LECTURE NOTES 5

Materials made up of normal matter (atoms, molecules, etc.) have some amazing electromagnetic properties!

Simplest kinds of electromagnetic properties:

A.) conductor (of electricity)
B.) $\updownarrow$ partial conductor/insulator
C.) non-conductor $\Rightarrow$ insulator

Why materials conduct vs. do not conduct electricity depends on microscopic (i.e. quantum) structure of materials & temperature (i.e. thermal/internal energy).

CONDUCTORS:

"normal" good conductors of electricity:
metals - gold, platinum, silver, copper...

Have finite DC resistance, R = V/I (Ohm's Law) @ finite temperatures, $T > 0\ K$.

"superconductors" - low temperature SC's such as lead (T_c ~4K) indium, niobium, ...
Hi-T_c SC's (e.g. T_c~77K): BSCO, YBCO .....

DC resistance vanishes below T_c (critical temp)

INSULATORS:
e.g. plastics, teflon, glass, rubber ….

PARTIAL CONDUCTORS:
e.g. doped plastics, semi-conductors (germanium, silicon, graphite….)

IONIC LIQUIDS:
e.g. salt water – can also conduct electricity
Acidic solutions – ions transport electrical charges – not electrons

An ***ideal/perfect*** conductor is a (hypothetical) material that would have an unlimited number of completely free electrons/free charges. No such things truly exist in nature, but $\exists$ many materials which do come (amazingly) close to an ideal/perfect conductor.

One important property of a conductor is that:

1) $\boxed{\vec{E}_{NET}(\vec{r}) \equiv 0 \text{ inside a conductor}}$

n.b. $\vec{E}_{NET}(\vec{r}) \equiv 0$ is a ***different*** physics statement than $\vec{E}(\vec{r}) \equiv 0$ inside a conductor!

If $\vec{E}_{NET}(\vec{r}) = 0$ inside a conductor, then free charges inside the conductor would move/be <u>accelerated</u> by this $\vec{E}_{\text{inside}}(\vec{r}) = 0$, because: $\vec{F}_{\text{inside}}(\vec{r}) = q_e\vec{E}_{\text{inside}}(\vec{r}) = m_e\vec{a}_e(\vec{r})$.

Indeed, this is ***precisely*** what happens inside a conductor e.g. when it is placed in a uniform external electric field $\vec{E}_{ext}(\vec{r}) = E_0\hat{x}$ – the free charges inside the conductor (electrically neutral!) re-distribute themselves to <u>create</u>/<u>produce</u> $\vec{E}_{\text{inside}}(\vec{r}) = 0$ on extremely short timescales of ~ femto → pico-seconds $\left(\sim 10^{-15} - 10^{-12} \text{ sec}\right)$.

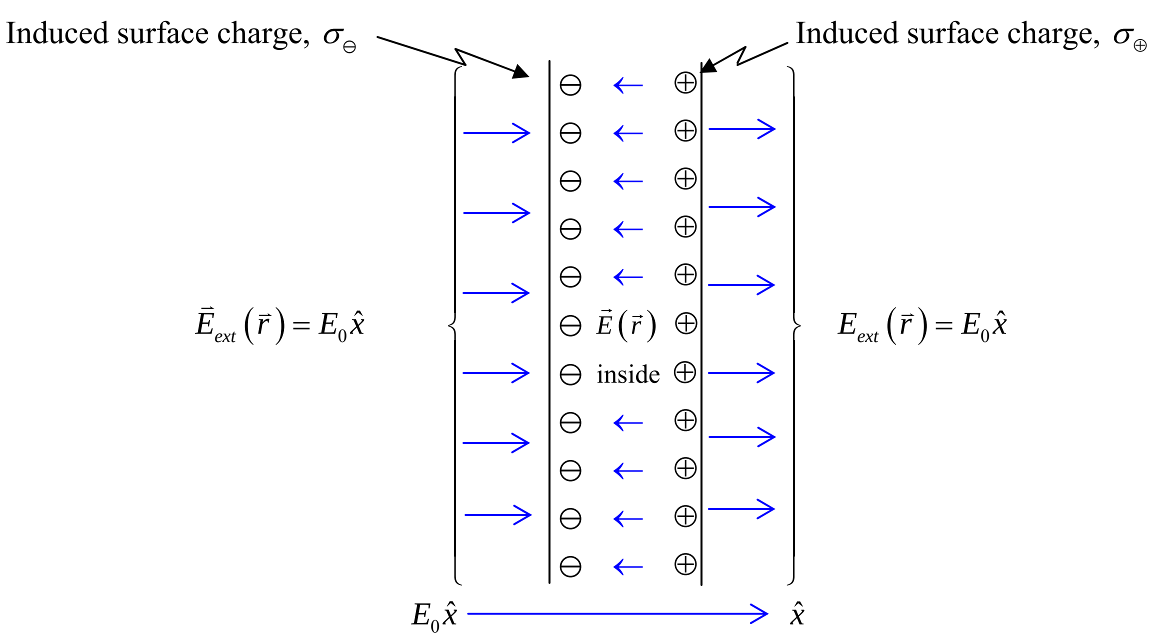
The redistributed free charges pile up on the surface(s) of the conductor in such a way as to produce $\vec{E}_{\text{inside}}(\vec{r}) = 0$. These ***induced*** charges produce an internal electric field of their own, which ***exactly*** cancels the external field, $\vec{E}_{ext}(\vec{r})$!

e.g. consider a block of metal in a uniform external electric field, $\vec{E}_{\text{ext}}(\vec{r}) = E_0\hat{x}$

$$\vec{E}_{\substack{\text{net}\\\text{inside}}}(\vec{r}) = \vec{E}_{ext}(\vec{r}) + \vec{E}_{\substack{\text{induced}\\\text{inside}}}(\vec{r}) = 0$$

$$\Rightarrow \vec{E}_{\substack{\text{induced}\\\text{inside}}}(\vec{r}) = -\vec{E}_{ext}(\vec{r}) = -E_0\hat{x}$$

Note that: $\vec{E}_{\substack{\text{net}\\\text{outside}}}(\vec{r}) = \vec{E}_{\text{ext}}(\vec{r})$ when a conductor is placed inside an initially uniform applied field, $\vec{E}_{\text{ext}}(\vec{r})$



Another important property of a conductor is that:

2) The volume free charge density, $\boxed{\rho^{free}_{inside}(\vec{r}) = 0}$ inside a conductor.

This follows from Gauss' Law (differential form):

$$\vec{\nabla} \cdot \vec{E}_{inside}(\vec{r}) = \rho^{free}_{inside}(\vec{r}) / \varepsilon_o$$

but if: $\vec{E}_{inside}(\vec{r}) = \vec{E}_{ext}(\vec{r}) + \vec{E}_{\substack{induced \\ inside}}(\vec{r}) = 0, \ \forall \vec{r}$ (i.e. everywhere inside conductor)

then: $\vec{\nabla} \cdot \vec{E}_{inside}(\vec{r}) = 0, \ \forall \vec{r}_{inside}$

$$\Rightarrow \rho^{free}_{inside}(\vec{r}) = 0, \ \forall \vec{r}_{inside}$$

3) Any ***induced*** charges on a conductor can ONLY reside on ***surface(s)*** of the conductor – as surface charge distributions, σ_{free}

(n.b. free surface charges minimize overall potential energy (i.e. maximize overall entropy of system!))

If $\exists$ induced free charges, and $\rho^{free}_{inside}(\vec{r}) = 0 \ \forall \vec{r}$, the only place(s) such induced free charges can reside is on the surface(s) of the conductor, as $\sigma_{free}(\vec{r})$.

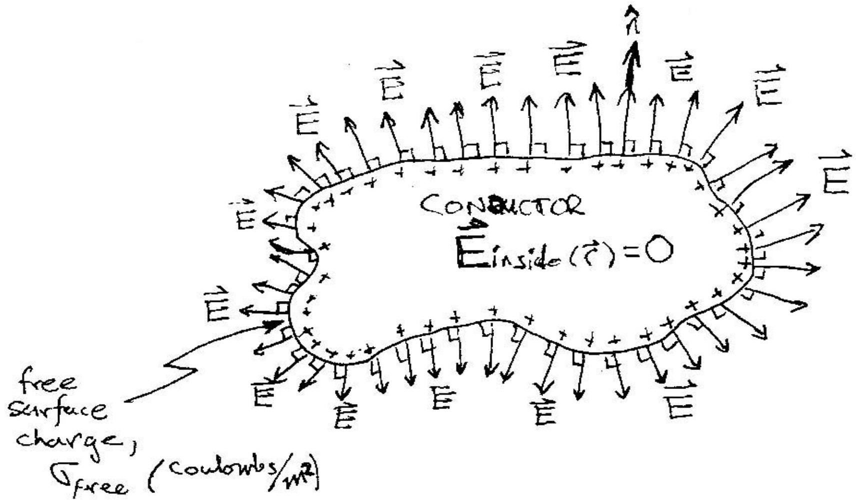
4) The entire volume & surface of a conductor is an equipotential.

If a & b are two arbitrary points, $\vec{r}_a$ & $\vec{r}_b$ on the surface of a conductor, the potential difference, $\Delta V_{ab} \equiv V_b - V_a = V(\vec{r}_b) - V(\vec{r}_a) = -\int_a^b \vec{E}(\vec{r}) \cdot d\vec{l} \overset{\text{must!!}}{=} 0 \Rightarrow V_a = V_b$

If $\Delta V_{ab} \neq 0$ then free charges *will* move!!

5) $\boxed{\text{\underline{Just} outside the surface of a conductor, } \vec{E}_{outside}\left(\vec{r} \text{ @ surface}\right) \text{ is \underline{perpendicular/normal} to the surface, i.e. } \vec{E}_{outside}\left(\vec{r} \text{ @ } surface\right) \parallel \hat{n}_{surface}}$

If $\exists$ an $\vec{E}_{\parallel}\left(\vec{r}\right) = \vec{E}_{tangential}\left(\vec{r}\right)$@ surface, $\Rightarrow$ free charge <u>*will*</u> move/flow!!!
This <u>can't</u> happen in <u>electrostatics</u>!!



<u>Example</u>: A conducting surface of total surface area, A_{cond} is charged with $\sigma_{free} = \sigma_o \left(\dfrac{Coulombs}{m^2}\right)$.
What is the electric field strength (i.e. electric field intensity) at the surface of the conductor, $\left|\vec{E}_{surface}\right|$?

$\Rightarrow$ Use (the integral form of) Gauss' Law – choose fully enclosing Gaussian surface *S* to be infinitesimally above the conducting surface (e.g. δh larger than conducting surface everywhere). Then take limit as $\delta h \to 0$:

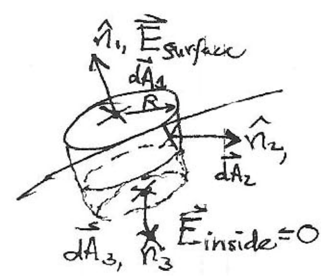
$$\oint_S \vec{E} \bullet d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma_{free} A_{cond}}{\varepsilon_0} \qquad \vec{E}_{surface}\left(\vec{r}\right) \parallel \hat{n} \text{ everywhere.}$$

$$= E_{surface} A_{cond} \qquad \boxed{\therefore \quad \left|\vec{E}_{surface}\right| = E_{surface} = \frac{\sigma_{free}}{\varepsilon_o} = \frac{\sigma_o}{\varepsilon_o} \; \left(Volts/m\right)}$$

n.b. If we had instead used e.g. a "shrunken" Gaussian <u>*pillbox*</u> on surface of conductor:

$$\oint_S \vec{E} \bullet d\vec{A} = \int_{S_1} \vec{E}_{surface} \bullet d\vec{A}_1 + \cancel{\int_{S_2} \overset{=0}{\vec{E}_{tangential}} \bullet d\vec{A}_2^{\,=0}} + \cancel{\int_{S_3} \overset{=0}{\vec{E}_{inside}} \bullet d\vec{A}_3} = \frac{Q_{encl}}{\varepsilon_o}$$

$$\pi R^2 E_{surface} = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma_o \pi R^2}{\varepsilon_o} \Rightarrow \boxed{E_{surface} = \frac{\sigma_{free}}{\varepsilon_o}\hat{n} = \frac{\sigma_o}{\varepsilon_o}\hat{n}}$$



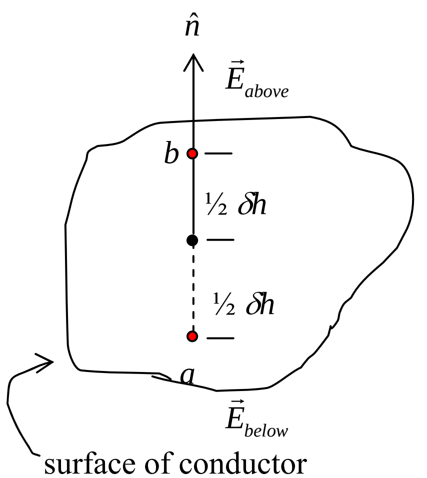
Get same answer! (We should!)

The Free Surface Charge σ_{free}
The Surface Electric field, $\vec{E}$ surface $(\vec{r})$
The Surface Potential, $V(r)$,
Electrostatic Force & Pressure
(Force per Unit Area) Acting on a Conductor

We have derived, using Gauss' Law: $\boxed{E_{surface}(\vec{r}) = \dfrac{\sigma_{free}}{\varepsilon_o} = \left|\vec{E}_{surface}(\vec{r})\right|}$

In vector notation: $\boxed{\vec{E}_{surface}(\vec{r}) = \dfrac{\sigma_{free}}{\varepsilon_o}\hat{n}}$ Where $\hat{n}$ is the *outward* pointing unit normal vector (outward = by convention).

From Griffiths Eqn's 2.34-2.37, p. 89-90:



$$\left[V_b^{above}(\vec{r}_b) - V_a^{below}(\vec{r}_a)\right]\Big|_{\lim\limits_{\delta h\to 0}} = -\int_a^b \vec{E}\bullet d\vec{\ell} = \int_a^b \vec{\nabla}V\bullet d\vec{\ell} = 0$$

(surface of conductor is an equipotential)

Gave: $\vec{\nabla}V^{above}(\vec{r}) - \cancel{\vec{\nabla}V^{below}(\vec{r})}^{=0} = -\dfrac{\sigma_{free}}{\varepsilon_0}\hat{n}$

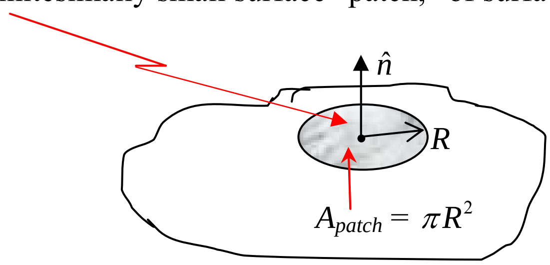
OR: $$\left.\frac{\partial V(\vec{r})}{\partial n}\right|_{surface} \equiv \vec{\nabla}V(\vec{r})\bullet\hat{n}\Big|_{surface} = -\frac{\sigma_{free}}{\varepsilon_o}$$

OR: $$\sigma_{free} = -\varepsilon_o \left.\frac{\partial V(\vec{r})}{\partial n}\right|_{surface}$$

If $\vec{E}_{surface}(\vec{r})$ or $V_{surface}(\vec{r})$ is known, then σ_{free} can be obtained (and/or vice-versa).

FORCE & PRESSURE ON A CONDUCTOR

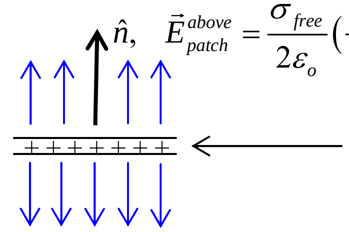
Consider an arbitrarily-shaped conductor, with $\sigma_{free}\left({coulombs}/{m^2}\right)$ free charge residing on its surface. Infinitesimally small surface "patch," of surface area $A_{patch} = \pi R^2$.



Now A_{patch} also has surface charge σ_{free} on it. The total charge of this patch is $Q_{patch} = \pi R^2 \sigma_{free}$.

<u>Edge-on view of infinitesimally small patch</u>:

$$\hat{n},\quad \vec{E}^{above}_{patch} = \frac{\sigma_{free}}{2\varepsilon_o}(+\hat{n}) = +\frac{\sigma_{free}}{2\varepsilon_o}\hat{n}$$



$$\sigma_{free} \text{ on } A_{patch} \; \left(=\pi R^2\right)$$

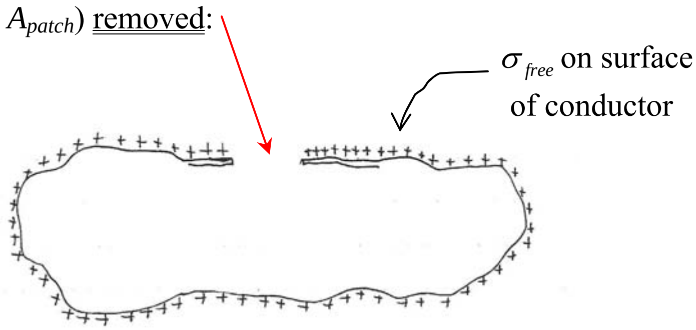
$$\vec{E}^{below}_{patch} = \frac{\sigma_{free}}{2\varepsilon_o}(-\hat{n}) = -\frac{\sigma_{free}}{2\varepsilon_o}\hat{n}$$

We have discussed before (Griffiths Ex. 2.4, p. 73-74 and/or P435 Lect. Notes 2 p. 9-12) that a surface charge σ has a (<u>net</u>) $\vec{E}$-field $\perp$ to surface on which σ resides, both above and below the surface.

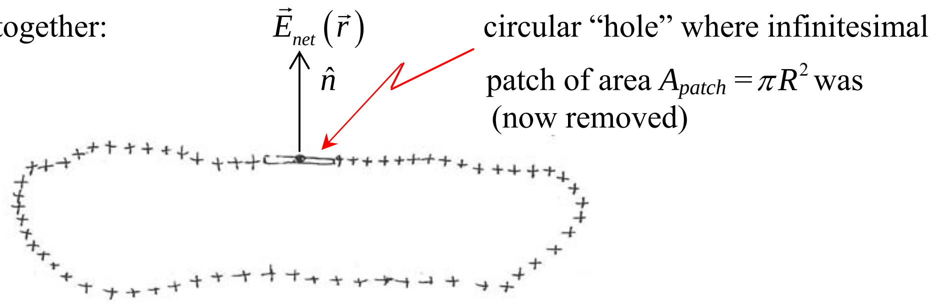
The transverse/tangential $\vec{E}$-field components (on a flat surface, from symmetry) were shown to cancel. (n.b. an <u>infinitesimally</u> small surface patch *is* <u>FLAT</u>.)

But we also said that $\vec{E}_{inside}(\vec{r}) = 0$ <u>inside a conductor</u>! How do we reconcile these two statements?

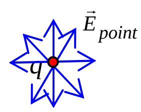
Consider an edge-on view of the (arbitrarily shaped conductor, but with the (infinitesimally) small surface patch (of area A_{patch}) <u>removed</u>:



Now consider just the electrostatic surface charge itself – i.e. mentally "erase" the presence of the conductor altogether:



The surface charge distribution is a (very!) special configuration that is an *equipotential* surface – but it is still a collection of individual point charges (@ the microscopic level). Each individual point charge q has its own radial electric field, $\vec{E}_{\text{point}}(\vec{r})$



We ask: what is the net electric field $\vec{E}_{\text{net}}(\vec{r})$ on the conductor's surface? (e.g. at the location of the "hole" where the patch of infinitesimal area $A_{patch} = \pi R^2$ was (which is now removed)).

It's hard to *rigorously* prove the following, for an *arbitrarily*-shaped charged conductor, but it can be rigorously proved (analytically) for symmetrically-shaped conductors – e.g. a charged sphere, with an infinitesimally small spherical cap removed, e.g. at its north pole. (see e.g. J.D. Jackson's book *Electrodynamics* for this proof…)

In general, for arbitrarily-shaped conductors, because their charged surfaces are equipotential surfaces, the net electric field, $\vec{E}_{net}(\vec{r}_{patch})$ on the conductor's surface, right at the "hole" location of the (missing/removed) patch, associated with the free charge everywhere else on the conductor's surface, is:

$$\vec{E}_{net}(\vec{r}_{patch}) = +\frac{\sigma_{free}}{2\varepsilon_o}\hat{n} \qquad (!!!)$$

infinitesimally above/below surface, @ the patch "hole" location:

$$\vec{E}_{net}(\vec{r}_{patch}) = \vec{E}_{net}^{above}(\vec{r}_{patch}) = \vec{E}_{net}^{below}(\vec{r}_{patch}) = +\frac{\sigma_{free}}{2\varepsilon_0}\hat{n}$$

Thus, we see that:

$$\vec{E}_{ToT}^{above\ outside}(\vec{r}_{patch}) = \vec{E}_{patch}^{above}(\vec{r}_{patch}) + \vec{E}_{net}^{above}(\vec{r}_{patch})$$

$$\vec{E}_{ToT}^{above}(\vec{r}_p) = \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} + \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} = \frac{\sigma_{free}}{\varepsilon_o}\hat{n}$$

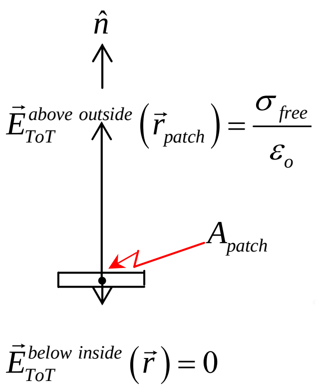
And we also see that:

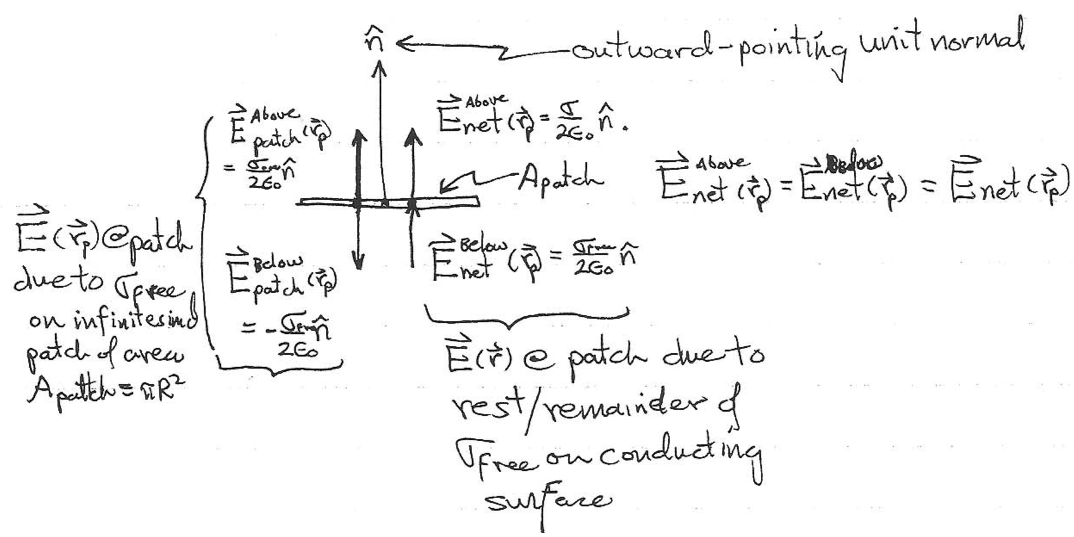
$$\vec{E}_{ToT}^{below\ inside}(\vec{r}_{patch}) = \vec{E}_{patch}^{below}(\vec{r}_{patch}) + \vec{E}_{net}^{below}(\vec{r}_{patch})$$

$$\vec{E}_{ToT}^{below}(\vec{r}_{patch}) = -\frac{\sigma_{free}}{2\varepsilon_o}\hat{n} + \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} = 0$$

Thus, the net electric field just above the surface of a conductor arises from *two* equal contributions – the $\vec{E}_{patch}$ from σ_{free} *on* the infinitesimal patch of area $A_{patch} = \pi R^2$ itself, *and* the net contribution from the free charge on the remainder of the conducting surface! Likewise, the internal $\vec{E}$ field inside the conductor = 0, because these two contributions cancel each other!!!

Thus, VECTORIALLY, in the region of the infinitesimally small patch of area $A_{patch} = \pi R$ on/at the surface of the conductor, we have:





$$\boxed{\begin{gathered}\vec{E}_{ToT}^{above\ outside}(\vec{r}_{patch}) = \vec{E}_{patch}^{above}(\vec{r}_{patch}) + \vec{E}_{net}^{above}(\vec{r}_{patch}) \\ \vec{E}_{ToT}^{above}(\vec{r}_p) = \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} + \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} = \frac{\sigma_{free}}{\varepsilon_o}\hat{n}\end{gathered}}$$

$$\boxed{\begin{gathered}\vec{E}_{ToT}^{below\ inside}(\vec{r}_{patch}) = \vec{E}_{patch}^{below}(\vec{r}_{patch}) + \vec{E}_{net}^{below}(\vec{r}_{patch}) \\ \vec{E}_{ToT}^{below}(\vec{r}_{patch}) = -\frac{\sigma_{free}}{2\varepsilon_o}\hat{n} + \frac{\sigma_{free}}{2\varepsilon_o}\hat{n} = 0\end{gathered}}$$

We are now in a position to ask: what is the net/total force, $\vec{F}_{patch}^{total}(\vec{r})$ acting on the infinitesimally small patch, of area $A_{patch} = \pi R^2$?

1) There can be no contribution(s) to the net/total force, $\vec{F}_{patch}^{total}(\vec{r})$ acting on the patch due to the free surface charge, σ_{free} (or associated $\vec{E}-fields$ $\vec{E}_{patch}^{above}(\vec{r}) + \vec{E}_{patch}^{below}(\vec{r})$) acting on the patch itself (analogous to trying to lift yourself up by pulling on your shoes!)

$$\vec{F}_{patch}^{total}(\vec{r}_{patch}) = Q_{patch} * \vec{E}_{at\ patch}(\vec{r}_{patch}) = Q_{patch} * \vec{E}_{net}(\vec{r}_{patch})$$

← net $\vec{E}$-field @ patch from σ_{free} elsewhere on conducting surface

$$\vec{F}_{patch}^{total}(\vec{r}_{patch}) = \sigma_{free}\, A_{patch} * \vec{E}_{net}(\vec{r}_{patch}) = \sigma_{free}\, A_{patch} * \left(\frac{\sigma_{free}}{2\varepsilon_o}\hat{n}\right)$$

$$\boxed{\vec{F}_{patch}^{total}\left(\vec{r}_{patch}\right)=\frac{\sigma_{free}^{2}\left(\vec{r}_{patch}\right)}{2\varepsilon_o}A_{patch}\hat{n}}$$

← NOTE: this is a force which points in the <u>outward</u> normal ($\perp$) direction!

If we sum up the infinitesimal force contributions from *<u>all</u>* the "patches" associated with the *<u>entirety</u>* of the conducting surface, the <u>magnitude</u> of the force is:

$$\left|\vec{F}_{conductor}\left(\vec{r}_{surface}\right)\right|=\left|\sum_{i=1}^{N}\vec{F}_{patch}^{total}\left(\vec{r}_{patch}\right)\right|=\oint_{conductor}\frac{\sigma_{free}^{2}\left(\vec{r}_{patch}\right)}{2\varepsilon_o}dA=\frac{\sigma_{free}^{2}}{2\varepsilon_o}A_{conductor}$$

n.b. Assumption made here is that $\sigma_{free}\left(\vec{r}\right)=\text{constant}$

Now, pressure ≡ force/unit area, i.e. $P = F/A$

Thus:

$$\boxed{\text{Then, electrostatic pressure, } P_{@surface}\left(\vec{r}_{surface}\right)=\frac{1}{2}\varepsilon_o\left(\frac{\sigma_{free}^{2}\left(\vec{r}_{surface}\right)}{\varepsilon_o^{2}}\right)=\frac{\varepsilon_o}{2}E_{@surface}^{2}\left(\vec{r}_{surface}\right)}$$

Since:

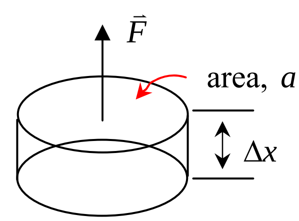
$$\boxed{\vec{E}_{surface}\left(\vec{r}_{patch}\right)=2\vec{E}_{net}\left(\vec{r}_{patch}\right)=2\frac{\sigma_{free}\left(\vec{r}_{patch}\right)}{2\varepsilon_o}\hat{n}=\left(\frac{\sigma_{free}\left(\vec{r}_{patch}\right)}{\varepsilon_o}\right)\hat{n}}$$

$$\boxed{\text{Then, electrostatic pressure, } P_{@surface}\left(\vec{r}_{surface}\right)=\frac{1}{2}\varepsilon_o\left(\frac{\sigma_{free}^{2}\left(\vec{r}_{surface}\right)}{\varepsilon_o^{2}}\right)=\frac{\varepsilon_o}{2}E_{@surface}^{2}\left(\vec{r}_{surface}\right)}$$

<u>n.b.</u> If electrostatic forces are able/capable of performing mechanical work W, then must do so at the expense of electrostatic energy (recall that total energy <u>must</u> <u>always</u> be conserved!)

n.b. metals are <u>elastic</u> solids!!

⇒ Suppose a small area, $\underline{\underline{a}}$ of the conductor is pulled into the electric field region by an infinitesimal distance, $\underline{\underline{\Delta x}}$. Then work done by field on area, $\underline{\underline{a}}$ is:



$$\Delta W\left(\vec{r}\right)=\underbrace{\vec{F}\left(\vec{r}\right)\bullet\Delta\vec{x}}_{\vec{F}\,\parallel\,\Delta\vec{x}}=\left(\frac{F\left(\vec{r}\right)}{a}\right)\left(a\Delta x\right)=P_{@surface}\left(\vec{r}\right)*\underbrace{a\Delta x}_{=\text{ volume, }\Delta V}=P_{@surface}\left(\vec{r}\right)\Delta V$$

$$\boxed{\therefore\ \Delta W\left(\vec{r}_{surface}\right)=P_{@surface}\left(\vec{r}_{surface}\right)\Delta V}$$

$$\boxed{P_{@surface}\left(\vec{r}_{surface}\right)=\frac{\Delta W\left(\vec{r}_{surface}\right)}{\Delta V}=Energy\ \ Density,\ u_E\left(\vec{r}_{surface}\right)}$$

Electrostatic Pressure, P (@ surface) = Volume Energy Density of Electrostatic Field, u_E (@surface)

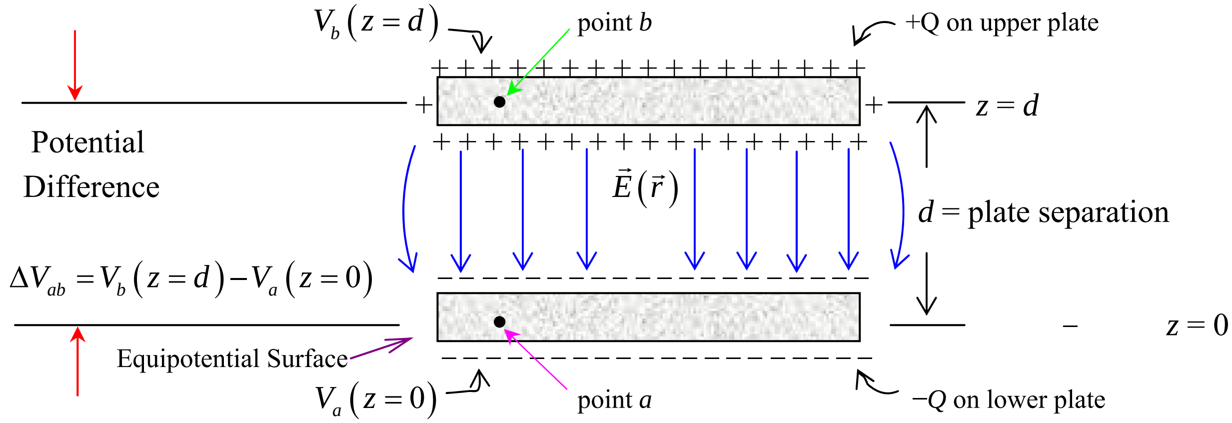
$$P_{@surface}\left(\vec{r}\ @surface\right)=U_{E\,@surface}\left(\vec{r}\ @surface\right)=\frac{1}{2}\varepsilon_o E_{@surface}^{2}\left(\vec{r}\ @surface\right)\quad\left(Joules/m^3\right)$$

n.b. Electrostatic field exerts a *<u>negative</u>* pressure on conductor – i.e. electrostatic force on the conductor <u>pulls</u> conductor *<u>into</u>* electrostatic field !!!

CAPACITORS

A capacitor is a device that enables the storage (long and/or short term) of electric charge, Q. Since there are electric fields associated with electric charge, a capacitor is also a device that enables the storage (long and/or short term) of electrical energy.

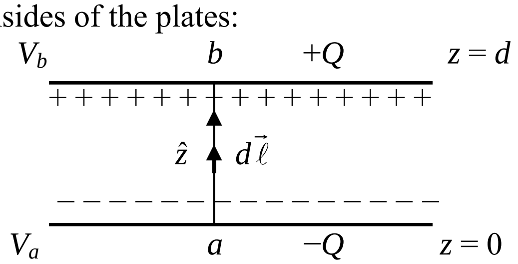
One can make a simple capacitor using e.g. two parallel, very thin, conducting plates, separated by a distance d and initially <u>uncharged</u>. Then, we put charge $+Q$ e.g. on the upper plate and put charge $-Q$ on the lower plate. A potential difference, $\Delta V_{ab} = V_b(z=d) - V_a(z=0)$ equipotential surface now exists between the two plates, as shown in the figure below:



A static electric field $\vec{E}(\vec{r})$ exists between the parallel plates (n.b. If the length × width ($L \times W$) dimensions are both large in comparison to the plate separation, d i.e. $L \gg d$ and $W \gg d$ then the electric field $\vec{E}(\vec{r})$ between the parallel plates will be nearly uniform inside the gap region – i.e. $\vec{E}(\vec{r}) \simeq E_0(-\hat{z}) = -E_0\hat{z}$ $\leftarrow$ constant

We know that $\Delta V_{ab} = V_b(z=d) - V_a(z=0) = -\int_a^b \vec{E}(\vec{r}) \cdot d\vec{\ell}$ and that $\int_a^b \vec{E}(\vec{r}) \cdot d\vec{\ell}$ is <u>independent of the path taken from</u> $a \to b$.

So let's simply take a path straight up along the $\hat{z}$ axis – i.e. one that just barely touches the insides of the plates:



we know:

$$\vec{E} = E_0(-\hat{z}) = -E_0\hat{z}$$

$$d\vec{\ell} = d\ell(+\hat{z}) = +d\ell\hat{z}$$

E = constant, $\neq$ *fcn* of (z)

$$\Delta V_{Ab} = V_b(z=d) - V_a(z=0) = -\int_a^b \vec{E} \cdot d\vec{\ell} = +Ed, \text{ for } \overbrace{L, W}^{\text{both}} \gg d$$

$$E = \frac{\Delta V_{ab}}{d}\left(\text{Volts}/m\right) = \text{ constant for } \overbrace{L, W}^{\text{both}} \gg d.$$

Then one can also see that if E = constant between conducting plates $\left(\text{for } \overbrace{L,W}^{\text{both}} \gg d\right)$ then $\Delta V(z) = Ez$, where z is measured from the bottom plate (z = 0).

From basic E&M principles, we know that $\vec{E}$ is <u>linearly</u> proportional to charge, Q $\left(cf: \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r^2}\right)\hat{r} \text{ for a point charge}\right)$.

Similarly, the potential, V is also <u>linearly</u> proportional to the charge, Q $\left(cf: \ \Delta V = -\int_c \vec{E} \cdot d\vec{\ell}\right)$

The constant of proportionality is known as the <u>capacitance</u>, C of the system, i.e. we <u>define</u> capacitance as the ratio of charge, Q to potential difference, ΔV associated with this system:

$$\boxed{C \equiv \frac{Q}{\Delta V}}$$

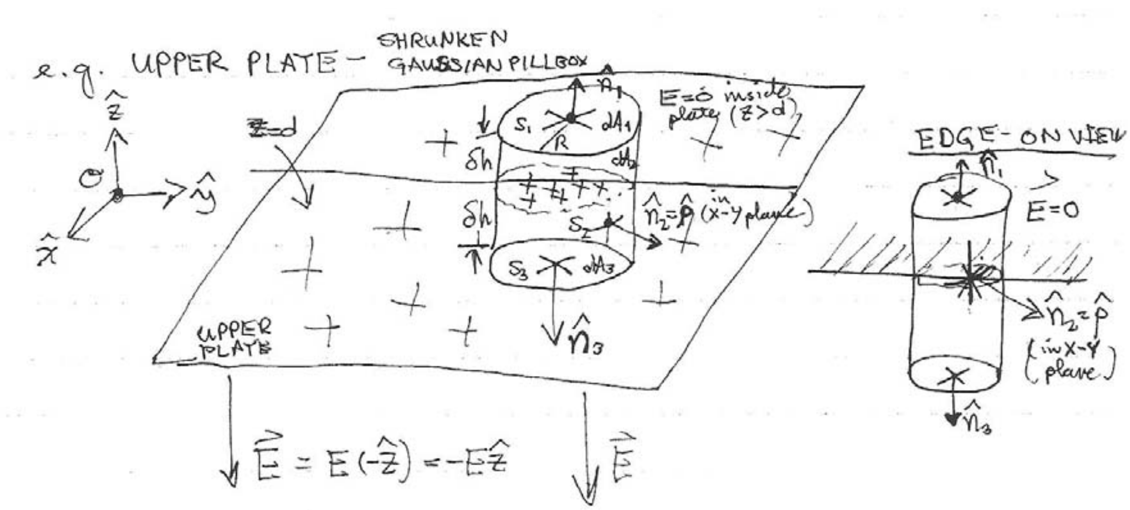
S.I. units of capacitance is <u>FARADS</u>, F

$$1 \text{ FARAD} = \frac{1 \text{ Coulomb}}{\text{per volt pot'l difference}}$$

Thus, for the case of the parallel plate capacitor, for $\overbrace{L,W}^{\text{both}} \gg d$ we see that:

capacitance of parallel plate capacitor $(L,W \gg d)$ $\quad C \equiv \frac{Q}{\Delta V_{AB}} = \frac{Q}{Ed}$

Using Gauss' Law (integral form) on (any) one of the parallel plates - e.g. the upper plate – use shrunken Gaussian pillbox as shown in figure below:



Then:

$$\oint_s \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o} \qquad d\vec{A}_1 = dA_1\hat{z}, \quad \vec{E} \parallel \hat{z}$$

$$= \int_{S_1} \vec{E} \cdot d\vec{A}_1 + \underbrace{\cancel{\int_{S_2} \vec{E} \cdot d\vec{A}_2}^{=0}}_{\substack{\vec{E} \perp d\vec{A}_2 \\ everywhere \\ on\ S_2}} + \int_{S_3} \vec{E} \cdot d\vec{A}_3 \qquad d\vec{A}_2 = dA_2\hat{\rho} \ \left(\text{in } x\text{-}y \text{ plane}\right)$$

$$d\vec{A}_3 = -dA_3\hat{z}$$

$$= \underbrace{\cancel{(0\hat{z}) \cdot (A_{disk}\hat{z})}^{=0}}_{E=0 \text{ inside upper plate } (z > d)} + (-E\hat{z}) \cdot (-A_{disk}\hat{z}) \qquad A_{S_1} = A_{S_3} = A_{disk} = \pi R^2$$

$$Q_{encl} = +\sigma A_{disk}$$

$$= +EA_{disk} = \frac{Q_{encl}}{\varepsilon_o} = +\frac{\sigma A_{disk}}{\varepsilon_o}$$

$$\therefore \quad E = \frac{\sigma}{\varepsilon_o}$$

$$\therefore \quad C = \frac{Q}{\Delta V_{ab}} = \frac{Q}{Ed} = \varepsilon_o \frac{Q}{\sigma d} \quad \text{but} \quad Q = \sigma A_{plate}$$

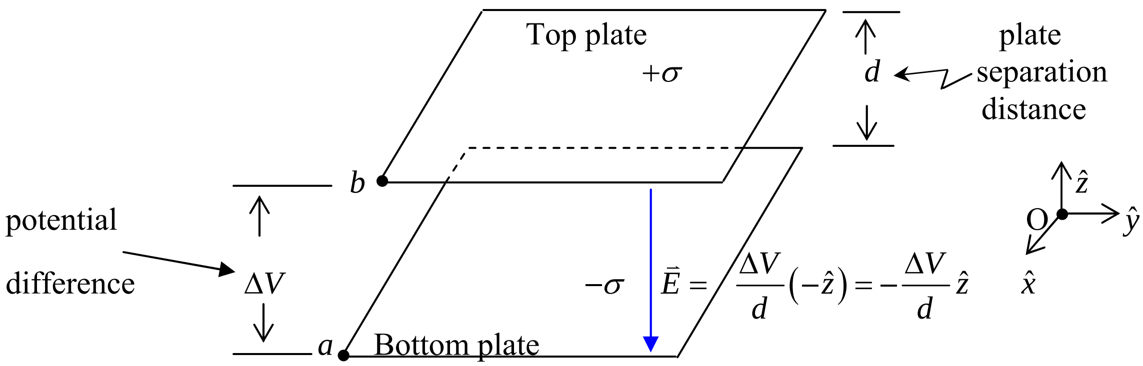
$$\boxed{\therefore \quad C = \frac{\varepsilon_o A_{plate}}{d} \quad \text{for parallel plate capacitor, with } \overbrace{L, W}^{\text{both}} \gg d}$$

$\Rightarrow$ Note that capacitance, C depends only on ε_o (electric permittivity of free space, 8.85×10^{-12} Farads/m) and (purely) geometrical factors (A_{plate} & d).

<u>Forces on a Parallel Plate Capacitor</u>

Suppose two parallel conducting plates, each of area $A = L \times W$ are separated by small distance $d \ll L, W$.

The potential difference between plates is initially ΔV. The <u>top</u> plate has surface charge density $+\sigma$, the bottom plate has surface charge density $-\sigma$.



$$\Delta V = \Delta V_{ab} = V_b^{top} - V_a^{bottom}$$

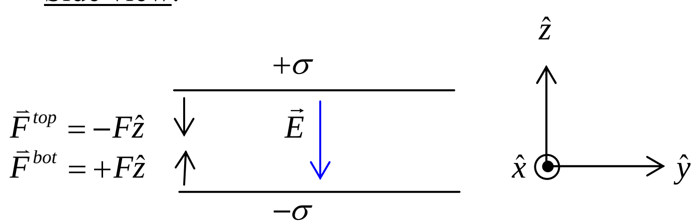
Neglecting fringe fields effects @ edges of parallel plate capacitor (valid approximation if $d \ll L, W$):

$$\text{Pressure} = \frac{\text{force}}{\text{per unit area}} = \text{field energy density } u_E \left(\frac{Joules}{m^3}\right)$$

$$P = \frac{F}{A} = u_E = \frac{1}{2}\varepsilon_o E^2 = \frac{1}{2}\varepsilon_o \left(\frac{\Delta V}{d}\right)^2$$

Attractive force on (each) plate: $F = P \cdot A = u_E \cdot A = \frac{1}{2}\varepsilon_o E^2 A = \frac{1}{2}\varepsilon_o \left(\frac{\Delta V}{d}\right)^2 A$ (Newtons)

<u>Side view</u>:



$$\vec{F}^{ToT}\begin{pmatrix}\parallel\text{-plate}\\ \text{capacitor}\end{pmatrix} = \vec{F}^{TOP} + \vec{F}^{BOT} = -F\hat{z} + F\hat{z} = 0$$

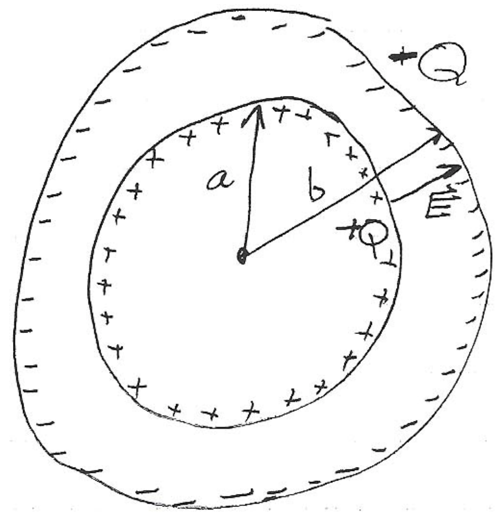
No <u>*net*</u> force acts on parallel plate capacitor − Newton's 1st Law: for every action, $\exists$ equal and opposite reaction.

Griffiths EXAMPLE 2.11

Find the capacitance, C of two concentric spherical metal shells, with radii a & b, $b > a$.

Place $+Q$ on inner sphere and place $-Q$ on outer sphere. Use Gauss' Law and show that

for $a \le r \le b$: $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r^2}\right)\hat{r}$



Then: $$\begin{cases} \Delta V_{ab} = -\int_a^b \vec{E}\bullet d\vec{\ell} \\ \quad = -\int_a^b \vec{E}\bullet d\vec{r} \\ \quad = -\frac{Q}{4\pi\varepsilon_o}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a}-\frac{1}{b}\right) \quad b > a \end{cases}$$

Then: $$\boxed{C_{\substack{concentric\\spheres}} \equiv \frac{Q}{\Delta V_{ab}} = 4\pi\varepsilon_o \frac{1}{\left(\frac{1}{a}-\frac{1}{b}\right)} = 4\pi\varepsilon_o \frac{ab}{(b-a)}}$$

Again, note that C depends only on ε_o and (purely) geometrical factors (radii a & b)

How much work, W is done in charging up a capacitor − e.q. (if it is initially uncharged)?

Charging an initially uncharged capacitor means individually removing electrons from the upper plate of the parallel-plate capacitor (inner sphere of concentric spherical capacitor) and transporting them to the lower plate of the parallel-plate capacitor (outer sphere of concentric spherical capacitor), respectively.

If $\Delta V_{ab} = \frac{W}{Q}$ (*Griffiths* 2.38, p.9) then $W = Q\Delta V_{ab}$

The infinitesimal amount of work dW needed to transport an infinitesimal amount of charge, dQ is $\underline{\underline{dW = \Delta V dQ}}$

But: $\Delta V = \frac{Q}{C}$ Therefore: $dW = \left(\frac{Q}{C}\right)dQ$

The total work done in charging a capacitor from $Q = 0$ to $Q = Q_{ToT}$ is:

$$W_{ToT} = \int dW = \int_{Q=0}^{Q=Q_{ToT}} \left(\frac{Q}{C}\right)dQ = \underbrace{\frac{1}{C}}_{\text{constant}}\int_{Q=0}^{Q=Q_{ToT}} QdQ = \frac{1}{2}\left(\frac{Q_{ToT}^2}{C}\right)$$

$$\boxed{\begin{array}{l} \text{Thus: } W_{ToT} = \frac{1}{2}\left(\frac{Q_{ToT}^2}{C}\right) \quad \text{but: } Q_{ToT} = C\Delta V \\ \therefore\ W_{ToT} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\left(\frac{Q^2}{C}\right) = \frac{1}{2}Q\Delta V \end{array}}$$

