LECTURE NOTES 4

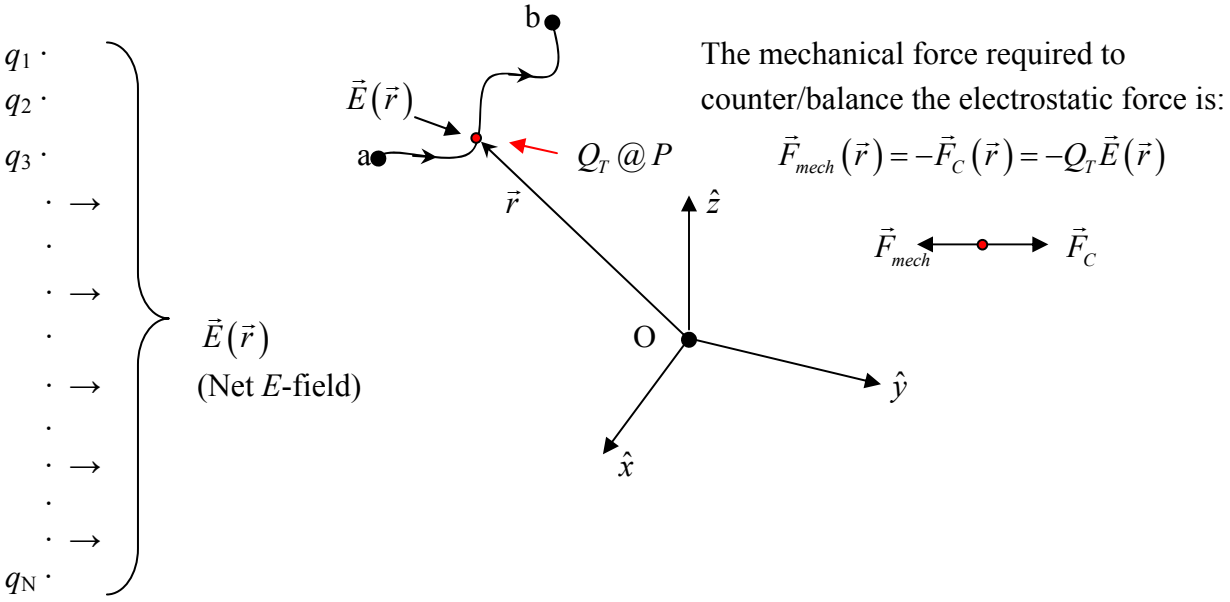
Work & Electrostatic Energy:

Consider the following situation:

A stationary / fixed configuration of source charges is used to generate a (net) electric field $\vec{E}(\vec{r})$.

A test charge Q_T is moved from point *a* to point *b* in this electric field $\vec{E}(\vec{r})$. How much mechanical work *W* is done on the test charge Q_T in moving it (slowly) from point *a* to point *b*?

At any point $\vec{r}$ along the path $a \rightarrow b$, the electrostatic force acting on Q_T is: $\vec{F}_C(\vec{r}) = Q_T \vec{E}(\vec{r})$



The mechanical force required to counter/balance the electrostatic force is:

$$\vec{F}_{mech}(\vec{r}) = -\vec{F}_C(\vec{r}) = -Q_T \vec{E}(\vec{r})$$

The mechanical work done on the test charge Q_T along the path $a \rightarrow b$ is:

$$\boxed{W = \int_a^b \vec{F}_{mech}(\vec{r}) \bullet \overline{dl} = -Q_T \int_a^b \vec{E}(\vec{r}) \bullet \overline{dl}}$$

But: $\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$ OR: $\Delta V_{ab} \equiv V(b) - V(a) = -\int_a^b \vec{E}(\vec{r}) \bullet \overline{dl}$

$$\boxed{\therefore\ W = Q_T \left[V(b) - V(a) \right] = Q_T \Delta V_{ab}}$$

The mechanical work done on the test charge Q_T along the path $a \rightarrow b$:

$$\vec{F}_{mech}(\vec{r}) = -\vec{F}_C(\vec{r}) = -Q_T \vec{E}(\vec{r})$$

$$W = Q_T \Delta V_{ab} = Q_T \left[V(b) - V(a) \right] = \int_a^b \vec{F}_{mech}(\vec{r}) \bullet \overline{dl} = -\int_a^b \vec{F}_C(\vec{r}) \bullet \overline{dl} = -Q_T \int_a^b \vec{E}(\vec{r}) \bullet \overline{dl}$$

Now, if point *a* is the reference point $\vec{r}_a = \infty$ where $V(\vec{r}_a) = V(\infty) = 0$ and point $\vec{r}_b = \vec{r}$

Then: $W = Q_T\left[V(\vec{r}) - \cancel{V(\infty)}\right] = Q_T V(\vec{r})$

i.e. the test charge Q_T is (slowly) mechanically moved from the point $\vec{r}_a = \infty$ to the point $\vec{r}_b = \vec{r}$. Then $W = Q_T V(\vec{r})$ is the mechanical work done on the charge Q_T in moving it along the path $a \rightarrow b$ from $\vec{r}_a = \infty$ to $\vec{r}_b = \vec{r}$.

Thus, now we also see that W = work done on charge Q_T is also equal to the POTENTIAL ENERGY, $P.E. = W = Q_T V(\vec{r})$ $P.E.$ is linearly proportional to the potential $V(r)$ (when referenced to $V(r_a = \infty)$).

POTENTIAL ENERGY = amount of work W it takes to create the system (Joules).

POTENTIAL *DIFFERENCE* = work W it takes to create the system *per unit charge* (Joules/Coulomb = Volts).

UNITS OF WORK, W = UNITS OF POTENTIAL ENERGY, $P.E.$ = JOULES (*SI* UNITS)

$W = P.E. = Q_T V(\vec{r})$ = Coulomb-Volts = Joules

i.e. 1 Coulomb x 1 Volt = 1 Joule

Fundamental unit of electric charge: $1e = 1.602\times10^{-19}$ Coulombs = $1.602\times10^{-19}\ C$

$\therefore$ 1 $\underbrace{\text{electron volt}}_{=eV}$ = 1.602×10^{-19} Joules ← energy conversion factor for eV $\Leftrightarrow$ Joules

Example 1:

An electron is initially infinitely far away from a proton, and is initially at rest (as is the proton). Assume the proton is fixed / rigidly attached to the head of a pin at the origin $O(x, y, z) = (0, 0, 0)$.

The electron is "released" at zero velocity and is attracted to the proton. What is its kinetic energy and speed when it is a distance of 1 meter from the proton?

Use Energy Conservation:

$$E_{TOT}^{INIT} = E_{TOT}^{FINAL} = 0$$

$$E_{TOT}^{INIT} = KE_e^{INIT} + PE_e^{INIT} = 0 \qquad E_{TOT}^{FINAL} = KE_e^{FINAL} + PE_e^{FINAL} = 0$$

$$KE_e^{INIT} = \tfrac{1}{2} m_e v_{init}^2 = 0 \qquad KE_e^{FINAL} = \tfrac{1}{2} m_e v_{final}^2 \geq 0$$

$$v_{init} = 0 \qquad v_{final} = ??$$

$Q_e = -e$ $\quad PE_e^{INIT} = Q_e V_p(\vec{r}_a = \infty) = -eV_p(\vec{r}_a = \infty) = -\frac{1}{4\pi\varepsilon_o}\frac{e^2}{\infty} = 0$ $\quad PE_e^{FINAL} = Q_e V_p(\vec{r}_b = 1m) = -eV_p(\vec{r}_b = 1m) = -\frac{1}{4\pi\varepsilon_o}\left(\frac{e^2}{1}\right) < 0$ where $V_p(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{Q_p}{r}$, with $Q_p = +e$.

So: $E_{TOT}^{INIT} = 0 = E_{TOT}^{FINAL} = KE_e^{FINAL} + PE_e^{FINAL} = 0$

Or: $KE_e^{FINAL} = -PE_e^{FINAL} = +\frac{1}{4\pi\varepsilon_o}\left(\frac{e^2}{1}\right)$

$r = 1$ meter

$e = 1.602\text{x}10^{-19}$ coulomb

$\varepsilon_o = 8.85\text{x}10^{-12}$ Farads/meter

$$\therefore KE_e^{FINAL} = \frac{1}{4\pi * 8.85\times10^{-12}}\frac{\left(1.6\times10^{-19}\right)^2}{1} \text{ Joules}$$

$KE_e^{FINAL} = 2.3\times10^{-28}$ Joules

But: $\boxed{1eV = 1.6\times10^{-19} \text{ Joules}}$ OR: 1 Joule $= \frac{1eV}{1.6\times10^{-19}}$

$\boxed{1 \text{ Joule} = 6.242\times10^{18} \text{ eV}}$

$\therefore \underline{\underline{KE_e^{FINAL} = 1.437\times10^{-9} eV}}$

Now $KE_e^{FINAL} = \frac{1}{2}m_e v_e^{2FINAL} = 2.3\times10^{-28} Joules$

But $m_e = 9.11\times10^{-31}\ kg$

1 Joule = 1 Newton-meter

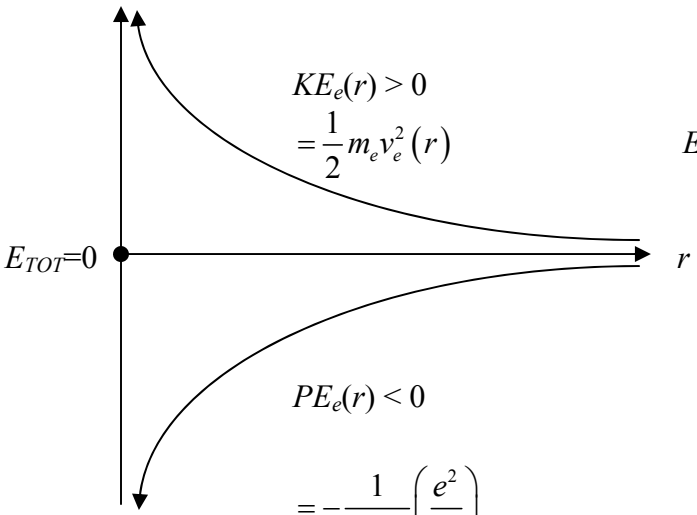
$$W = \int_a^b \vec{F}_{mech} \bullet \vec{dl}$$

$$v_e^{FINAL} = \sqrt{\frac{2KE_e^{FINAL}}{m_e}} = \sqrt{\frac{2 * 2.3\times10^{-28} Joules}{9.11\times10^{-31} kg}} = \sqrt{504.94\left(m/s\right)^2} \simeq 22.5\ m/s$$

$v_e^{FINAL} \simeq 22.5\ m/sec$ at $r = 1m$

NOTE: $v_e^{FINAL}(r = 1m) \ll c$ (speed of light)

(22.5 m/s) $\quad (c = 3\times10^8$ m/s)



$$E_{TOT}(r) = KE_e(r) + PE_e(r) = 0$$

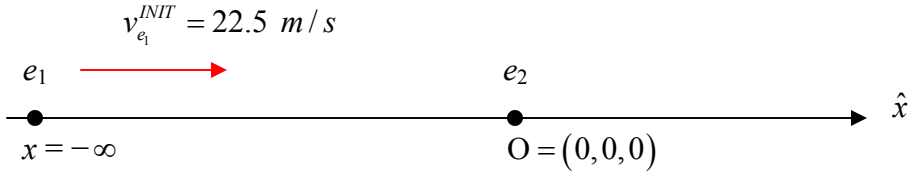
i.e. $KE_e(r) = -PE_e(r)$

$$\frac{1}{2} m_e v_e^2(r) = +\frac{1}{4\pi\varepsilon_o}\left(\frac{e^2}{r}\right)$$

$$\boxed{v_e(r) = \sqrt{\frac{1}{2\pi\varepsilon_o}\left(\frac{e^2}{m_e r}\right)}}$$

Example 2:

An electron is located at $x = -\infty$ and has an initial velocity $\vec{v} = 22.5\ m/s\ \hat{x}$ and is initially infinitely far away from another electron, which is rigidly fixed at the origin $O(x,y,z) = (0,0,0)$ at rest.



What is the distance of closest approach of e_1 to e_2? Why is there a *minimum* distance?

We know that $E_{e_1}^{TOT\,INIT} = E_{e_1}^{TOT\,FINAL}$ (Energy is conserved in this process)

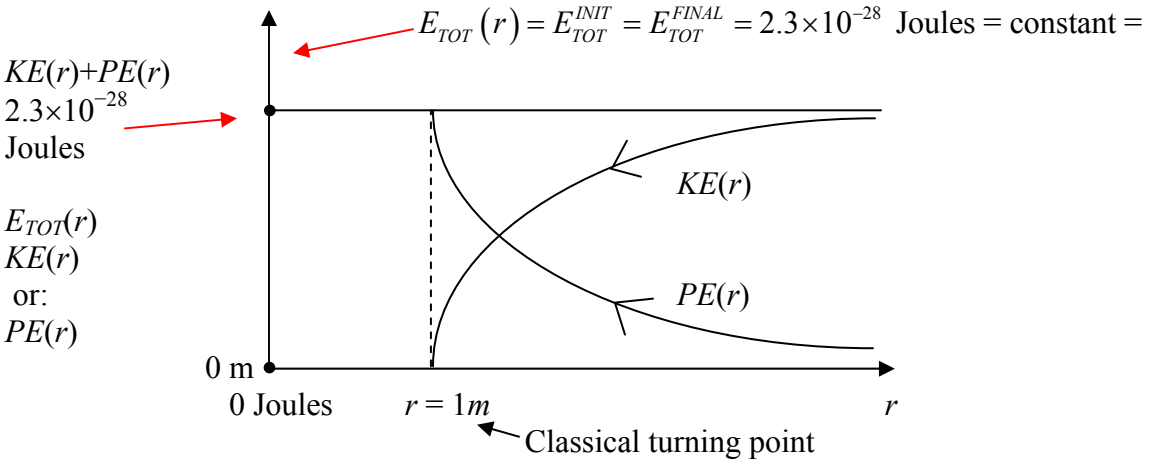
$$E_{e_1}^{TOT\,INIT} = KE_{e_1}^{INIT} + PE_{e_1}^{INIT} \qquad E_{e_1}^{TOT\,FINAL} = KE_{e_1}^{FINAL} + PE_{e_1}^{FINAL}$$

$$= \frac{1}{2} m_e v_{INIT}^2 + 0 = 2.3\times10^{-28}\ Joules \qquad = 0 + \frac{1}{4\pi\varepsilon_o}\frac{e^2}{r^2}$$

v_{INIT}: 22.5 m/s; repulsive; r: closest approach

Solve for $r_{\substack{closest\\approach}}$:

$$\therefore\ r_{\substack{closest\\approach}} = \sqrt{\frac{1}{2\pi\varepsilon_o}\left(\frac{e^2}{m_e v_{init}^2}\right)} = \sqrt{\frac{1}{2\pi * 8.85\times10^{-12}}\frac{\left(1.6\times10^{-19}\right)^2}{9.11\times10^{-31}\times(22.5)^2}} = 1\ meter\ !$$



What happens:

e_1 comes in from $r = \infty$, is decelerated and stopped at $r = 1m$ (= distance of closest approach), and then e_1 is accelerated back out to $r = \infty$. $v_{e_1}^{final} = 22.5\ m/s$ at $r = \infty$ again.

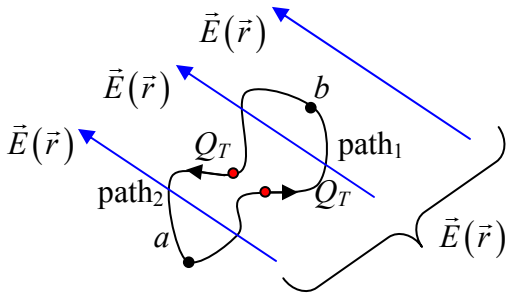
How much *mechanical work* is done in taking a charge Q_T around a closed path / closed contour C in an external electric field $\vec{E}(\vec{r})$?

$$W = \oint_C \int_a^a \vec{F}_{mech}(\vec{r})\bullet\overline{dl} = \oint_C \int_a^a \left(-\vec{F}_C(\vec{r})\right)\bullet\overline{dl} = -\oint_C \int_a^a \vec{F}_C(\vec{r})\bullet\overline{dl}$$

$$= -Q_T \oint_C \int_a^a E(\vec{r})\bullet\overline{dl} = 0 \qquad \text{because:} \qquad \oint \vec{E}(\vec{r})\bullet\overline{dl} = 0$$

NO work is done on closed contour, because the electrostatic Coulomb Force, $\vec{F}_C(\vec{r})$ is a *conservative* force.

$\therefore$ Mechanical work done on test charge Q_T is *independent* of the path:



$\vec{E}(\vec{r})$ is produced by some (arbitrary) source charge distribution.

Note: $\oint_C \int_a^a = \int_a^b + \int_b^a$

path$_1$ path$_2$

path$_1$ + path$_2$ are arbitrary because $\vec{F}_C(\vec{r})$ is conservative (Coulomb Pot'l is $1/r$, central pot'l - *no* θ,φ dependence.) $\vec{F}_C(\vec{r}) = -Q_T\vec{\nabla}V(\vec{r})$

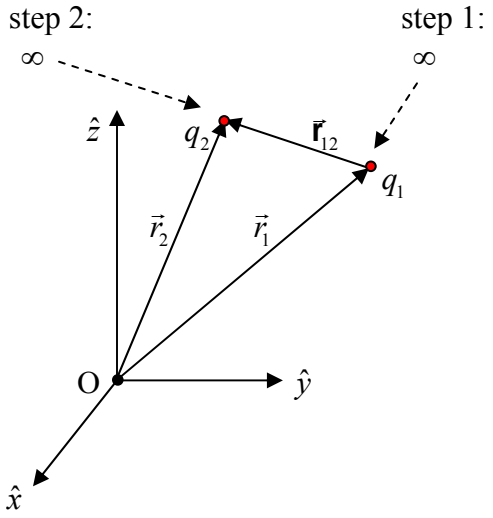
ELECTROSTATIC ENERGY OF ASSEMBLY OF A POINT CHARGE DISTRIBUTION

How much work does it take to assemble a collection of point charges – bringing them in from infinity, one by one? Bringing in the first charge q_1 takes NO work ($W_1 = 0$), since there is no electric field present, initially.

Now bring in the 2nd charge q_2 from infinity. The work done in bringing in q_2 from infinity is:

$$\boxed{W_2 = \frac{q_2}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{12}}\right) = q_2 V_1(\vec{r}_2)} \quad \text{with} \quad \boxed{V_1(\vec{r}_2) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{12}}\right)} \quad \text{where } \vec{\mathbf{r}}_{12} = \vec{r}_2 - \vec{r}_1 \text{ and}$$

$\mathbf{r}_{12} = |\vec{\mathbf{r}}_{12}| = |\vec{r}_2 - \vec{r}_1| =$ separation distance between $\vec{r}_2$ & $\vec{r}_1$ (i.e. between charges q_1 & q_2)



Now bring in the 3rd charge from infinity. This requires work $W_3 = q_3 V_{1,2}(\vec{r}_3)$ where $V_{1,2}(\vec{r}_3)$ is the (total) potential due to charges q_1 & q_2:

$$V_{1,2}(\vec{r}_3) = V_1(\vec{r}_3) + V_2(\vec{r}_3) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{13}} + \frac{q_2}{\mathbf{r}_{23}}\right)$$

$$\therefore\ W_3 = q_3 V_{1,2}(\vec{r}_3) = \frac{q_3}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{13}} + \frac{q_2}{\mathbf{r}_{23}}\right)$$

Similarly for the 4th charge: $V_{1,2,3}(\vec{r}_4) = V_1(\vec{r}_4) + V_2(\vec{r}_4) + V_3(\vec{r}_4) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{14}} + \frac{q_2}{\mathbf{r}_{24}} + \frac{q_2}{\mathbf{r}_{34}}\right)$

$$\therefore\ W_4 = q_4 V_{1,2,3}(\vec{r}_4) = \frac{q_4}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{14}} + \frac{q_2}{\mathbf{r}_{24}} + \frac{q_3}{\mathbf{r}_{34}}\right)$$

The *total* work necessary to assemble the first 4 charges is thus:

$$W_{TOT} = W_1 + W_2 + W_3 + W_4 = 0 + q_2 V_1(\vec{r}_2) + q_3 V_{1,2}(\vec{r}_3) + q_4 V_{1,2,3}(\vec{r}_4)$$
$$= \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_2}{\mathbf{r}_{12}} + \frac{q_1 q_3}{\mathbf{r}_{13}} + \frac{q_1 q_4}{\mathbf{r}_{14}} + \frac{q_2 q_3}{\mathbf{r}_{23}} + \frac{q_2 q_4}{\mathbf{r}_{24}} + \frac{q_3 q_4}{\mathbf{r}_{34}} \right)$$

We can generalize this relation for *N* charges as:

$$W_{TOT} = \frac{1}{4\pi\varepsilon_o} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i \\ j>i}}^{N} \left(\frac{q_i q_j}{\mathbf{r}_{ij}} \right) = \frac{1}{8\pi\varepsilon_o} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(\frac{q_i q_j}{\mathbf{r}_{ij}} \right)$$

so we don't count same pair twice!

double-counts pairs - but factor of 8 (vs. 4) takes care of this!

$$W_{TOT} = \frac{1}{2} \sum_{i=1}^{N} q_i \underbrace{\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_o} \left(\frac{q_j}{\mathbf{r}_{ij}} \right) \right)}_{= V(\vec{r}_i)} = \frac{1}{2} \sum_{i=1}^{N} q_i V(\vec{r}_i)$$

Notice W_{TOT} can be < 0 or can be > 0, depending on signs of q_i & q_j

e.g. if all $q_i q_j$ terms > 0, $W_{TOT} > 0$ (for pair-wise repulsive forces)
if all $q_i q_j$ terms < 0, $W_{TOT} < 0$ (for pair-wise attractive forces)

ELECTROSTATIC ENERGY OF A CONTINUOUS CHARGE DISTRIBUTION

For a discrete / discretized charge distribution:

$$W_{TOT} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i) = \frac{1}{2}\sum_{i=1}^{N} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_o}\left(\frac{q_i}{\mathbf{r}_{ij}}\right)\right)$$

For a continuum volume charge density, $\rho(\vec{r})$: $\sum_{i=1}^{N} q_i V(\vec{r}_i) \Rightarrow \int_V dq V(\vec{r}) = \int_V \rho(\vec{r}) V(\vec{r})\, d\tau$

$$W_{TOT} = \frac{1}{2}\oint_V \rho(\vec{r}) V(\vec{r})\, d\tau$$

For a continuum line charge density, $\lambda(\vec{r})$:

$$W_{TOT} = \frac{1}{2}\int_C \lambda(\vec{r}) V(\vec{r})\, dl$$

For a continuum surface charge density, $\sigma(\vec{r})$:

$$W_{TOT} = \frac{1}{2}\int_S \sigma(\vec{r}) V(\vec{r})\, dA$$

Using Gauss' Law:

$$\rho(r) = \varepsilon_o \vec{\nabla}\bullet\vec{E}$$

$$W_{TOT} = \frac{1}{2}\int_V \rho(\vec{r}) V(\vec{r})\, d\tau = \frac{\varepsilon_o}{2}\int_V \left(\vec{\nabla}\bullet\vec{E}\right) V(\vec{r})\, d\tau$$

Now use integration by parts to transfer the derivative from $\vec{E}$ to V (and also use the divergence theorem):

$\int_V \vec{\nabla}\bullet\left(f\,\vec{A}\right) d\tau = \int_V f\left(\vec{\nabla}\bullet\vec{A}\right) d\tau + \int_V \vec{A}\bullet\left(\vec{\nabla} f\right) d\tau = \oint_S f\vec{A}\bullet d\vec{a}$ (see Griffith's eqn. 1.59, p. 37)

Or: $\int_V f\left(\vec{\nabla}\bullet\vec{A}\right) d\tau = -\int_V \vec{A}\bullet\left(\vec{\nabla} f\right) d\tau + \oint_S f\vec{A}\bullet d\vec{a}$

Thus: $W_{TOT} = \frac{\varepsilon_o}{2}\left[-\int_V \vec{E}(\vec{r})\bullet\underbrace{\left(\vec{\nabla}V(\vec{r})\right)}_{=-\vec{E}(\vec{r})} d\tau + \oint_S V(\vec{r})\vec{E}(\vec{r})\bullet d\vec{A}\right]$

$$W_{TOT} = \frac{\varepsilon_o}{2}\left[-\int_V \vec{E}(\vec{r})\bullet\left(\vec{\nabla}V(\vec{r})\right)d\tau + \oint_S V(\vec{r})\,\vec{E}(\vec{r})\bullet d\vec{A}\right]$$
$$= \frac{\varepsilon_o}{2}\left[\int_V \left(\vec{E}(\vec{r})\bullet\vec{E}(\vec{r})\right)d\tau + \oint_S V(r)\,\vec{E}(\vec{r})\bullet d\vec{A}\right]$$
$$= \frac{\varepsilon_o}{2}\left[\int_V E^2(\vec{r})\,d\tau + \oint_S V(r)\,\vec{E}(\vec{r})\bullet d\vec{A}\right]$$

Now integration volume V and enclosing surface of integration S are *arbitrary*, as long as all charges are contained within the volume V and enclosed by the surface S.

So we can take $\begin{pmatrix} S \\ V \end{pmatrix} \rightarrow$ all space (i.e. ∞ - volume & surface) *without any loss of generality*.

Then the surface integral over all space $\oint_{\substack{S \\ All\ Space}} V(r)\,\vec{E}(\vec{r})\bullet d\vec{A}$ *vanishes*, because for a *localized* charge distribution i.e. one which has finite spatial extent {with characteristic size ~ d}, far away from the localized charge distribution, if there is a net electric charge associated with the localized charge distribution, then $V(r \gg d) \sim 1/r$ and $\vec{E}(r \gg d) = -\vec{\nabla}V(r \gg d) \sim 1/r^2$.

If the localized charge distribution has no net electric charge, then for $r \gg d$ the potential $V(r \gg d)$ and hence the electric field $\vec{E}(r \gg d)$ will fall of *more rapidly* than $1/r$ and $1/r^2$, respectively. {Please see/read P435 Lecture notes 8 (and/or Griffiths 3.4, p. 146-159) for further details/understanding of these large-r dependencies.} Thus, we see that both $V(r=\infty) = 0$ and $\vec{E}(r=\infty) = 0$, hence $\oint_{\substack{S \\ All\ Space}} V(r)\,\vec{E}(\vec{r})\bullet d\vec{A} = 0$ and thus:

$$\therefore\ W_{TOT} = \frac{\varepsilon_o}{2}\int_{\substack{V \\ all \\ space}} E^2(\vec{r})\,d\tau \qquad \text{(n.b. } \geq 0 \text{ always!)}$$

However, note that for a single point charge, since $\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\frac{1}{r^2}\hat{r}$

Then: $W_{TOT}^{q} = \frac{\varepsilon_o}{2}\frac{q^2}{(4\pi\varepsilon_o)^2}\int_{\substack{V \\ all \\ space}} \frac{1}{r^4} r^2 dr \sin\theta d\theta d\varphi = \frac{q^2}{8\pi\varepsilon_o}\int_0^\infty \frac{1}{r^2}dr = \infty$!!!

The electrostatic energy associated with a point electric charge is infinite (i.e. singular)!

The origin $r = 0$ cannot truly be included here – this formula breaks down – because it mathematically describes only classical physics – and e.g. we know that quantum mechanics takes over at distance scales of:

$$r \leq \lambda\!\!\!^{-}_e = \frac{\hbar c}{m_e c^2} \simeq \frac{197\,MeV - fm}{0.511} \simeq 385.5\,fm \simeq 385.5\times 10^{-15}\,m$$

$\lambda\!\!\!^{-}_e = \lambda_e / 2\pi = \dfrac{\hbar c}{m_e c^2}$ = "reduced" Compton wave length of electron, here h = Planck's constant,

and $\hbar = \dfrac{h}{2\pi}$, $\hbar c \simeq 197 MeV - fm$ $\left(1\ MeV = 10^6 eV\right)$, electron rest mass energy,

$m_e c^2 = 511 keV = 0.511 MeV$

And $1 fm = 1 femto - meter = 1 fermi = 10^{-15} m$

Actually, we know that for *atoms & molecules*, quantum mechanics takes over at distance scales of ~ Bohr radius (H-atom): $a_o = 0.53\text{Å} = 0.53 \times 10^{-10} m = 0.053\ nm \gg \lambda\!\!\!^{-}_e = 385.5\ fm \simeq 385.5 \times 10^{-15} m$

The classical formula for W^q_{TOT} *breaks down / is invalid* for point charges, for $r \le \lambda\!\!\!^{-}_e = \dfrac{\hbar c}{m_e c^2} \simeq 385.5\ fm$

At an even *shorter* distance scale, that of the so-called Planck distance $L_P = \sqrt{\hbar G_N / c^3} \simeq 1.6 \times 10^{-35} m$, with corresponding Planck time $t_P = L_P / c = \sqrt{\hbar G_N / c^5} \simeq 5.4 \times 10^{-44} s$, the "classical", continuous nature of ***space-time*** itself becomes "foam-like" – thus, we certainly have no reason to believe in the validity of Coulomb's law below the Planck distance scale.

$$W_{TOT} = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} \vec{E}(\vec{r}) \bullet \vec{E}(\vec{r})\, d\tau \quad \text{(Joules)}$$

$$= \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} E^2(\vec{r})\, d\tau > 0 \ \text{(always)}$$

This formula works fine/is valid for *continuous* charge distributions: $\lambda(\vec{r}), \sigma(\vec{r}), \rho(\vec{r})$

$\vec{E}(\vec{r})$ is total/net electric field arising from charge distribution, $\rho(\vec{r})$.

Define: $u_E(\vec{r}) \equiv \dfrac{\varepsilon_o}{2}\left(\vec{E}(\vec{r}) \bullet \vec{E}(\vec{r})\right) = \dfrac{\varepsilon_o}{2} E^2(\vec{r}) = \dfrac{1}{2}\rho(\vec{r}) V(\vec{r})$ = *electrostatic energy density* = Joules/meter^3

$$W_{TOT} = \int\limits_{\substack{V \\ all \\ space}} u_E(\vec{r})\, d\tau = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} E^2(\vec{r})\, d\tau$$

Physically, says that the energy density, $u_E(\vec{r}) = \frac{\varepsilon_o}{2} E^2(\vec{r})$ is associated *entirely* with the electric field, $\vec{E}(\vec{r})$.

$$= \frac{1}{2} \oint\limits_{\substack{V \\ all \\ space}} \rho(\vec{r}) V(\vec{r})\, d\tau$$

Physically, says that the energy density, $u_E(\vec{r}) = \frac{1}{2}\rho(\vec{r}) V(\vec{r})$ is due to product (i.e *mix*) of volume charge density, $\rho(\vec{r})$ and electrostatic potential, $V(\vec{r})$.

Electrostatic energy density: $u_E(\vec{r}) = \dfrac{\varepsilon_o}{2} E^2(\vec{r}) = \dfrac{1}{2}\rho(\vec{r}) V(\vec{r})$ (Joules/meter^3)

ELECTROSTATIC ENERGY AND THE SUPERPOSITION PRINCIPLE

Note that since W (work) / $P.E.$ (potential energy) is quadratic in the electric field, i.e.

$$W = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} E^2(\vec{r})\, d\tau = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} \vec{E}(\vec{r}) \bullet \vec{E}(\vec{r})\, d\tau$$

then for example, if we double $\vec{E}$, i.e. $\vec{E} \to 2\vec{E}$, the work needed / done to assemble the now-doubled charge distribution / stored potential energy in the electrostatic field quadruples (i.e increases 4x).

Thus, work done to assemble the charge distribution / potential energy stored in the electrostatic field of the charge distribution does NOT obey the principle of linear superposition.

This is not surprising, as there exist many other physical examples where the superposition principle does not hold – e.g. the superposition of two (or more) acoustic / sound waves, or e.g. the superposition of two (or more) light waves – sound/light overall *intensities* (proportional to square of overall/total amplitude) result from interference effects at the amplitude level.

Here, e.g. let $\vec{E}_{TOT}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$

Then: $$W_{TOT} = P.E. = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} \left(\vec{E}_{TOT}(\vec{r}) \bullet \vec{E}_{TOT}(\vec{r})\right) d\tau$$

$$= \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} \left(\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})\right) \bullet \left(\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})\right) d\tau$$

$$= \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} \left(E_1^2(\vec{r}) + 2\vec{E}_1(\vec{r}) \bullet \vec{E}_2(\vec{r}) + E_2^2(\vec{r})\right) d\tau$$

$$W_{TOT} = P.E. = W_1 + W_2 + \varepsilon_o \int\limits_{\substack{V \\ all \\ space}} \left(\vec{E}_1(\vec{r}) \bullet \vec{E}_2(\vec{r})\right) d\tau$$

Clearly, the principle of linear superposition is ***not*** (always) obeyed for W_{TOT} – there exists an *additional cross-term* involving the product the two electric fields: $\vec{E}_1(\vec{r}) \bullet \vec{E}_2(\vec{r})$.