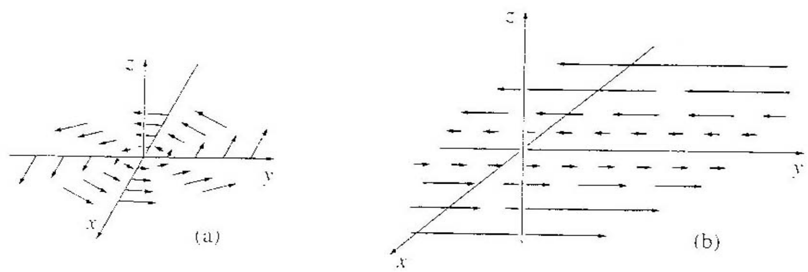
LECTURE NOTES 3

<u>The Scalar Electric Potential</u>, $V(\vec{r})$

The electric field $\vec{E}(\vec{r})$ is a *<u>very special type of vector point function/vector field</u>*, which for <u>electrostatics</u>, the CURL of $\vec{E}(\vec{r})$ = <u>zero</u>, i.e. $\vec{\nabla}\times\vec{E}(\vec{r})=0$. The physical meaning of the curl of a vector field is as follows:

For an arbitrary vector field $\vec{A}(\vec{r})$, if $\vec{\nabla}\times\vec{A}(\vec{r})\neq 0$ for all (or some) points in space, $\vec{r}$ then the vector $\vec{A}(\vec{r})$ rotates/circulates/ swirls, or shears in some manner in that region of space − e.g. the velocity field of a <u>whirlpool</u>, $\vec{v}_{\text{wp}}(\vec{r})$, or that associated e.g. with a wind <u>shear field</u>, $\vec{v}_{\text{shear}}(\vec{r})$:

<u>Curl of Whirlpool Field $\vec{\nabla}\times\vec{v}_{\text{wp}}(\vec{r})\neq 0$:</u> <u>Curl of Shear Field $\vec{\nabla}\times\vec{v}_{\text{shear}}(\vec{r})\neq 0$:</u>



<u>In Cartesian Coordinates</u>: $\vec{v}(\vec{r})=v_x(\vec{r})\hat{x}+v_y(\vec{r})\hat{y}+v_z(\vec{r})\hat{z}$

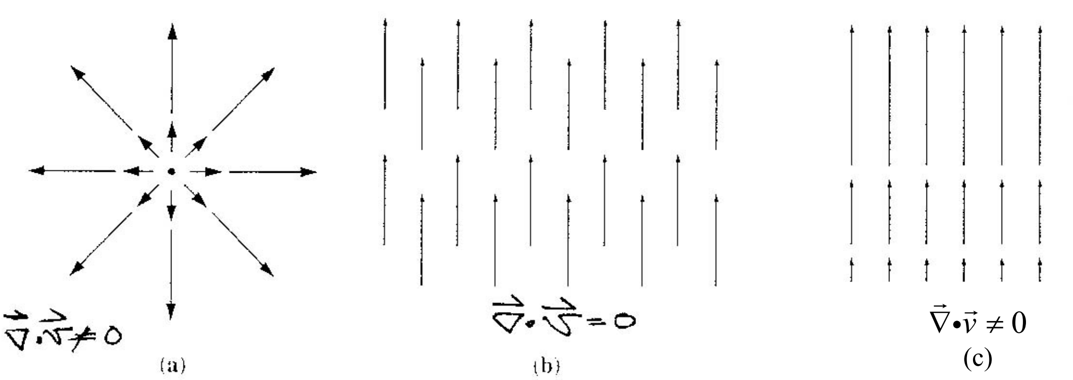
<u>Curl</u>: $\vec{\nabla}\times\vec{v}(\vec{r})=\left(\frac{\partial v_z(\vec{r})}{\partial y}-\frac{\partial v_y(\vec{r})}{\partial z}\right)\hat{x}+\left(\frac{\partial v_x(\vec{r})}{\partial z}-\frac{\partial v_z(\vec{r})}{\partial x}\right)\hat{y}+\left(\frac{\partial v_y(\vec{r})}{\partial x}-\frac{\partial v_x(\vec{r})}{\partial y}\right)\hat{z}$

$\vec{\nabla}\times\vec{v}(\vec{r})$ is a <u>vector</u> quantity. Compare this with

<u>Divergence</u>: $\vec{\nabla}\bullet\vec{v}(\vec{r})$ (= measure of how much the vector $\vec{v}$ spreads out, or *<u>diverges</u>* in space):

<u>In Cartesian Coordinates</u>: $\vec{\nabla}\bullet\vec{v}(\vec{r})=\frac{\partial v_x(\vec{r})}{\partial x}+\frac{\partial v_y(\vec{r})}{\partial y}+\frac{\partial v_z(\vec{r})}{\partial z}$

$\vec{\nabla}\bullet\vec{v}(\vec{r})$ is a <u>scalar</u> quantity (i.e. number).



We saw in the previous lecture (P435 Lect. Notes 2, p.15) by use of Stokes' Theorem, (for electrostatics) that:

$$\oint_S \vec{\nabla}\times\vec{E}(\vec{r})\bullet d\vec{A} = \oint_C \vec{E}(\vec{r})\bullet d\vec{\ell} = 0$$

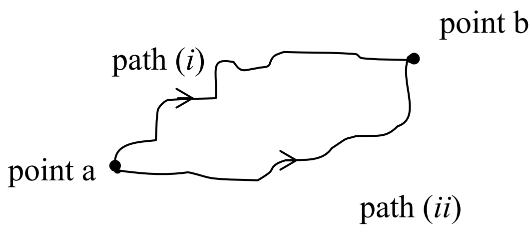
(closed surface) (closed contour)

from which there are two implications (assuming $\vec{E}(\vec{r}) \neq 0$ everywhere):

1.) $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ everywhere (for arbitrary closed surface *S*).

2.) $\oint_C \vec{E}(\vec{r})\bullet d\vec{\ell} = 0$ implies path independence of this (arbitrary) closed contour, *C*.

i.e./e.g. taking path (i) in figure below gives identical result as taking path (ii) in the figure below:



$$\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell} = \int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell} = \int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}$$

path (i) path (ii) any path

because $\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}$ is *independent* of the path taken from point $a \to b$.

We now define a *scalar point function*, $V(\vec{r})$ known as the electric potential, as follows:

Electric Potential (integral version)	$V(\vec{r}) \equiv -\int_{O_{ref}}^{r} \vec{E}(\vec{r})\bullet d\vec{\ell}$

By convention, the point $r = O_{ref}$ is taken to be a *standard reference point* of electric potential, $V(\vec{r})$ where $V(\vec{r} = O_{ref}) = 0$ (usually $r = \infty$).

SI Units of Electric Potential = Volts

If $V(r = O_{ref}) = 0$ @ the reference point, O_{ref} then $V(\vec{r})$ depends only on point $\vec{r}$.

The electric potential difference between two points a & b is thus:

$$V(b) - V(a) = \left(-\int_{O_{ref}}^{b} \vec{E}(\vec{r})\bullet d\vec{\ell}\right) - \left(-\int_{O_{ref}}^{a} \vec{E}(\vec{r})\bullet d\vec{\ell}\right)$$

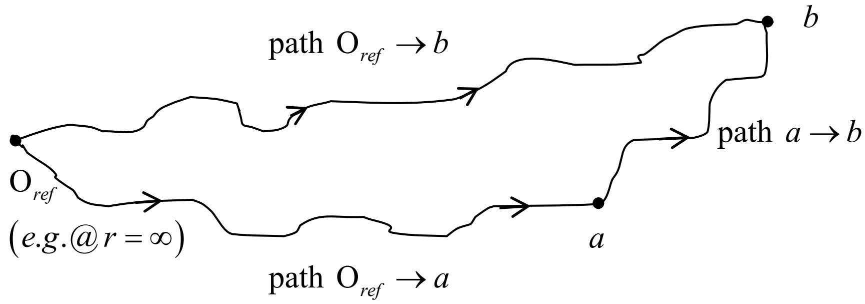
$$= -\int_{O_{ref}}^{b} \vec{E}(\vec{r})\bullet d\vec{\ell} + \int_{O_{ref}}^{a} \vec{E}(\vec{r})\bullet d\vec{\ell}$$

$$= -\int_{O_{ref}}^{b} \vec{E}(\vec{r})\bullet d\vec{\ell} - \int_{a}^{O_{ref}} \vec{E}(\vec{r})\bullet d\vec{\ell}$$

$$= -\int_{a}^{O_{ref}} \vec{E}(\vec{r})\bullet d\vec{\ell} - \int_{O_{ref}}^{b} \vec{E}(\vec{r})\bullet d\vec{\ell}$$

Thus:

$\Delta V_{ab} \equiv V(\vec{r}=b) - V(\vec{r}=a) = -\int_a^b \vec{E}(\vec{r}) \bullet d\vec{\ell}$	Integral Version



The <u>fundamental theorem for gradients</u> states that:

$$\text{Potential difference: } \Delta V_{ab} \equiv V(r=b) - V(r=a) = \int_a^b \vec{\nabla} V(\vec{r}) \bullet d\ell = -\int_a^b \vec{E}(\vec{r}) \bullet d\vec{\ell}$$

This is true for <u>any</u> end-points <u>a</u> & <u>b</u> (and any contour from a → b). Thus the two <u>*integrands*</u> <u>***must***</u> be equal:

$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$	Differential Version

Now (for <u>electrostatics</u>): $\vec{\nabla} \times \vec{E}(\vec{r}) = 0$, Thus: $\vec{\nabla} \times \vec{E}(\vec{r}) = \vec{\nabla} \times \left(-\vec{\nabla} V(\vec{r})\right) = -\vec{\nabla} \times \vec{\nabla} V(\vec{r}) = 0$

See inside front cover of Griffiths, $\vec{\nabla} \times \vec{\nabla} f(\vec{r}) = 0$, where $f(\vec{r})$ is a scalar point function.

For <u>Electrostatic problems</u>, $\vec{\nabla} \times \vec{E}(\vec{r}) = 0$ will <u>always</u> be true. For such problems, this means that (both) $\vec{F}(\vec{r}) = Q_T\ \vec{E}(\vec{r})$ and $\vec{E}(\vec{r})$ can be expressed as the (negative) gradient of a <u>scalar point function</u>, $V(\vec{r})$,

i.e.

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

$$\vec{F}(\vec{r}) = -Q_T \vec{\nabla}(\vec{r})$$

A scalar point function ($V(\vec{r})$ here) is one which is a scalar quantity (not a vector quantity) who's numerical value depends on position in space, $\vec{r}$ – e.g. a continuous/well-behaved function which is mathematically <u>defined at every point</u> $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

⇒ Knowing $V(\vec{r})$ enables you to specify/calculate $\vec{E}(\vec{r})$!!

What is the physical meaning of the electric potential, $V(\vec{r})$?

SI units of $V(\vec{r}) = \underline{\underline{\text{Volts}}} = \dfrac{\text{Newton-meters}}{\text{Coulomb}}$

Also

SI units of $\vec{E}(\vec{r}) = \dfrac{\text{Newtons}}{\text{Coulomb}} \left(= \dfrac{\vec{F}(\vec{r})}{Q_T} \right)$ but SI units of $\vec{E}(\vec{r}) = \text{Volts}/\text{meter}$ (since $\vec{E} = -V(\vec{r})$)

The electric field, $\vec{E}(\vec{r})$ is the (negative) spatial gradient of electric potential, $V(\vec{r})$

In Cartesian coordinates, $\vec{\nabla} V(\vec{r}) = \dfrac{\partial V(\vec{r})}{\partial x}\hat{x} + \dfrac{\partial V(\vec{r})}{\partial y}\hat{y} + \dfrac{\partial V(\vec{r})}{\partial z}\hat{z}$

Why is $\vec{E}(\vec{r})$ specified as negative gradient of scalar quantity, the electric potential??

Because of the way we define (by convention) the reference point for absolute voltage/potential, when $r = \infty$.

Consider our point charge problem (again) {n.b. choose local origin @ the point charge}:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)} \qquad \vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)\right) = -\frac{1}{4\pi\varepsilon_o} q \vec{\nabla}\left(\frac{1}{r}\right)$$

n.b. $V(\vec{r})$ for a point charge has no θ or φ-dependence

In spherical-polar coordinates: $\vec{\nabla} = \dfrac{\partial}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \varphi}\hat{\varphi}$

$$\therefore\ \vec{\nabla}\left(\frac{1}{r}\right) = \left\{\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{\varphi}\right\}\left(\frac{1}{r}\right)$$

$$= \underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}_{\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}}\hat{r} + \overbrace{\frac{1}{r}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)}_{\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)=0}}^{=0}\hat{\theta} + \overbrace{\frac{1}{r\sin\theta}\underbrace{\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\right)}_{\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\right)=0}}^{=0}\hat{\varphi}$$

$$\therefore\ \vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = +\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)\hat{r}$$

$V(r) \sim \dfrac{1}{r}$ and thus both $\vec{E}(\vec{r}), \vec{F}_C(\vec{r}) \sim \dfrac{1}{r^2}$ for point electric charge. The electrostatic potential $V(\vec{r})$ associated with a point charge q is a central potential; it varies as ~ $1/r$.

Note that $V(\vec{r})$ has no θ- and/or φ-dependence.

⇒ The Coulomb force is a <u>central</u> force (as is e.g. the gravitational force). Thus, the Coulomb force is a <u>*conservative*</u> force, like gravity, <u>*because*</u> $\vec{F}(\vec{r}) = -Q_T\vec{\nabla}V(\vec{r})$ <u>*can*</u> be written as the (negative) gradient of scalar point function, $V(\vec{r})$.

Let's plot $V(\vec{r})$ for a point charge Q. For definiteness' sake, we will plot $V(\vec{r})$ for $Q = +e$ and $Q = -e$ (e = magnitude of charge of an electron or proton, i.e. 1.602 x 10^{-19} Coulombs). n.b. again, we choose the local origin to be located at the point charge.

Electric Potentials & Fields:

$$V^{\oplus}(\vec{r}) = \frac{+e}{4\pi\varepsilon_o}\left(\frac{1}{r}\right) \qquad V^{\ominus}(\vec{r}) = \frac{-e}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)$$

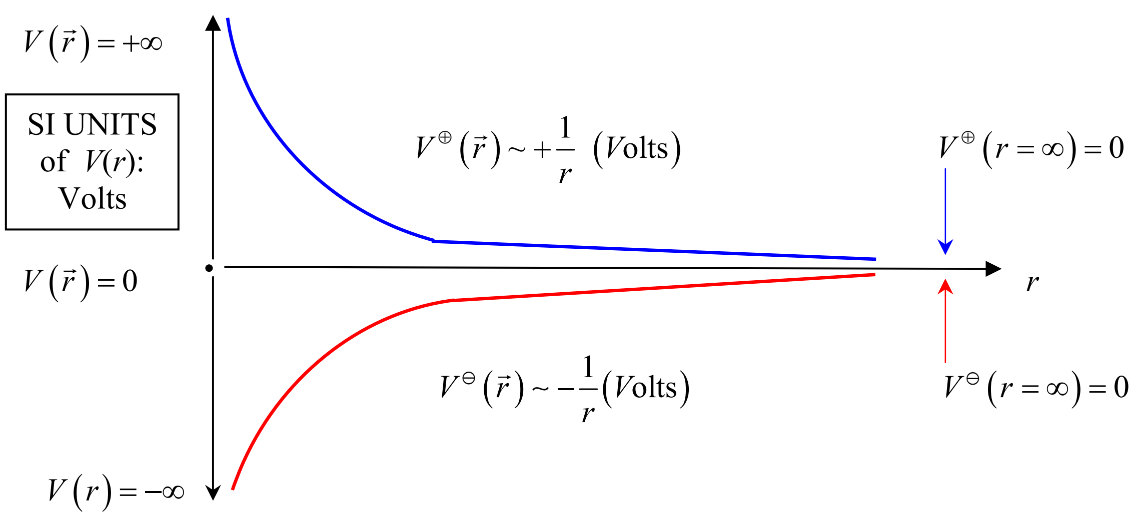
$$\vec{E}^{\oplus}(\vec{r}) = -\vec{\nabla}V^{\oplus}(\vec{r}) \qquad \vec{E}^{\ominus}(\vec{r}) = -\vec{\nabla}V^{\ominus}(\vec{r})$$

$$= +\frac{e}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\hat{r} \qquad = -\frac{e}{4\pi\varepsilon_o}\left(\frac{1}{r^2}\right)\hat{r}$$

Radial outward Lines of $\vec{E}(\vec{r})$	Radial inward Lines of $\vec{E}(\vec{r})$

Thus, we see that by defining $\vec{E}(\vec{r})$ as the <u>negative</u> gradient, this also <u>simultaneously</u> defines the <u>convention</u> that lines of $\vec{E}$ point <u>outward</u> from ⊕ charge, and point <u>inward</u> for ⊖ charge.

Graph of the electrostatic potential $V(\vec{r})$ for ⊕ and ⊖charges:



$r = \infty$ is the reference point, where $V(r = \infty) = 0$

Equipotentials:

Note that from e.g. $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)$ (i.e. potential for a point charge, q) that for r = constant, e. g. $r = R$, then $V(r = R) =$ constant.

Thus, for a point charge, q, there exist "imaginary" surfaces – concentric spheres of varying radii $r = R_1 < R_2 < R_3 < \ldots R_N$ whose spherical surfaces are surfaces of *constant* potential (i.e. potential = same value, in Volts everywhere on one of these surfaces, R_N, where $N = 1, 2, 3, \ldots$)

These "imaginary" surfaces of constant potential are known as equipotential surfaces – projecting them onto a 2-D surface, contours of constant potential can be seen.

Note that the equipotentials/contours of constant electrostatic potential $V(\vec{r})$ are *everywhere perpendicular* to lines of $\vec{E}(\vec{r})$! e. g. for a +ve point charge, $+q$ looks like a contour map! (It is!!!)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{+q}{r}\right)$$
$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{+q}{r^2}\right)\hat{r}$$

Equipotential Surfaces:

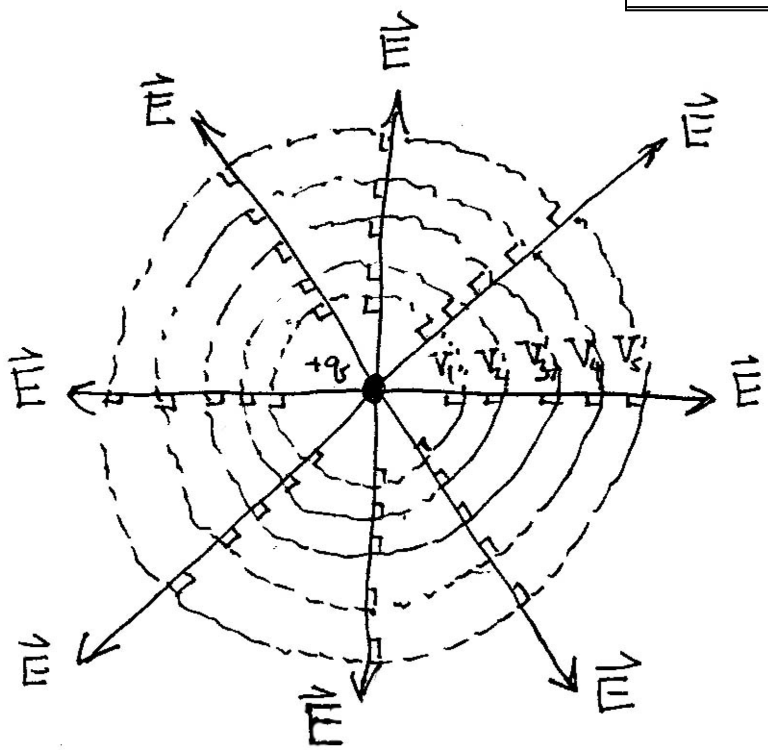
$r = R_1 \quad V_1(R_1) = +100V$

$r = R_2 \quad V_2(R_2) = +80V$

$r = R_3 \quad V_3(R_3) = +65V$

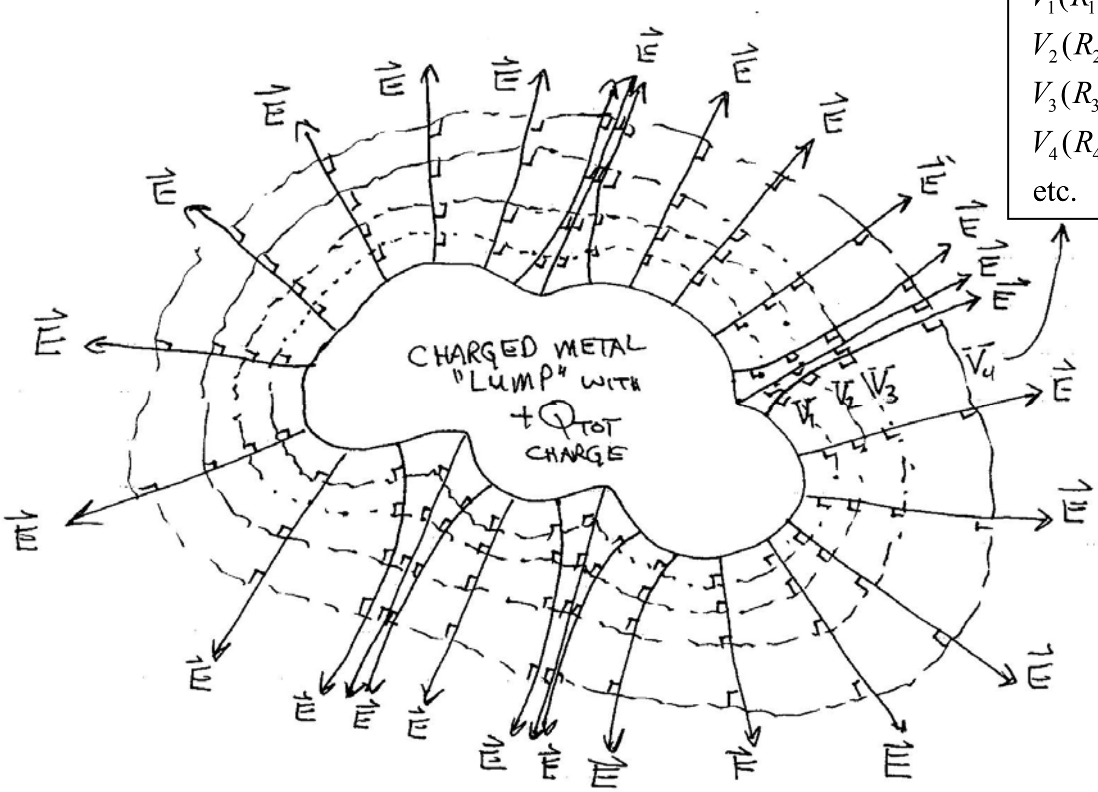
$r = R_4 \quad V_4(R_4) = +55V$

$r = R_5 \quad V_5(R_5) = +50V$



Equipotential surfaces exist for (arbitrary) electrostatic charge distributions – e.g. a charged lump of metal:

e.g.
$V_1(R_1) = +\ 100V$
$V_2(R_2) = +\ 80V$
$V_3(R_3) = +\ 65V$
$V_4(R_4) = +\ 55V$
etc.



Close in to the actual conducting surface (itself an equipotential!), the equipotentials outside the charge distribution "follow" the shape of the conducting surface. However, note that the further away from the conducting surface that the equipotential surface is, it becomes "smoother"/less bumpy/less wiggly. Very far away, the equipotential surface is nearly spherical in shape, independent of the shape of the actual object, at least for reasonable/small *h* x *w* x *l* aspect ratios.

$\Rightarrow$ This has ramifications for the ability to measure/infer shape of object by mapping out equipotential surfaces. Lose detailed information about the geometrical shape of the object, the farther away one gets!!

- Neglecting internal stress/strains, plate tectonics, tidal effects, earth's rotation, etc. The earth's surface is an equipotential of the earth's gravito-electric field!
- Indeed, sea level (if you also don't think too much about details of this) is also an equipotential of the earth's gravito-electric field! (neglecting tidal effects, Global warming, ice ages, El Nino, La Nina…)
- We define atmospheric pressure = 1 ATM (@ T = 20°C) and use this pressure as our "reference standard" to which other pressures are related, defined by pressure differences from sea level pressure (can also do this relative to zero pressure also, i.e. absolute pressure) because we have developed technology of vacuum pumps.
- Similarly, in the "real world" we find that only potential ***differences*** have practical meaning. We cannot physically measure $V(r = \infty) = 0$ because it's in outer space (somewhere)…

- For convenience, scientists have defined the earth's surface (everywhere) as our (local) electrostatic zero of potential, $V_{\text{earth}} = 0$. (But in reality, this is also not true….see comment below…)

- In practice, people drive a ~ 6' long copper-coated steel rod nearly entirely into the ground – that rod is *defined* as $V_{\text{earth}} = 0$, from which (relative) potential difference (voltage difference) measurements can be made.

- In reality, $\exists$ (there can/sometimes do exist) huge ground currents in the earth (from magnetic storms/ solar flares) – "earth ground" potential = 0 is not ideally so at every point on the planet at all times!

- Also, what is the electrostatic potential difference between: (earth – moon)? (earth – mars)? (earth – venus)? (earth – sun)?? We know these potential differences are in fact huge, because powerful electric fields exist in/throughout the solar system, e.g. driven by the sun's solar wind & solar flares (!)

$$\Delta V_{sun-earth} = V_{earth} - V_{sun} = -\int_{sun}^{earth} \vec{E}_{sun}(\vec{r}) \cdot d\vec{\ell}'$$

If $\langle \vec{E}_{sun}(\vec{r}) \rangle \sim 100\, V/m \qquad d_{sun-earth} \simeq 1.5 \times 10^{11}\, m$

Then $\Delta V_{sun-earth} \sim 1.5 \times 10^{13}$ Volts = 15 Tera-Volts (a lot!!)

The electrostatic potential (i.e. "voltage") is analogous e.g. to the pressure of a gas:

Electrostatic potential differences between two points in space, ΔV_{ab} (due to gradients in electrostatic potential) create electric fields, $\vec{E}(\vec{r})$ which in turn can accelerate charges $\left(\vec{F}(\vec{r}) = Q\vec{E}(\vec{r}) = m\vec{a}(\vec{r})\right)$ causing them to move – in turn producing electric currents,

$$I = \frac{dQ}{dt} \quad (\text{Amperes} = \text{Coulombs/sec})$$

Likewise, pressure differences/pressure gradients can cause mass flow. In a gas (or a fluid, more generally speaking) – mass currents = mass flow:

$$\text{“}I_m = \frac{dm}{dt}\text{”}$$

The Electrostatic Potential $V(\vec{r})$ and the Superposition Principle

We have seen that the Coulomb Force, $\vec{F}_c(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_T Q_S}{r^2}\hat{r}$ and the electrostatic field, $\vec{E}(\vec{r})$ obey the principle of superposition:

$$\vec{F}_{C\,NET}(\vec{r}) = \sum_{i=1}^{N}\vec{F}_{C\,i}(\vec{r}) = \vec{F}_{C\,1}(\vec{r}) + \vec{F}_{C\,2}(\vec{r}) + \vec{F}_{C\,3}(\vec{r}) + \ldots \vec{F}_{C\,N}(\vec{r})$$

$$\vec{E}_{NET}(\vec{r}) = \vec{F}_{C\,NET}(\vec{r})\Big/Q_T = \sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \ldots \vec{E}_N(\vec{r})$$

The above relations hold/are valid for any arbitrary electrostatic charge distributions: $q_i(r_i)$, $\lambda(\vec{r})$, $\sigma(\vec{r})$, $\rho(\vec{r})$, etc.

Since $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ or $\Delta V_{ab}(\vec{r}) = -\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}$, where $\Delta V_{ab} = V_b - V_a$

If we integrate from a common reference point, $a = \mathrm{O}_{ref}$ (It doesn't matter which point is taken as the common reference point, because $V_{\mathrm{O}_{ref}}(r = \mathrm{O}_{ref}) = 0$ will be the same in each expression (as we saw above)…

Thus, we can show that electrostatic potential also obeys the principle of superposition:

$$\Delta V_{NET} \equiv V_{NET}(\vec{r}) - V_{\mathrm{O}_{ref}}(r = \mathrm{O}_{ref})$$

$$\Delta V_{NET} = \sum_{i=1}^{N}\Delta V_i = \left(V_1(\vec{r}) - V_{\mathrm{O}_{ref}}(\mathrm{O}_{ref})\right) + \left(V_2(\vec{r}) - V_{\mathrm{O}_{ref}}(\mathrm{O}_{ref})\right) + \ldots\left(V_N(\vec{r}) - V_{\mathrm{O}_{ref}}(\mathrm{O}_{ref})\right)$$

Adding $V_{\mathrm{O}_{ref}}(\mathrm{O}_{ref}) = 0$ to LHS and RHS, we see that:

$$V_{NET}(\vec{r}) = \sum_{i=1}^{N}V_i(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) + V_3(\vec{r}) + \ldots V_N(\vec{r})$$

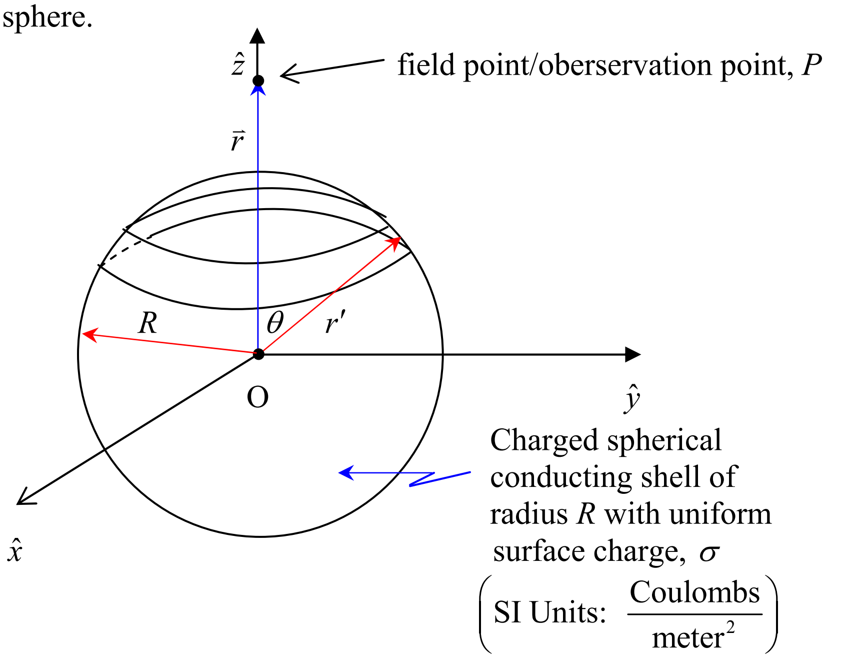
Note that this is a <u>scalar</u> (i.e. ordinary) numerical sum, not a <u>vector</u> sum!

Griffiths Example 2.7

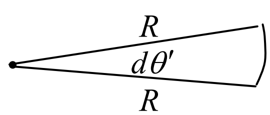
Electrostatic Potential, $V(\vec{r})$ and Electric Field $\vec{E}(\vec{r})$ of a uniformly charged spherical (conducting) shell of radius, *R*:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_{sphere}\frac{\sigma(r')\,dA'}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_o}\int_{sphere}\frac{\sigma\,dA'}{\mathscr{r}^2} \quad \text{and} \quad \vec{\mathscr{r}} = \vec{r} - \vec{r}',\ \mathscr{r} = |\vec{\mathscr{r}}| = |\vec{r} - \vec{r}'|$$

$\sigma(r' = R) = \sigma =$ constant on sphere.



$$\boxed{Q_{ToT} = \sigma A_{sphere} = 4\pi\sigma R^2}$$



Note that we can calculate $V(\vec{r})$ from $V(\vec{r}) = \int_{sphere} dV'(\vec{\mathscr{r}})$ where $dV'(\vec{\mathscr{r}}) =$ potential @ *P* due to infinitesimal annular charged strip $\sigma\left(C/m^2\right)$ and annular area $dA' = 2\pi R^2 \sin\theta' d\theta$ (← note that $V(\vec{r})$ has no φ-dependence)

$$dV'(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{dQ}{\mathscr{r}} \quad \leftarrow \quad dQ = \sigma dA'$$

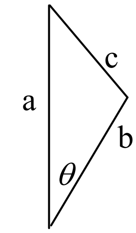
i.e.

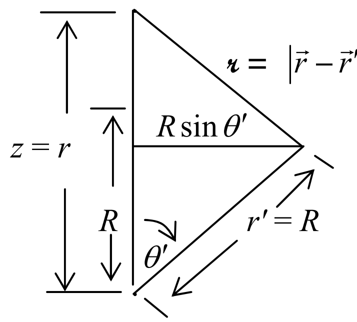
$$= \frac{1}{4\pi\varepsilon_o}\frac{\sigma dA'}{\mathscr{r}}$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{\sigma\left(2\pi R^2 \sin\theta' d\theta'\right)}{\mathscr{r}}$$

Now, use law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$ to find $\mathscr{r}$:

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta'$$





$$\mathscr{r} = \left|\vec{r} - \vec{r}'\right| = \sqrt{z^2 + R^2 - 2zR\cos\theta'}$$

$$\therefore\ V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left(2\pi\sigma R^2\right)\int_{\theta'=0}^{\theta'=\pi}\frac{\sin\theta' d\theta'}{\sqrt{z^2 + R^2 - 2zR\cos\theta'}}$$

Recall that $d\cos\theta' = \sin\theta' d\theta'$

Make a change of variables: define $u = \cos\theta'$

If $u \equiv \cos\theta'$, then when $\theta' = 0$, $\cos\theta' = 1$, $u = 1$

when $\theta' = \pi$, $\cos\theta' = -1$, $u = -1$

Note also: $du = d\left(\cos\theta'\right)$

Then: $V(\vec{r}) = \dfrac{\sigma R^2}{2\varepsilon_o}\displaystyle\int_{\theta'=0}^{\theta'=\pi}\frac{d\left(\cos\theta'\right)}{\sqrt{z^2 + R^2 - 2zR\cos\theta'}}$

becomes: $V(\vec{r}) = \dfrac{\sigma R^2}{2\varepsilon_o}\displaystyle\int_{u=+1}^{u=-1}\frac{du}{\sqrt{z^2 + R^2 - 2zRu}} = -\dfrac{\sigma R^2}{2\varepsilon_o}\displaystyle\int_{u=-1}^{u=+1}\frac{dU}{\sqrt{z^2 + R^2 - 2zRU}}$

Now: $\displaystyle\int\frac{dx}{\sqrt{a - bx}} = \int\left[a - bx\right]^{-1/2}dx = \frac{2}{b}\left[a - bx\right]^{1/2} = \frac{2}{b}\sqrt{a - bx}$

where: $a = z^2 + R^2$ and $b = 2zR$.

<u>Thus</u>: $$V(\vec{r}) = -\frac{\sigma R^2}{2\varepsilon_o}\left(\frac{2}{2zR}\right)\sqrt{z^2+R^2-2zRu}\,\Big|_{u=-1}^{u=+1}$$

$$= -\frac{\sigma R^2}{2\varepsilon_o z}\left\{\sqrt{z^2+R^2-2zR}-\sqrt{z^2+R^2+2zR}\right\}$$

$$= +\frac{\sigma R}{2\varepsilon_o z}\left\{\sqrt{z^2+R^2+2zR}-\sqrt{z^2+R^2-2zR}\right\}$$

$$= \frac{\sigma R}{2\varepsilon_o z}\left\{\sqrt{z^2+2zR+R^2}-\sqrt{z^2-2zR+R^2}\right\}$$

$$= \frac{\sigma R}{2\varepsilon_o z}\left\{\underbrace{\sqrt{(z+R)(z+R)}}_{\text{always positive}}-\underbrace{\sqrt{(\pm(z-R))*(\pm(z-R))}}_{\textit{must}\text{ take positive root}}\right\}$$

always positive *must* take <u>positive</u> root:

If $z > R$: $(z-R)*(z-R)$

If $z < R$: $(-(z-R))*(-(z-R))$

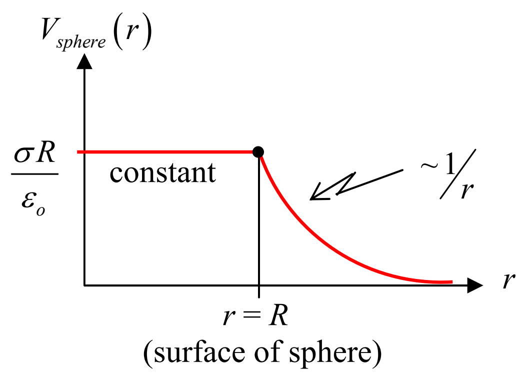
$$V_{z>R}^{\text{outside}}(\vec{r}=z\hat{z}) = \frac{\sigma R}{2\varepsilon_o z}\{(z+R)-(z-R)\} = \frac{\sigma R^2}{\varepsilon_o z} \quad (z>R)$$

$$V_{(z<R)}^{\text{inside}}(\vec{r}=z\hat{z}) = \frac{\sigma R}{2\varepsilon_o z}\{(z+R)-(R-z)\} = \frac{\sigma R}{\varepsilon_o} \quad (z<R) \quad \Leftarrow V_{(z<R)}^{\text{inside}}(\vec{r}=z\hat{z}) = \text{constant!!!}$$

$(\neq 0!!)$

n.b. surface of <u>charged</u> sphere is an <u>*equipotential*</u>: $\Rightarrow V(r=R) = \frac{\sigma R}{\varepsilon_0} \neq 0!$

Let $r = z$, then:



We calculated that the total electric charge on the surface of the sphere is: $Q_{\text{sphere}} = 4\pi\sigma R^2$

<u>Then</u>:
$$\begin{cases} V^{\text{outside}}_{(z>R)}\left(\vec{r} = z\hat{z}\right) = \dfrac{\sigma R^2}{\varepsilon_o z} = \dfrac{4\pi}{4\pi}\dfrac{\sigma R^2}{\varepsilon_o z} = \dfrac{4\pi\sigma R^2}{4\pi\varepsilon_o z} = \dfrac{Q_{\text{sphere}}}{4\pi\varepsilon_0 z} \\ V^{\text{inside}}_{z<R}\left(\vec{r} = z\hat{z}\right) = \dfrac{\sigma R}{\varepsilon_o} = \dfrac{4\pi R}{4\pi R}\dfrac{\sigma R}{\varepsilon_o} = \dfrac{4\pi\sigma R^2}{4\pi\varepsilon_o R} = \dfrac{Q_{\text{sphere}}}{4\pi\varepsilon_o R} = \text{constant} \neq 0 \end{cases}$$

Now since the sphere has rotational invariance, then more generally, we can replace z with r (= radial distance of field point, P from the center of the sphere, then $V(r)$ will have only r-dependence, no θ- or φ-dependence)

<u>Then</u>:
$$\begin{cases} V^{\text{outside}}_{(r>R)}\left(\vec{r}\right) = \dfrac{\sigma R^2}{\varepsilon_o r} = \dfrac{4\pi\sigma R^2}{4\pi\varepsilon_o r} = \dfrac{Q_{\text{sphere}}}{4\pi\varepsilon_o r} \\ V^{\text{inside}}_{(r<R)}\left(\vec{r}\right) = \dfrac{\sigma R}{\varepsilon_o} = \dfrac{4\pi\sigma R^2}{4\pi\varepsilon_o R} = \dfrac{Q_{\text{sphere}}}{4\pi\varepsilon_o R} = \text{constant} \neq 0 \end{cases}$$

n.b. for $r > R$, this is same $V(r)$ as for point charge, with $q = Q_{\text{sphere}}$!!!

Then electric field, $\vec{E}\left(\vec{r}\right) = -\vec{\nabla} V\left(\vec{r}\right)$

In spherical coordinates, $\vec{\nabla} = \dfrac{\partial}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\hat{\theta}$

<u>Then</u>: $\vec{E}^{\text{outside}}_{(r>R)}\left(\vec{r}\right) = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q_{\text{sphere}}}{r^2}\hat{r}$ ← same as for $\vec{E}\left(\vec{r}\right)$ for point charge, $q = Q_{\text{sphere}}$

$\vec{E}^{\text{inside}}_{(r<R)}\left(\vec{r}\right) = 0$ <u>because</u>: $V^{\text{inside}}_{(r<R)}\left(\vec{r}\right) =$ constant, i.e. no <u>gradient</u> of $V^{\text{inside}}_{(r<R)}\left(\vec{r}\right)$ for $r < R$!!!

POISSON'S EQUATION & LAPLACE'S EQUATION

Since the (electrostatic) electric field $\vec{E}(\vec{r})$ can be written as the negative gradient of a scalar point function – the electrostatic potential, $V(\vec{r})$, i.e. $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

Then with $\vec{\nabla}\bullet\vec{E}(\vec{r}) = \dfrac{\rho_{encl}(\vec{r})}{\varepsilon_o}$ and $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$

We see that: $\vec{\nabla}\times\vec{E}(\vec{r}) = -\vec{\nabla}\times\vec{\nabla}V(\vec{r}) = 0$

$$\vec{\nabla}\bullet\vec{E}(\vec{r}) = -\vec{\nabla}\bullet\vec{\nabla}V(\vec{r}) = -\nabla^2 V(\vec{r}) = \frac{\rho_{encl}(\vec{r})}{\varepsilon_o}$$

<u>Or</u>: $$\nabla^2 V(\vec{r}) = -\frac{\rho_{encl}(\vec{r})}{\varepsilon_o} \Leftarrow \text{Poisson's Equation}$$

Laplacian Operator = $\nabla^2 = \vec{\nabla}\bullet\vec{\nabla}$ $\Leftarrow$ n.b. scalar quantity!

<u>Cartesian Coordinates</u>: $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

<u>Cylindrical Coordinates</u>: $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

<u>Spherical Coordinates</u>: $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

Poissons' Equation is a linear, <u>inhomogeneous</u> 2nd order differential equation.

In regions of space where the volume charge density, $\rho(\vec{r}) = 0$, then Poisson's equation $\Rightarrow$ Laplace's Equation $\nabla^2 V(\vec{r}) = 0$ $\Leftarrow$ linear <u>homogenous</u> 2nd order differential equation.

We will discuss and use these two differential equations (much) more in the near future….

Usually in an electrostatics problem, for example:

1) A charge distribution $q(\vec{r}), \sum q_i(\vec{r}), \lambda(\vec{r}), \sigma(\vec{r})$ and/or $\rho(\vec{r})$ is <u>specified</u> (i.e. given) afore-hand, and you are asked to calculate e.g. $\vec{E}(\vec{r})$. Generally speaking, it's best (i.e. easiest) to calculate $V(\vec{r})$ <u>*first*</u> (as an intermediate step), and <u>*then*</u> calculate $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

THUS:

$$\left\{\begin{matrix}\text{charge distribution}\\ q, \sum_{i=1}^{N} q_i, \lambda, \sigma, \rho\end{matrix}\right\} \Rightarrow V(\vec{r}) = \left\{\begin{matrix} \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathfrak{r}},\ \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathfrak{r}_i},\ \text{or} \\ \frac{1}{4\pi\varepsilon_o}\int_C \frac{\lambda(r')\,d\ell'}{\mathfrak{r}},\ \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(r')\,dA'}{\mathfrak{r}} \\ \frac{1}{4\pi\varepsilon_o}\int_V \frac{\rho(r')\,d\mathfrak{r}'}{\mathfrak{r}} \end{matrix}\right\} \Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$$

OR:

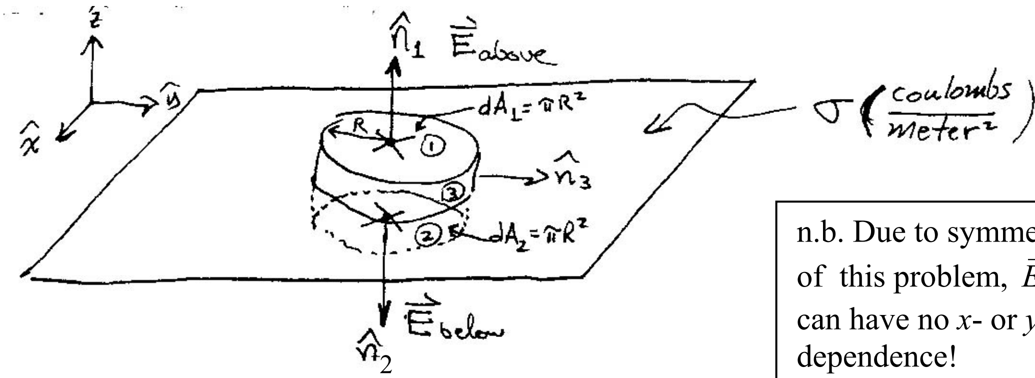
$$\vec{E}(\vec{r}) = \left\{\begin{matrix} \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathfrak{r}^2},\ \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathfrak{r}_i^2}\hat{\mathfrak{r}}\ \text{ or} \\ \frac{1}{4\pi\varepsilon_o}\int_C \frac{\lambda(r')\,\hat{\mathfrak{r}}\,d\ell'}{\mathfrak{r}^2},\ \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(r')\,\hat{\mathfrak{r}}\,dA'}{\mathfrak{r}^2} \\ \frac{1}{4\pi\varepsilon_o}\int_V \frac{\rho(r')\,\hat{\mathfrak{r}}\,d\tau'}{\mathfrak{r}^2} \end{matrix}\right\}$$

2) On the other hand, if $V(\vec{r})$ is <u>specified</u> (i.e. given). then we can use Poisson's equation $\nabla^2 V(\vec{r}) = -\dfrac{\rho(\vec{r})}{\varepsilon_o}$ to find $\rho(\vec{r})$.

3) If $\vec{E}(\vec{r})$ is <u>given/specified</u>, then use $\Delta V(\vec{r}) = -\int_c \vec{E}(\vec{r})\bullet d\vec{\ell}'$ to find $V(\vec{r})$ and then use $\vec{\nabla}\bullet\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o$ to find $\rho(\vec{r})$.

ELECTROSTATIC BOUNDARY CONDITIONS

Consider a 2-dimensional infinite-planar surface/sheet charge distribution – then place e.g. a cylindrical Gaussian pillbox centered on the charged planar surface as shown in the figure below:



n.b. Due to symmetry of this problem, $\vec{E}(\vec{r})$ can have no x- or y-dependence!

Gauss' Law: $\oint_{\text{Gaussian surface, S}} \vec{E} \bullet d\vec{A}' = \frac{Q_{encl}}{\varepsilon_0}$

Now shrink height of cylindrical Gaussian pillbox to be infinitesimally above/below charged sheet,
i.e. let $h_{pillbox} \to 0$.

Then: vanishes for $h_{pillbox} \to 0$

$$\oint_{\text{Gaussian surface, S}} \vec{E} \bullet d\vec{A}' = \int_{S_1} \vec{E} \bullet d\vec{A}_1 + \int_{S_2} \vec{E} \bullet d\vec{A}_2 + \int_{S_3} \vec{E} \bullet d\vec{A}_3$$

$d\vec{A}_1 = \pi R^2 \hat{n}_1 = \pi R^2 \hat{z}$ $\qquad$ $d\vec{A}_2 = \pi R^2 \hat{n}_2 = -\pi R^2 \hat{z} = -d\vec{A}_1$

$\vec{E}_{above} = E\hat{z}$ $\qquad$ $\vec{E}_{below} = -E\hat{z}$

$$\oint_{\text{Gausian surface}} \vec{E} \bullet d\vec{A}' = \pi R^2 E \Big(\underbrace{\hat{z} \cdot \hat{z}}_{=1} \Big) + \pi R^2 E \Big(\underbrace{-\hat{z} \cdot -\hat{z}}_{=1} \Big) = \frac{Q_{encl}}{\varepsilon_o} = \frac{\pi R^2 \sigma}{\varepsilon_o}$$

Or: $E = \dfrac{\sigma}{2\varepsilon_o}$, as we obtained previously.

However, what we *really* want to point out is that $\vec{E}$ is discontinuous across a charged interface.

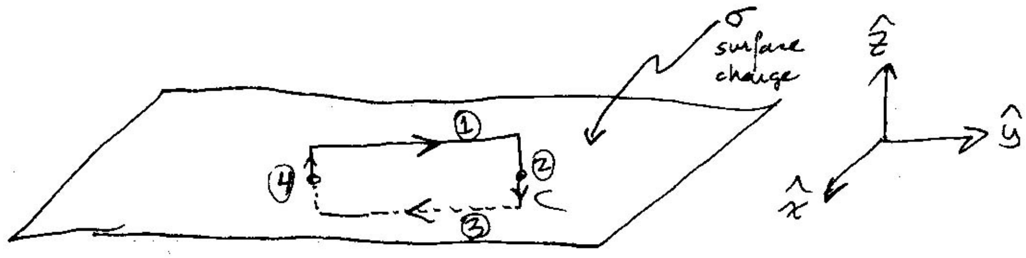
For the "shrunken" Gaussian pillbox, we can write Gauss' Law as:

$$\int \vec{E}_{above} \bullet d\vec{A}_1 + \int \vec{E}_{below} \bullet d\vec{A}_2 = \int \vec{E}_{above} \bullet d\vec{A}_1 - \int \vec{E}_{be;pw} \bullet d\vec{A}_1$$
$$= \int \left(\vec{E}_{above} - \vec{E}_{below} \right) \bullet d\vec{A} = \int \left(\vec{E}_{above} - \vec{E}_{below} \right) \bullet \hat{n}_1 dA$$

Now: $\left(\vec{E}_{above} - \vec{E}_{below} \right) \bullet \hat{n}_1 = E^{\perp}_{above} - E^{\perp}_{below} = \sigma / \varepsilon_o$

i.e. the perpendicular (normal) components of $\vec{E}$ are *discontinuous* across a charged surface/interface (with surface charge, σ) by an amount: $\boxed{E^{\perp}_{above} - E^{\perp}_{below} = \sigma / \varepsilon_o}$

What about the tangential components of $\vec{E}$ across a charged surface/interface?



We know that: $\oint_c \vec{E}(\vec{r})\cdot d\vec{l}' = 0$

Shrink contour, C such that height h of vertical (②,④) portions shrink to infinitesimal size, i.e. $h \to 0$.

$$\oint_C \vec{E}\cdot d\vec{\ell} = \underset{①}{\int} \vec{E}_{above}\cdot d\vec{\ell}_1 + \underset{②}{\int} \vec{E}_2\cdot d\vec{\ell}_2 + \underset{③}{\int} \vec{E}_{below}\cdot d\vec{\ell}_3 + \underset{④}{\int} \vec{E}_4\cdot d\vec{\ell}_4 = 0$$

$$= \underset{①}{\int} \vec{E}_{above}\cdot d\vec{\ell}_1 + \underset{③}{\int} \vec{E}_{below}\cdot d\vec{\ell}_3 \qquad \begin{cases} d\vec{\ell}_1 = dy\,\hat{y} \\ d\vec{\ell}_3 = -dy\,\hat{y} = -d\vec{\ell}_1 \end{cases}$$

$$= \underset{①}{\int} \vec{E}_{above}\cdot d\vec{\ell}_1 - \underset{③}{\int} \vec{E}_{below}\cdot d\vec{\ell}_1$$

$$= \int \left(\vec{E}_{above} - \vec{E}_{below} = 0\right)\cdot d\vec{\ell}_1 = 0 \quad \Rightarrow \quad \left(\vec{E}_{above} - \vec{E}_{below}\right)\cdot d\vec{\ell}_1 = 0$$

Now: $\vec{E}_{above}\cdot d\ell_1 = E^{\parallel}_{above} d\ell_1$ and $\vec{E}_{below}\cdot d\vec{\ell}_1 = E^{\parallel}_{below} d\ell_1$

$\therefore$ $\boxed{E^{\parallel}_{above} - E^{\parallel}_{below} = 0}$ OR: $\boxed{E^{\parallel}_{above} = E^{\parallel}_{below}}$

i.e. the tangential component of $\vec{E}$ is always continuous across an interface

$\Rightarrow$ Potential $\Delta V = V_{above} - V_{below} = -\int_a^b \vec{E}(\vec{r})\cdot d\vec{\ell} = 0$

Thus, V is (also) always continuous across an interface: $V_{above} = V_{below}$

point a is located infinitesimally *below* the interface
point b is located infinitesimally *above* the interface

<u>Since</u>: $\left\{\begin{array}{l} E_{above}^{\perp} - E_{below}^{\perp} = \sigma/\varepsilon_o \\ E_{above}^{\parallel} - E_{below}^{\parallel} = 0 \end{array}\right\} \Rightarrow \vec{E}_{above} - \vec{E}_{below} = \dfrac{\sigma}{\varepsilon_o}\hat{n}$ where $\hat{n}$ points from "below" to "above".

<u>But</u>: $\vec{E} = -\vec{\nabla} V$, thus: $\vec{\nabla} V_{above} - \vec{\nabla} V_{below} = -\dfrac{\sigma}{\varepsilon_o}\hat{n}$

Or, more specifically, if $\hat{n}$ is the <u>unit</u> <u>outward</u> normal of interface, at/on the interface,

<u>Then</u>: $\vec{\nabla} V_{above} - \vec{\nabla} V_{below} = -\left(\dfrac{\sigma}{\varepsilon_o}\right)\hat{n}$ can be written as: $\boxed{\left.\dfrac{\partial V_{above}}{\partial n}\right|_{interface} - \left.\dfrac{\partial V_{below}}{\partial n}\right|_{interface} = -\left(\dfrac{\sigma}{\varepsilon_o}\right)}$

<u>Where</u>: $\left.\dfrac{\partial V(\vec{r})}{\partial n}\right|_{interface} = \vec{\nabla} V(\vec{r}) \bullet \hat{n}\Big|_{interface} =$ ***normal*** derivative of the potential, $V(r)$ on the interface.

= spatial gradient in the direction ***perpendicular*** (***normal***) to the interface, on the interface.

