

Thus $\boxed{W_{actual} = \frac{1}{2} W_{ICP}}$ i.e. $\boxed{W_{actual} = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q^2}{(2d)}}$

For the *actual* problem we can obtain W_{actual} by calculating the work required to bring $+q$ in from infinity to a distance d above the grounded ∞-conducting plane. The mechanical force required to oppose the electrical force of attraction is:

$$\vec{F}_{mech} = -\vec{F}_E = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2z)^2}\hat{z} \quad \text{(along } \hat{z} \text{ axis)}$$

$$W_{actual} = \int_\infty^d \vec{F}_{mech} \bullet d\vec{l} = \frac{1}{4\pi\varepsilon_o}\int_\infty^d \frac{q^2}{(2z)^2}dz = \frac{q^2}{16\pi\varepsilon_o}\int_\infty^d \frac{dz}{z^2}$$

$$= \frac{q^2}{16\pi\varepsilon_o}\left(\frac{-1}{z}\right)\Bigg|_\infty^d = -\frac{q^2}{16\pi\varepsilon_o d}$$

$\boxed{W_{actual} = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q^2}{(2d)}}$ Same answer as that obtained above!

IMPORTANT NOTES / COMMENTS ON IMAGE CHARGE PROBLEMS

1.) Image charges are always located outside of regions(s) where $V(\vec{r})$ and $\vec{E}(\vec{r})$ are to be calculated!!

→ Image charges ***cannot / must not*** be located inside region where $V(\vec{r})$ and $\vec{E}(\vec{r})$ are to be calculated (no longer the same problem!!)

2.) W_{ICP} (all space) = $2\times W_{actual}$ (half space).

→ In general, this is not true ∀ image charge problems. Be careful here! Depends on detailed geometry of conducting surfaces.

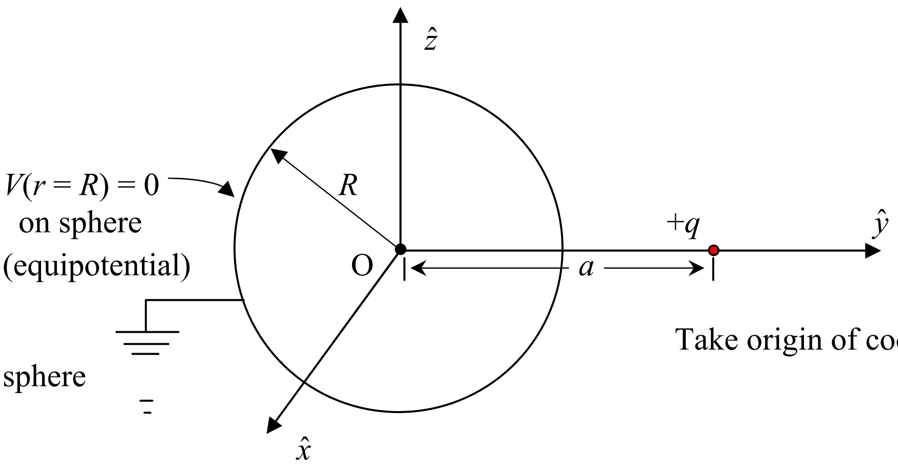
3.) Depending on nature of problem, image charge(s) **may or may not** be opposite charge sign!!

4.) Depending on nature of problem, image charge(s) **may or may not** be same strength as original charge Q.

Image Charge Problem

Example 2: (Griffiths Example 3.2 p. 124-126)

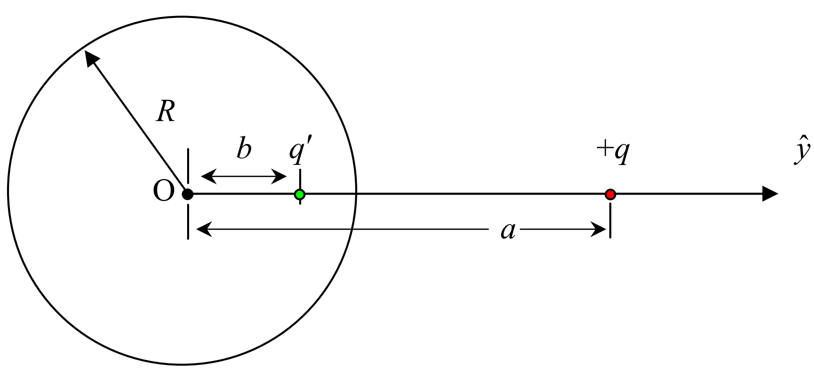
Point charge $+q$ situated a distance a away from the center of a grounded conducting sphere of radius $R < a$. Find the potential outside the sphere.



Take origin of coordinates to be @ center of sphere

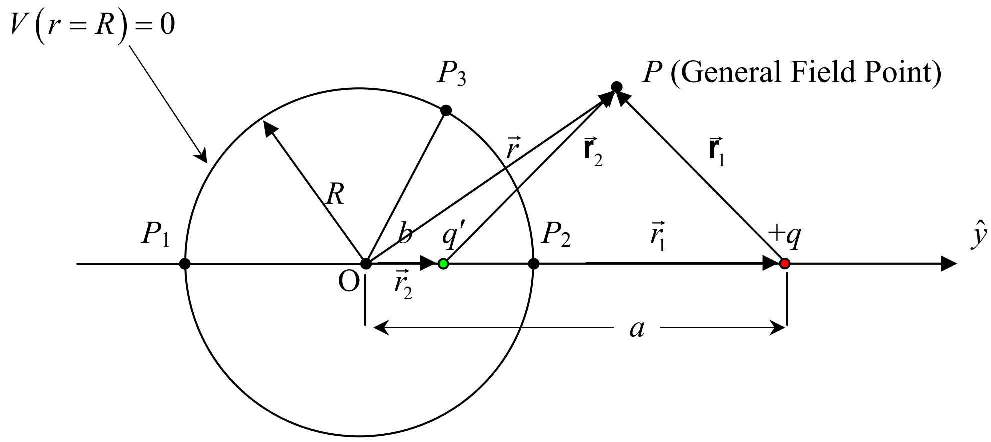
From spherical *and* $\hat{y}$ axial symmetry (rotational invariance) of problem, if solution for image charge q' is to exist, it must be:

1.) *inside* spherical conductor ($r < R$)
2.) q' image charge *must* lie along $\hat{y}$ axis (i.e. along line from charge $+q$ to center of sphere).
3.) because $V(r = R) = 0$ on sphere, q' *must* be opposite charge sign of $+q$.
4.) want to replace grounded conducting sphere with equipotential $V(r = R) = 0$ by use of image charge q' at distance b away from center of sphere:



NOTE: Two points on the surface of sphere where the potential $V_{TOT}(r = R) = 0$ is easy to calculate - is on the $\hat{y}$ axis at the field points P_1 and P_2:

$$\boldsymbol{\mathfrak{r}}_1 = \vec{r} - \vec{r}_1 \quad \text{and} \quad \boldsymbol{\mathfrak{r}}_2 = \vec{r} - \vec{r}_2$$



In general: $$V_{TOT}(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\boldsymbol{\mathfrak{r}}_1} + \frac{q'}{\boldsymbol{\mathfrak{r}}_2}\right)$$

At point P_1: $\vec{r}_1 = a\hat{y}$, $\vec{r} = R(-\hat{y}) = -R\hat{y}$, $\boldsymbol{\mathfrak{r}}_1 = \vec{r} - \vec{r}_1 = -R\hat{y} - a\hat{y} = -(R + a)\hat{y}$

$V_{P_1}(r = R) = 0$ $\qquad \boldsymbol{\mathfrak{r}}_1 = |\boldsymbol{\mathfrak{r}}_1| = (R + a)$

$\vec{r}_2 = b\hat{y}$, $\vec{r} = R(-\hat{y}) = -R\hat{y}$, $\boldsymbol{\mathfrak{r}}_2 = \vec{r} - \vec{r}_2 = -R\hat{y} - b\hat{y} = -(R + b)\hat{y}$

$\boldsymbol{\mathfrak{r}}_2 = |\boldsymbol{\mathfrak{r}}_2| = (R + b)$

$$V_{P_1}(r = R) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\boldsymbol{\mathfrak{r}}_1} + \frac{q'}{\boldsymbol{\mathfrak{r}}_2}\right)$$

$$= \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{(R + a)} + \frac{q'}{(R + b)}\right) = 0 \;\Rightarrow\; \boxed{\frac{q}{(R + a)} = -\frac{q'}{(R + b)}}$$

Relation #1

At point P_2: $\vec{r}_1 = a\hat{y}$, $\vec{r} = R(+\hat{y}) = +R\hat{y}$, $\boldsymbol{\mathfrak{r}}_1 = \vec{r} - \vec{r}_1 = R\hat{y} - a\hat{y} = (R - a)\hat{y}$

$V_{P_2}(r = R) = 0$ $\qquad \boldsymbol{\mathfrak{r}}_1 = |\boldsymbol{\mathfrak{r}}_1| = (a - R) \qquad (a > R)$!!

$\vec{r}_2 = b\hat{y}$, $\vec{r} = R(+\hat{y}) = +R\hat{y}$, $\boldsymbol{\mathfrak{r}}_2 = \vec{r} - \vec{r}_2 = R\hat{y} - b\hat{y} = (R - b)\hat{y}$

$\boldsymbol{\mathfrak{r}}_2 = |\boldsymbol{\mathfrak{r}}_2| = (R - b)$

$$V_{P_2}(r = R) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{(R - a)} + \frac{q'}{(R - b)}\right) = 0 \;\Rightarrow\; \boxed{\frac{q}{(a - R)} = -\frac{q'}{(R - b)}}$$

Relation #2

We now have two equations (Relations # 1 & 2), and we have two unknowns: q' and b. Solve equations simultaneously!

- First, we eliminate q':

From Relation #1 we have: $q' = -\left[\dfrac{R+b}{R+a}\right]q$

From Relation #2 we have: $q' = -\left[\dfrac{R-b}{a-R}\right]q$

$\therefore \quad \left[\dfrac{R+b}{R+a}\right] = \left[\dfrac{R-b}{a-R}\right]$ OR: $(R+b)(a-R) = (R+a)(R-b)$

$$-R^2 + \cancel{aR} + ab - \cancel{bR} = R^2 + \cancel{aR} - \cancel{bR} - ab$$

$$-2R^2 + 2ab = 0$$

OR: $ab = R^2$

OR: $\boxed{b = R^2/a}$

Then: $q' = -\left[\dfrac{R+b}{R+a}\right]q$

$$= -\left[\frac{R + R^2/a}{R+a}\right]q = -R\left[\frac{1+R/a}{R+a}\right]q = -\left(\frac{R}{a}\right)\frac{a\left[1+R/a\right]}{\left[R+a\right]}q$$

$$= -\left(\frac{R}{a}\right)\frac{\left[a+R\right]}{\left[R+a\right]}q = -\left(\frac{R}{a}\right)\cancel{\frac{\left[R+a\right]}{\left[R+a\right]}}q = -\left(\frac{R}{a}\right)q$$

Thus: $\boxed{q' = -\left(\dfrac{R}{a}\right)q}$

CHECK:

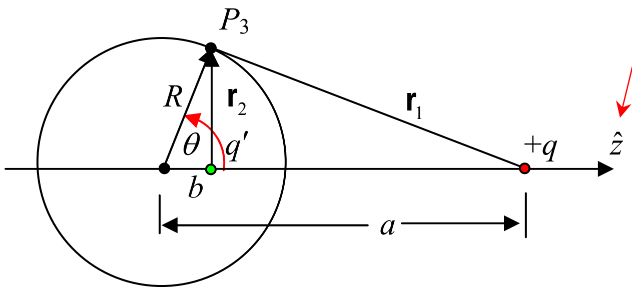
Does $q' = -\left(\frac{R}{a}\right)q$, located at $\vec{r}_2 = b\hat{y} = \left(\frac{R^2}{a}\right)\hat{y}$ satisfy the B.C. that $V(r=R)=0$ for any $r = R$?

$$V_{TOT}(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q}{\mathbf{r}_1} + \frac{q'}{\mathbf{r}_2}\right\}$$

At an arbitrary field point P_3 anywhere on the surface of sphere, $r = R$:

$$\mathbf{r}_1 = \sqrt{a^2 + R^2 - 2aR\cos\theta} \qquad \text{and} \qquad \mathbf{r}_2 = \sqrt{b^2 + R^2 - 2bR\cos\theta}$$

Note that we changed to $\hat{z}$ axis here in order to define (& use) the polar angle, θ !!!



Then: $V_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left\{\frac{q}{\mathbf{r}_1} + \frac{q'}{\mathbf{r}_2}\right\}$ with: $\left\{ \begin{array}{l} q' = -\left(\frac{R}{a}\right)q \\ b = \left(\frac{R^2}{a}\right) \end{array} \right\}$

$$V_{TOT}(r=R) = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{\left(R/a\right)}{\sqrt{b^2 + R^2 - 2bR\cos\theta}}\right\}$$

$$= \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{\left(R/a\right)}{\sqrt{\left(R^2/a\right)^2 + R^2 - 2\left(R^2/a\right)R\cos\theta}}\right\}$$

$$= \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{1}{\left(a/R\right)\sqrt{\left(R^2/a\right)^2 + R^2 - 2\left(R^3/a\right)\cos\theta}}\right\}$$

$$V_{TOT}(r=R)=\frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2+R^2-2aR\cos\theta}}-\frac{1}{\sqrt{\left(a/R\right)^2\left(R^4/a^2\right)+a^2-2aR\cos\theta}}\right\}$$

$$=\frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2+R^2-2aR\cos\theta}}-\frac{1}{\sqrt{a^2+R^2-2aR\cos\theta}}\right\}$$

$$=0 \quad \forall \quad \theta,\varphi \quad (@\, r=R)$$ <u>YES</u>!!!

The scalar potential for an arbitrary point *<u>outside</u>* the grounded, conducting sphere ($r > R$) is:

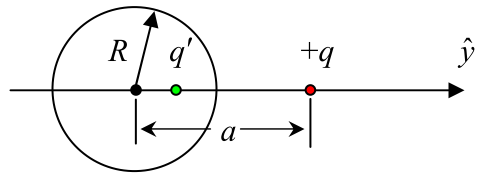
$$V_{TOT}(\vec{r})=\frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{\sqrt{a^2+r^2-2ar\cos\theta}}-\frac{\left(R/a\right)}{\sqrt{\left(R^2/a\right)^2+r^2-2\left(R^2/a\right)r\cos\theta}}\right\}$$

Then: $\vec{E}_{TOT}(\vec{r})=-\vec{\nabla}V_{TOT}(\vec{r})$ and thus: $E_r(\vec{r})=-\dfrac{\partial V_{TOT}(\vec{r})}{\partial r}$

And thus: $\sigma_{free}(r=R)=-\varepsilon_o\left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial n}\right|_{r=R}=-\varepsilon_o\left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial r}\right|_{r=R}=+\varepsilon_o E_r(\vec{r})\big|_{r=R}$

Total charge on surface of the conducting sphere: $Q_{free}^{\substack{total\\sphere}}=\int_{sphere}\sigma_{free}\,dA=-\left(\dfrac{R}{a}\right)q$

Image Charge Problem



Example #3: Point charge $+q$ near a *charged* conducting sphere of radius R.

(variation on image charge Example #2)

→ Use the superposition principle for image charges!!

Step 1: Replace the conducting sphere by an image charge $q' = -\left(R/a\right)q$ located at

$\vec{r}_{q'} = b\hat{y} = \left(R^2/a\right)\hat{y}$ (same as in Example #2)

→ This makes surface of sphere an equipotential surface $V(r = R) = 0$.

Step 2: Add a *second* image charge q'' at *center* of sphere to raise potential on surface of sphere to achieve required potential $V(r = R) = V$ (positive or negative constant potential on sphere)

Note: q'' is also on same axis ($\hat{y}$) as q and q'.

Then: $\sigma_{free}^{TOT} = \sigma_{free}(q') + \sigma_{free}(q'')$

Surface free charge Density due to q'

surface free charge density due to q''

Then: $V_{TOT}(\vec{r}) = V_q(\vec{r}) + V_{q'}(\vec{r}) + V_{q''}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\left\{\dfrac{q}{\mathfrak{r}_q} + \dfrac{q'}{\mathfrak{r}_{q'}} + \dfrac{q''}{\underbrace{\mathfrak{r}_{q''}}_{=r}}\right\}$

But: $q' = -\left(R/a\right)q$ and $Q_{sphere} = q' + q''$

Then: $\vec{E}_{TOT}(\vec{r}) = -\vec{\nabla}V_{TOT}(\vec{r})$ and $\sigma_{free}^{TOT} = -\varepsilon_o \left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial n}\right|_{r=R} = -\varepsilon_o \left.\dfrac{\partial V_{TOT}(\vec{r})}{\partial r}\right|_{r=R} = +\varepsilon_o E_r(\vec{r})\big|_{r=R}$

Since: $E_r(\vec{r}) = -\dfrac{\partial V_{TOT}(\vec{r})}{\partial r}$

