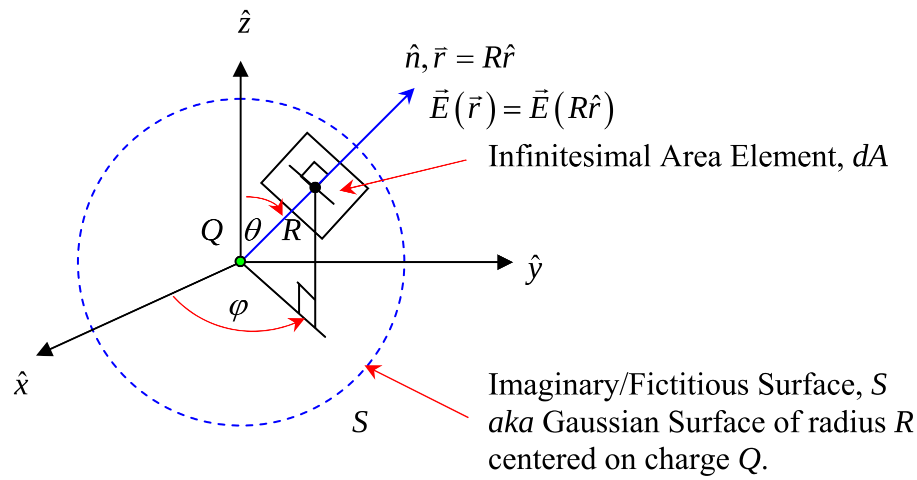
LECTURE NOTES 2

Gauss' Law / Divergence Theorem

Consider an imaginary / fictitious surface enclosing / surrounding e.g. a point charge (or a small charged conducting object). For simplicity, use an imaginary sphere of radius *R* centered on charge *Q* at origin:



Area element *dA* is a VECTOR quantity: $d\vec{A} = dA\hat{n} = dA\hat{r}$. By convention, $\hat{n}$ is outward-pointing unit normal vector at area element *dA*. In this particular case (because of spherical symmetry of problem): $\hat{n} = \hat{r}$

FLUX OF ELECTRIC FIELD LINES (through surface *S*): $\boxed{\Phi_E \equiv \int_S \vec{E}(\vec{r}) \cdot d\vec{A}}$

Φ_E = "measure" of "number of *E*-field "lines" passing through surface *S*, (SI Units: Volt-meters).

TOTAL ELECTRIC FLUX (Φ_E^{TOT}) associated with any closed surface *S*, is a measure of the (total) charge enclosed by surface *S*.

n.b. charge outside of surface *S* will contribute *nothing* to total electric flux Φ_E (since *E*-field lines pass through one portion of the surface *S* and out another – no net flux!)

Consider our point charge *Q* at origin. Calculate the flux of $\vec{E}$ passing through a sphere of radius *r*: (see above picture)

$$\Phi_E = \oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = r\frac{Q}{4\pi\varepsilon_o}\int_S \left(\frac{1}{\cancel{r^2}}\hat{r}\right) \cdot \underbrace{\left(\cancel{r^2}\sin\theta d\theta d\varphi \hat{r}\right)}_{\substack{=d\vec{A} \\ \text{infinitesimal vector} \\ \text{area element for} \\ \text{sphere of radius r}}}$$

n.b. Vector area element of sphere of radius, *r* is $\boxed{d\vec{A} = dA\hat{r} = \left(r^2 \sin\theta d\theta d\varphi\right)\hat{r}}$ in spherical-polar coordinates.

Thus: $\Phi_E = \dfrac{Q}{4\pi\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\int_{\varphi=o}^{\varphi=2\pi}\sin\theta d\theta d\varphi\underbrace{(\hat{r}\bullet\hat{r})}_{=1} = \dfrac{2\pi Q}{\underbrace{4\pi}_{2}\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\sin\theta d\theta$

$$= \frac{2Q}{2\varepsilon_o} = \frac{Q}{\varepsilon_o}$$

$\therefore$ <u>Gauss' Law (in Integral Form)</u>: $\boxed{\Phi_E = \oint_s \vec{E}(\vec{r})\bullet d\vec{A} = \dfrac{Q_{enclosed}}{\varepsilon_o}}$

Electric flux through closed surface S = (electric charge enclosed by surface S)/ ε_o

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If $\exists$ (= there exists) lots of <u>discrete</u> charges q_i (ALL <u>enclosed</u> by imaginary / fictitious / Gaussian surface S), we know from principle of superposition that:

$$\vec{E}_{NET}(\vec{r}) = \sum_{i=1}^{N}\vec{E}_i(\vec{r})$$

Then: $\Phi_E^{NET} = \oint_S \vec{E}_{NET}(\vec{r})\bullet d\vec{A} = \sum_{i=1}^{N}\left(\oint_S \vec{E}_i(\vec{r})\bullet d\vec{A}\right) = \sum_{i=1}\dfrac{q_i}{\varepsilon_o} = \dfrac{1}{\varepsilon_o}\sum_{i=1} q_i = \dfrac{Q_{encl}}{\varepsilon_o}$

If $\exists$ volume charge density $\rho(\vec{r}\,')$, then: $Q_{encl} = \int_v \rho(\vec{r}\,')d\tau'$

Then using the <u>DIVERGENCE THEOREM:</u>

$$\Phi_E = \oint_S \vec{E}(\vec{r})\bullet d\vec{A} = \int_v \overbrace{\left(\vec{\nabla}\bullet\vec{E}(\vec{r})\right)}d\tau' = \frac{Q_{encl}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int_v \overbrace{\rho(\vec{r})}d\tau'$$

This relation holds for <u>*any*</u> volume $v \Rightarrow$ the <u>*integrands*</u> of $\int_v(\;\;)d\tau'$ <u>*must*</u> be equal, i.e.:

$\therefore$ <u>Gauss' Law (in Differential Form)</u>: $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o}$

The DIVERGENCE OF $\vec{E}(\vec{r})$: $\vec{\nabla}\cdot\vec{E}(\vec{r})$

Calculate $\vec{\nabla}\cdot\vec{E}(\vec{r})$ directly from $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int_{\substack{v\\ all\\ space}} \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\rho(\vec{r}')\,d\tau'$

n.b. now extended over *all* space!

Remember that $\boldsymbol{r}$ is NOT a constant!

$\boldsymbol{r} \equiv \vec{r} - \vec{r}'$

field point P ($\vec{r}$); source point S ($\vec{r}'$)

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\cdot\left[\frac{1}{4\pi\varepsilon_o}\int_{\substack{v\\ all\\ space}}\left(\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right)\rho(\vec{r}')\,d\tau'\right] = \frac{1}{4\pi\varepsilon_o}\int_{\substack{v\\ all\\ space}}\vec{\nabla}\cdot\left(\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right)\rho(\vec{r}')\,d\tau'$$

Now: $\vec{\nabla}\cdot\left(\dfrac{\hat{r}}{r^2}\right) = 4\pi\underbrace{\delta^3(\vec{r})}_{\substack{3-D\\ Dirac\\ \delta-fcn.}}$ (see equation 1.100, Griffiths p. 50)

Thus: $\vec{\nabla}\cdot\left(\dfrac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = 4\pi\delta^3(\boldsymbol{r})$ or : $\vec{\nabla}\cdot\left(\dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = 4\pi\delta^3(\vec{r}-\vec{r}')$

$$\therefore\ \vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\cancel{4\pi}\,\varepsilon_o}\int_{\substack{v\\ all\\ space}}\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}')\rho(\vec{r}')\,d\tau' = \frac{\rho(\vec{r})}{\varepsilon_o}$$

Gauss' Law in Differential Form:

$$\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o}}$$

Gauss' Law in Integral Form:

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o},\ \text{thus: } \int_V\left(\vec{\nabla}\cdot\vec{E}(\vec{r}')\right)d\tau' = \int_v\left(\frac{\rho(\vec{r}')}{\varepsilon_o}\right)d\tau' = \frac{1}{\varepsilon_o}\int_v\rho(\vec{r}')\,d\tau' = \frac{1}{\varepsilon_o}Q_{encl}$$

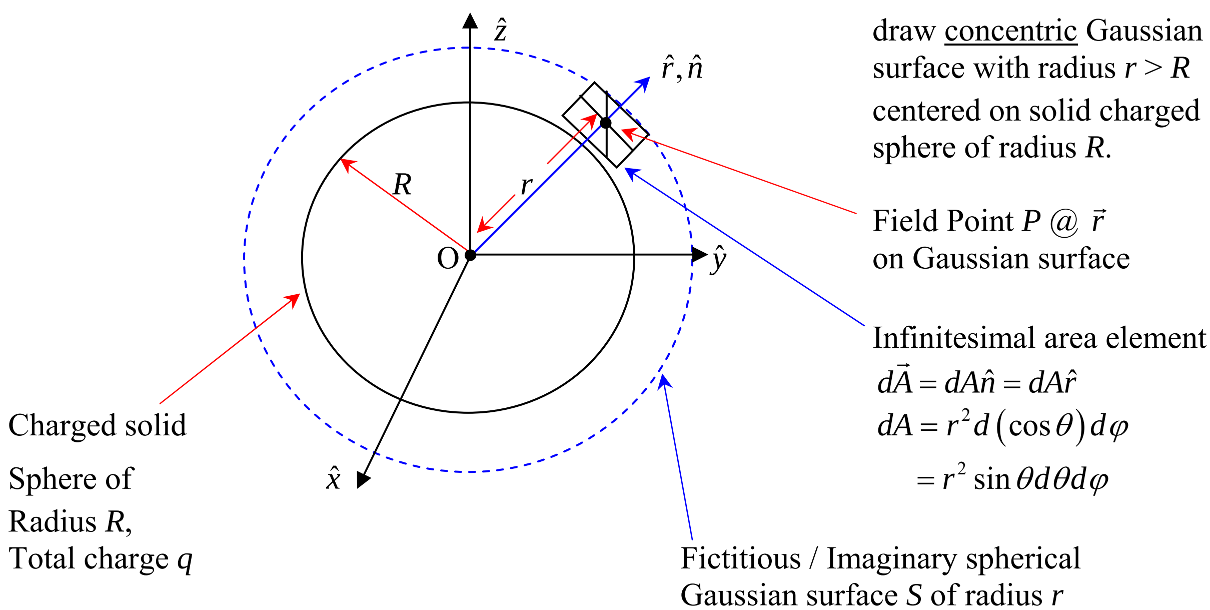
Now apply/use the Divergence Theorem on the volume integral associated with $\vec{\nabla}\cdot\vec{E}(\vec{r}')$:

$$\int_v\left(\vec{\nabla}\cdot\vec{E}(\vec{r}')\right)d\tau' = \oint_S\vec{E}(\vec{r})\cdot d\vec{A} = \frac{1}{\varepsilon_o}\int_v\rho(\vec{r}')\,d\tau' = \frac{1}{\varepsilon_o}Q_{encl}$$

Thus we obtain: $\boxed{\oint_S\vec{E}(\vec{r}')\cdot d\vec{A}' = \dfrac{Q_{encl}}{\varepsilon_o}}$ Gauss' Law in Integral Form

APPLICATIONS OF GAUSS' LAW - very explicit, detailed derivation -

Griffiths Example 2.2: Find / determine the electric field intensity $\vec{E}(\vec{r})$ outside a uniformly charged solid sphere of radius R and total charge q:



Gauss' Law: $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{1}{\varepsilon_o} Q_{encl} = \frac{1}{\varepsilon_o} q = \frac{q}{\varepsilon_o}$

n.b. by symmetry of sphere: $\vec{E}_{sphere}(r > R) = E(r)\hat{r}$ i.e. E must be radial!!

$\vec{E}(\vec{r}) = E(\vec{r})\hat{r}$ $d\vec{A} = dA\hat{n} = dA\hat{r}$ (for Gaussian sphere)

$$\therefore\ \vec{E}(\vec{r}) \cdot d\vec{A} = \left(E(\vec{r})\hat{r}\right) \cdot \left(dA\hat{r}\right) = E(\vec{r})\, dA \underbrace{\left(\hat{r} \cdot \hat{r}\right)}_{=1} = E(\vec{r})\, dA$$

NOTE: $E(\vec{r}) = \left|\vec{E}(\vec{r})\right| \Leftarrow$ n.b. Here again, by symmetry, the magnitude of $\vec{E}$ is constant $\forall$ (for all)/for any fixed r!!! (on the Gaussian spherical surface).

$$\therefore\ \oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \oint_S E(\vec{r})\, dA = q/\varepsilon_o$$

$$= E(\vec{r}) \oint_S dA = E(\vec{r})\left(4\pi r^2\right) = q/\varepsilon_o$$

$$\therefore\ E(\vec{r}) = q/4\pi\varepsilon_o r^2 = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2} \quad \text{or:} \quad \boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o r^2}\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}}$$

= Electric field outside a charged sphere of radius R at radial distance $r > R$ from center of sphere.

n.b. the electric field (for $r > R$) for charged sphere is equivalent / identical to that of a point charge q located at the origin!!!

GAUSS' LAW AND SYMMETRY

Use of (Geometrical / Reflection) symmetry (and any / all kinds of symmetry arguments in general) can be extremely powerful in terms of simplifying seemingly complicated problems!!

⇒ Learn skill of recognizing symmetries and applying symmetry arguments to solve problems!

Examples of use of Geometrical Symmetries and Gauss' Law

a) Charged sphere – use concentric Gaussian sphere and spherical coordinates
b) Charged cylinder – use coaxial Gaussian cylinder and cylindrical coordinates
c) Charged box / Charged plane – use appropriately co-located Gaussian "pillbox" (rectangular box) and rectangular coordinates
d) Charged ellipse – use concentric Gaussian ellipse and elliptical coordinates
e) Charged planar equilateral triangle } Think about
f) Charged pyramid } these!!

APPLICATIONS OF GAUSS' LAW (CONTINUED) - very explicit detailed derivation

Griffiths Example 2.3 Consider a long cylinder (e.g. plastic rod) of length L and radius S that carries a volume charge density ρ that is proportional to the distance from the axis s of the cylinder / rod – i.e.

$$\rho(s) = ks \quad \left(\frac{coulombs}{(meter)^3}\right)$$

k = proportionality constant $\left(\frac{coulombs}{(meter)^4}\right)$

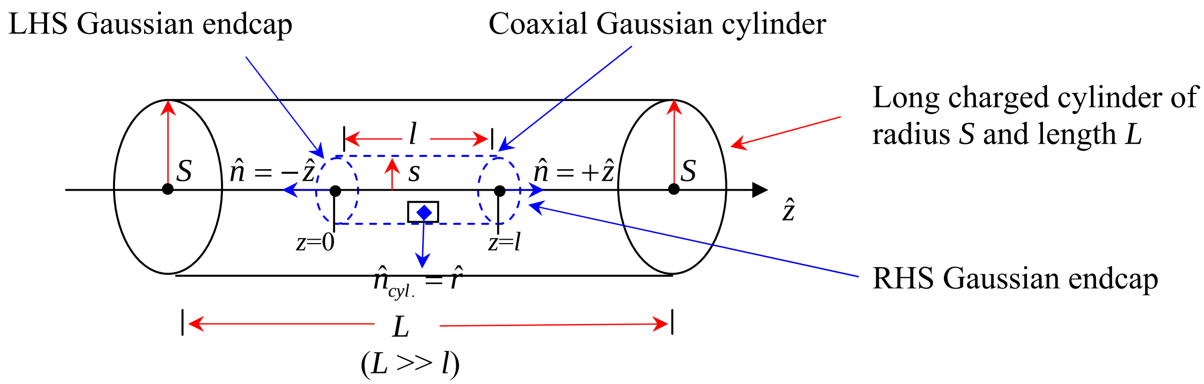
a) Determine the electric field $\vec{E}(\vec{r})$ inside this long cylinder / charged plastic rod
- Use a coaxial Gaussian cylinder of length l and radius s: (with $l \ll L$)

Gauss' Law $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_o}$

Enclosed charge: $Q_{encl} = \int_v \rho(s') d\tau' = \int_v (ks')(s'ds'd\varphi dz) \Leftarrow$ integral over Gaussian surface

$$Q_{encl} = \int_{s'=0}^{s'=s} \int_{\varphi=0}^{\varphi=2\pi} \int_{z=0}^{z=l} (ks')(s'ds'd\varphi dz) = 2\pi kl \int_{s'=0}^{s'=s} s'^2 ds'$$

$$Q_{encl} = \frac{2}{3}\pi k l s^3$$



Cylindrical Symmetry $\Rightarrow \vec{E}(\vec{r}) = E(\vec{r})\hat{r}$ (i.e. $\vec{E}$ points radially outward, $\perp$ to z-axis.)

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{\substack{\text{cylindrical}\\ \text{portion of}\\ \text{Gaussian}\\ \text{surface}}} \vec{E}(\vec{r})\cdot d\vec{A}_{cyl.} + \int_{\substack{\text{LHS endcap}\\ \text{portion of}\\ \text{Gaussian}\\ \text{surface}}} \vec{E}(\vec{r})\cdot d\vec{A}_{\substack{LHS\\ endcap}} + \int_{\substack{\text{RHS endcap}\\ \text{portion of}\\ \text{Gaussian}\\ \text{surface}}} \vec{E}(\vec{r})\cdot d\vec{A}_{\substack{RHS\\ endcap}}$$

Again, from cylindrical symmetry (here):
$E(\vec{r}) = |\vec{E}(\vec{r})|$ = constant on cylindrical Gaussian surface – i.e. fixed $r = |\vec{r}| = s$

What are $d\vec{A}_{cyl.}$, $d\vec{A}_{\substack{LHS\\ endcap}}$, and $d\vec{A}_{\substack{RHS\\ endcap}}$???

$d\vec{A}_{cyl.} = \underbrace{sdld\varphi}\,\hat{r} \leftarrow (\hat{n}_{cyl.} = \hat{r})$

infinitesimal surface area element of Gaussian cylinder

$d\vec{A}_{\substack{LHS\\ endcap}} = sdsd\varphi(-\hat{z}) = -sdsd\varphi\,\hat{z} \leftarrow (\hat{n}_{\substack{LHS\\ endcap}} = -\hat{z})$

$d\vec{A}_{\substack{RHS\\ endcap}} = sdsd\varphi(+\hat{z}) = +sdsd\varphi\,\hat{z} \leftarrow (\hat{n}_{\substack{RHS\\ endcap}} = +\hat{z})$

$$\therefore \oint_{\substack{S\\ \text{Gaussian}\\ \text{cylinder}}} \vec{E}(\vec{r})\cdot d\vec{A} = \underbrace{\int_{\substack{\text{Cyl.}\\ \text{Gaussian}\\ \text{surface}}} (E(\vec{r})\hat{r})\cdot(sdld\varphi\hat{r})}_{\hat{r}\cdot\hat{r}=1} + \underbrace{\int_{\substack{\text{LHS}\\ \text{Gaussian}\\ \text{endcap}}} (E(\vec{r})\hat{r})\cdot(-sdsd\varphi\hat{z})}_{\hat{r}\cdot\hat{z}=0} + \underbrace{\int_{\substack{\text{RHS}\\ \text{Gaussian}\\ \text{endcap}}} (E(\vec{r})\hat{r})\cdot(+sdsd\varphi\hat{z})}_{\hat{r}\cdot\hat{z}=0}$$

Note(s):
$E(\vec{r}) = |\vec{E}(\vec{r})|$ = constant on cylindrical Gaussian surface (fixed $r = s$)
$\vec{E}(\vec{r}) = E(\vec{r})\hat{r}$ by symmetry of charged cylinder
On LHS and RHS endcaps $\vec{E}(\vec{r})$ is ***not*** constant, because r is changing there - (but $\vec{E}$ still points in $\hat{r}$ direction! However, note that $\hat{r}\cdot\hat{r} = 1$ and $\hat{r}\cdot(\pm\hat{z}) \equiv 0 \Rightarrow$ Gaussian endcap terms do *not* contribute!!!

$$\therefore \oint_{\substack{S\\ \text{Gaussian}\\ \text{cylinder}}} \vec{E}(\vec{r})\cdot d\vec{A} = \int_{\substack{\text{cylindrical}\\ \text{Gaussian}\\ \text{surface}}} \overbrace{E(\vec{r})}^{\text{Constant here}}\, sdld\varphi = E(\vec{r})\,s\int_{z=0}^{z=l}\int_{\varphi=0}^{\varphi=2\pi} dld\varphi = E(\vec{r})\,sl(2\pi) = 2\pi slE(\vec{r})$$

Putting this all together now: $\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}$ where (here): $Q_{encl} = \frac{2}{3}\pi kls^3$

$$\cancel{2\pi}\cancel{s}\cancel{l}E(\vec{r}) = \frac{\cancel{2\pi}ks^{\cancel{4}^3}\cancel{l}}{3\varepsilon_o}$$ or:

inside
$\vec{E}_{in}(\vec{r}) = \frac{ks^2}{3\varepsilon_o}\hat{r}$
$(s = r < S)$

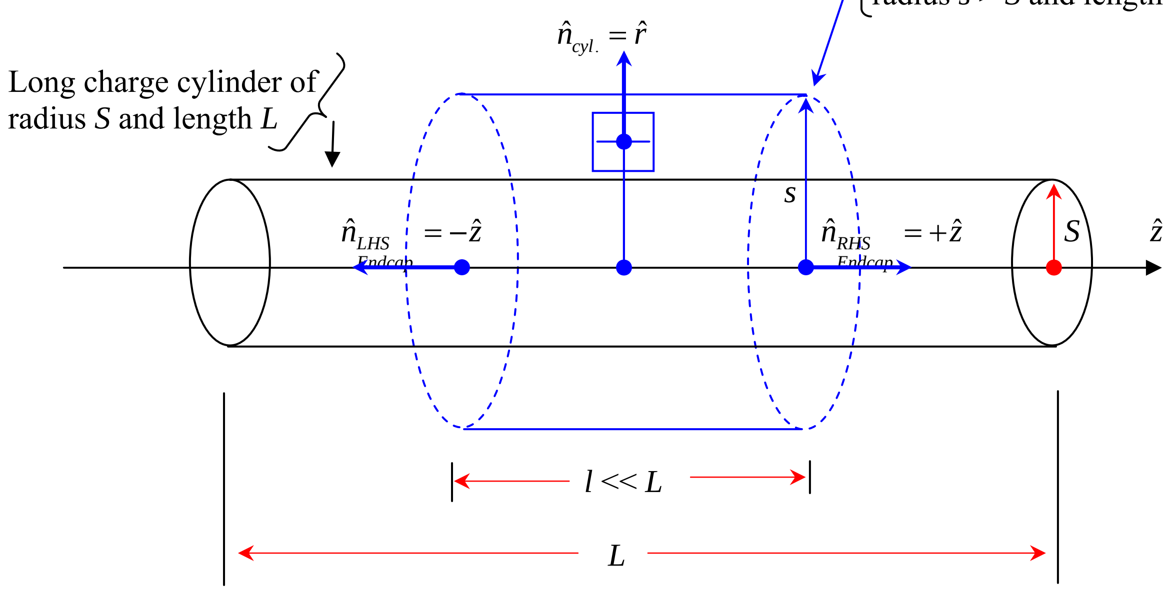
n.b. $(\hat{r} \equiv \hat{s})$ ← as used in Griffith's book, page 73

b) Find ELECTRIC FIELD $\vec{E}(\vec{r})$ <u>outside</u> of this long cylinder / charged plastic rod

Again, use Coaxial Gaussian cylinder of length l ($<< L$) and radius s ($> S$):

<u>Gauss' Law:</u> $\oint_S \vec{E}(\vec{r})\bullet d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}$

Enclosed charge (for $s > S$): $Q_{encl} = \frac{2}{3}\pi klS^3$



Again, from symmetry of long cylinder $\vec{E}(\vec{r}) = E(\vec{r})\hat{r} =$ constant (radial) direction!!

$r = s$ (fixed radius)

$$\oint_S \vec{E}(\vec{r})\bullet d\vec{A} = \int_{\substack{cylindrical\\Gaussian\\surface}} \vec{E}(\vec{r})\bullet d\vec{A}_{cyl} + \int_{\substack{LHS\\Gaussian\\endcap}} \vec{E}(\vec{r})\bullet d\vec{A}_{\substack{LHS\\endcap}} + \int_{\substack{RHS\\Gaussian\\endcap}} \vec{E}(\vec{r})\bullet d\vec{A}_{\substack{RHS\\endcap}}$$

$$d\vec{A}_{cyl} = sdld\varphi\,\hat{r} \qquad d\vec{A}_{\substack{LHS\\endcap}} = sdsd\varphi(-\hat{z}) = -sdsd\varphi\hat{z} = \left|d\vec{A}_{\substack{LHS\\endcap}}\right|(-\hat{z})$$

$$= \left|d\vec{A}_{cyl}\right|\hat{r} = dA_{cyl}\hat{r} \qquad d\vec{A}_{\substack{RHS\\endcap}} = sdsd\varphi(+\hat{z}) = +sdsd\varphi\hat{z} = \left|d\vec{A}_{\substack{RHS\\endcap}}\right|(+\hat{z})$$

<u>Now</u>: $\hat{r}\bullet\hat{r} = 1$ and $\hat{r}\bullet(\pm\hat{z}) \equiv 0$

Then:

$$\oint_S \vec{E}(\vec{r})\bullet d\vec{A} = \int_{\substack{cylindrical\\Gaussian\\surface}} \left(E(\vec{r})\hat{r}\right)\bullet\left(dA_{cyl}\hat{r}\right) + \int_{\substack{LHS\\Gaussian\\endcap}} \left(E(\vec{r})\hat{r}\right)\bullet\underbrace{\left(-dA_{\substack{LHS\\endcap}}\hat{z}\right)}_{=0} + \int_{\substack{RHS\\endcap}} \left(E(\vec{r})\hat{r}\right)\bullet\underbrace{\left(dA_{\substack{RHS\\endcap}}\hat{z}\right)}_{=0}$$

$$= E(\vec{r})\int_{z=0}^{z=l}\int_{\varphi=0}^{\varphi=2\pi} s\,dl\,d\varphi = 2\pi s l E(\vec{r})$$

$\therefore$ Electric field outside charged rod ($s = r > S$) : $E_{out}(\vec{r}) = \dfrac{2\pi k l S^3}{3\bullet 2\pi s l \varepsilon_o}\hat{r} = \dfrac{kS^3}{3s\varepsilon_o}$

ELECTRIC FIELD (INSIDE/OUTSIDE) vs. radial distance s	LONG CHARGED CYLINDER (radius S, $\rho(s) = ks$)

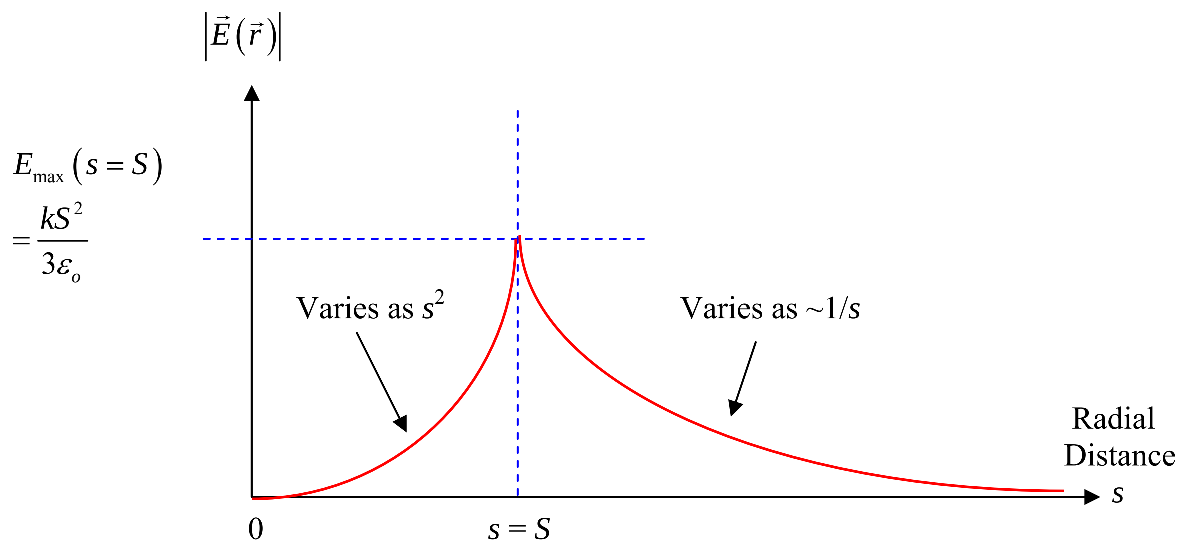
Inside ($s < S$):

$$\boxed{\vec{E}_{in}(\vec{r}) = \frac{ks^2}{3\varepsilon_o}\hat{s}}$$

Outside ($s > S$):

$$\boxed{\vec{E}_{out}(\vec{r}) = \frac{kS^3}{3\varepsilon_o}\left(\frac{1}{s}\right)\hat{s}} \quad (\hat{s} = \hat{r})$$

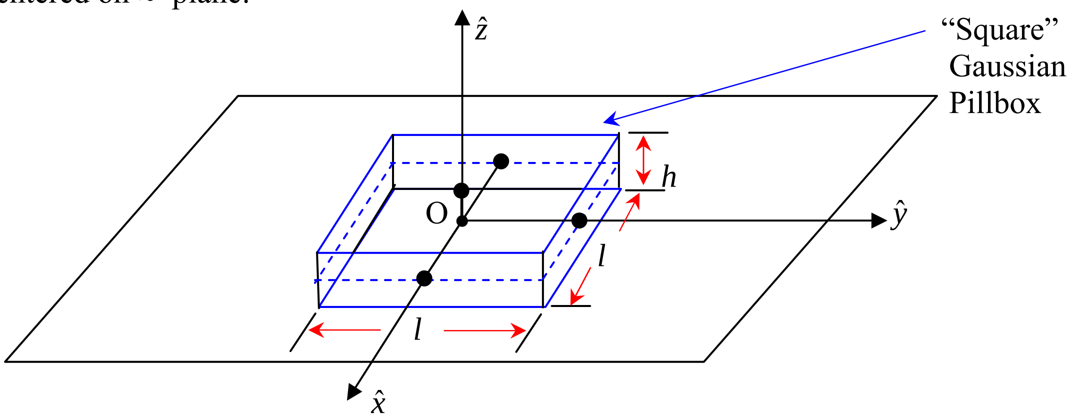
Make a plot of $\left|\vec{E}(\vec{r})\right|$ vs. radial distance s:



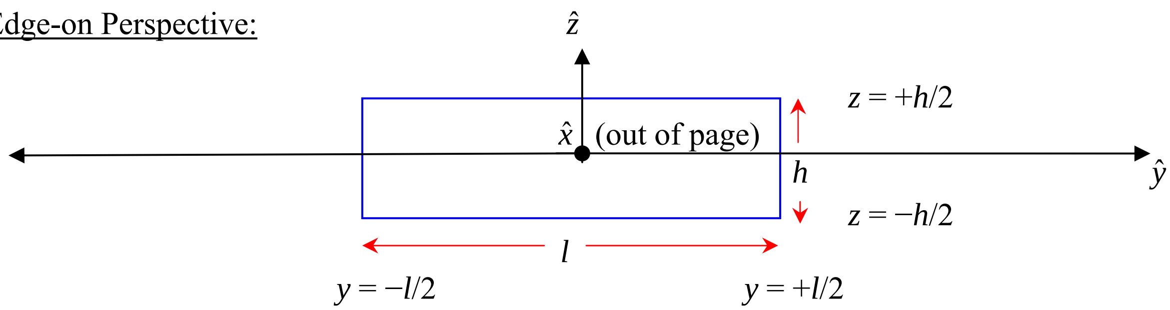
APPLICATIONS OF GAUSS' LAW - very explicit / detailed derivation –

Griffiths Example 2.4: An infinite plane carries uniform charge σ (coulombs / meter^2). Find the electric field a distance $z = z_o$ above (or below) the plane.

Use Gaussian Pillbox centered on ∞-plane:



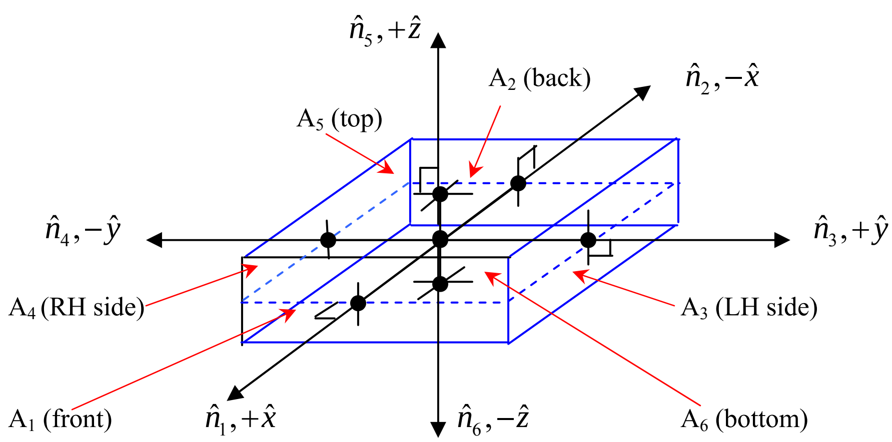
Edge-on Perspective:



Again, from the symmetry associated with ∞-plane,

$$\vec{E}(\vec{r}) = E(\vec{r})\hat{z} = E(z)\hat{z} \text{ (above plane),} \qquad = -E(z)\hat{z} \text{ (below plane)}$$

The Gaussian Pillbox has 6 sides – and thus has six outward unit normal vectors: :



Then:

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_{A_1} \vec{E}(\vec{r})\cdot d\vec{A}_1 + \int_{A_2} \vec{E}(\vec{r})\cdot d\vec{A}_2 + \int_{A_3} \vec{E}(\vec{r})\cdot d\vec{A}_3 + \int_{A_4} \vec{E}(\vec{r})\cdot d\vec{A}_4 + \int_{A_5} \vec{E}(\vec{r})\cdot d\vec{A}_5 + \int_{A_6} \vec{E}(\vec{r})\cdot d\vec{A}_6$$

$$d\vec{A}_1 = +dydz\,\hat{x} \qquad d\vec{A}_2 = dydz(-\hat{x}) = -dydz\,\hat{x}$$
$$d\vec{A}_3 = +dxdz\,\hat{y} \qquad d\vec{A}_4 = dxdz(-\hat{y}) = -dxdz\,\hat{y}$$
$$d\vec{A}_5 = +dxdy\,\hat{z} \qquad d\vec{A}_6 = dxdy(-\hat{z}) = -dxdy\,\hat{z}$$

for $z > 0$: $\vec{E}(\vec{r}) = +E(z)\hat{z}$

for $z < 0$: $\vec{E}(\vec{r}) = E(z)(-\hat{z}) = -E(z)\hat{z}$

} Again, by symmetry (of plane) n.b. $E(z)$ = constant (at least for fixed z).

Now because $\vec{E}(r) = \pm E(z)\hat{z}$ for $\begin{Bmatrix} z>0 \\ z<0 \end{Bmatrix}$ respectively, we must break up integrals over z into two separate regions: $\int_{z=-h/2}^{z=+h/2} dz = \int_{z=-h/2}^{z=0} dz + \int_{z=0}^{z=+h/2} dz$

Then:

$$\begin{aligned}\oint_S \vec{E}(\vec{r})\cdot d\vec{A} = &\int_{y=-l/2}^{y=+l/2}\int_{z=-h/2}^{z=+h/2} \vec{E}(\vec{r})\cdot(dydz\,\hat{x}) + \int_{y=-l/2}^{y=+l/2}\int_{z=-h/2}^{z=+h/2} \vec{E}(\vec{r})\cdot(-dydz\,\hat{x}) \\ &+\int_{x=-l/2}^{x=+l/2}\int_{z=-h/2}^{z=+h/2} \vec{E}(\vec{r})\cdot(dxdz\,\hat{y}) + \int_{x=-l/2}^{x=+l/2}\int_{z=-h/2}^{z=+h/2} \vec{E}(\vec{r})\cdot(-dxdz\,\hat{y}) \\ &+\int_{x=-l/2}^{x=+l/2}\int_{y=-l/2}^{y=+l/2} \vec{E}(\vec{r})\cdot(dxdy\,\hat{z}) + \int_{x=-l/2}^{x=+l/2}\int_{y=-l/2}^{y=+l/2} \vec{E}(\vec{r})\cdot(-dxdy\,\hat{z})\end{aligned}$$

$$\begin{aligned}\oint_S \vec{E}(\vec{r})\bullet d\vec{A} = &\int_{y=-l/2}^{y=+l/2}\left[\int_{z=-h/2}^{z=0}\left(-E(z)\,\hat{z}\cdot\hat{x}\right)dydz + \int_{z=0}^{z=+h/2}\left(+E(z)\,\hat{z}\cdot\hat{x}\right)dydz\right] \leftarrow \text{side } A_1 \text{ (front)} \\ &+\int_{y=-l/2}^{y=+l/2}\left[\int_{z=-h/2}^{z=0}\left(-E(z)\,\hat{z}\cdot-\hat{x}\right)dydz + \int_{z=0}^{z=+h/2}\left(+E(z)\,\hat{z}\cdot-\hat{x}\right)dydz\right] \leftarrow \text{side } A_2 \text{ (back)} \\ &+\int_{x=-l/2}^{x=+l/2}\left[\int_{z=-h/2}^{z=0}\left(-E(z)\,\hat{z}\cdot\hat{y}\right)dxdz + \int_{z=0}^{z=+h/2}\left(+E(z)\,\hat{z}\cdot\hat{y}\right)dxdz\right] \leftarrow \text{side } A_3 \text{ (RHS)} \\ &+\int_{x=-l/2}^{x=+l/2}\left[\int_{z=-h/2}^{z=0}\left(-E(z)\,\hat{z}\cdot-\hat{y}\right)dxdz + \int_{z=0}^{z=+h/2}\left(+E(z)\,\hat{z}\cdot-\hat{y}\right)dxdz\right] \leftarrow \text{side } A_4 \text{ (LHS)} \\ &+\underbrace{\int_{x=-l/2}^{x=+l/2}\int_{y=-l/2}^{y=+l/2}\left(-E(z)\,\hat{z}\cdot-\hat{z}\right)dxdy}_{\text{side } A_6 \text{ (bottom)}} + \underbrace{\int_{x=-l/2}^{x=+l/2}\int_{y=-l/2}^{y=+l/2}\left(E(z)\,\hat{z}\cdot\hat{z}\right)dxdy}_{\text{side } A_5 \text{ (top)}}\end{aligned}$$

Now: $(\hat{z}\cdot\hat{x}) = 0 \qquad (\hat{z}\cdot\hat{y}) = 0 \qquad (\hat{x}\cdot\hat{z}) = 0 \qquad (\hat{y}\cdot\hat{z}) = 0$ etc.

And: $(\hat{x}\cdot\hat{x}) = 1 \qquad (\hat{y}\cdot\hat{y}) = 1 \qquad (\hat{z}\cdot\hat{z}) = 1$

$\therefore$ Because $\hat{x} \perp \hat{y} \perp \hat{z}$, no contributions to $\oint_S \vec{E} \bullet d\vec{A}$ (here) from 4 sides of Gaussian Pillbox (i.e. A_1, A_2, A_3 and A_4)

$\Rightarrow$ Only remaining / non-zero contributions are from bottom and top surfaces of Gaussian Pillbox because $\hat{n}_5 = -\hat{z}$ and $\hat{n}_6 = +\hat{z}$ which are $\parallel$ (or anti-parallel) to $E(z)\hat{z}$

Thus, we only have (here):

$$\oint_S \vec{E}(\vec{r}) \bullet d\vec{A} = \int_{x=-l/2}^{x=+l/2} \int_{y=-l/2}^{y=+l/2} \left(-E(z)\hat{z} \bullet (-\hat{z})\right) dxdy \quad \leftarrow \text{ side } A_6 \text{ (bottom)}$$

$$+\int_{x=-l/2}^{x=+l/2} \int_{y=-l/2}^{y=+l/2} \left(+E(z)\hat{z} \bullet \hat{z}\right) dxdy \quad \leftarrow \text{ side } A_5 \text{ (top)}$$

These integrals are not over z, and $E(z)$ = constant for z = fixed = z_o

$\therefore$ can pull $E(z)$ outside integral, $\hat{z} \bullet \hat{z} = 1$ $\quad -\hat{z} \bullet \hat{z} = -1$ $\quad$ etc.

$$\therefore \oint_S \vec{E}(\vec{r}) \bullet d\vec{A} = +E(z) \int_{x=-l/2}^{x=+l/2} \int_{y=-l/2}^{y=+l/2} dxdy \quad \leftarrow \text{ side } A_6 \text{ (bottom)}$$

$$+E(z) \int_{x=-l/2}^{x=+l/2} \int_{y=-l/2}^{y=+l/2} dxdy \quad \leftarrow \text{ side } A_5 \text{ (top)}$$

$$= E(z)l^2 + E(z)l^2 = 2E(z)l^2$$

<u>But</u>: $l^2 = l \times l \equiv \text{A} =$ surface area of top and bottom surfaces of Gaussian Pillbox

<u>Now</u>: $\oint_S \vec{E}(\vec{r}) \bullet d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_o}$ What is Q_{encl} (by Gaussian Pillbox)?

$$Q_{encl} = \sigma\left(\frac{Coulombs}{meter^2}\right) \times \text{A}\left(meters^2\right) = \sigma l^2 \left(Coulombs\right)$$

$$\therefore \oint_S \vec{E}(\vec{r}) \bullet d\vec{A} = \frac{Q_{encl}}{\varepsilon_o} \Rightarrow 2E(z)\cancel{l^2} = \sigma \cancel{l^2} / \varepsilon_o$$

<u>or:</u>

$$\boxed{E(z) = \left(\frac{1}{2}\right)\sigma/\varepsilon_o = \frac{\sigma}{2\varepsilon_o}}$$

<u>Vectorially:</u>

$$\boxed{\vec{E}(z) = \left(\sigma/2\varepsilon_o\right)\begin{Bmatrix} +\hat{z}, \text{ for } z>0 \\ -\hat{z}, \text{ for } z<0 \end{Bmatrix}}$$

<u>NOTE:</u> $\left|\vec{E}(z)\right|$ = constant!!

<u>No</u> z – dependence for charged ∞ plane!

$\vec{E}(\vec{r})$ from ∞ - plane (slight return):

Note that in the initial process of setting up the Gaussian Pillbox, if we'd shrunk the height *h* of the Pillbox to be infinitesimally small, i.e. $h \to \delta h$ and then took the limit $\delta h \to 0$, the contributions to $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A}$ from (infinitesimally small) sides of (A_1, A_2, A_3 and A_4) Gaussian Pillbox would (formally) have vanished (i.e. = 0) independently of whether integrand $\left(\vec{E}(\vec{r}) \cdot d\vec{A}\right)$ vanished on these sides (or not). Only top and bottom surfaces contribute to $\oint_S \vec{E}(\vec{r}) \cdot d\vec{A}$ then (here).

So using this "trick" of the shrinking Pillbox at a surface / boundary very often can be useful, to simplify doing the problem.

This explicitly shows that (sometimes) there *is* more than one way to correctly do / solve a problem – equivalent methods *may* exist.

→ It is very important, conceptually-speaking to have a (very) clear / good understanding of how to do these Gauss' Law-type problems the "long' way and the "short" way!

The Curl of $\vec{E}(\vec{r})$: $\left(\vec{\nabla}\times\vec{E}(\vec{r})\right)$

First, study / consider simplest possible situation: point charge *at origin*: $\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)\hat{r}$

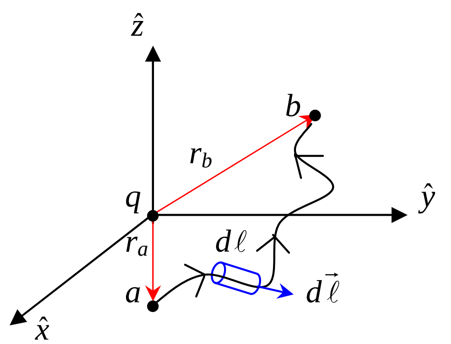
(note: $\boldsymbol{r}\equiv\vec{r}-\vec{r}'=\vec{r}$ here because $\vec{r}'=0$ - charge q located at origin!!!)

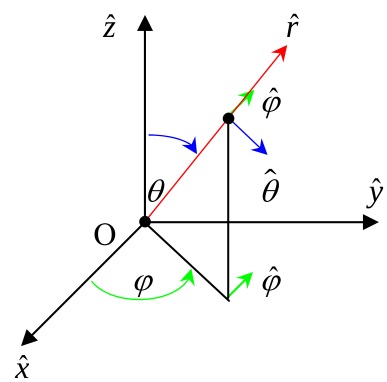
Thus (here), $\vec{E}(\vec{r})$ is radial (i.e. in $\hat{r}-$direction) due to spherical symmetry of problem (rotational invariance), thus static $\vec{E}$-field has no rotation/swirl/whirl $\Rightarrow$ no curl! (Read Griffith's Ch. 1 on curl)

$\Rightarrow\vec{\nabla}\times\vec{E}(\vec{r})=0$ (*must* = 0)

Let's calculate:

Line integral $\int_a^b\vec{E}(\vec{r})\cdot d\vec{\ell}$ as shown in figure below:





In spherical coordinates: $d\vec{\ell}=dr\hat{r}+rd\theta\hat{\theta}+r\sin\theta d\varphi\hat{\varphi}$

$$\vec{E}(\vec{r})\cdot d\vec{\ell}=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)\hat{r}\cdot\left\{dr\hat{r}+rd\theta\hat{\theta}+r\sin\theta\varphi\hat{\varphi}\right\}$$

Again:

$\hat{r}\cdot\hat{r}=1$	$\hat{r}\cdot\hat{\theta}=0$	$\hat{r}\cdot\hat{\varphi}=0$
$\hat{\theta}\cdot\hat{\theta}=1$	$\hat{\theta}\cdot\hat{r}=0$	$\hat{\theta}\cdot\hat{\varphi}=0$
$\hat{\varphi}\cdot\hat{\varphi}=1$	$\hat{\varphi}\cdot\hat{r}=0$	$\hat{\varphi}\cdot\hat{\theta}=0$

($\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ are mutually orthogonal basis vectors (form *ortho-normal* basis))

$$\therefore\ \vec{E}(\vec{r})\cdot d\vec{\ell}=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)dr$$

Thus: $$\int_a^b\vec{E}(\vec{r})\cdot d\vec{\ell}=\frac{1}{4\pi\varepsilon_o}\int_a^b\frac{q}{r^2}dr=\frac{-1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)\Bigg|_{r_a}^{r_b}=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r_a}-\frac{q}{r_b}\right)=\frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

r_a = distance from origin O to point a. r_b = distance from origin O to point b.

The line integral $\oint\vec{E}(\vec{r})\cdot d\vec{\ell}$ around a *closed* contour C is zero!

i.e. $\boxed{\oint_C \vec{E}(\vec{r})\cdot d\vec{\ell} = 0}$ This is not a trivial result! (Not true $\forall$ vectors!!)

(But *is* true for *static* $\vec{E}$-fields)

Use Stokes' Theorem (See Griffiths, Ch. 1.3.5, p. 34 and Appendix A-5)

$$\boxed{\int_S \left(\vec{\nabla}\times\vec{E}(\vec{r})\right)\cdot d\vec{A} = \oint_C \vec{E}(\vec{r})\cdot d\vec{\ell} = 0}$$

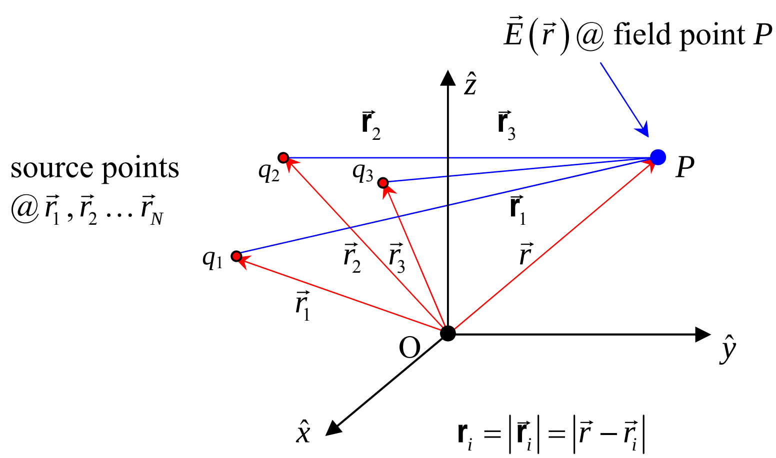
arbitrary closed surface S — arbitrary closed contour C (on S)

Since $\int_S \left(\vec{\nabla}\times\vec{E}(\vec{r})\right)\cdot d\vec{A} = 0$ must be / is true for *arbitrary* closed surface S, this can *only* be true for all $\forall$ closed surfaces S IFF (if and only if): $\boxed{\vec{\nabla}\times\vec{E}(\vec{r}) = 0}$

Can use the Principle of Superposition to show that:

$$\vec{E}_{TOT}(\vec{r}) = \sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathbf{r}_i^2}\hat{\mathbf{r}}_i \quad \leftarrow i = 1,2,3\ldots N \text{ discrete charges, and } \vec{\mathbf{r}}_i = (\vec{r}-\vec{r}_i)$$

$$= \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \vec{E}_3(\vec{r}) + \ldots + \vec{E}_N(\vec{r})$$



Then: $\vec{\nabla}\times\vec{E}_{TOT}(\vec{r}) = \vec{\nabla}\times\sum_{i=1}^{N}\vec{E}_i(\vec{r}) = \sum_{i=1}^{N}\left(\vec{\nabla}\times\vec{E}_i(\vec{r})\right)$

$$= \sum_{i=1}^{N}\vec{\nabla}\times\left(\frac{1}{4\pi\varepsilon_o}\left(\frac{q_i}{\mathbf{r}_i^2}\right)\hat{\mathbf{r}}_i\right) = 0 \quad \Leftarrow \text{n.b. all individual terms} = 0\ !!!$$

or: $\vec{\nabla}\times\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}q_i\vec{\nabla}\times\left(\frac{1}{\mathbf{r}_i^2}\right)\hat{\mathbf{r}}_i = 0$

It can be shown that $\boxed{\vec{\nabla}\times\vec{E}(\vec{r}) = 0}$

FOR ANY STATIC CHARGE DISTRIBUTION
STATIC = NO TIME DEPENDENCE / VARIATION

$\vec{\nabla}\times\vec{E}(\vec{r})=0$ <u>HOLDS FOR:</u>

- Static Discrete/Point Charges $q(\vec{r})$
- Static Line Charges $\lambda(\vec{r})$
- Static Surface Charges $\sigma(\vec{r})$
- Static Volume Charges $\rho(\vec{r})$

} <u>All</u> <u>*Static*</u> Charge Distributions

Again, this *<u>not</u>* trivial (we'll see why, soon. . .)

One other (very important) point about the mathematical & geometrical nature of vector fields:

The nature of a (physically-realizable) vector field $\vec{A}(\vec{r})$ is fully specified if *<u>both</u>* its divergence $\vec{\nabla}\cdot\vec{A}(\vec{r})$ and its curl $\vec{\nabla}\times\vec{A}(\vec{r})$ are known.

This is a consequence of the so-called <u>Helmholtz</u> <u>theorem</u> – see/read <u>Appendix *B*</u> of Griffiths book.

The Helmholtz theorem also has an important <u>corollary</u>:

Any differentiable vector function $\vec{A}(\vec{r})$ that goes to zero faster than $1/r$ as $r\to\infty$ can be expressed as the gradient of a scalar plus the curl of a vector:

$$\boxed{\vec{A}(\vec{r})=\vec{\nabla}\left(-\frac{1}{4\pi}\int_{v'}\frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{\mathscr{r}}d\tau'\right)+\vec{\nabla}\times\left(\frac{1}{4\pi}\int_{v'}\frac{\vec{\nabla}'\times\vec{A}(\vec{r}')}{\mathscr{r}}d\tau'\right)}$$

For the case of <u>electrostatics</u>: $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r})=\rho(\vec{r})/\varepsilon_o}$ and $\boxed{\vec{\nabla}\times\vec{E}(\vec{r})=0}$

<u>Thus</u>:

$$\vec{E}(\vec{r})=\vec{\nabla}\left(-\frac{1}{4\pi}\int_{v'}\frac{\vec{\nabla}'\cdot\vec{E}(\vec{r}')}{\mathscr{r}}d\tau'\right)+\vec{\nabla}\times\left(\frac{1}{4\pi}\int_{v'}\frac{\cancel{\vec{\nabla}'\times\vec{E}(\vec{r}')}}{\mathscr{r}}d\tau'\right)$$
$$=-\frac{1}{4\pi\varepsilon_o}\vec{\nabla}\left(\int_{v'}\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau'\right)=-\vec{\nabla}V(\vec{r})$$

i.e. $\boxed{\vec{E}(\vec{r})=-\vec{\nabla}V(\vec{r})}$ with $\boxed{V(\vec{r})\equiv\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau'}$ = Electrostatic <u>Potential</u> [SI Units: Volts]

This result is valid e.g. in electrostatics for <u>localized</u> (i.e. finite spatial extent) charge distributions.

For <u>infinite-expanse</u> charge distributions (n.b. these are unphysical/artificial!), we must appeal to (more sophisticated) mathematical formalisms than the Helmholtz theorem…