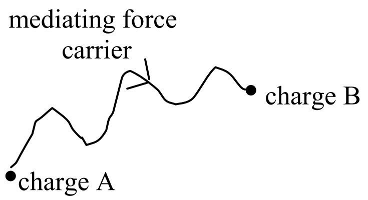
LECTURE NOTES 1

Introduction:

- In this course, we will study/investigate the nature of the ELECTROMAGNETIC INTERACTION (at {very} low energies, i.e. $E \sim 0$ GeV, {1 GeV = 10^9 electron volts = 1.602×10^{-10} Joules}).
- The electromagnetic interaction is ONE of FOUR known FORCES (or INTERACTIONS) of Nature:
  1) Electromagnetic Force – binds electrons & nuclei together to form atoms
     - binds atoms together to form $\underbrace{\text{molecules}}_{\text{gases}}$, liquids, solids. . . .
  2) Strong Force – binds protons & neutrons together to form nuclei
  3) Weak Force – responsible for radioactivity (e.g. β decay) (weak force important @ high energies)
  4) Gravity – binds matter together to form stars, planets, solar systems, galaxies, etc.

- At the MICROSCOPIC (i.e. QUANTUM) LEVEL (elementary particle physics) the forces of nature are mediated by the exchange of a "force-carrying" particle e.g. between two "charged" particles:



Quantum Field Theory	Force	Force Mediator	Force Type	Mass of force mediator	Range of force mediator	Intrinsic spin of force mediator	Charge associated w/ force
QED	1) EM	single PHOTON	attractive & repulsive	≡ 0.000	∞	$1\hbar$	$\pm e$
QCD	2) STRONG	octet of GLUONs	attractive & repulsive	≡ 0.000	~ 1fm	$1\hbar$	$\left\{\begin{matrix} r, g, b \\ \bar{r}, \bar{g}, \bar{b} \end{matrix}\right\}$
QWD	3) WEAK	$W^{\pm}$, Z^o	attractive & repulsive	$M_W \approx 80.4$ GeV/c^2 $M_Z \approx 91.2$ GeV/c^2	~ 1fm	$1\hbar$	$\pm g_W$
QGD	4) GRAVITY	single GRAVITON	attractive only	≡ 0.000	∞	$2\hbar$	MASS, m (unquantized)

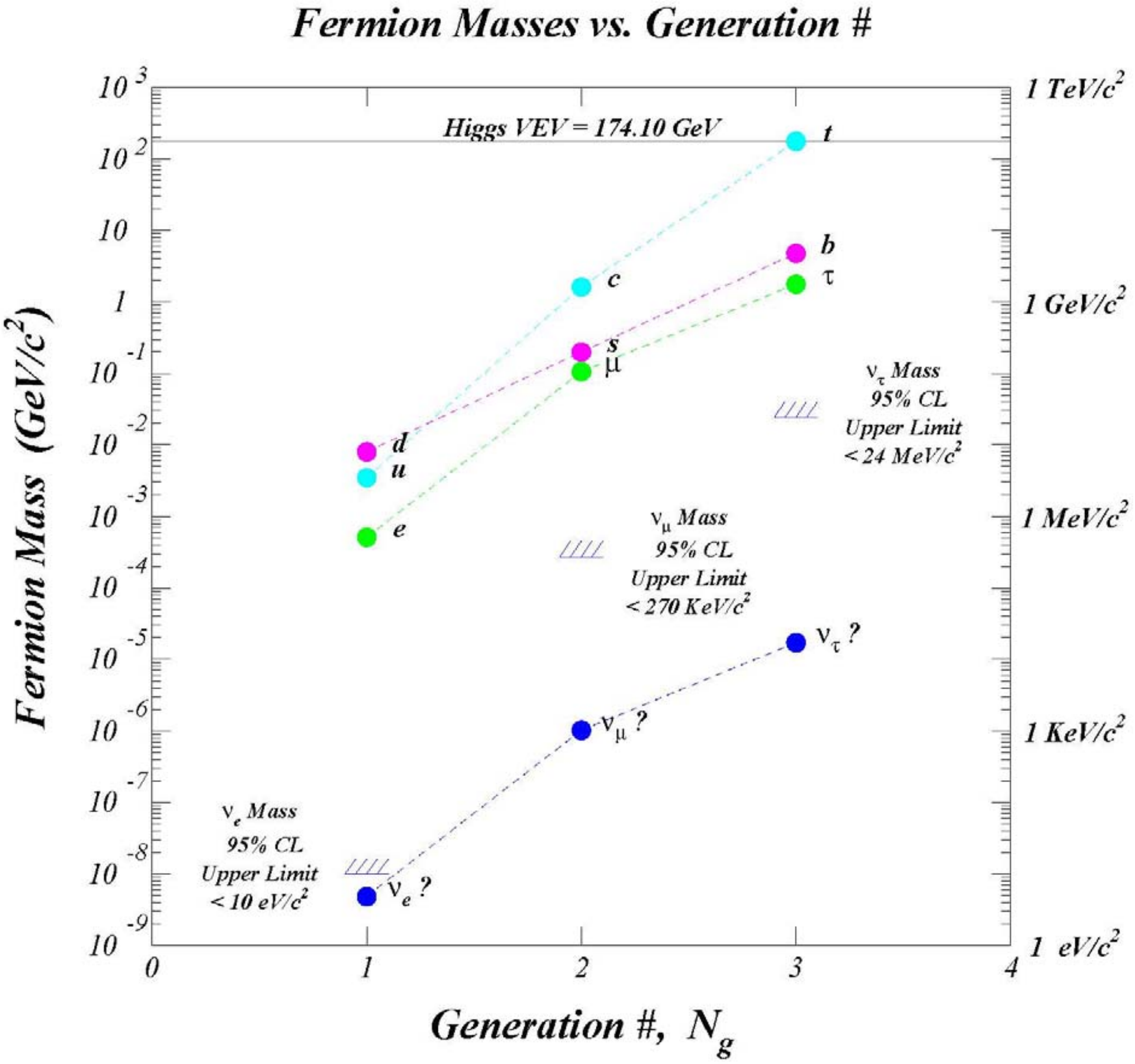
At high energies, QED & QWD unify to become a single force, known as the ELECTROWEAK FORCE

$\hbar$ = Planck's constant divided by 2π = $h/2\pi$ = 1.0546 x 10^{-34} Joule – seconds

m_{proton}= 0.93 GeV/c^2 = 1.67262158×10^{-27} kg

1 fm = 1 femto-meter = 1 Fermi = 10^{-15}meters

Pattern of Masses for Fundamental, Spin-½ Matter Particles - Fermions



"doublets" of quarks: $\begin{pmatrix}u\\ d\end{pmatrix}\ \begin{pmatrix}c\\ s\end{pmatrix}\ \begin{pmatrix}t\\ b\end{pmatrix}\ \begin{array}{l}\leftarrow \text{ have electric charge } +2/3\\ \leftarrow \text{ have electric charge } -1/3\end{array}\Bigg\}$ fractional electric charge charge!!!

Each quark comes in 3 strong ("color") charges: red, green, blue

"doublet" of anti-quarks: $\begin{pmatrix}\overline{u}\\ \overline{d}\end{pmatrix}\ \begin{pmatrix}\overline{c}\\ \overline{s}\end{pmatrix}\ \begin{pmatrix}\overline{t}\\ \overline{b}\end{pmatrix}\ \begin{array}{l}\leftarrow \quad q = -2/3\\ \leftarrow \quad q = +1/3\end{array}\Bigg\}$ with 3 anti-color charges:
red, green, blue
(i.e. anti-red, anti-green, anti-blue)

Questions:

Why are there 3 generations of quarks & leptons? Internal Quantum #? Why not just one? Are there more? (seemingly not…)

What physics is responsible for the observed *pattern* of quark/lepton masses?

Why are there *four* forces of nature? Why not just one? Are there more forces?

Note that ALL 4 fundamental forces of nature have both electric & magnetic fields!!!

"Magnetic" field arises from *motion* of "electric" charge in space – relativity & space-time involved here!

FORCE	"ELECTRIC" FIELD	"MAGNETIC" FIELD
EM	EM – electric	EM – magnetic
STRONG	chromo–electric	chromo–magnetic
WEAK	weak–electric	weak–magnetic
GRAVITY	gravito–electric	gravito–magnetic

$\underbrace{\text{gravito–magnetic}}$

Nordvedt Effect
e.g. affects motion of
moon's orbit around
earth (very small)

no motion/movement

Electric Field – time-averaged field (macroscopic) present for static charges exchanging virtual quanta associated w/given force

Magnetic Field – time averaged field (macroscopic) arises/associated w/moving charges – motional effect

$\underbrace{\text{Magnetic field}}_{\text{Any/all/each of 4 fundamental forces of nature}}$ arises from motion of $\underbrace{\text{charge}}_{\text{any/all/each of 4 fundamental forces of nature}}$

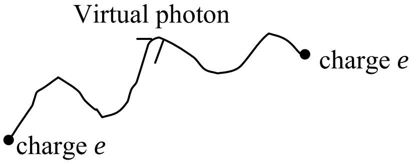
Any/all/each of
4 fundamental
forces of nature

any/all/each of
4 fundamental
forces of nature

$\overbrace{\text{Magnetic field}}$ results from $\overbrace{\text{charge}}$ + space-time structure of our universe!!

At microscopic level, EM force mediated by (virtual) photons

- two electrically charged particles "know" about each other by exchanging virtual photons.



Virtual photons carry linear momentum, $p_{\gamma_v}\left(= \overset{\text{Planck's constant}}{h} \Big/ \underset{\text{DeBroglie wavelength}}{\lambda_{\gamma_v}}\right)$

but have *zero total energy*:

$\boxed{E_{\gamma_v}^2 = p_{\gamma_v}^2 c^2 + m_{\gamma_v}^2 c^4 = 0}$ c = speed of light = 3 × 10^8 m/sec

$$\boxed{\begin{array}{l}\text{Real Photons}\\ \text{(e.g. visible light):}\\ E_{\gamma_R}^2 = p_{\gamma_R}^2 c^2 \quad m_{\gamma_R} = 0\\ p_{\gamma_R} = h/\lambda_{\gamma_R}\ ,\ E_{\gamma_R} = hf_{\gamma_R} > 0\end{array}}$$

If $E_{\gamma_v} = 0$, then $p_{\gamma_v}^2 c^2 = -m_{\gamma_v}^2 c^4$

i.e. $p_{\gamma_v} = \pm i m_{\gamma_v} c^2$ $i=\sqrt{-1}$ complex!

If $E_{\gamma_v} = 0$ then: $E_{\gamma_v} = hf_{\gamma_v} = 0 \Rightarrow f_{\gamma_v} = 0$ virtual photons have zero frequency, but have non-zero DeBroglie wavelength, $\lambda_{\gamma_v} > 0$!

FORCE: $\vec{F} = m\vec{a}$ (Newton's 2nd Law)

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{\Delta\vec{p}}{\Delta t} = \frac{d\left(m_{\gamma_v}\vec{v}_{\gamma_v}\right)}{dt} = m_{\gamma_t}\frac{d\vec{v}_{\gamma_r}}{dt} = m_{\gamma_t}\vec{a}$$

electric charges emit & absorb virtual photons (lots of them!!!)

− each such photon carries with it momentum, P_{γ_v}

− depending on sign of momentum (emitted/absorbed), a net force will result, acting on each charged particle

Like charges – repulsive: $\vec{F}_{e_1^+} \longleftarrow \bullet\, e_1^+ \qquad e_2^+\, \bullet \longrightarrow \vec{F}_{e_2^+}$

Opposite charges – attract: $e_1^+ \bullet \underset{\vec{F}_{e_1^+}}{\longrightarrow} \quad \underset{\vec{F}_{e_2^-}}{\longleftarrow} \bullet\, e_2^-$

n.b. Your own body *can* sense virtual photons!!!

- Get your comb out, comb your hair several times - charges up comb via static electricity
- Bring comb near to e.g. hair on your forearm & feel the pull on forearm hairs from electric charge on comb (works best in winter/dry conditions).

ELECTROSTATIC FIELDS IN A VACUUM

COULOMB'S LAW

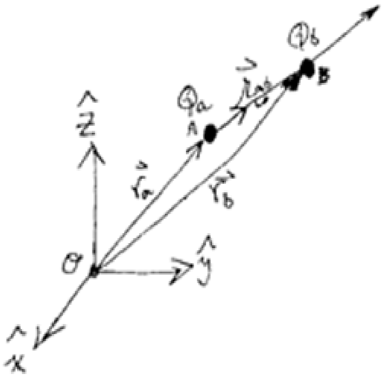
It has been experimentally observed (Charles Augustin Coulomb, 1785) that the net, time-averaged force (i.e. summed over many, many virtual photons) between two stationary point charges Q_a & Q_b:

1) Acts along the line joining the two point charges, Q_a & Q_b (i.e. radial force!)
2) Is linearly proportional to the *product* of the two point charges, $Q_a * Q_b$
   (n.b. Force is *charge-signed*!)
   – Net force is *repulsive* if Q_a is *same* sign as Q_b.
   – Net force is *attractive* if Q_a is *opposite* sign as Q_b.
3) Is inversely proportional to the square of the separation distance, $\boldsymbol{r}_{ab} \equiv |\vec{r}_b - \vec{r}_a| = |\triangle\vec{r}_{ab}|$ between the two point charges.

The net force exerted by point charge Q_a ON point charged Q_b is given by:

$$\vec{F}_{ab} = \underset{\substack{\uparrow \\ \text{constant of} \\ \text{proportionality}}}{K} \frac{Q_a Q_b}{\boldsymbol{r}_{ab}^2} \underset{\substack{\uparrow \\ \text{unit vector}}}{\hat{\boldsymbol{r}}_{ab}} \quad \text{(SI UNITS – Newtons)}$$

(points from Q_a at $\vec{r}_a$ to Q_b at $\vec{r}_b$)



$$\underset{\looparrowleft}{\vec{\boldsymbol{r}}_{ab}} \equiv \vec{r}_b - \vec{r}_a = \underset{\looparrowleft}{\triangle\vec{r}_{ab}}$$

$$\hat{\boldsymbol{r}}_{ab} \equiv \frac{\vec{\boldsymbol{r}}_{ab}}{|\vec{\boldsymbol{r}}_{ab}|} = \frac{\vec{\boldsymbol{r}}_{ab}}{\boldsymbol{r}_{ab}} = \text{ unit vector pointing from point } A \text{ to point } B.$$

$\underset{\looparrowleft}{\vec{F}_{ab}}$ is a radial force, one which points from (to) point A to (from) point B, depending on sign of the charge product (Q_aQ_b)

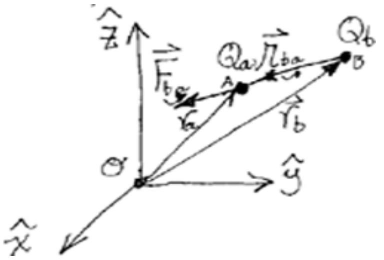
$Q_a Q_b < 0$ is *attractive* force ($F < 0$)
$Q_a Q_b > 0$ is *repulsive* force ($F > 0$)

The NET force exerted by point charge Q_b ON point charge Q_a:

$$\underset{\circlearrowleft}{\vec{F}_{ba}} = K\frac{Q_b Q_a}{\boldsymbol{r}_{ba}^2}\underset{\circlearrowleft}{\hat{\boldsymbol{r}}_{ba}}$$

$\vec{F}_{ba}$ is radial force, point from (to) point B to (from) point A, depending on sign of charge product ($Q_a Q_b$)

$Q_a\, Q_b < 0$ is *attractive* force ($F < 0$)
$Q_b\, Q_a > 0$ is *repulsive* force ($F > 0$)



$$\boldsymbol{r}_{ba} \equiv \vec{r}_a - \vec{r}_b = \Delta\vec{r}_{ba} \qquad \hat{\boldsymbol{r}}_{ba} \equiv \frac{\boldsymbol{r}_{ba}}{\left|\boldsymbol{r}_{ba}\right|} = \frac{\boldsymbol{r}_{ba}}{\boldsymbol{r}_{ba}} = -\hat{\boldsymbol{r}}_{ab}$$

$$\vec{F}_{ab} = K\frac{Q_a\ Q_b}{\boldsymbol{r}_{ab}^2}\hat{\boldsymbol{r}}_{ab} \qquad \vec{F}_{ba} = K\frac{Q_b\ Q_a}{\boldsymbol{r}_{ba}^2}\hat{\boldsymbol{r}}_{ba}$$

Now $\mathbf{r}_{ab} = \mathbf{r}_{ba}$, since $\boldsymbol{r}_{ab} \equiv \left|\vec{r}_b - \vec{r}_a\right| = \left|\Delta\vec{r}_{ab}\right| \quad and \quad \boldsymbol{r}_{ba} = \left|\vec{r}_a - \vec{r}_b\right| = \left|\Delta\vec{r}_{ba}\right|$

but note that $\underset{\circlearrowleft}{\vec{r}_{ba}} = -\underset{\circlearrowleft}{\vec{r}_{ab}}$

and/or $\underset{\circlearrowleft}{\boldsymbol{r}_{ba}} = -\underset{\circlearrowleft}{\boldsymbol{r}_{ab}}$ since $\left(\vec{r}_b - \vec{r}_a\right) = -\left(\vec{r}_a - \vec{r}_b\right) = \Delta\vec{r}_{ab} = -\Delta\vec{r}_{ba}$

thus, we see that: $\underset{\circlearrowleft}{\vec{F}_{ab}} = -\underset{\circlearrowleft}{\vec{F}_{ba}}$

This is Newton's 1st Law: For every action, there is equal and opposite reaction.

SI units for electric charge Q: Coulombs (C)

Fundamental unit of electric charge, $Q_e = 1.602 \times 10^{-19}$ Coulombs

Question: What is the physics that dictates (specifies/determines) the value of e??
i.e. Why is $e = 1.602 \times 10^{-19}$ Coulombs?

What is K? $K = \dfrac{1}{4\pi\varepsilon_o}$ in SI units

ε_o = electric permittivity of free space = $8.8542 \times 10^{-12} \dfrac{\text{Coulombs}^2}{\text{Newton -m}^2}$ $\left\{ = \dfrac{\text{Farads}}{\text{meter}} \right\}$

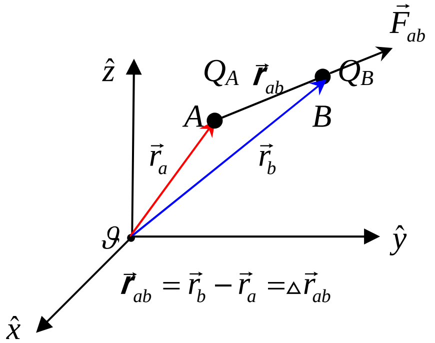
(Farad is SI unit of capacitance)

Question: If free space is truly empty, how can it have *any* measurable physical properties associated with it???

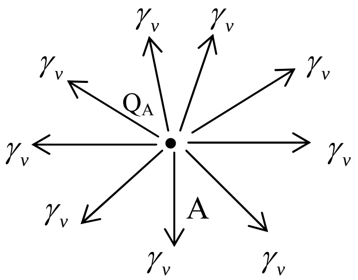
Answer: Free space is NOT empty!!! It is "filled" with virtual particle-anti-particle pairs!! (e.g. e^+-e^-, μ^+-μ^-, $q-\bar{q}$, W^+W^-, etc. pairs) existing for short time(s), as allowed by the Heisenberg Uncertainty Principle – can "violate" energy (momentum) conservation only for time interval $\Delta t \le \hbar/\Delta E$ (and over a distance of $\Delta x \le \hbar/\Delta p_x$).

ε_0 is the macroscopic, time-averaged (over many many such virtual pairs) electric permittivity of (quantum) vacuum - the physical vacuum behaves like a dielectric medium!!!

Thus: $\vec{F}_{ab} = \dfrac{1}{4\pi\varepsilon_o} \dfrac{Q_a Q_b}{\boldsymbol{r}_{ab}^2} \hat{\boldsymbol{r}}_{ab}$



Factor of 4π = "flux factor" for solid angle associated with flux of virtual photons emitted by point charge!!! Virtual photons "emitted" from Q_A are emitted into 4π steradians @ point *A*:



Force decreases as $1/\boldsymbol{r}^2$

Just like/analogous to real Photons emitted from e.g. 100 watt light bulb - Intensity decreases as $1/\mathbf{r}^2$ from light source.

Note the similarity between Coulomb’s Law and Newton’s Law of Gravity:

$$\boxed{\vec{F}_C = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{Q_aQ_b}{r^2}\hat{r}} \leftrightarrow \boxed{\vec{F}_G = G_N\frac{M_aM_b}{r^2}\hat{r}}$$

Newton’s constant, $G_N = 6.673 \text{ x } 10^{-11} m^3 kg^{-1} s^{-2}$

Or can define:

$$G_E \equiv \frac{1}{4\pi\varepsilon_0}$$

Define:

$$G_N \equiv \frac{1}{4\hbar\varepsilon_0^g} \qquad \varepsilon_o^g \equiv \frac{1}{4\pi G_N}$$

$$\boxed{\vec{F}_C = G_E\frac{Q_aQ_b}{r^2}\hat{r}} \longleftrightarrow \text{ then } \boxed{\vec{F}_G = \left(\frac{1}{4\pi\varepsilon_o^g}\right)\frac{M_aM_b}{r^2}\hat{r}}$$

Coulomb’s Constant

$\boxed{\text{Coulomb's Law}}$ $\quad \vec{F}_C = \frac{1}{4\pi\varepsilon_o}\frac{Q_aQ_b}{r^2}\hat{r}$

“nothing”

Note that if dielectric properties of free space (vacuum) were different than they are, then Coulomb’s Law, i.e. the force between electrically charged particles would be different. Consider a universe in which we could change the EM properties of the vacuum at will:

$\ell im(\varepsilon_o \to 0): \vec{F}_C \to \infty$!! “strong” electromagnetism

$\ell im(\varepsilon_o \to \infty): \vec{F}_C \to 0$!! “weak” electromagnetism

(assuming this doesn’t also affect value of fundamental electric charge, *e*)

Note further/we shall see that: c = speed of light = $1/\sqrt{\varepsilon_o\mu_o} = 3 \text{ x } 10^8\ m/\sec$

μ_o = magnetic permeability of free space = $4\pi \text{ x } 10^{-7}$ Newtons/Ampere

1 Ampere of electric <u>*current*</u> = 1 Coulomb/sec ($I = dQ/dt$)

Thus:

$$\left.\begin{array}{l} \ell im(\varepsilon_o \to 0) \Rightarrow c \to \infty \\ \ell im(\varepsilon_o \to \infty) \Rightarrow c \to 0 \end{array}\right\} \text{ If } \mu_o \text{ is unchanged}$$

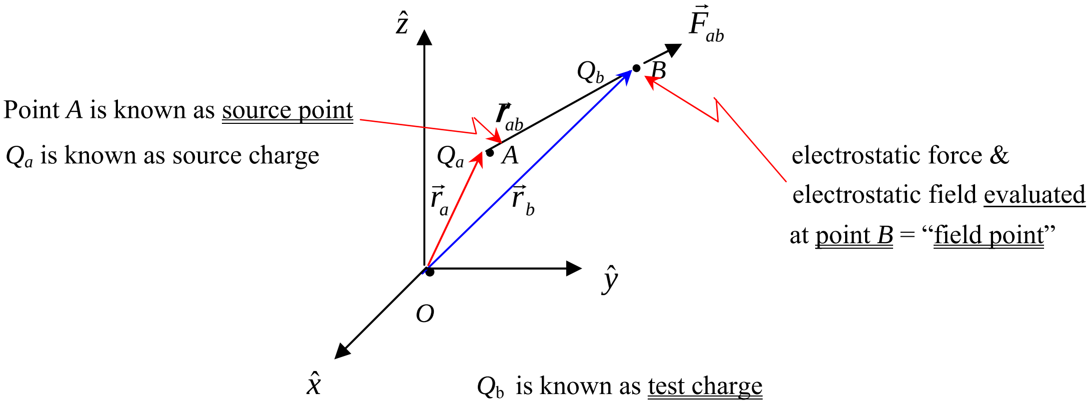
THE ELECTRIC FIELD $\vec{E}$ (Vector Quantity!!)
(Also known as the Electric Field Intensity)

We've introduced/discussed the net/time averaged force, $\vec{F}$ e.g. of Q_a acting on Q_b:

$$\boxed{\vec{F}_{ab} = \frac{1}{4\pi\varepsilon_o}\frac{Q_a Q_b}{\mathbf{r}_{ab}^2}\hat{\mathbf{r}}_{ab}}$$

We now introduce the concept of a net/time averaged electrostatic field, $\vec{E}_a$, due to Q_a, at a (separation) distance, $|\mathbf{r}_b - \mathbf{r}_a|$ from Q_a (i.e. at Q_b), which is ***defined*** in terms of the ratio of the net/time averaged force $\vec{F}_{ab}(\vec{r}_b)$ to the strength of the test charge Q_b used as a probe:

$$\boxed{\vec{E}_a(\vec{r}_b) \equiv \vec{F}_{ab}(\vec{r}_b)/Q_b} \qquad \text{or:} \qquad \boxed{\vec{F}_{ab}(\vec{r}_b) = Q_b\vec{E}_a(\vec{r}_b)}$$



Point A is known as source point

Q_a is known as source charge

electrostatic force & electrostatic field evaluated at point B = "field point"

Q_b is known as test charge

$\vec{r}_a$ points from the local origin, **O** to point A where the source charge Q_a is located.

$\vec{r}_b$ points from the local origin, **O** to point B where the test charge Q_b is located.

$\vec{r}_b$ points from the local origin, **O** to point B where the electric field (net/time averaged) due to Q_a is to be evaluated (i.e. by experimentally *measuring* $\vec{F}_{ab}$, and knowing (*apriori*) Q_a and Q_b).

$$\boxed{\vec{F}_{ab}(\vec{r}_b) = Q_b\vec{E}_a(\vec{r}_b) = \frac{1}{4\pi\varepsilon_o}\frac{Q_a Q_b}{\mathbf{r}_{ab}^2}\hat{\mathbf{r}}_{ab} = \frac{1}{4\pi\varepsilon_o}\frac{Q_a Q_b}{|\vec{r}_b - \vec{r}_a|^3}(\vec{r}_b - \vec{r}_a)}$$

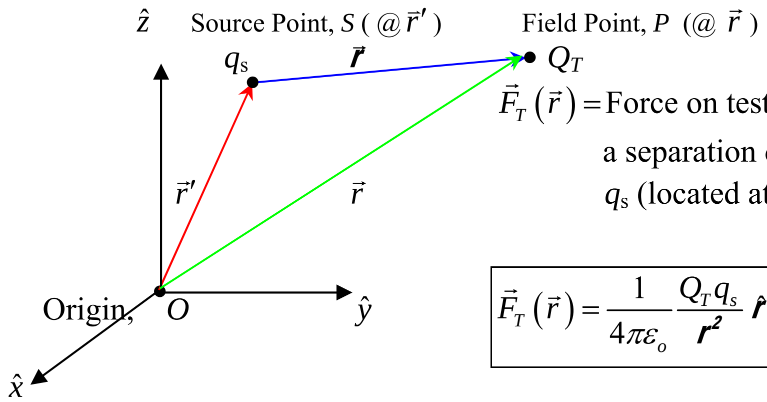
Then:

$$\boxed{\vec{E}_a(\vec{r}_b) = \frac{1}{4\pi\varepsilon_o}\frac{Q_a}{\mathbf{r}_{ab}^2}\hat{\mathbf{r}}_{ab} = \frac{1}{4\pi\varepsilon_o}\frac{Q_a}{|\vec{r}_b - \vec{r}_a|^3}(\vec{r}_b - \vec{r}_a)}$$

Very often, we will be considering situations in electrostatics where we use one charge, Q_T to TEST for the presence/existence of "source" charge(s) q_s.

We want to know e.g. the electric field due to q_s, a separation distance, $\boldsymbol{r}$ from it:

vector $\boxed{\vec{\boldsymbol{r}} \equiv (\vec{r} - \vec{r}')}$ with magnitude: $\boxed{\boldsymbol{r} = |\vec{r} - \vec{r}'|}$



$\vec{F}_T(\vec{r}) =$ Force on test charge Q_T (at field point $\vec{r}$), a separation distance $\boldsymbol{r}$ from source charge q_s (located at source point, $\vec{r}'$):

$$\boxed{\vec{F}_T(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{Q_T q_s}{\boldsymbol{r}^2}\hat{\boldsymbol{r}} = \frac{1}{4\pi\varepsilon_o}\frac{Q_T q_s}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')}$$

n.b. primed quantities (e.g. $\vec{r}'$) *always* refer to source (charge) distribution.
unprimed quantities (e.g. $\vec{r}$) refer to field/observation point.

$\vec{E}(\vec{r}) =$ Electrostatic field $(@\text{ point } \vec{r})$ due to source charge q_s a distance $\boldsymbol{r} = |\vec{r} - \vec{r}'|$ away from q_s:

$$\boxed{\vec{E}(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_s}{\boldsymbol{r}^2}\hat{\boldsymbol{r}} = \underbrace{\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_s}{\boldsymbol{r}^2}\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_s}{|\vec{r} - \vec{r}'|^2}\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_s}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')}_{\text{cumbersome notation, but very explicit!!!}}}$$

$\boxed{\vec{F}_T(\vec{r}) = Q_T\vec{E}(\vec{r})}$ $\boxed{\text{Obviously, SI Units of } \vec{E}(\vec{r}) \text{ are } Newtons/C \text{ (also } \equiv \text{ volts}/m)}$

Units of E = force per unit charge (N/C)

from dimensional analysis

A Detail:

A more rigorous definition of electric field intensity, $\vec{E}(\vec{r})$ is given by: $\vec{E}(\vec{r}) \equiv \lim_{Q_T \to 0}\left(\frac{\vec{F}(\vec{r})}{Q_T}\right)$

We really *do* need this limiting process – *experimentally/in real life*, the presence of a finite-singed test charge Q_T necessarily perturbs the source charge distribution that one is attempting to measure!! This is especially true for spatially-extended source charge distributions. As the test charge is made smaller and smaller, the perturbing effect on the original/unperturbed source charge distribution is made smaller and smaller. In the limit $Q_T \to 0$, the *true* source charge distribution is obtained.

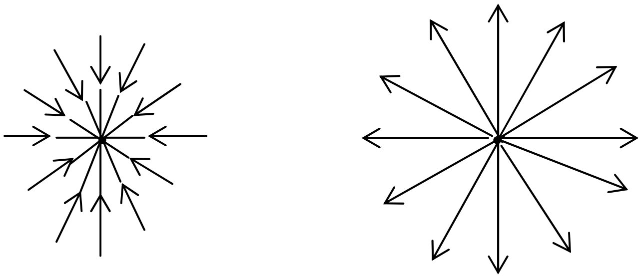
THIS IS VERY IMPORTANT TO KEEP THIS IN MIND!!! IT IS NOT A TRIVIAL POINT!!!

Usually, we might think of e.g. $Q_T = 1\ e$ and e.g. $q_s = 10^{19}\ e$, thus $q_s >> Q_T$, and thus perturbing effects are negligible (in this case).

We have shown that: $\vec{E}(\vec{r}) \equiv \frac{\vec{F}(r)}{Q_T} = \frac{1}{4\pi\varepsilon_o}\frac{q_s}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}$ and thus: $\vec{F}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q_s Q_T}{\boldsymbol{r}^2}\hat{\boldsymbol{r}} = Q_T\vec{E}(\vec{r})$

$$\left.\begin{array}{l}\text{If } \vec{F}(\vec{r}) \text{ is a radial force}\\ \text{then } \vec{E}(\vec{r}) \text{ is also radial}\end{array}\right\} \text{for point source charge, } q_s$$

Convention: direction of electric field lines for $q_s = +e$ and $q_s = -e$

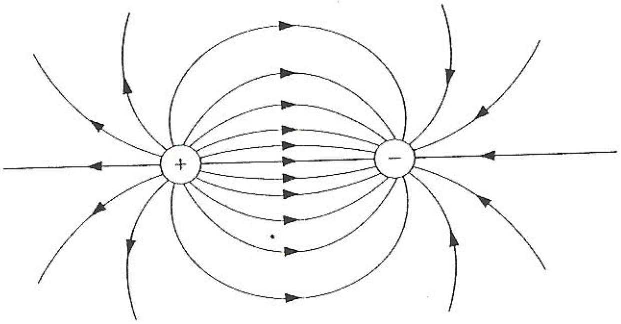


$q_s = -e$
inward

$q_s = +e$
outward

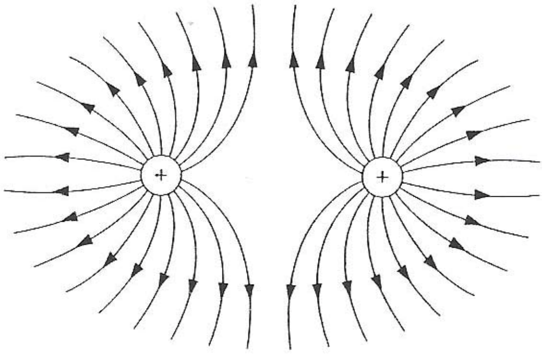
ELECTRIC FIELD LINES

Associated with Two Point Charges



Equal but opposite charges

Figure 2.13



Equal charges

Figure 2.14

THE PRINCIPLE of LINEAR SUPERPOSITION

-VERY IMPORTANT-

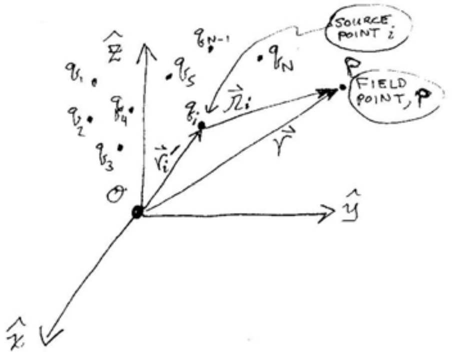
Assuming we are always in $\underset{Q_T \to 0}{\ell im}\left(\frac{\vec{F}(\vec{r})}{Q_T}\right)$(i.e. $Q_T << q_s$) regime, then suppose we have N discrete point source charges: $q_1,\ q_2,\ q_3,\ q_4 \ldots\ q_N$

What is the (total or net) force, $\vec{F}_{ToT}(\vec{r})$ due to all of the N source charges?

Vectorially, we know that $\vec{F}_{ToT}(\vec{r}) = \vec{F}_1(\vec{r}) + \vec{F}_2(\vec{r}) + \vec{F}_3(\vec{r}) + \ldots\ \vec{F}_N(\vec{r}) = \sum_{i=1}^{N} \vec{F}_i(\vec{r})$. More explicitly:

$$\boxed{\begin{aligned}\vec{F}_{TOT}(\vec{r}) &= \vec{F}_1(\vec{r}) + \vec{F}_2(\vec{r}) + \vec{F}_3(\vec{r}) + \ldots \vec{E}_N(\vec{r}) = \sum_{i=1}^{N} \vec{F}_i(\vec{r}) \\ &= \left\{\frac{Q_T}{4\pi\varepsilon_o}\right\}\left\{\frac{q_1}{\mathbf{r}_1^{\ 2}}\hat{\boldsymbol{r}}_1 + \frac{q_2}{\mathbf{r}_2^{\ 2}}\hat{\boldsymbol{r}}_2 + \frac{q_3}{\mathbf{r}_3^{\ 2}}\hat{\boldsymbol{r}}_3 + \ldots \frac{q_N}{\mathbf{r}_N^{\ 2}}\hat{\boldsymbol{r}}_N\right\} = \frac{Q_T}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathbf{r}_i^{\ 2}}\hat{\boldsymbol{r}}_i\end{aligned}}$$

where: $\boldsymbol{\mathrm{r}} \equiv (\vec{r} - \vec{r}_i) = \vartriangle\vec{r}_i$



What is (total or net) electric field intensity, $\vec{E}_{TOT}(\vec{r})$ due to all of the N source charges?

We know that: $\vec{F}_{TOT}(\vec{r}) = Q_T\vec{E}_{TOT}(\vec{r})$ or: $\vec{E}_{TOT}(\vec{r}) \equiv \vec{F}_{TOT}(\vec{r})/Q_T$

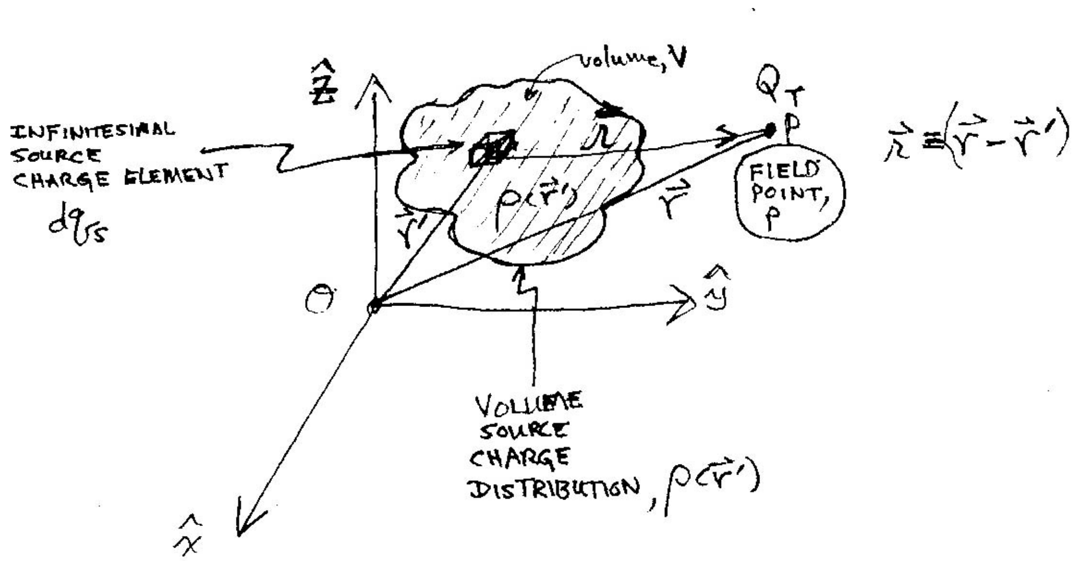
$$\therefore \boxed{\begin{aligned}\vec{E}_{TOT}(\vec{r}) &= \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \vec{E}_3(\vec{r}) + \ldots \vec{E}_N(\vec{r}) = \sum_{i=1}^{N} \vec{E}_i(\vec{r}) \\ &= \left\{\frac{1}{4\pi\varepsilon_o}\right\}\left\{\frac{q_1}{\mathbf{r}_1^{\ 2}}\hat{\boldsymbol{r}}_1 + \frac{q_2}{\mathbf{r}_2^{\ 2}}\hat{\boldsymbol{r}}_2 + \frac{q_3}{\mathbf{r}_3^{\ 2}}\hat{\boldsymbol{r}}_3 + \ldots \frac{q_N}{\mathbf{r}_N^{\ 2}}\hat{\boldsymbol{r}}_N\right\} = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathbf{r}_i^{\ 2}}\hat{\boldsymbol{r}}_i\end{aligned}}$$

We can extend the use of the principle of linear superposition to mathematically describe the net/total force + net/total electric field intensity at the field point, $\vec{r}$ for <u>arbitrary</u> <u>continuous</u> charge distributions:

$$\vec{F}_{TOT}(\vec{r}) = \frac{Q_T}{4\pi\varepsilon_o}\int\left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)dq_s \quad \text{and} \quad \vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int\left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)dq_s$$

$$\text{where:} \quad \vec{\boldsymbol{r}} = (\vec{r} - \vec{r}'), \quad \boldsymbol{r} = |\vec{r} - \vec{r}'|, \quad \hat{\boldsymbol{r}} = \vec{\boldsymbol{r}}/\boldsymbol{r}$$

Then for volume, surface & line charge source distributions:



A.) VOLUME CHARGE DISTRIBUTIONS: Volume Charge Density, $\rho(\vec{r}')$:
(e.g. inside cylinders, spheres, boxes, etc.)

$$\boxed{dq_s = \rho(\vec{r}')\,d\tau' : \quad \vec{F}_{TOT}(\vec{r}) = \frac{Q_T}{4\pi\varepsilon_o}\int_v \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\rho(\vec{r}')\,d\tau'}$$

Coulombs/m^3

$$\boxed{\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_v \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\rho(\vec{r}')\,d\tau'}$$

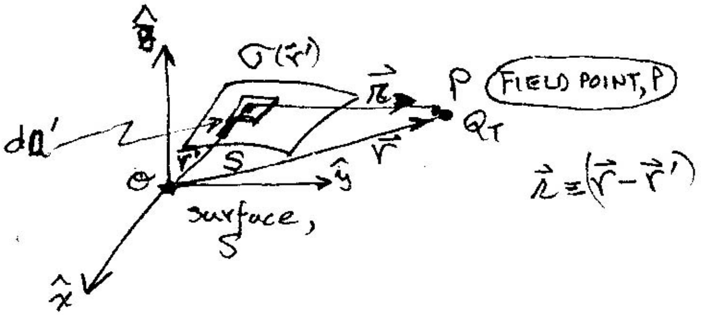
B.) SURFACE CHARGE DISTRIBUTIONS: Surface Charge Density, $\sigma(\vec{r}')$:
(e.q. on surfaces of cylinders, spheres, boxes, etc.)

$$\boxed{dq_s = \sigma(\vec{r}')\,da' : \quad \vec{F}_{TOT}(\vec{r}) = \frac{Q_T}{4\pi\varepsilon_o}\int_S \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\sigma(\vec{r}')\,da'}$$

Coulombs/m^2

$$\boxed{\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_S \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\sigma(\vec{r}')\,da'}$$

where: $\vec{\boldsymbol{r}} = (\vec{r} - \vec{r}')$, $\boldsymbol{r} = |\vec{r} - \vec{r}'|$, $\hat{\boldsymbol{r}} = \vec{\boldsymbol{r}}/\boldsymbol{r}$



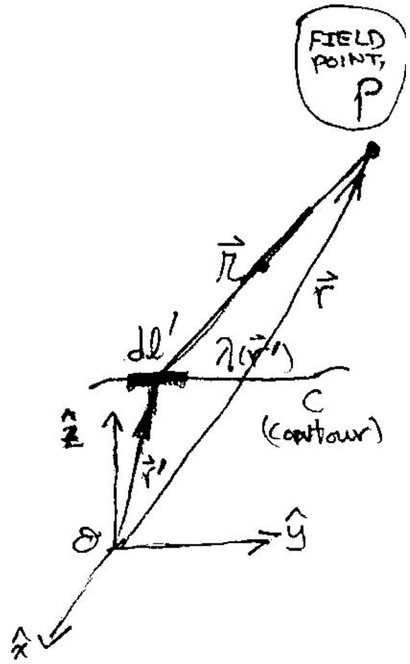
C). LINE CHARGE DISTRIBUTIONS: Linear Charge Density, $\lambda(\vec{r}')$:
(e.q. wire)

$$\boxed{dq_s = \lambda(\vec{r}')\,d\ell' : \quad \vec{F}_{TOT}(\vec{r}) = \frac{Q_T}{4\pi\varepsilon_o}\int_C \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\lambda(\vec{r}')\,d\ell'}$$

Coulombs/m

$$\boxed{\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_C \left(\frac{1}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)\lambda(\vec{r}')\,d\ell'}$$

where: $\boldsymbol{r} = (\vec{r} - \vec{r}')$, $\boldsymbol{r} = |\vec{r} - \vec{r}'|$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/\boldsymbol{r}$



Thus, a complete description of all possible charge distributions, consisting of discrete and continuous charge distributions:

$$\boxed{\vec{F}_{TOT}(\vec{r}) = \frac{Q_T}{4\pi\varepsilon_o}\left\{\sum_{i=1}^{N}\frac{q_i}{\boldsymbol{r}_i^2}\hat{\boldsymbol{r}}_i + \int_V\left(\frac{\rho(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)d\tau' + \int_S\left(\frac{\sigma(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)da' + \int_C\left(\frac{\lambda(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)d\ell'\right\}}$$

$$\boxed{\vec{E}_{TOT}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left\{\sum_{i=1}^{N}\frac{q_i}{\boldsymbol{r}_i^2}\hat{\boldsymbol{r}}_i + \int_V\left(\frac{\rho(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)d\tau' + \int_S\left(\frac{\sigma(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)da' + \int_C\left(\frac{\lambda(\vec{r}')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\right)d\ell'\right\}}$$

Please Note:
For all integrals (above), when integrals over $d\tau'$, da', and/or $d\ell'$ are carried out, $\vec{F}_{TOT}(\vec{r})$ and thus $\vec{E}_{TOT}(\vec{r})$ have NO $\vec{r}'$ (*i.e.* source-position) dependence - it has been integrated over/integrated out!!!

$\vec{F}_{TOT}(\vec{r})$ and $\vec{E}_{TOT}(\vec{r}) \equiv \vec{F}_{TOT}(\vec{r})/Q_T$ are functions of the field point variable $\vec{r}$ ONLY
i.e. they are *not* functions of $\vec{r}'$ (source point{s}) !!!

PLEASE work/grind through example 2.1 Griffiths p. 62-63) on your own to better learn/understand this!

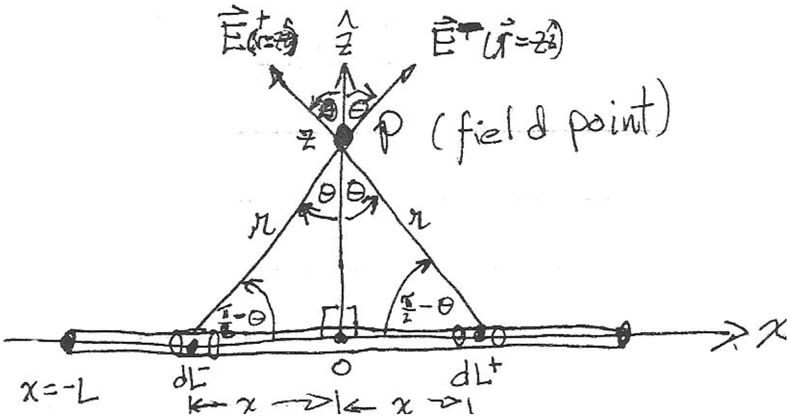
ACTIVE "LEARNING BY DOING"

EXAMPLE 2.1 p. 62 Griffiths

- very explicit detailed derivation-

Find the electric field intensity, $E(r)$ a distance z above mid-point of a straight line segment of length $2L$, which carries a uniform line charge λ (Coulombs/meter) (n.b. $Q_{TOT} = 2\lambda L$)

Here, $\vec{E}(\vec{r} = z\hat{z}) = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int_C \dfrac{\lambda(r')}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}\,d\ell'$



Notice the *symmetry* of this problem – contribution to net $\vec{E}(r)$ @ field point, $\mathcal{P}$ from infinitesimal line charges $d\lambda$ associated with infinitesimal line segments, dL located at $\pm x$ such that $\hat{x}$ components of net electric field @ field point, $\mathcal{P}$ cancel each other:

$$\boxed{\begin{aligned}\cos\theta &= \frac{z}{\boldsymbol{r}} = \frac{z}{\sqrt{x^2+z^2}}\\ \sin\theta &= \frac{x}{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+z^2}}\end{aligned}}$$

$$\begin{aligned}d\vec{E}_{NET}(\vec{r}=z\hat{z}) &= d\vec{E}^+(\vec{r}=z\hat{z}) + d\vec{E}^-(\vec{r}=z\hat{z})\\ &= \left\{\left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{\lambda dL}{\boldsymbol{r}^2}\right)\left[\left(-\sin\theta\hat{x}\right)+\left(\cos\theta\hat{z}\right)\right]\right\} \leftarrow\ d\vec{E}^+\left(\underset{\sim}{r}=z\hat{z}\right)\\ &+\left\{\left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{\lambda dL}{\boldsymbol{r}^2}\right)\left[\left(+\sin\theta\hat{x}\right)+\left(\cos\theta\hat{z}\right)\right]\right\} \leftarrow\ d\vec{E}^-\left(\vec{r}=z\hat{z}\right)\\ &=2\left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{\lambda dL}{\boldsymbol{r}^2}\right)\cos\theta\hat{z}\end{aligned}$$

Now only need to integrate this expression over x from $0 \le x \le L$:

$$\vec{E}_{NET}\left(\vec{r}=z\hat{z}\right)=\int_0^L d\vec{E}\left(\vec{r}=z\hat{z}\right)=\left\{\int_0^L d\vec{E}^{+}\left(\vec{r}=z\hat{z}\right)+\int_0^L d\vec{E}^{-}\left(\vec{r}=z\hat{z}\right)\right\}=\int_0^L 2\left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{\lambda dL}{\boldsymbol{r}^2}\right)\cos\theta\,\hat{z}$$

$$=2\left(\frac{1}{4\pi\varepsilon_o}\right)\lambda\int_0^L\left[\frac{1}{\left(x^2+z^2\right)}\right]\left[\frac{z}{\sqrt{x^2+z^2}}\right]dx\,\hat{z}$$

$$=\left(\frac{2\lambda z}{4\pi\varepsilon_o}\right)\int_0^L\frac{1}{\left(x^2+z^2\right)^{3/2}}\,dx\,\hat{z}$$

$$=\left(\frac{2\lambda z}{4\pi\varepsilon_o}\right)\left[\frac{x}{z^2\sqrt{x^2+z^2}}\right]\Bigg|_0^L\hat{z}=\left(\frac{2\lambda}{4\pi\varepsilon_o}\right)\left[\frac{\cancel{z}}{z^{\cancel{2}}}\frac{L}{\sqrt{L^2+z^2}}\right]\hat{z}=\left(\frac{2\lambda L}{4\pi\varepsilon_o}\right)\left(\frac{1}{z\sqrt{z^2+L^2}}\right)\hat{z}$$

If $z >> L$; then (Taylor Series Expansion) $\sqrt{1+\varepsilon}\approx\left(1+\frac{\varepsilon}{2}\right)\approx 1$ for $\varepsilon=\left(\frac{L}{z}\right)^2 \ll 1$

$$\vec{E}_{NET}\left(\vec{r}=z\hat{z}\right)\approx\frac{2\lambda L}{4\pi\varepsilon_o z^2}\hat{z}=\frac{Q_{TOT}}{4\pi\varepsilon_o z^2}\hat{z}\quad\leftarrow\text{ same } E\text{-field as that due to a point charge, } q!$$

If $L \to \infty$ (i.e. infinite straight wire): use the same Taylor series expansion, but for $L \gg z$:

i.e. $\sqrt{1+\varepsilon}\approx\left(1+\frac{\varepsilon}{2}\right)\approx 1$ for $\varepsilon=\left(\frac{z}{L}\right)^2 \ll 1$

Then: $\vec{E}_{NET}\left(\vec{r}=z\hat{z}\right)\approx\frac{1}{4\pi\varepsilon_o}\frac{2\lambda}{z}\hat{z}=\frac{\lambda}{2\pi\varepsilon_o z}\hat{z}$

The E-field is actually in the radial $\left(\hat{\rho}\right)$ direction for an infinite straight wire – in cylindrical coordinates:

$$\vec{E}_{NET}\left(\vec{r}\right)\approx\frac{1}{4\pi\varepsilon_o}\frac{2\lambda}{\rho}\hat{\rho}=\frac{\lambda}{2\pi\varepsilon_o\rho}\hat{\rho}$$